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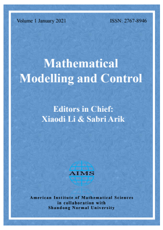


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***Research article***

# Coupled time-fractional hemivariational inequalities in frictional thermo-viscoelastic contact problem

**Abdelhafid Ouaanabi**[1,*], **Mohammed Alaoui**[1], **Mustapha Bouallala**[1,2] **and El Hassan Essoufi**[1]

[1] University Hassan, Laboratory MSDTE, 26000 Settat, Morocco

[2] Cadi Ayyad University, Polydisciplinary faculty, Modeling and Combinatorics Laboratory, Department of Mathematics and Computer Science B. P. 4162, Safi, Morocco

* **Correspondence:** Email: ouaanabi@gmail.com.

**Abstract:** This study examines a quasistatic frictional contact problem involving a thermo-viscoelastic body interacting with a thermally conductive foundation. The constitutive behavior is described by a fractional Kelvin–Voigt model utilizing the Caputo derivative. Heat conduction is modeled through time-fractional displacement and temperature parameters. The contact, friction, and heat exchange are governed by Clarke's subdifferential boundary conditions. The problem is weakly formulated as a system of two coupled time-fractional hemivariational inequalities. The existence of solutions is established by reducing the system to a single time-fractional hemivariational inequality, leveraging recent advances in the theory of time-fractional hemivariational inequalities.

**Keywords:** thermo-viscoelastic materials; Caputo derivative; frictional contact problem; weak solvability; Clarke's subdifferential; hemivariational inequality

## 1. Introduction

Contact problems involving friction, deformation, and heat exchange are of significant interest in the study of materials and mechanical systems. These problems become particularly complex when the interacting bodies exhibit time-dependent behaviors, such as viscoelasticity, and when thermal effects play a crucial role in the overall, dynamics. In many real-world applications, materials exhibit not only elastic but also viscoelastic responses, with frictional and thermal interactions significantly influencing the system's performance and stability. Consequently, the mathematical modeling and analysis of such systems are both challenging and essential for understanding and predicting their behavior.

In the literature, several theoretical results have been established, particularly in [1–3], using the theory of variational or hemivariational inequalities combined with fixed–point arguments. More recent advancements, as seen in [4], have incorporated the piezoelectric effect. In this study, we extend these works by employing a fractional Kelvin–Voigt constitutive law, which is expressed as

$$\sigma(t) = \mathcal{A}\varepsilon({}_0^C D_t^\alpha u(t)) + \mathcal{F}\varepsilon(u(t)) - \theta(t)\mathcal{M}, \tag{1.1}$$

where $0 < \alpha < 1$ is a material constant, and $\sigma$, $\mathcal{A}$, $\mathcal{F}$, and $\mathcal{M}$ represent the stress, viscosity, elasticity, and thermal expansion tensors, respectively. Here, ${}_0^C D_t^\alpha u$ denotes the Caputo fractional derivative of order $\alpha$ applied to the displacement $u$. For a better understanding of how the Eq (1.1) is derived from the following classical form:

$$\sigma(t) = \mathcal{A}\varepsilon(\dot{u}(t)) + \mathcal{F}\varepsilon(u(t)) - \theta(t)\mathcal{M}, \tag{1.2}$$

we refer the reader to [5, 6]. Additionally, we model heat conduction using a time-fractional version of Fourier's law for the temperature field $\theta$, given by:

$${}_0^C D_t^\alpha \theta(t) + \operatorname{div} q(t) = R\left({}_0^C D_t^\alpha u(t)\right) + q_{th}(t), \tag{1.3}$$

where $R$ is a linear function.

The origins of fractional calculus can be traced back to the late 17th century, with contributions from Gottfried Leibniz, Guillaume de l'Hôpital, and Johann Bernoulli, laying the foundation for this field [7, 8]. Significant developments occurred in the mid-19th century following the works of Joseph Liouville and Bernhard Riemann, which spurred various advancements in the theory [9]. These developments have since been extensively explored in modern research, as evidenced by Bazhlekova's study on fractional evolution equations in Banach spaces [7] and Kostić's contributions to abstract Volterra integro-differential equations [10]. One notable application of fractional calculus is in the mechanical modeling of rubber-like materials. References [11, 12] highlight models that incorporate specific viscoelastic materials, where fractional constitutive laws such as the Kelvin–Voigt and the fractional Maxwell models are considered. In [13], the authors examined a general quasistatic frictionless contact problem for a viscoelastic body governed by the fractional Kelvin–Voigt law, with the contact condition expressed via the Clarke subdifferential of a nonconvex and nonsmooth functional.

However, the domain of fractional calculus extends far beyond contact mechanics, the subject of this article. For instance, these operators have been successfully applied in diverse fields such as fluid dynamics, diffusion processes, and materials science. The Caputo and Caputo-Fabrizio time fractional operators have gained significant attention in heat transfer analysis due to their ability to model memory effects and non-local behaviors in thermal processes. These operators generalize classical time derivatives, offering a more accurate representation of complex systems where past history influences the present state, which is particularly useful in anomalous diffusion and fractional heat conduction problems. Recent studies, such as those in [14–16], underscore their relevance in various applications, including thermal conductivity in materials and heat distribution in heterogeneous media. The growing body of work in this field highlights the increasing importance of fractional calculus in enhancing our understanding of thermal processes and improving predictive models in heat transfer analysis.

This paper focuses on the analysis of a quasistatic frictional contact problem involving a thermo-viscoelastic body, governed by the constitutive relation (1.1). The contact, friction, and heat exchange processes are modeled using subgradients of nonconvex and nonsmooth potentials. We present a weak formulation of the problem, expressed in terms of displacement and temperature fields, leading to a system of two coupled time-fractional hemivariational inequalities. Subsequently, we establish the existence and regularity of weak solutions.

While previous research on similar models has often employed the Banach fixed point theorem to prove the existence of weak solutions (see, e.g., [5]), the key contribution of this paper lies in introducing a new approach that avoids the use of fixed point arguments. Specifically, we combine the two inequalities from the variational formulation into a single fractional hemivariational inequality. This is then solved using an abstract result for time-fractional hemivariational inequalities, by constructing suitable product spaces, which represents the main novelty of this work.

The remainder of the paper is organized as follows. Section 2 introduces some notation and preliminary material, including an abstract result on the existence of solutions for an elliptic time-fractional hemivariational inequality. In Section 3, we present the mechanical model for a thermo-viscoelastic fractional contact problem. Section 4 introduces additional notation, outlines the assumptions on the problem's data, derives the variational formulation, and states our main existence result. Finally, in Section 5, we provide proof of the main result.

## 2. Preliminaries on fractional calculus

In this section, we present some well-known results related to fractional calculus, which can be found in many monographs and papers, such as [12, 17].

For a real Banach space $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$, we denote its dual space by $\mathcal{B}^*$, and the duality pairing between $\mathcal{B}^*$ and $\mathcal{B}$ is represented as $\langle\cdot,\cdot\rangle_{\mathcal{B}^*\times\mathcal{B}}$. For $1 \le p \le \infty$ and $m = 1, 2, \ldots$, we use the standard notation for the spaces $L^p(0, T; \mathcal{B})$ and $W^{m,p}(0, T; \mathcal{B})$, where $0 < T < +\infty$. We also use the notation $\mathcal{L}(\mathcal{B}, \mathcal{B}^*)$ to denote the space of bounded linear operators from space $\mathcal{B}$ to its dual space $\mathcal{B}^*$, equipped with the standard norm $\|\cdot\|_{\mathcal{L}(\mathcal{B},\mathcal{B}^*)}$. Finally, we denote by $C(0,T;\mathcal{B})$ the space of continuous functions from $[0,T]$ to $\mathcal{B}$.

### 2.1. Riemann–Liouville fractional integral

The Riemann–Liouville fractional integral is an important tool in fractional calculus. It is defined for a function $f \in L^1(0,T;X)$, where $X$ is a Banach space and $(0,T)$ is a finite time interval.

**Definition 2.1** (Riemann–Liouville fractional integral)**.** *Let $X$ be a Banach space and $(0,T)$ be a finite time interval. The Riemann–Liouville fractional integral of order $\alpha > 0$ for a given function $f \in L^1(0,T;X)$ is defined as follows:*

$$ {}_0I_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\, ds, \quad \textit{for a.e. } t \in (0,T), $$

*where $\Gamma$ denotes the Gamma function, which is defined by*

$$ \Gamma(\alpha) = \int_0^{+\infty} t^{\alpha-1} e^{-t}\, dt. $$

*To complement the definition, we set ${}_0I_t^0 = I$, where $I$ is the identity operator, which means that*

$$ {}_0I_t^0 f(t) = f(t) \quad \textit{for a.e. } t \in (0,T). $$

### 2.2. Caputo fractional derivative

The Caputo fractional derivative is another essential concept in fractional calculus, which is widely used in various applications, especially in the modeling of anomalous diffusion and viscoelastic materials.

**Definition 2.2** (Caputo derivative of order $0 < \alpha \le 1$)**.** *Let $X$ be a Banach space, $0 < \alpha \le 1$, and $(0,T)$ be a finite time interval. For a given function $f \in AC(0,T;X)$, the Caputo fractional derivative of $f$ is defined by*

$$ {}_0^CD_t^\alpha f(t) = {}_0I_t^{1-\alpha} f'(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} f'(s)\, ds, $$

*for a.e. $t \in (0,T)$. The notation $AC(0,T;X)$ refers to the space of all absolutely continuous functions from $(0,T)$ into $X$.*

*It is clear that if $\alpha = 1$, the Caputo derivative reduces to the classical first-order derivative:*

$$ {}_0^CD_t^1 f(t) = If'(t) = f'(t), \quad \textit{for a.e. } t \in (0,T). $$

### 2.3. Properties of fractional calculus operators

We present some well-known properties related to the Riemann–Liouville fractional integral and the Caputo derivative, which are essential in understanding the behavior of these operators.

**Proposition 2.3.** *Let $X$ be a Banach space. If $\alpha, \beta > 0$ and $0 < \gamma \le 1$, we have the following properties:*

*(1)* ${}_0I_t^\alpha\, {}_0I_t^\beta u(t) = {}_0I_t^{\alpha+\beta} u(t)$ *if* $u \in L^1(0,T;X)$.
*(2)* ${}_0I_t^\gamma\, {}_0^CD_t^\gamma u(t) = u(t) - u(0)$ *if* $u \in AC(0,T;X)$.
*(3)* ${}_0^CD_t^\gamma\, {}_0I_t^\gamma u(t) = u(t)$ *if* $u \in L^1(0,T;X)$.

### 2.4. Generalized time-fractional hemivariational inequality

Let $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ be a Banach space, and let $\Psi : \mathcal{B} \longrightarrow \mathbb{R}$ be a locally Lipschitz function. The (Clarke) generalized directional derivative of $\Psi$ at $x \in \mathcal{B}$ in the direction $\lambda \in \mathcal{B}$, denoted by $\Psi^0(x;\lambda)$, is defined as

$$ \Psi^0(x;\lambda) = \limsup_{y \longrightarrow x, \omega \downarrow 0} \frac{\Psi(y+\omega\lambda) - \Psi(y)}{\omega}. $$

The (Clarke) generalized gradient of $\Psi$ at $x$, denoted by $\partial\Psi(x)$, is a subset of $\mathcal{B}^*$ defined by

$$ \partial\Psi(x) = \{\zeta \in \mathcal{B}^* \mid \Psi^0(x;\lambda) \ge (\zeta,\lambda)_{\mathcal{B}^*\times\mathcal{B}} \text{ for all } \lambda \in \mathcal{B}\}. $$

We now introduce a general elliptic time-fractional hemivariational inequality within the framework of an evolution triple of spaces. For this inequality, we will present a result regarding the existence of solutions. Let $\mathcal{V} \subset \mathcal{H} \subset \mathcal{V}^*$ be an evolution triple of spaces, where $(\mathcal{V}, \|\cdot\|_{\mathcal{V}})$ is a reflexive and separable Banach space, $(\mathcal{H}, \|\cdot\|_{\mathcal{H}})$ is a separable Hilbert space, and the embedding $\mathcal{V} \to \mathcal{H}$ is dense and continuous. We denote the embedding operator between $\mathcal{V}$ and $\mathcal{H}$ by $i$, and assume that it is compact. The dual mapping $i^* : \mathcal{H} \to \mathcal{V}^*$ associated with $i$ is also linear, continuous, and compact. Given $0 < T < +\infty$, we consider the standard Bochner-Lebesgue function space $\overline{\mathcal{V}} = L^2(0,T;\mathcal{V})$. The reflexivity of $\mathcal{V}$ ensures that both $\overline{\mathcal{V}}$ and its dual space $\overline{\mathcal{V}}^* = L^2(0,T;\mathcal{V}^*)$ are reflexive Banach spaces. Furthermore, let $\overline{\mathcal{H}} = L^2(0,T;\mathcal{H})$. By identifying $\overline{\mathcal{H}}$ with its dual, we obtain the continuous embeddings $\overline{\mathcal{V}} \subset \overline{\mathcal{H}} \subset \overline{\mathcal{V}}^*$. Let $\mathcal{X}$ be another separable and reflexive Banach space, with $\overline{\mathcal{X}} = L^2(0,T;\mathcal{X})$ and $\overline{\mathcal{X}}^* = L^2(0,T;\mathcal{X}^*)$. Given bounded linear operators $A, B : \mathcal{V} \to \mathcal{V}^*$, $M : \mathcal{V} \to \mathcal{X}$, a locally Lipschitz functional $J : \mathcal{X} \to \mathbb{R}$, $F \in \overline{\mathcal{V}}^*$, and $u_0 \in \mathcal{V}$, we consider the generalized hemivariational inequality of the following form.

**Problem 2.4.** *Find $u \in \overline{\mathcal{V}}$ such that for all $v \in \mathcal{V}$, a.e. $t \in (0,T)$,*

$$(A_0^C D_t^\alpha u(t), v)_{\mathcal{V}^*\times\mathcal{V}} + (Bu(t), v)_{\mathcal{V}^*\times\mathcal{V}} + J^0(Mu(t); Mv) \geq (F(t), v)_{\mathcal{V}^*\times\mathcal{V}}, \qquad u(0) = u_0.$$

To study this problem, we need the following hypotheses on the data.

**H(A)** $A \in \mathcal{L}(\mathcal{V}, \mathcal{V}^*)$ is coercive, i.e., there exists a constant $m_A > 0$ such that

$$(Av, v)_{\mathcal{V}^*\times\mathcal{V}} \geq m_A \|v\|_{\mathcal{V}}^2 \quad \text{for all } v \in V.$$

**H(B)** $B \in \mathcal{L}(\mathcal{V}, \mathcal{V}^*)$.

**H(J)** $J : \mathcal{X} \to \mathbb{R}$ is locally Lipschitz, and there exists $m_J > 0$ such that

$$\|\partial J(v)\|_{\mathcal{X}^*} \leq m_J(1 + \|v\|_{\mathcal{X}}) \quad \text{for all } v \in \mathcal{X}.$$

**H(M)** $M \in \mathcal{L}(\mathcal{V}, \mathcal{X})$ is compact.

**H(F)** $F \in C(0, T; \mathcal{V}^*)$.

Under all these considerations, we have the following existence result.

**Theorem 2.5.** *Under the hypotheses **H(A)**, **H(B)**, **H(J)**, **H(M)**, and **H(F)**, Problem 2.4 has at least one solution $u \in W^{1,2}(0, T; \mathcal{V})$.*

This theorem was established in [6]. While we do not present the proof details here, we highlight that it is based on the Rothe method and a result concerning a class of nonlinear evolutionary abstract inclusions with a pseudononmonotone multivalued term characterized by the Clarke generalized gradient.

## 3. Physical setting and classical formulation

Consider a thermo-viscoelastic body that initially occupies a bounded domain $\Omega \subset \mathbb{R}^d$ ($d = 2, 3$) with a smooth boundary $\Gamma = \partial\Omega$. The body is subjected to body forces with density $f_0$ and a heat source $q_{th}$ in $\Omega$, along with mechanical and thermal boundary conditions. To specify these conditions, we partition $\Gamma$ into three measurable, disjoint subsets: $\Gamma_D$, $\Gamma_N$, and $\Gamma_C$, with $meas(\Gamma_D) > 0$. The body is assumed to be clamped on $\Gamma_D$, and the temperature is set to zero on $\Gamma_D \cup \Gamma_N$. Additionally, surface tractions of density $f_N$ are applied on $\Gamma_N$. On the contact surface $\Gamma_C$, the body may experience frictional contact with a thermally conductive obstacle, referred to as the foundation.

Let $T > 0$, and denote by $[0, T]$ the time interval of interest, with $x \in \Omega \cup \Gamma$ and $t \in [0, T]$ representing the spatial and temporal variables, respectively. For simplicity, we occasionally omit the explicit dependence of various functions on $x$. Throughout this paper, the indices $i$ and $j$ range from 1 to $d$, and the summation convention for repeated indices is adopted.

The space of symmetric second-order tensors in $\mathbb{R}^d$ is denoted by $\mathbb{S}^d$. In addition, we define the inner product and its associated norm on $\mathbb{R}^d$ and $\mathbb{S}^d$ by

$$u \cdot v = u_i v_i, \quad \|v\| = \sqrt{v \cdot v}, \qquad \forall u, v \in \mathbb{R}^d,$$
$$\sigma \cdot \tau = \sigma_{ij}\tau_{ij}, \quad \|\tau\| = \sqrt{\tau \cdot \tau}, \qquad \forall \sigma, \tau \in \mathbb{S}^d.$$

We denote by $\nu$ the unit outward normal on the boundary $\Gamma$, and we shall adopt the usual notation for normal and tangential components of vectors and tensors.

$$u = u_\nu \nu + u_\tau, \quad u_\nu = u \cdot \nu, \qquad \forall u \in \mathbb{R}^d,$$
$$\sigma\nu = \sigma_\nu \nu + \sigma_\tau, \qquad \sigma_\nu = (\sigma\nu) \cdot \nu, \quad \forall \sigma \in \mathbb{S}^d.$$

To define our problem, we introduce the following notations: $u : \Omega \times (0, T) \to \mathbb{R}^d$ represents the displacement field, $\theta : \Omega \times (0, T) \to \mathbb{R}$ the temperature field, $\sigma : \Omega \times (0, T) \to \mathbb{S}^d$ the stress tensor, and $q : \Omega \times (0, T) \to \mathbb{R}^d$ the heat flux vector. Additionally, let $\varepsilon(u) = (u_{i,j} + u_{j,i})/2$ denote the linearized strain tensor, where a comma in the subscript indicates differentiation with respect to the corresponding spatial variable.

The classical model for the fractional contact problem with Coulomb friction in thermo-viscoelasticity over the finite time interval $(0, T)$ is given as follows:

**Problem 3.1.** *Find a displacement field $u : \Omega \times (0, T) \longrightarrow \mathbb{R}^d$ and a temperature field $\theta : \Omega \times (0, T) \longrightarrow \mathbb{R}$ such that for all*

$t \in (0, T)$

$$\sigma(t) = \mathcal{A}\varepsilon({}_0^C D_t^\alpha u(t)) + \mathcal{F}\varepsilon(u(t)) - \theta(t)\mathcal{M} \quad \text{in } \Omega, \quad (3.1)$$

$$q(t) = -\mathcal{K}\nabla\theta(t) \quad \text{in } \Omega, \quad (3.2)$$

$$-\operatorname{Div}\sigma(t) = f_0(t) \quad \text{in } \Omega, \quad (3.3)$$

$${}_0^C D_t^\alpha \theta(t) + \operatorname{div} q(t) = R({}_0^C D_t^\alpha u(t)) + q_{th}(t) \quad \text{in } \Omega, \quad (3.4)$$

$$u(t) = 0 \quad \text{on } \Gamma_D, \quad (3.5)$$

$$\sigma(t)\nu = f_N(t) \quad \text{on } \Gamma_N, \quad (3.6)$$

$$\theta(t) = 0 \quad \text{on } \Gamma_D \cup \Gamma_N, \quad (3.7)$$

$$-\sigma_\nu(t) \in \partial j_\nu(u_\nu(t)) \quad \text{on } \Gamma_C, \quad (3.8)$$

$$-\sigma_\tau(t) \in \partial j_\tau(u_\tau(t)) \quad \text{on } \Gamma_C, \quad (3.9)$$

$$q(t) \cdot \nu \in \partial j_c(\theta(t)) \quad \text{on } \Gamma_C, \quad (3.10)$$

$$u(0) = u_0, \ \theta(0) = \theta_0 \quad \text{in } \Omega. \quad (3.11)$$

We now explain the equations and boundary conditions from (3.1) to (3.11). Equations (3.1) and (3.2) represent the constitutive laws for thermo-viscoelastic materials, where $\mathcal{A}$ denotes the viscosity tensor, $\mathcal{F}$ the elasticity tensor, $\mathcal{M}$ the thermal expansion tensor, and $\mathcal{K}$ the thermal conductivity tensor. Equations (3.3) and (3.4) describe the equation of motion and the equilibrium condition for the heat flux field, respectively. The operators Div and div represent the divergence for tensor and vector fields, respectively, defined as:

$$\operatorname{Div}\sigma = (\sigma_{ij,j}) \quad \text{and} \quad \operatorname{div} D = D_{i,i}.$$

The linear function $R$ captures the effect of the displacement field on temperature; in [18], the following form was used: $R(\zeta) = -\mathcal{M} \cdot \zeta$. Conditions (3.5)–(3.7) represent the mechanical and thermal boundary conditions, respectively, with their physical interpretations discussed in the second paragraph of this section. The contact condition (3.8) is known as the multivalued normal compliance condition, governed by the subdifferential of a nonconvex potential $j_\nu$, which has been addressed in several papers; see, e.g., [19–21]. (3.9) is the friction condition where $j_\tau$ is a prescribed function. For a thorough discussion on the friction law (3.9), we refer to [22, Section 6.3]. The relation in (3.10) represents the heat exchange between $\Gamma_C$ and the foundation, where $j_c$ is a prescribed function. For examples of frictional models leading to subdifferential boundary conditions of the form (3.10), we refer to [23]. Finally, (3.11) specifies the initial conditions of the problem, where $u_0$ and $\theta_0$ are given functions representing the initial displacement and initial temperature, respectively.

## 4. Variational formulation and main existence result

In this section, we derive a weak formulation of Problem 3.1. In order to achieve this, we must introduce some notations.

Let us denote by $H$, $\mathbf{H}^1(\Omega)$, and $\mathcal{H}$ the following spaces:

$$H = [L^2(\Omega)]^d,$$
$$\mathbf{H}^1(\Omega) = [H^1(\Omega)]^d,$$
$$\mathcal{D} = \left\{\sigma = (\sigma_{ij});\ \sigma_{ij} = \sigma_{ji} \in L^2(\Omega)\right\}.$$

The spaces $H$, $\mathbf{H}^1(\Omega)$, and $\mathcal{D}$ are real Hilbert spaces endowed with the following inner products:

$$(u, v)_H = \int_\Omega u_i v_i dx, \quad \forall u, v \in H,$$
$$(\sigma, \tau)_{\mathcal{D}} = \int_\Omega \sigma_{ij}\tau_{ij} dx, \ \ \forall \sigma, \tau \in \mathcal{D},$$
$$(u, v)_{\mathbf{H}^1(\Omega)} = (u, v)_H + (\varepsilon(u), \varepsilon(v))_{\mathcal{D}}, \ \ \forall u, v \in \mathbf{H}^1(\Omega).$$

The associated norms in $H$, $\mathcal{D}$, and $\mathbf{H}^1(\Omega)$ are denoted by $\|\cdot\|_H$, $\|\cdot\|_{\mathcal{D}}$ and $\|\cdot\|_{\mathbf{H}^1(\Omega)}$, respectively.

Keeping in mind the condition (3.5), we introduce the closed subspace of $\mathbf{H}^1(\Omega)$

$$V = \{v \in \mathbf{H}^1(\Omega) \mid v = 0 \ \text{ on } \Gamma_D\}.$$

Over the space $V$, we define the following inner product:

$$(u, v)_V = (\varepsilon(u), \varepsilon(v))_{\mathcal{D}}, \ \ \forall u, v \in V,$$

and its associated norm

$$\|v\|_V = \|\varepsilon(v)\|_{\mathcal{D}}. \quad (4.1)$$

Since $meas(\Gamma_D) > 0$, the following Korn's inequality holds: There exists $c_k > 0$ depending only on $\Omega$ and $\Gamma_D$ such that

$$\|\varepsilon(v)\|_{\mathcal{D}} \geq c_k \|v\|_{\mathbf{H}^1(\Omega)}, \quad \forall v \in V. \quad (4.2)$$

It follows from (4.1) and (4.2) that $\|\cdot\|_V$ is equivalent to $V$ to the usual norm $\|\cdot\|_{\mathbf{H}^1(\Omega)}$, therefore $(V, \|\cdot\|_V)$ is a real Hilbert space.

For simplicity, for an element $\omega \in \mathbf{H}^1(\Omega)$, we still denote by $\omega$ its trace $\gamma(\omega)$ on $\Gamma$. By the trace theorem, there exists a constant $c_0 > 0$ depending only on $\Omega$, $\Gamma_D$, and $\Gamma_C$ such that

$$\|v\|_{[L^2(\Gamma_C)]^d} \leq c_0 \|v\|_V, \quad \forall v \in V.$$

Next, for the temperature field, keeping in mind (3.7) we introduce the closed functions subspace of $H^1(\Omega)$

$$Q = \{\eta \in H^1(\Omega) \mid \eta = 0 \quad \text{on} \quad \Gamma_D \cup \Gamma_N\}.$$

Over $Q$, we consider the following inner product:

$$(\theta, \eta)_Q = (\nabla\theta, \nabla\eta)_H, \quad \forall \theta, \eta \in Q, \tag{4.3}$$

and the associated norm

$$\|\eta\|_Q = \|\nabla\eta\|_H. \tag{4.4}$$

Since $meas(\Gamma_D) > 0$, Friedrichs-Poincaré inequality holds; therefore, there exists a constant $c_p > 0$ such that

$$\|\nabla\eta\|_H \geq c_p \|\eta\|_{H^1(\Omega)}, \quad \forall \eta \in Q. \tag{4.5}$$

It follows from (4.4)-(4.5) that $\|\cdot\|_Q$ is equivalent on $Q$ to the usual norm $\|\cdot\|_{H^1(\Omega)}$ and then $(Q, \|\cdot\|_Q)$ is a real Hilbert space. Moreover, by the trace theorem, there exists a constant $c_1 > 0$ depending only on $\Omega$, $\Gamma_D$ and $\Gamma_C$ such that

$$\|\eta\|_{L^2(\Gamma_C)} \leq c_1 \|\eta\|_Q, \quad \forall \eta \in Q. \tag{4.6}$$

Finally, we recall the Gelfand triples $V \subset H \subset V^*$ and $Q \subset L^2(\Omega) \subset Q^*$, Let us denote

$$(u, v)_{V^* \times V} = (u, v)_H, \quad \forall u \in H, v \in V, \tag{4.7}$$

$$(\theta, \eta)_{Q^* \times Q} = (\theta, \eta)_{L^2(\Omega)}, \quad \forall \theta \in L^2(\Omega), \eta \in Q. \tag{4.8}$$

To study Problem 3.1, we need the following assumptions on its data.

$(H_{\mathcal{A}})$ The viscosity tensor $\mathcal{A} : \Omega \times \mathbb{S}^d \longrightarrow \mathbb{S}^d$ satisfies

(1) $\mathcal{A}(x, \zeta) = \bar{\mathcal{A}}(x)\zeta$ for all $\zeta \in \mathbb{S}^d$ and a.e. $x \in \Omega$.
(2) $\bar{\mathcal{A}}(x) = (\bar{\mathcal{A}}_{ijkl}(x))$ with $\bar{\mathcal{A}}_{ijkl} \in L^\infty(\Omega)$.
(3) There exists $m_{\mathcal{A}} > 0$ such that $(\mathcal{A}(x, \zeta), \zeta) \geq m_{\mathcal{A}} \|\zeta\|^2$ for all $\zeta \in \mathbb{S}^d$ and a.e. $x \in \Omega$.

$(H_{\mathcal{F}})$ The elasticity tensor $\mathcal{F} : \Omega \times \mathbb{S}^d \longrightarrow \mathbb{S}^d$ satisfies

(1) $\mathcal{F}(x, \zeta) = \bar{\mathcal{F}}(x)\zeta$ for all $\zeta \in \mathbb{S}^d$ and a.e. $x \in \Omega$.
(2) $\bar{\mathcal{F}}(x) = (\bar{\mathcal{F}}_{ijkl}(x))$ with $\bar{\mathcal{F}}_{ijkl} \in L^\infty(\Omega)$.

$(H_{\mathcal{K}})$ The thermal conductivity tensor $\mathcal{K} : \Omega \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$ satisfies

(1) $\mathcal{K}(x, \zeta) = \bar{\mathcal{K}}(x)\zeta$ for all $\zeta \in \mathbb{R}^d$ and a.e. $x \in \Omega$.
(2) $\bar{\mathcal{K}}(x) = (\bar{\mathcal{K}}_{ij}(x))$ with $\bar{\mathcal{K}}_{ij} \in L^\infty(\Omega)$.

$(H_{\mathcal{M}})$ The thermal expansion tensor $\mathcal{M} : \Omega \times \mathbb{R} \longrightarrow \mathbb{S}^d$ satisfies

(1) $\mathcal{M}(x, \zeta) = \bar{\mathcal{M}}(x)\zeta$ for all $\zeta \in \mathbb{R}$ and a.e. $x \in \Omega$.
(2) $\bar{\mathcal{M}}(x) = (\bar{\mathcal{M}}_{ij}(x))$ with $\bar{\mathcal{M}}_{ij} \in L^\infty(\Omega)$.

$(H_R)$ The function $R : \mathbb{R}^d \longrightarrow \mathbb{R}$ satisfies

(1) $R(\zeta) \in L^2(\Omega)$ for all $\zeta \in \mathbb{R}^d$.
(2) There exists $M_R$ such that $|R(\zeta)| \leq M_R$ for all $\zeta \in \mathbb{R}^d$.
(3) There exists $L_R > 0$ such that $|R(\zeta_1) - R(\zeta_2)| \leq L_R \|\zeta_1 - \zeta_2\|$ for all $\zeta_1, \zeta_2 \in \mathbb{R}^d$.

$(H_{j_\nu})$ The functional $j_\nu : \Gamma_C \times \mathbb{R} \longrightarrow \mathbb{R}$ satisfies

(1) $j_\nu(\cdot, r)$ is measurable on $\Gamma_C$ for all $r \in \mathbb{R}$.
(2) There exists $e_\nu \in L^2(\Gamma_C)$ such that $j_\nu(\cdot, e_\nu(\cdot)) \in L^1(\Gamma_C)$.
(3) $j_\nu(x, \cdot)$ is locally Lipschitz on $\mathbb{R}$ for a.e. $x \in \Gamma_C$.
(4) There exist $c_{0\nu} \geq 0$ and $c_{1\nu}$ such that $|\partial j_\nu(x, r)| \leq c_{0\nu} + c_{1\nu}|r|$ for all $r \in \mathbb{R}$ and a.e. $x \in \Gamma_C$.

$(H_{j_\tau})$ The functional $j_\tau : \Gamma_C \times \mathbb{R}^d \longrightarrow \mathbb{R}$ satisfies

(1) $j_\tau(\cdot, r)$ is measurable on $\Gamma_C$ for all $r \in \mathbb{R}^d$.
(2) There exists $e_\tau \in [L^2(\Gamma_C)]^d$ such that $j_\tau(\cdot, e_\tau(\cdot)) \in L^1(\Gamma_C)$.
(3) $j_\tau(x, \cdot)$ is locally Lipschitz on $\mathbb{R}$ for a.e. $x \in \Gamma_C$.
(4) There exist $c_{0\tau} \geq 0$ and $c_{1\tau} \geq 0$ such that $\|\partial j_\tau(x, r)\| \leq c_{0\nu} + c_{1\tau}\|r\|$ for all $r \in \mathbb{R}^d$ and a.e. $x \in \Gamma_C$.

$(H_{j_c})$ The functional $j_c : \Gamma_C \times \mathbb{R} \longrightarrow \mathbb{R}$ satisfies

(1) $j_c(\cdot, r)$ is measurable on $\Gamma_C$ for all $r \in \mathbb{R}$.
(2) There exists $e_c \in L^2(\Gamma_C)$ such that $j_c(\cdot, e_c(\cdot)) \in L^1(\Gamma_C)$.
(3) $j_c(x, \cdot)$ is locally Lipschitz on $\mathbb{R}$ for a.e. $x \in \Gamma_C$.

(4) There exists $c_{0c} \geq 0$ such that $|\partial j_c(x,r)| \leq c_{0c}$ for all $r \in \mathbb{R}$ and a.e. $x \in \Gamma_C$.

$(H_D)$ The forces, the traction, the heat source density, and the initial conditions are assumed to satisfy the following regularity conditions.

$$f_0 \in C(0,T;H), \qquad f_N \in C(0,T;[L^2(\Gamma_N)]^d),$$
$$q_{th} \in C(0,T;L^2(\Omega)), \qquad u_0 \in V, \quad \theta_0 \in Q.$$

Next, we define the elements $f(t) \in V^*$ and $q_c(t) \in Q^*$ by

$$(f(t),w)_{V^*\times V} = (f_0(t),w)_H + (f_N(t),w)_{[L^2(\Gamma_N)]^d}, \tag{4.9}$$
$$(q_c(t),\eta)_{Q^*\times Q} = (q_{th}(t),\eta)_{L^2(\Omega)}, \tag{4.10}$$

for all $w \in V$ and $\eta \in Q$.

Now, by utilizing Green's formula and the definition of the Clarke subdifferential, we obtain the following variational formulation of Problem 3.1 expressed in terms of displacement and temperature fields.

**Problem 4.1.** *Find a displacement field $u : (0,T) \longrightarrow V$ and a temperature field $\theta : (0,T) \longrightarrow Q$ such that for a.e. $t \in (0,T)$ and all $w \in V$, $\eta \in Q$*

$$(\mathcal{A}\varepsilon({}_0^C D_t^\alpha u(t)),\varepsilon(w))_{\mathcal{D}} + (\mathcal{F}\varepsilon(u(t)),\varepsilon(w))_{\mathcal{D}} - (\theta(t)\mathcal{M},\varepsilon(w))_{\mathcal{D}} + \int_{\Gamma_C} j_\nu^0(u_\nu(t);w_\nu)d\Gamma + \int_{\Gamma_C} j_\tau^0(u_\tau(t);w_\tau)d\Gamma \geq (f(t),w)_{V^*\times V}, \tag{4.11}$$

$$({}_0^C D_t^\alpha \theta(t),\eta)_{Q^*\times Q} + (\mathcal{K}\nabla\theta(t),\nabla\eta)_H - (R({}_0^C D_t^\alpha u(t)),\eta)_{L^2(\Omega)} + \int_{\Gamma_C} j_c^0(\theta(t);\eta)d\Gamma \geq (q_c(t),\eta)_{Q^*\times Q}, \tag{4.12}$$

$$u(0) = u_0, \qquad \theta(0) = \theta_0. \tag{4.13}$$

Our primary result concerning the existence of solutions is as follows:

**Theorem 4.2.** *Assume that conditions $(H_\mathcal{A})$, $(H_\mathcal{F})$, $(H_\mathcal{M})$, $(H_\mathcal{K})$, $(H_R)$, $(H_{j_\nu})$, $(H_{j_\tau})$, $(H_{j_c})$, and $(H_D)$ are satisfied, along with the following smallness condition:*

$$\min\left\{\frac{2m_\mathcal{A} - L_R}{2}, \frac{2-L_R}{2}\right\} > 0. \tag{4.14}$$

*Then, Problem 4.1 has at least one solution $(u,\theta)$ that satisfies the following regularity conditions:*

$$u \in W^{1,2}(0,T;V) \quad and \quad \theta \in W^{1,2}(0,T;Q). \tag{4.15}$$

It is important to note that once the displacement field $u$ and the temperature field $\theta$ are determined by solving Problem 4.1, the stress $\sigma$ and the heat flux $q$ can be computed using the thermo-viscoelastic constitutive laws (3.1)-(3.2).

## 5. Proof of Theorem 4.2

To solve Problem 4.1, we will utilize Theorem 2.5 within the following functional framework: $\mathcal{V} = V \times Q$, $\mathcal{H} = H \times L^2(\Omega)$, and $\mathcal{X} = [L^2(\Gamma_C)]^{d+1}$, equipped with the canonical inner products.

We introduce operators $A : \mathcal{V} \longrightarrow \mathcal{V}^*$ and $B : \mathcal{V} \longrightarrow \mathcal{V}^*$ given by

$$(AX,Y)_{\mathcal{V}^*\times\mathcal{V}} = (\mathcal{A}\varepsilon(u),\varepsilon(w))_{\mathcal{D}} + (\theta,\eta)_{Q^*\times Q} - (R(u),\eta)_Q, \tag{5.1}$$

$$(BX,Y)_{\mathcal{V}^*\times\mathcal{V}} = (\mathcal{F}\varepsilon(u),\varepsilon(w))_{\mathcal{D}} + (\mathcal{K}\nabla\theta,\nabla\eta)_H - (\theta\mathcal{M},w)_{\mathcal{D}}, \tag{5.2}$$

for all $X = (u,\theta) \in \mathcal{V}$ and $Y = (w,\eta) \in \mathcal{V}$. Let the function $J : \Gamma_C \times \mathcal{X} \longrightarrow \mathbb{R}$ be defined by

$$J(x,X) = \sum_{i=1}^{3} \int_{\Gamma_C} j_i(x,X)d\Gamma, \tag{5.3}$$

where

$$j_1(x,X) = j_\nu(x,u_\nu),$$
$$j_2(x,X) = j_\tau(x,u_\tau),$$
$$j_3(x,X) = j_c(x,\theta),$$

for all $X = (u,\theta) \in \mathcal{X}$ and a.e. $x \in \Gamma_C$. Moreover, let $M = \gamma$, where $\gamma : \mathcal{V} \to \mathcal{X}$ represents the trace operator. We also define the elements $X_0 \in \mathcal{V}$ and $F(t) \in \mathcal{V}^*$ by

$$X_0 = (u_0,\theta_0), \tag{5.4}$$
$$(F(t),Y)_{\mathcal{V}^*\times\mathcal{V}} = (f(t),w)_{V^*\times V} + (q_c(t),\eta)_{Q^*\times Q}, \tag{5.5}$$

for all $Y = (w,\eta) \in \mathcal{V}$ and a.e. $t \in (0,T)$. We have the following equivalence result.

**Lemma 5.1.** *The pair $X(t) = (u(t),\theta(t)) \in \mathcal{V}$ is a solution of Problem 4.1 if and only if for all $Y \in \mathcal{V}$ and a.e. $t \in (0,T)$*

$$(A_0^C D_t^\alpha X(t),Y)_{\mathcal{V}^*\times\mathcal{V}} + (BX(t),Y)_{\mathcal{V}^*\times\mathcal{V}} + J^0(MX(t);MY) \geq (F(t),Y)_{\mathcal{V}^*\times\mathcal{V}}, \tag{5.6}$$

$$X(0) = X_0. \tag{5.7}$$

*Proof.* Let $u(t) \in V$ and $\theta(t) \in Q$ be the solutions of (4.11) and (4.12) combined with the initial conditions $u(0) = u_0$ and $\theta(0) = \theta_0$, respectively, for a.e. $t \in (0,T)$. We add (4.11) to (4.12) to find that $(u(t), \theta(t)) \in \mathcal{V}$ is a solution of (5.6)-(5.7). Conversely, let $X(t) = (u(t), \theta(t))$ be a solution of the fractional hemivariational inequality (5.6) coupled with the initial condition (5.7) for a.e. $t \in (0,T)$. By choosing $Y = (w, 0)$ in (5.6), where $w$ is an arbitrary element of $V$, we obtain that $u(t) \in V$ is a solution to (4.11) for a.e. $t \in (0,T)$. Moreover, if we choose $Y = (0, \eta)$ in (5.6), where $\eta$ is an arbitrary element of $Q$, we find that $\theta(t) \in Q$ is the solution of (4.12) for a.e. $t \in (0,T)$, which finishes the proof. □

We will now proceed to demonstrate the existence of solutions for the problem described by (5.6)-(5.7). To accomplish this, we will check that the hypotheses **H(A)**, **H(B)**, **H(J)**, **H(M)**, and **H(F)** are satisfied. Given that the tensor $\mathcal{A}$ and the function $R$ are both bounded and linear, we can assert that the operator $A \in \mathcal{L}(\mathcal{V}, \mathcal{V}^*)$. Additionally, under the smallness assumption stated in (4.14), we assert that $A$ is coercive. Indeed, let $X = (u, \theta) \in \mathcal{V}$. It follows from $(H_{\mathcal{A}})$ and $(H_R)$ that for a.e. $t \in (0,T)$

$$(AX, X)_{\mathcal{V}^* \times \mathcal{V}} \geq m_{\mathcal{A}} \|u\|_V^2 + \|\theta\|_Q^2 - L_R \|u\|_V \|\theta\|_Q . \quad (5.8)$$

Therefore, by the inequality $ab \leq \frac{1}{2}(a^2 + b^2)$ for all $a, b \in \mathbb{R}$, we obtain

$$(AX, X)_{\mathcal{V}^* \times \mathcal{V}} \geq m_{\mathcal{A}} \|u\|_V^2 + \|\theta\|_Q^2 - \frac{L_R}{2} \|u\|_V^2 - \frac{L_R}{2} \|\theta\|_Q^2 . \quad (5.9)$$

Then

$$(AX, X)_{\mathcal{V}^* \times \mathcal{V}} \geq \frac{2m_{\mathcal{A}} - L_R}{2} \|u\|_V^2 + \frac{2 - L_R}{2} \|\theta\|_Q^2 . \quad (5.10)$$

Hence we deduce the coercivity of $A$ with

$$m_A = \min \left\{ \frac{2m_{\mathcal{A}} - L_R}{2}, \frac{2 - L_R}{2} \right\}.$$

On the other hand, since the tensors $\mathcal{F}$ and $\mathcal{K}$ are bounded and linears, the operator $B \in \mathcal{L}(\mathcal{V}, \mathcal{V}^*)$.

Next, it is clear that $j_i(x, \cdot)$ is locally Lipschitz on $\mathcal{X}$ for almost every $x \in \Gamma_C$ with $i = 1, 2, 3$. Therefore, $J(x, \cdot)$ is also locally Lipschitz. Given the regularity of $j_\nu(x, \cdot)$, $j_\tau(x, \cdot)$, and $j_c(x, \cdot)$, we can conclude that $j_i(x, \cdot)$ is regular for $i = 1, 2, 3$. By using this regularity and applying Proposition 5.6.33 from [22], we have

$$\partial j_i(x, X) \subseteq \partial_u j_i(x, X) \times \partial_\theta j_i(x, X), \quad (5.11)$$

for all $X = (u, \theta) \in \mathcal{X}$ and almost every $x \in \Gamma_C$, where $\partial_u j_i$ and $\partial_\theta j_i$ denote the partial generalized gradients of $j_i(x, (\cdot, \theta))$ and $j_i(x, (u, \cdot))$, respectively, for $i = 1, 2, 3$. Next, by utilizing [20, Proposition 2] and [24, Proposition 5.6.23], we derive from (5.11):

$$\partial j_1(x, X) \subseteq \partial j_\nu(x, u_\nu)_\nu \times \{0\}, \quad (5.12)$$

$$\partial j_2(x, X) \subseteq \partial j_\tau(x, u_\tau)_\tau \times \{0\}, \quad (5.13)$$

$$\partial j_3(x, X) \subseteq \{0\} \times \partial j_c(x, \theta), \quad (5.14)$$

$$\partial J(x, X) \subseteq \int_{\Gamma_C} (\partial j_\nu(x, u_\nu)_\nu + \partial j_\tau(x, u_\tau)_\tau) \times \partial j_c(x, \theta) \, d\Gamma, \quad (5.15)$$

for all $X = (u, \theta) \in \mathcal{X}$ and almost every $x \in \Gamma_C$. From (5.12)–(5.14) and the hypotheses $(H_{j_\nu})$, $(H_{j_\tau})$ and $(H_{j_c})$ we have:

$$\|\partial j_1(x, X)\|_{\mathcal{X}} \leq |\partial j_\nu(x, u_\nu)| \leq c_{0\nu} + c_{1\nu}|u_\nu| \leq c_{0\nu} + c_{1\nu}\|X\|_{\mathcal{X}},$$
$$\|\partial j_2(x, X)\|_{\mathcal{X}} \leq \|\partial j_\tau(x, u_\tau)\| \leq c_{0\tau} + c_{1\tau}\|u_\tau\| \leq c_{0\tau} + c_{1\tau}\|X\|_{\mathcal{X}},$$
$$\|\partial j_3(x, X)\|_{\mathcal{X}} \leq |\partial j_c(x, \theta)| \leq c_{0c},$$

for all $X = (u, \theta) \in \mathcal{X}$ and almost every $x \in \Gamma_C$. From the above, we deduce that the functional $J$ defined in (5.3) satisfies **H(J)** with

$$m_J = \max \left\{ c_{0c} \operatorname{meas}(\Gamma_C)^2 (c_{0\nu} + c_{0\tau}), c_0 c_{0c} \sqrt{\operatorname{meas}(\Gamma_C)^3} (c_{1\nu} + c_{1\tau}) \right\}.$$

From the regularity assumption $(H_D)$, we can deduce that $F$ satisfies **H(F)**. Additionally, it is evident, as shown in [24, Theorem 3.9.34], that the trace operator fulfills **H(M)**.

All the assumptions in Theorem 2.5 have been verified. Therefore, we can confidently state that the time-fractional hemivariational inequality (5.6) combined with initial condition (5.7) has at least one solution $X$ that satisfies the regularity

$$X \in W^{1,2}(0, T; \mathcal{V}). \quad (5.16)$$

By exploiting the equivalence result stated in Lemma 5.1, we conclude that $(u, \theta)$ is a solution to Problem 4.1 that satisfies the regularity conditions (4.15), which finishes the proof.

## 6. Conclusions

In this work, we have investigated a quasistatic frictional contact problem for a thermo-viscoelastic body whose material behavior incorporates fractional time effects through the Caputo derivative. The model couples mechanical and thermal processes and includes nonmonotone contact, friction, and heat-exchange laws expressed via Clarke's subdifferential. By formulating the problem as a system of two coupled time-fractional hemivariational inequalities, we captured both the hereditary nature of the material and the nonsmooth features of the contact conditions.

A key contribution of this study is the reduction of the coupled weak formulation to a single time-fractional hemivariational inequality. This reformulation enabled the use of recent developments in the theory of time-fractional hemivariational inequalities to establish the existence of weak solutions. The results obtained extend classical frameworks of contact mechanics to fractional viscoelasticity and provide a rigorous mathematical foundation for future analytical and numerical studies of fractional thermo-mechanical systems with nonmonotone boundary behavior.

## Author contributions

Authors contributed equally to this work. All authors have read and approved the final version of the manuscript for publication.

## Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

## Conflict of interest

The authors declare that they have no conflicts of interests in this paper.

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