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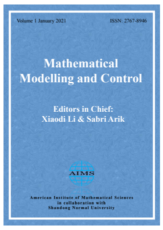

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Research article

From DK-STP to a set of Lie bracket

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Abstract: In this paper, semi-tensor product (STP) and related properties of dimension keeping semi-tensor product (DK-STP) are analyzed. The commutativity and anticommutativity of DK-STP are studied by means of matrix mapping, and sufficient conditions for both are obtained. The structure matrix of the Lie bracket of non-square matrices (NSM) is discussed, and some properties are derived. The correspondences between the special Lie subalgebras of square matrix and Lie subalgebras of NSM are discussed through a homomorphism.

Keywords: semi-tensor product; dimension keeping semi-tensor product; (anti) commutativity; Lie bracket; structure matrix

1. Introduction

The semi-tensor product (STP) of matrices is a generalization of conventional matrix product. It has become a necessary tool in the study of finite value systems, such as Boolean networks [1–5], finite games [6] and control systems [7, 8]. In addition, it is also a powerful tool to deal with multilinear mappings. Constrained least square solutions to Sylvester equations have been obtained via STP method in [9]. STP was used to investigate finite algebra extensions of $\mathbb{R}$ [10, 11] and general Boolean-type algebras [12]. In [13, 14], commutativity of algebras are discussed. For a finite dimensional algebra, commutativity is a problem that we have been exploring. As a special kind of algebra, Lie algebra is also often discussed, such as commutativity [15].

In the study of STP, defining the Lie bracket of nonsquare matrices (NSM) poses significant challenges due to the dimensional changes of matrices resulting from STP operations, which prevents the satisfaction of algebraic closure. Previously, we were limited to defining the Lie bracket for square matrices to meet the closure requirement of the algebra. To address this issue, we introduce a new matrix multiplication method based on STP, termed dimension-keeping semi-tensor product (DK-STP) [16, 17]. It is demonstrated that this operation maintains dimensionality and satisfies the closure of the algebra. The Lie bracket for NSM can be defined, allowing for the establishment of a structure matrix. This research contributes to the understanding of the Lie bracket of NSM and enhances the theoretical framework in this area. For the new DK-STP, a significant feature is that it can make two $n \times m$ matrices after the operation, and the obtained is still a $n \times m$ matrix. This leads us to consider how it is similar to normal matrix product properties. To begin, there is obviously no way to satisfy commutativity in the broad sense. Therefore, we consider if we can add some conditions that make commutativity true. We then refer to the matrix mapping, starting from the bridge matrix to obtain a sufficient condition, under which the commutativity is true. Then, we can give a counterexample to show that necessity is not true.

The treatment of Lie brackets was historically confined to operations involving square matrices. However, with the integration of the innovative DK-STP framework, we have now expanded the realm of applicability to encompass NSM, enabling the execution of Lie bracket operations that transcend conventional dimensional constraints. By arranging the resulting Lie bracket in a matrix format with a column-wise organization, we obtain a structure matrix that encapsulates the essential algebraic features of this operation. Consequently, we embark on an analysis of the algebraic properties inherent in this structural matrix, thereby contributing to the mathematical discourse on Lie algebra and its extensions beyond traditional matrix domains.

The rest of this paper is organized as follows: Section 2 reviews some necessary preliminaries, including (i) STP and DK-STP of matrices; (ii) Structure matrix of a finite dimensional algebra; (iii) Structure matrix of Lie bracket and matrix mapping. Section 3 introduces sufficient conditions about (anti) commutativity of DK-STP. Section 4 discusses some properties of dimension keeping Lie bracket by its structure matrix and special Lie subalgebras under the dimension keeping Lie bracket. Section 5 introduces homomorphisms of Lie algebras for NSM and square matrices. Section 6 is a brief conclusion with suggestions of some problems for further study.

Before ending this section, a list of notations is presented as follows:

- $\mathbb{R}^n$: n dimensional Euclidean space.
- $\mathcal{M}_{m\times n}$: the set of $m \times n$ real matrices.
- I_n: the identity matrix.
- δ_n^i: the i-th column of I_n.
- $[\delta_n^{i_1}, \delta_n^{i_2}, \cdots, \delta_n^{i_n}]$ is briefly denoted by $\delta_n[i_1, i_2, \cdots, i_n]$.
- $1_n := (\underbrace{1, 1, \cdots, 1}_{n})^T$.

2. Preliminaries

The section will be divided into four parts. The first part introduces the basic definitions and some basic properties of STP and DK-STP. The second part introduces the structure matrix of finite dimensional algebras. The third part presents the construction process of the structure matrix of square matrix with Lie bracket. In the fourth part, we give the concept of matrix mapping, which will help us to analyze the construction process of Lie bracket in NSM.

2.1. STP and DK-STP

On the basis of classical matrix multiplication, we use the Kronecker product to define STP. This allows us to obtain its definition. The next definition will introduce the concept of STP in [10,18]. One advantage of STP is that commutativity can be overcome to a certain extent by defining a swap matrix.

Definition 2.1. *Let $A \in \mathcal{M}_{n\times m}$ and $B \in \mathcal{M}_{p\times q}$, and $t = lcm(n, p)$ be the least common multiple of n and p. The STP of A and B, denoted by $A \ltimes B$, is defined as*

$$A \ltimes B := (A \otimes I_{t/n})(B \otimes I_{t/p}),$$

where $\otimes$ is the Kronecker product.

Definition 2.2. *Define a swap matrix $W_{[m,n]} \in \mathcal{M}_{mn\times mn}$ as follows $W_{[n,p]} := \delta_{np}[1, n+1, \cdots, (p-1)n+1, 2, n+2, \cdots, (p-1)n+2, \cdots, n, 2n, \cdots, pn]$.*

It can also be expressed as

$$W_{[n,p]} = [I_p \otimes \delta_n^1, I_p \otimes \delta_n^2, \cdots, I_p \otimes \delta_n^n].$$

Record briefly as $W_{[n]} = W_{[n,n]}$. The following lemma shows how to swap a vector with a vector.

Lemma 2.1. *[10] Let $x \in \mathbb{R}^n, y \in \mathbb{R}^p$. The swap matrix can be used to achieve the commutativity*

$$W_{[n,p]} x \ltimes y = y \ltimes x.$$

In [16], a new STP is proposed, called DK-STP and denoted by $\bar{\times}$. Dimension keeping means if both matrices are of the same dimension, then their product remains to be of the same dimension. Due to this special property, we can expand the matrix operation further. The definition of DK-STP is expressed as follows:

Definition 2.3. *The DK-STP of A and B is $A \bar{\times} B$, and it is expressed as follows*

$$A \bar{\times} B := (A \otimes 1_{t/n}^T)(B \otimes 1_{t/p}),$$

where $A \in \mathcal{M}_{m\times n}, B \in \mathcal{M}_{p\times q}, t = lcm(n, p)$.

We rewrite the definition of DK-STP and introduce a bridge matrix. This allows for easier calculation of DK-STP and can also be used to prove some algebraic properties. The proposal of the bridge matrix will play an important role in our later proof of (anti) commutativity. The results in this section build upon the work of [16].

Definition 2.4. *Let $A \in \mathcal{M}_{m\times n}$, $B \in \mathcal{M}_{p\times q}, t = lcm(n, p)$. Then*

$$A \barwedge B := A\psi_{n\times p}B,$$

where

$$\psi_{n\times p} = (I_n \otimes 1^T_{t/n})(I_p \otimes 1_{t/p}) \in \mathcal{M}_{n\times p},$$

is called a bridge matrix of dimension $n \times p$.

Remark 2.1. *Regarding bridge matrices, when multiplying matrices of different dimensions, the bridge matrix depends solely on the dimensions. For operations involving matrices of the same dimension, the bridge matrix remains the same. Next, we provide the following supplementary information regarding the properties of bridge matrices.*

- $\psi^T_{n\times p} = \psi_{p\times n}$.
- $||\psi_{p\times n}||^2_F = ||\psi_{n\times p}||^2_F$.

The norm referred to here is the Frobenius norm. The above two properties can be easily derived from their definitions, which aids in our further understanding of bridge matrices. Additionally, all elements within the bridge matrix are positive values.

From Definition 2.4, Proposition 2.1 can be derived.

Proposition 2.1. *Let $A, B, C \in \mathcal{M}_{m\times n}$, we can calculate the following proposition by the bridge matrix.*

(i) *Distributivity*

$$A \barwedge (B + C) = A \barwedge B + A \barwedge C,$$

$$(B + C) \barwedge A = B \barwedge A + C \barwedge A.$$

(ii) *Associativity*

$$(A \barwedge B) \barwedge C = A \barwedge (B \barwedge C).$$

In the previous research, we usually use STP operation to perform matrix operation, which is a special kind of matrix operation. It does not need to satisfy the dimensional compatibility of the matrix, which provides convenience for our calculation. In this article, we introduce the DK-STP operation. DK-STP is a specialized variant of the standard STP operation. While STP employs the identity matrix to increase dimensionality, DK-STP utilizes a row vector for this purpose. This distinction leads to differences in how dimensions are altered during the operations. In STP, calculations can be performed across different dimensions. Consequently, when two matrices of the same dimension are operated upon, the resulting dimension cannot be preserved. In DK-STP, the dimensionality remains unchanged after operations involving matrices of the same dimension, which facilitates the subsequent satisfaction of algebraic closure.

2.2. Finite dimensional algebra and Lie algebra

Next, we will define finite dimensional algebras using the structure matrix of a finite dimensional algebra to analyze their properties. This approach relies on STP and is detailed in reference [19].

Definition 2.5. *A n-dimensional algebra $\mathcal{A} = (V, *)$ is defined in n-dimensional vector space V, and it is defined a multiplication: $V * V = V$, which satisfies the distributive law for vectors, i.e.,$x, y, x \in V$, $a, b \in \mathbb{R}$*

$$(ax + by) * z = a(x * z) + b(y * z),$$

$$z * (ax + by) = a(z * x) + b * (z * y).$$

We now consider the structure matrix of a finite dimensional algebra, which we find very useful for analyzing its algebraic properties. Here, we introduce the n-dimensional basis and use it to construct the structure matrix of the finite dimensional algebra.

The next definition reads as follows:

Definition 2.6. *[10, 19, 20] Let $\{e_1, \cdots, e_n\}$ be a fixed basis for V and assume*

$$e_i * e_j = \sum_{k=1}^{n} \alpha^k_{ij} e_k, \quad i, j = 1, \cdots, n.$$

Then, $\left\{\alpha^k_{ij}\right\}$ is called structure constant of $\mathcal{A}$.

$$M_V = \begin{bmatrix} \alpha^1_{11} & \cdots & \alpha^1_{1n} & \cdots & \alpha^1_{n1} & \cdots & \alpha^1_{nn} \\ \alpha^2_{11} & \cdots & \alpha^2_{1n} & \cdots & \alpha^2_{n1} & \cdots & \alpha^2_{nn} \\ \vdots & & \vdots & & \vdots & & \vdots \\ \alpha^n_{11} & \cdots & \alpha^n_{1n} & \cdots & \alpha^n_{n1} & \cdots & \alpha^n_{nn} \end{bmatrix}$$

is called structure matrix of $\mathcal{A}$.

In this discussion, we are considering n-dimensional algebras. If we consider their column arrangements for

NSM, which allow us to transform matrix problems into vector problems, we can then use finite dimensional algebras to analyze further. Therefore, the column arrangement of a matrix is defined as

$$V_c(A) = (a_{11}, \cdots, a_{m1}, \cdots, a_{1n}, \cdots, a_{mn}).$$

Let $x = \sum_{i=1}^{n} x_i e_i$, $y = \sum_{i=1}^{n} y_i e_i$. We simplified them to $x = (x_1 \pounds \cdots \pounds x_n)^T$, $y = (y_1 \pounds \cdots \pounds y_n)^T$ in a fixed basis $\{e_1, \cdots, e_n\}$.

Proposition 2.2. *[19] Let $z = x * y$. Then, we can get z in the form of the coefficient vector*

$$z = M_V \ltimes x \ltimes y = M_V xy.$$

As we mentioned earlier that STP is a generalization of classical matrix multiplication, we can omit the multiplication symbol $\ltimes$. We will omit all $\ltimes$ in the discussion that follows. When two matrices do not satisfy the condition of dimension matching, it must be the $\ltimes$.

The some properties of an algebra are determined by its structure matrix, so the study of algebraic properties can correspond to the study of its structure matrix. Some basic property definitions are given below.

Definition 2.7. *[10] $\mathcal{A} = (V, *)$ is a finite dimensional algebra*

(i) *$\mathcal{A}$ is commutative if*

$$x * y = y * x, \quad \forall x, y \in V.$$

(ii) *$\mathcal{A}$ is anti-commutative if*

$$x * y = -y * x, \quad \forall x, y \in V.$$

(iii) *$\mathcal{A}$ is associative if*

$$(x * y) * z = x * (y * z), \quad \forall x, y, z \in V.$$

After we have some basic algebraic properties, we can use the structure matrix to analyze the related properties. The relevant properties of an algebra can be revealed through the equations which are satisfied by structure matrices. The conclusion reads as follows:

Lemma 2.2. *[10] $\mathcal{A} = (V, *)$ is a n-dimensional algebra.*

(i) *V is symmetric if and only if*

$$M_V(W_{[n]} - I_{n^2}) = 0.$$

(ii) *V is skew-symmetric if and only if*

$$M_V(W_{[n]} + I_{n^2}) = 0.$$

(iii) *V is associative if and only if*

$$M_V(M_V \otimes I_n - I_n \otimes M_V) = 0.$$

After analyzing the algebraic properties of the foundation, The definitions of the Lie algebra and Lie bracket are given as follows.

Definition 2.8. *[19] $\mathcal{A}$ is Lie algebra, if satisfied*

(i) *Skew-symmetric*

$$X * Y = -Y * X, \quad X, Y \in V.$$

(ii) *Jacobi equation*

$$(X * Y) * Z + (Y * Z) * X + (Z * X) * Y = 0, \quad X, Y, Z \in V.$$

We first give the definition of Lie bracket in $\mathcal{M}_{n\times n}$ as follows:

$$[A, B] = AB - BA, \quad A, B \in \mathcal{M}_{n\times n}. \tag{1}$$

The lie bracket over $\mathcal{M}_{n\times n}$ is a type of Lie algebra known as a general linear algebra $gl(n, \mathbb{R})$.

Next, we will use DK-STP to define the Lie brackets over $\mathcal{M}_{m\times n}$, which is an extension of the square case.

Definition 2.9. *[16] Consider $\mathcal{M}_{m\times n}$. Using $\bar{\times}$, a Lie bracket over $\mathcal{M}_{m\times n}$ is defined as*

$$[A, B]_{\bar{\times}} = A \,\bar{\times}\, B - B \,\bar{\times}\, A, \quad A, B \in \mathcal{M}_{m\times n}, \tag{2}$$

where $[A, B]_{\bar{\times}}$ is called dimension keeping Lie bracket.

Remark 2.2. *The above definition of the Lie bracket is an extension of the square case, if $m = n$, then we get the familiar square case. The Lie bracket operation on $\mathcal{M}_{m\times n}$ is a Lie algebra, and we call it $gl(m \times n, \mathbb{R})$. This definition is similar to the definition on a square matrix.*

2.3. Structure matrix of Lie bracket

Next, we consider the structure matrix of Lie algebra. Since Lie algebras must satisfy the skew-symmetric and Jacobi equation, we naturally obtain some equations for the structure matrix. However, it should be noted that the operation here is not on finite dimensional vectors. Therefore, by defining a suitable matrix base and considering the column arrangements of matrices, we can convert them into finite dimensional vector problems. This allows us to obtain some conclusions regarding related properties. To construct the structure matrix of $gl(n,\mathbb{R})$, we use the following lemma to establish the relationship between column arrangements.

Lemma 2.3. *[19] Let $A \in \mathcal{M}_{m\times n}, B \in \mathcal{M}_{n\times p}$. Then,*

$$V_c(AB) = \Psi_{mnp} V_c(A) V_c(B),$$

where

$$\Psi_{mnp} = \begin{bmatrix} I_m \otimes (\delta_p^1 \delta_n^1)^T & \cdots & I_m \otimes (\delta_p^1 \delta_n^n)^T \\ I_m \otimes (\delta_p^2 \delta_n^1)^T & \cdots & I_m \otimes (\delta_p^2 \delta_n^n)^T \\ \vdots & \ddots & \vdots \\ I_m \otimes (\delta_p^p \delta_n^1)^T & \cdots & I_m \otimes (\delta_p^p \delta_n^n)^T \end{bmatrix}.$$

For convenience, Ψ_{nnn} is expressed as Ψ_n.

Consider the structure matrix of $gl(n,\mathbb{R})$. $\{M_{IJ} | I = 1, 2, \cdots, n; J = 1, 2, \cdots, n\}$ represents a set of bases where

$$(M_{IJ})_{ij} = \begin{cases} 1, & i = I \text{ or } j = J. \\ 0, & others. \end{cases}$$

Now, construct the structure matrix of Lie bracket.

$$[A, B] = AB - BA.$$

Next, we calculate matrix column arrangement of AB and BA,

$$V_c(AB) = \Psi_n V_c(A) V_c(B),$$

$$V_c(BA) = \Psi_n V_c(B) V_c(A) = \Psi_n W_{[n^2]} V_c(A) V_c(B).$$

Then, the structure matrix of $gl(n,\mathbb{R})$ is represented as

$$M_{gl(n,\mathbb{R})} = \Psi_n (I_{n^4} - W_{[n^2]}).$$

Lemma 2.4. *[19] The structure matrix of $gl(n,\mathbb{R})$ should meet the following equations:*

(i) Skew-symmetric

$$M_{gl(n,\mathbb{R})}(I_{n^2} + W_{[n,n]}) = 0.$$

(ii) Jacobi equation

$$M^2_{gl(n,\mathbb{R})}(I_{n^3} + W_{[n,n^2]} + W_{[n^2,n]}) = 0.$$

2.4. Matrix mapping

Here we write the matrix mapping in column arrangement, which gives us a way to transform a matrix problem into a vector problem.

Theorem 2.1. *[19] Let $A \in \mathcal{M}_{m\times n}, B \in \mathcal{M}_{p\times q}, C \in \mathcal{M}_{m\times p}, D \in \mathcal{M}_{n\times q}, Z \in \mathcal{M}_{n\times p}$. Π is a linear mapping: $Z \to \Pi(Z)$, M^c_Π is called the structure matrix of matrix column arrangement in Π, that is,*

$$V^c(\Pi(Z)) = M^c_\Pi V^c(Z).$$

(i) Let $\Pi : Z \to AZ$. Then,

$$M^c_\Pi = I_p \otimes A.$$

(ii) Let $\Pi : Z \to ZB$. Then,

$$M^c_\Pi = B^T \otimes I_n.$$

(iii) Let $\Pi : Z \to AZB$. Then,

$$M^c_\Pi = B^T \otimes A.$$

3. (Anti) Commutativity of DK-STP

Starting from classical matrix theory, we have learned that commutativity in classical matrix operations can not be satisfied, and only in special cases can commutativity be established. STP is proposed through the swap matrix can achieve a class of STP operations commutativity, so DK-STP also has a similar commutative property. Due to the special dimension relationship of DK-STP. If $A \in \mathcal{M}_{n\times m}$, $B \in \mathcal{M}_{p\times q}$, then to achieve commutativity there must be

$$A \,\bar{\times}\, B = B \,\bar{\times}\, A.$$

$A \,\bar{\times}\, B \in \mathcal{M}_{n\times q}$, $B \,\bar{\times}\, A \in \mathcal{M}_{p\times m}$, that must be satisfied $n = p, q = m$. There could be commutativity, so we consider a case of $A, B \in \mathcal{M}_{n\times m}$. Next, we will use matrix mapping to analyze the commutativity of DK-STP, which benefits from the fact that both of them have the same bridge matrix. In this way, we can express a sufficient condition for commutativity in the form of Kronecker product. The bridge matrix serves as a bridge that connects the two items that are swapped.

Theorem 3.1. *Let $A, B \in \mathcal{M}_{n\times m}$. Then,*

(i) *A, B gratify commutativity*

$$A \ltimes B = B \ltimes A,$$

if

$$B^T \otimes A = A^T \otimes B.$$

(ii) *A, B gratify anticommutativity*

$$A \ltimes B = -B \ltimes A,$$

if

$$B^T \otimes A + A^T \otimes B = 0.$$

Proof. (i) Given the following two matrix mapping:

$$A \ltimes B = A\psi_{m\times n}B,$$

$$B \ltimes A = B\psi_{m\times n}A,$$

it is noted that $\psi_{m\times n}$ is their common part. Thus, we can treat the two equations as two matrix mappings starting from $\psi_{m\times n}$. The matrix mappings are presented as follows:

$$\psi_{m\times n} \longmapsto A\psi_{m\times n}B,$$

$$\psi_{m\times n} \longmapsto B\psi_{m\times n}A.$$

By Theorem 2.1, the matrix column arrangement of $A\psi_{m\times n}B$ and $B\psi_{m\times n}A$ can be obtained,

$$V_c(A\psi_{m\times n}B) = (B^T \otimes A)V_c(\psi_{m\times n}),$$

$$V_c(B\psi_{m\times n}A) = (A^T \otimes B)V_c(\psi_{m\times n}).$$

If

$$B^T \otimes A = A^T \otimes B,$$

then the conclusion is obvious.

(ii) The proof of (ii) is similar to (i). □

Consider Theorem 3.1, we have only demonstrated the sufficiency of the theorem, but the necessity is not satisfied. Next, we will provide a counterexample to demonstrate that the necessity is false.

Example 3.1. (i) *Take two matrices*

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 & 0 \\ 1 & -2 & 0 \end{bmatrix}.$$

We can calculate

$$A \ltimes B = A\psi_{3\times 2}B = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$B \ltimes A = B\psi_{3\times 2}A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

where

$$\psi_{3\times 2} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix}^T.$$

We can verify $A \ltimes B = B \ltimes A$, where $B^T \otimes A$ and $A^T \otimes B$ can be expressed

$$B^T \otimes A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, A^T \otimes B = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

It can be verified by computation

$$B^T \otimes A \neq A^T \otimes B.$$

(ii) *Take two matrices*

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 & 0 \\ 1 & -2 & 0 \end{bmatrix}.$$

We can calculate

$$A \ltimes B = A\psi_{3\times 2}B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$B \ltimes A = B\psi_{3\times 2}A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

$A \ltimes B = -B \ltimes A$ can be confirmed, and next we calculate $B^T \otimes A$ and $A^T \otimes B$.

$$B^T \otimes A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, A^T \otimes B = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

It can be verified by computation,

$$B^T \otimes A + A^T \otimes B \neq 0.$$

Take some matrices to verify Theorem 3.1, which gives us a sufficient condition to confirm whether matrices are exchanged under DK-STP. Meanwhile it is possible to construct commutative matrices through this condition.

Example 3.2. *Consider two matrices*

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix},\ B = \begin{bmatrix} 2 & 4 & 6 \\ 4 & 6 & 8 \end{bmatrix}.$$

Through calculation, we can find

$$B^T \otimes A = A^T \otimes B.$$

Further calculation can also be obtained

$$A \bar{\times} B = A\psi_{3\times 2}B = \begin{bmatrix} 40 & 64 & 88 \\ 58 & 94 & 130 \end{bmatrix},$$

$$B \bar{\times} A = B\psi_{3\times 2}A = \begin{bmatrix} 40 & 64 & 88 \\ 58 & 94 & 130 \end{bmatrix}.$$

Remark 3.1. *(i) It is easy to verify if the condition is valid, the commutativity (i) of DK-STP is correct. More deeply, if the commutativity (i) is valid, the matrix must meet dimension condition in order to establish commutativity, that is, the elements in the matrix must establish*

$$a_{ij} = \lambda b_{ij},\ \ \lambda \in \mathbb{R},\ \ \ i, j = 1, \cdots, n,$$

then

$$A = \lambda B.$$

If elements of A and B are 0, then the corresponding position $a_{ij} = b_{ij} = 0$.

(ii) For the commutativity (ii), we need matrix A,B satisfy $B^T \otimes A + A^T \otimes B = 0$*, this is a sufficient condition. Similar to the treatment of (1) above, we correspond to the relationship between the internal elements of the two matrices. The elements in the matrix must establish*

$$a_{ij} = \lambda b_{ij},\ \ \lambda \in \mathbb{R},\ \ \ i, j = 1, \cdots, n,$$

then

$$A + \lambda B = 0.$$

If elements of A and B are 0, then the corresponding position $a_{ij} = b_{ij} = 0$.

4. Structure matrix of dimension keeping Lie bracket

In this section, we will study the structure matrix of dimension keeping Lie bracket. We consider the structure matrix of the mapping between vectors. By utilizing Lie bracket in NSM, we can determine the column arrangement of a matrix and obtain its structure matrix.

Theorem 4.1. *Under the action of DK-STP, the structure matrix* M_V *is generated from the mapping of two n-dimensional vector spaces*

$$M_V = I_n \otimes \psi_{1\times n},$$

then

$$x \bar{\times} y = (I_n \otimes \psi_{1\times n})xy,\ \ \ x, y \in \mathcal{M}_{n\times 1}.$$

Proof. In the n-dimensional space mapping, we analyze the structure matrix M_V. Let us start with the form of the bridge matrix.

$$\psi_{n\times p} = (I_n \otimes 1^T_{t/n})(I_p \otimes 1_{t/p}) \in \mathcal{M}_{n\times p},\ \ \ t = lcm(n, p).$$

Then, $\psi_{1\times n}$ can be expressed as follows:

$$\psi_{1\times n} = (I_1 \otimes 1^T_n)(I_n \otimes 1_1) = 1^T_n.$$

Definition of structure matrix is used to compute the elements of structure matrix from the basis of n-dimensional vector,

$$e_i \bar{\times} e_j = \delta^i_n,\ \ \ i, j = 1, 2, \cdots, n.$$

From this expression, we can see that only the first position basis actually affects the result of the calculation. Therefore, the structure matrix is divided into blocks so that the results of each block can be expressed,

$$M_V = \begin{bmatrix} M_1 & M_2 & \cdots & M_n \end{bmatrix},$$

$$M_i = \begin{bmatrix} \delta^i_n & \delta^i_n & \cdots & \delta^i_n \end{bmatrix}.$$

In the end, structure matrix M_V can be expressed as

$$M_V = diag(1^T_n, 1^T_n, \cdots, 1^T_n) = I_n \otimes \psi_{1\times n}.$$

The DK-STP satisfies the associative law, which can be proved by using the bridge matrix. The structure matrix of the DK-STP is considered here. The corresponding proof can also be obtained by the structure matrix, which only needs to prove that the structure matrix satisfies the algebraic associative condition Lemma 2.2 (*iii*). This is easy to verify.

In Lemma 2.3, we get the structure matrix of $gl(n, \mathbb{R})$. The following theorem will get the structure matrix of $gl(m \times n, \mathbb{R})$. $\square$

Theorem 4.2. *The structure matrix of dimension keeping Lie bracket is*

$$M_{gl(m\times n,\mathbb{R})} = \Psi_{mmn}(\psi_{m\times n}\otimes I_m)(I_{m^2n^2} - W_{[mn]}).$$

Proof. After extending the matrix operation of Lie bracket from classical matrix operation to DK-STP, Lie bracket can be defined for NSM, which is great significance to NSM operations.

$\{M_{IJ}|I = 1, 2, \cdots, m; J = 1, 2, \cdots, n\}$ represents a set of bases, where

$$(M_{IJ})_{ij} = \begin{cases} 1 & i = I \quad or \quad j = J, \\ 0 & \quad others. \end{cases}$$

By Lemma 2.1 and Theorem 2.1, we have

$$V_c(A \bar{\times} B) = \Psi_{mmn}(\psi_{m\times n}\otimes I_m)V_c(A)V_c(B),$$

$$V_c(B \bar{\times} A) = \Psi_{mmn}(\psi_{m\times n}\otimes I_m)W_{[mn]}V_c(A)V_c(B).$$

Then, we can find

$$V_c([A, B]_{\bar{\times}}) = \Psi_{mmn}(\psi_{m\times n}\otimes I_m)(I_{m^2n^2} - W_{[mn]})V_c(A)V_c(B),$$

and

$$M_{gl(m\times n,\mathbb{R})} = \Psi_{mmn}(\psi_{m\times n}\otimes I_m)(I_{m^2n^2} - W_{[mn]}).$$

We know $gl(m \times n, \mathbb{R})$ is a Lie algebra. Then, skew-symmetric and Jacobi equation are established in $gl(m \times n, \mathbb{R})$. Thus, the structure matrix need to satisfy the following equations. □

Corollary 4.1. *The structure matrix of dimension keeping Lie bracket $M_{gl(m\times n,\mathbb{R})}$ must satisfies the following two equations.*

(*i*) *Skew-symmetric*

$$M_{gl(m\times n,\mathbb{R})}(I_{m^2n^2} + W_{[mn]}) = 0.$$

(*ii*) *Jacobi equation*

$$M^2_{gl(m\times n,\mathbb{R})}(I_{(mn)^3} + W_{[mn,(mn)^2]} + W_{[(mn)^2,mn]}) = 0.$$

Proof. (*i*) The expression for the anticommutation is as follows:

$$[A, B]_{\bar{\times}} = -[B, A]_{\bar{\times}}, \quad A, B \in \mathcal{M}_{m\times n}.$$

Then, the following equation can be deduced:

$$V_c([A, B]_{\bar{\times}}) = -V_c([B, A]_{\bar{\times}}).$$

By Theorem 4.2, we can compute

$$M_{gl(m\times n,\mathbb{R})}V_c(A)V_c(B) = -M_{gl(m\times n,\mathbb{R})}V_c(B)V_c(A),$$

$$M_{gl(m\times n,\mathbb{R})}(I_{m^2n^2} + W_{[mn]})V_c(A)V_c(B) = 0.$$

Since A, B is arbitrary, the conclusion is obvious.

(*ii*) The expression for the Jacobi equation is as follows:

$$[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0, \quad X, Y, Z \in \mathcal{M}_{m\times n}.$$

It is written in the form of structure matrix and can be obtained by calculation.

$$M^2_{gl(m\times n,\mathbb{R})}(I_{(mn)^3} + W_{[mn,(mn)^2]} + W_{[(mn)^2,mn]})V_c(X)V_c(Y)V_c(Z) = 0.$$

Since X, Y, Z is arbitrary, the conclusion is obvious. □

Next, we give some related special subalgebras for dimension keeping Lie bracket. Under the action of DK-STP, there are related Lie subalgebras for NSM.

Example 4.1. *Take the coefficients $n = 2$, $m = 3$ in Theorem 4.2, we verify its relevant properties by calculating the structure matrix of $gl(3 \times 2, \mathbb{R})$.*

Consider the definition of Ψ_{mmn}, we can give

$$\Psi_{332} = \begin{bmatrix} I_3\otimes(\delta_2^1\delta_3^1)^T & I_3\otimes(\delta_2^1\delta_3^2)^T & I_3\otimes(\delta_2^1\delta_3^3)^T \\ I_3\otimes(\delta_2^2\delta_3^1)^T & I_3\otimes(\delta_2^2\delta_3^2)^T & I_3\otimes(\delta_2^2\delta_3^3)^T \end{bmatrix}.$$

Then, we can get parts of the structure matrix of $gl(3\times 2, \mathbb{R})$,

$$\psi_{2\times 3}\otimes I_3 = \begin{bmatrix} 2&0&0&1&0&0&0&0&0 \\ 0&2&0&0&1&0&0&0&0 \\ 0&0&2&0&0&1&0&0&0 \\ 0&0&0&1&0&0&2&0&0 \\ 0&0&0&0&1&0&0&2&0 \\ 0&0&0&0&0&1&0&0&0 \end{bmatrix}$$

and

$$W_{[6]} = [I_6\otimes\delta_6^1, I_6\otimes\delta_6^2, \cdots, I_6\otimes\delta_6^6].$$

Then, we can verify Corollary 4.1 by calculating the following:

$$M_{gl(3\times 2,\mathbb{R})} = \Psi_{3\times 3\times 2}(\psi_{3\times 2}\otimes I_3)(I_{36} - W_{[6]}),$$

$$M_{gl(3\times 2,\mathbb{R})}(I_{36} + W_{[6]}) = 0,$$

$$M^2_{gl(3\times 2,\mathbb{R})}(I_{216} + W_{[6,36]} + W_{[36,6]}) = 0.$$

Through this example, we find that the structure matrix of $gl(3 \times 2, \mathbb{R})$ *indeed satisfies the equation relationship in the Corollary 4.1.*

5. Homomorphisms of Lie algebra

Next, we use homomorphisms of Lie algebras to describe some Lie subalgebras and present some subalgebras under DK-STP.

Define

$$\varphi(A) := A\psi_{n\times m}, A \in \mathcal{M}_{m\times n}.$$

Taking $A, B \in \mathcal{M}_{m\times n}$, we can get

$$\varphi(A + B) = \varphi(A) + \varphi(B),$$

$$\varphi(A \bar{\ltimes} B) = \varphi(A)\varphi(B).$$

From the above relationship, we can get the following formula:

$$\varphi([A, B]_{\bar{\ltimes}}) = [\varphi(A), \varphi(B)].$$

Therefore, we can obtain

$$\varphi : gl(m \times n, \mathbb{R}) \to gl(m, \mathbb{R}),$$

that is $gl(m \times n, \mathbb{R}) \simeq gl(m, \mathbb{R})$.

Through this mapping, we have achieved a Lie algebra homomorphism from NSM to square matrices. Thus for the subalgebra set of square matrices, we can also define the set of corresponding nonsquare algebras.

Example 5.1. *Some special Lie subalgebras under the dimension keeping Lie bracket are obtained as follows:*

(i) *Nonsquare orthogonal algebra*

$$o(m \times n, \mathbb{R}) = \left\{X \in gl(m \times n, \mathbb{R}) | (X\psi_{n\times m})^T = -X\psi_{n\times m}\right\}.$$

Proof. If $X, Y \in o(m \times n, \mathbb{R})$, then

$$\begin{aligned}
[[X, Y]_{\bar{\ltimes}}\psi_{n\times m}]^T =& ((X \bar{\ltimes} Y - Y \bar{\ltimes} X)\psi_{n\times m})^T \\
=& ((X\psi_{n\times m}Y - Y\psi_{n\times m}X)\psi_{n\times m})^T \\
=& (Y\psi_{n\times m})^T (X\psi_{n\times m})^T - (X\psi_{n\times m})^T (Y\psi_{n\times m})^T \\
=& -(X\psi_{n\times m}Y - Y\psi_{n\times m}X)\psi_{n\times m} \\
=& -[X, Y]_{\bar{\ltimes}}\psi_{n\times m}.
\end{aligned}$$

The algebraic closure is valid in the sense of dimension keeping Lie bracket. From the perspective of the homomorphism,

$$\begin{aligned}
(\varphi([X, Y]_{\bar{\ltimes}}))^T &= (\varphi(X)\varphi(Y) - \varphi(Y)\varphi(X))^T \\
&= \varphi(Y)^T\varphi(X)^T - \varphi(X)^T\varphi(Y)^T \\
&= -\varphi([X, Y]_{\bar{\ltimes}}),
\end{aligned}$$

that is, $[X, Y]_{\bar{\ltimes}} \in o(m \times n, \mathbb{R})$. Consider the definition of $\varphi(X)$. Thus, we know that

$$o(m \times n, \mathbb{R}) \simeq o(m, \mathbb{R}).$$

From nonsquare orthogonal algebra to square orthogonal algebra is a Lie algebra homomorphism.

(ii) Nonsquare special linear algebra

$$sl(m \times n, \mathbb{R}) = \{X \in gl(m \times n, \mathbb{R}) | tr(X\psi_{n\times m}) = 0\}.$$

Proof. If $X, Y \in sl(m \times n)$, then

$$\begin{aligned}
tr([X, Y]_{\bar{\ltimes}}\psi_{n\times m}) =& tr((X \bar{\ltimes} Y - Y \bar{\ltimes} X)\psi_{n\times m}) \\
=& tr((X\psi_{n\times m}Y - Y\psi_{n\times m}X)\psi_{n\times m}) \\
=& 0.
\end{aligned}$$

(iii) Nonsquare unitary algebra

$$u(m \times n, \mathbb{C}) = \left\{X \in gl(m \times n, \mathbb{C}) | \overline{(X\psi_{n\times m})}^T = -X\psi_{n\times m}\right\}.$$

Proof. If $X, Y \in u(m \times n)$, then

$$\begin{aligned}
\overline{[[X, Y]_{\bar{\ltimes}}\psi_{n\times m}]}^T =& \overline{((X \bar{\ltimes} Y - Y \bar{\ltimes} X)\psi_{n\times m})}^T \\
=& \overline{((X\psi_{n\times m}Y - Y\psi_{n\times m}X)\psi_{n\times m})}^T \\
=& \overline{(Y\psi_{n\times m})}^T\,\overline{(X\psi_{n\times m})}^T - \overline{(X\psi_{n\times m})}^T\,\overline{(Y\psi_{n\times m})}^T \\
=& (-Y\psi_{n\times m})(-X\psi_{n\times m}) - (-X\psi_{n\times m})(-Y\psi_{n\times m}) \\
=& -(X\psi_{n\times m}Y - Y\psi_{n\times m}X)\psi_{n\times m} \\
=& -[X, Y]_{\bar{\ltimes}}\psi_{n\times m}.
\end{aligned}$$

This means that through this homomorphism, a correspondence between NSM operations and square matrices operations is generated. Thus, for Example 5.1, their algebraic relationships can be proven through this correspondence, which are presented as follows:

$$sl(m \times n, \mathbb{R}) \simeq sl(m, \mathbb{R}),$$

$$u(m \times n, \mathbb{C}) \simeq u(m, \mathbb{C}).$$

The above formulas all express a kind of homomorphism correspondence from NSM to square matrices.

6. Conclusions

In this paper, we filled in the gap in the structure matrix of the Lie bracket for NSM and solved the (anti) commutative problem of DK-STP to a certain extent. The conditions that the structure matrix of dimension keeping Lie bracket must satisfy are analyzed from the perspective of the structure matrix. Finally, we provide examples of special Lie subalgebras.

As an extension of traditional matrix multiplication, STP is undoubtedly a very convenient new tool for studying algebraic problems. The study in this paper is only the tip of the iceberg for the extension of STP, and there are many algebraic properties worth studying. For example, there is space for more discussion on Killing type, group mapping, group isomorphism, and so on. This is actually derived from the algebraic closure of DK-STP for NSM, which provides us with a method to study the operation of NSM. This aspect may be used in the matrix equation, which is worth our in-depth exploration.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

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