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Research article

Positive radial solutions for a p-Monge-Ampère problem

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Abstract: In this paper, by virtue of fixed point theory, we investigate a p-Monge-Ampère problem and establish several existence results for positive radial solutions when the nonlinearity satisfies some $(p-1)n$-superlinear and $(p-1)n$-sublinear conditions.

Keywords: p-Monge-Ampère problem; fixed point theory; positive radial solution

1. Introduction

In this paper, we study the p-Monge-Ampère problem:

$$\begin{cases} \det\left(D\left(|Du|^{p-2}Du\right)\right) = f(|x|, -u), \ x \in B, \\ u = 0, \ x \in \partial B, \end{cases} \tag{1.1}$$

where $p \geq 2$, $B := \{x \in \mathbb{R}^n : |x| < 1\}$, and $n \geq 2$ is an integer. Let $|x| = t$ and $v = -u$. Then, from [1, 2] we can transform (1.1) into the following boundary value problem

$$\begin{cases} t^{1-n}\left(\frac{1}{n}(-v')^{(p-1)n}\right)' = f(t, v), 0 < t < 1, \\ v'(0) = v(1) = 0. \end{cases} \tag{1.2}$$

Consequently, we obtain

$$v(t) = \int_t^1 \left(\int_0^\tau ns^{n-1} f(s, v(s))ds\right)^{\frac{1}{(p-1)n}} d\tau, t \in [0, 1]. \tag{1.3}$$

In this paper, we always assume that the nonlinearity f satisfies the following condition:

(H1) $f : [0, 1] \times \mathbb{R}^+ \to \mathbb{R}^+$ is continuous, $\mathbb{R}^+ := [0, +\infty)$.

A new operator introduced in [3] is p-Monge-Ampère operator, which is denoted by $\det(D(|Du|^{p-2}Du))$, and when $p = 2$, this operator is just the Monge-Ampère operator. We refer the reader to some related results [1–20] and the references therein. For example, in [6] the authors used the method of moving planes to study the monotonicity of positive solutions for the parabolic equation

$$u_t(x, t) - D_s^\theta u(x, t) = f(u(x, t)), \ (x, t) \in \mathbb{R}_+^n \times \mathbb{R},$$

where D_s^θ is called the Monge-Ampère operator, and it is defined by

$$D_s^\theta u(x, t) = \inf_{A \in \mathcal{A}} \left\{ P.V. \int_{\mathbb{R}^n} \frac{u(x, t) - u(y, t)}{\left|A^{-1}(y - x)\right|^{n+2s}} dy \right\},$$

where $t > 0, 0 < s < 1$, P.V. is the Cauchy principal value, and $\mathcal{A} = \{A \mid A$ is $n \times n$ symmetric positive definite matrix, $\det A = 1, \lambda_{\min}(A) \geq \theta > 0\}$. Here, $\lambda_{\min}(A)$ is the smallest eigenvalue of matrix A.

In [13], the authors studied the uniqueness of nontrivial convex solutions for a system of Monge-Ampère equations

$$\begin{cases} \det D^2 u = \gamma |v|^p, \ \text{in } \Omega, \\ \det D^2 v = \mu |u|^{n^2/p}, \ \text{in } \Omega, \\ u = v = 0, \ \text{on } \partial\Omega, \end{cases}$$

where γ, μ are parameters, $\Omega \subset \mathbb{R}^n$ is a bounded, smooth, and uniformly convex domain, and p is close to $n \geq 2$.

Fixed point theory is an important method to find the existence of solutions for nonlinear problems, see for example [2, 4, 8–12, 16–18, 22] and the references therein. In [11], the authors used eigenvalue theory to study the singular p-Monge-Ampère problem

$$\begin{cases} \det\left(D\left(|Du|^{p-2}Du\right)\right) = \mu h(|x|)f(-u) \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases}$$

where μ is a parameter, and Ω is the open unit ball in $\mathbb{R}^n$. They not only obtained the nontrivial solutions for this equation, but considered the dependence of these solutions on the parameter μ. Some more related works, please refer to [2, 8, 9].

In [17], the authors studied the k-Hessian type system with the gradients

$$\begin{cases} S_k\left(\sigma\left(D^2u_i + \alpha\,|\nabla u_i|\,I\right)\right) = \varphi_i\left(|x|, -u_1, -u_2, \ldots, -u_n\right), \text{in}\Omega, \\ u_i = 0, \ \text{ on } \partial\Omega, \ i = 1, 2, \ldots, n. \end{cases}$$

They used $\mathbb{R}_+^n$-monotone matrices and fixed point theory, combining some basic inequality techniques, to obtain several existence results regarding the existence of negative k-convex radial solutions.

Motivated by the aforementioned works, in this paper we use fixed point theory to study the existence and multiplicity of positive radial solutions for (1.1). We obtain the following results: when the nonlinearity f grows $(p-1)n$-superlinearly at ∞, at least one solution is obtained, and when f grows $(p-1)n$-sublinearly at ∞, at least one solution and at least three solutions are derived.

2. Preliminaries

In this section, we first provide some basic notations and lemmas, which are used in the following section. Define two functions as follows:

$$G(t,s) = \begin{cases} n^{\frac{1}{(p-1)n}} s^{\frac{n-1}{(p-1)n}}(1-s), 0 \le t \le s \le 1, \\ n^{\frac{1}{(p-1)n}} s^{\frac{n-1}{(p-1)n}}(1-t), 0 \le s \le t \le 1, \end{cases} \tag{2.1}$$

and

$$\overline{G}(t,s) = \begin{cases} n^{\frac{q}{(p-1)n}} s^{\frac{q(n-1)}{(p-1)n}}(1-s), 0 \le t \le s \le 1, \\ n^{\frac{q}{(p-1)n}} s^{\frac{q(n-1)}{(p-1)n}}(1-t), 0 \le s \le t \le 1, \end{cases} \tag{2.2}$$

where $q \ge (p-1)n$. Then, we have the following lemma:

Lemma 2.1. *Let $\varphi(s) = G(s,s)$, and $\overline{\varphi}(s) = \overline{G}(s,s)$, $s \in [0,1]$. Then, there exist $\kappa_1 := \int_0^1 (1-t)\varphi(t)dt$ and $\kappa_2 := \int_0^1 \overline{\varphi}(t)dt$ such that*

$$\int_0^1 G(t,s)\varphi(t)dt \ge \kappa_1\varphi(s),$$

$$\int_0^1 \overline{G}(t,s)\overline{\varphi}(t)dt \le \kappa_2\overline{\varphi}(s), s \in [0,1].$$

Proof. We can easily prove that

$$G(t,s) \ge (1-t)G(s,s), \quad \overline{G}(t,s) \le \overline{G}(s,s), \ t,s \in [0,1].$$

Hence, we have

$$\int_0^1 G(t,s)\varphi(t)dt \ge \int_0^1 (1-t)\varphi(s)\varphi(t)dt = \kappa_1\varphi(s),$$

and

$$\int_0^1 \overline{G}(t,s)\overline{\varphi}(t)dt \le \int_0^1 \overline{\varphi}(s)\overline{\varphi}(t)dt = \kappa_2\overline{\varphi}(s), \ s \in [0,1].$$

This completes the proof. □

Let $E := C[0,1]$. Then, E is a real Banach space with the norm $\|v\| = \max_{t\in[0,1]} |v(t)|$. Moreover, let a set be $P_0 = \{v \in E : v(t) \ge 0, t \in [0,1]\}$. Then, P_0 is a cone on E. Note that (1.3), we can define an operator $A : P_0 \to P_0$ as follows:

$$(Av)(t) = \int_t^1 \left(\int_0^\tau ns^{n-1}f(s,v(s))ds\right)^{\frac{1}{(p-1)n}} d\tau, t \in [0,1].$$

We easily find that if there exists $v^* \in P_0\backslash\{0\}$ such that $Av^* = v^*$, then this v^* is a positive radial solution for (1.1).

Lemma 2.2. *Assume that v is a nonnegative, concave, decreasing function on* $[0,1]$. *Then, there exists $\kappa_3 := \int_0^1 (1-t)\overline{\varphi}(t)dt$ such that*

$$\int_0^1 v(t)\varphi(t)dt \ge \kappa_1\|v\|, \quad \int_0^1 v(t)\overline{\varphi}(t)dt \ge \kappa_3\|v\|.$$

Proof. Noting that v is concave and it reaches its maximum at $t = 0$, we can obtain

$$\begin{aligned} \int_0^1 v(t)\varphi(t)dt &= \int_0^1 v(1\times t + 0\times(1-t))\varphi(t)dt \\ &\ge \int_0^1 [tv(1) + (1-t)v(0)]\varphi(t)dt \\ &\ge \|v\|\int_0^1 (1-t)\varphi(t)dt, \end{aligned}$$

and

$$\int_0^1 v(t)\overline{\varphi}(t)dt = \int_0^1 v(1 \times t + 0 \times (1-t))\overline{\varphi}(t)dt$$
$$\geq \int_0^1 [tv(1) + (1-t)v(0)]\overline{\varphi}(t)dt$$
$$\geq \|v\| \int_0^1 (1-t)\overline{\varphi}(t)dt.$$

This completes the proof. □

Lemma 2.3. *(see [21]). Suppose that $\Omega \subset E$ is a bounded open set, P is a cone on E, and $A : \overline{\Omega} \cap P \to P$ is a completely continuous operator. If there exists $v_0 \in P \setminus \{0\}$ such that $v - Av \neq \lambda v_0$ for all $\lambda \geq 0$, $v \in \partial\Omega \cap P$, then the fixed point index $i(A, \Omega \cap P, P) = 0$.*

Lemma 2.4. *(see [21]). Suppose that $\Omega \subset E$ is a bounded open set with $0 \in \Omega$, P is a cone on E, and $A : \overline{\Omega} \cap P \to P$ is a completely continuous operator. If $v \neq \lambda Av$ for all $v \in \partial\Omega \cap P$, $0 \leq \lambda \leq 1$, then the fixed point index $i(A, \Omega \cap P, P) = 1$.*

Lemma 2.5. *(see [22]). Let γ be nonnegative and continuous on* $[0, 1]$. *Then,*

(i) $\left(\int_a^b \gamma(t)dt\right)^\alpha \geq (b-a)^{\alpha-1} \int_a^b \gamma^\alpha(t)dt,\ 0 < \alpha \leq 1,$

(ii) $\left(\int_a^b \gamma(t)dt\right)^\alpha \leq (b-a)^{\alpha-1} \int_a^b \gamma^\alpha(t)dt,\ \alpha \geq 1.$

Let E be a real Banach space with a cone P. A map $\widetilde{\beta} : P \to \mathbb{R}^+$ is said to be a nonnegative continuous concave functional on P if $\widetilde{\beta}$ is continuous and

$$\widetilde{\beta}(tu + (1-t)v) \geqslant t\widetilde{\beta}(u) + (1-t)\widetilde{\beta}(v)\ u, v \in P, t \in [0, 1].$$

Let $\widetilde{a}, \widetilde{b}$ be two numbers with $0 < \widetilde{a} < \widetilde{b}$, and $\widetilde{\beta}$ be a nonnegative continuous concave functional on P. We define the following convex sets:

$$P_{\widetilde{a}} = \{v \in P : \|v\| < \widetilde{a}\}, \quad \partial P_{\widetilde{a}} = \{v \in P : \|v\| = \widetilde{a}\},$$
$$\overline{P}_{\widetilde{a}} = \{v \in P : \|v\| \leqslant \widetilde{a}\},$$
$$P(\widetilde{\beta}, \widetilde{a}, \widetilde{b}) = \{v \in P : \widetilde{a} \leqslant \widetilde{\beta}(v), \|v\| \leqslant \widetilde{b}\}.$$

Lemma 2.6. *(see [23]). Let $A : \overline{P}_{\widetilde{c}} \to \overline{P}_{\widetilde{c}}$ be completely continuous, and $\widetilde{\beta}$ a nonnegative continuous concave functional on P such that $\widetilde{\beta}(v) \leqslant \|v\|$ for $v \in \overline{P}_{\widetilde{c}}$. Suppose that there exist $0 < \widetilde{d} < \widetilde{a} < \widetilde{b} \leqslant \widetilde{c}$ such that*

(i) $\{v \in P(\widetilde{\beta}, \widetilde{a}, \widetilde{b}) : \widetilde{\beta}(v) > \widetilde{a}\} \neq \emptyset$ and $\widetilde{\beta}(Av) > \widetilde{a}$ for $v \in P(\widetilde{\beta}, \widetilde{a}, \widetilde{b})$,

(ii) $\|Av\| < \widetilde{d}$ for $\|v\| \leqslant \widetilde{d}$,

(iii) $\widetilde{\beta}(Av) > \widetilde{a}$ for $v \in P(\widetilde{\beta}, \widetilde{a}, \widetilde{c})$ with $\|Av\| > \widetilde{b}$.

Then, A has at least three fixed points v_1, v_2, v_3 in $\overline{P}_{\widetilde{c}}$ such that

$$\|v_1\| < \tilde{d}, \quad \widetilde{a} < \widetilde{\beta}(v_2) \quad \text{and} \quad \|v_3\| > \tilde{d}, \quad \widetilde{\beta}(v_3) < \widetilde{a}.$$

3. Main results

We now list our assumptions for f.

(H2) There exists $d_1 > \kappa_1^{(1-p)n}$ such that

$$\liminf_{v \to +\infty} \frac{f(t, v)}{v^{(p-1)n}} \geq d_1$$

uniformly on $t \in [0, 1]$;

(H3) There exists $d_2 \in \left(0, \kappa_2^{\frac{(1-p)n}{q}}\right)$ (q is as in (2.2)) such that

$$\limsup_{v \to 0^+} \frac{f(t, v)}{v^{(p-1)n}} \leq d_2$$

uniformly on $t \in [0, 1]$;

(H4) There exists $d_3 > \kappa_1^{(1-p)n}$ such that

$$\liminf_{v \to 0^+} \frac{f(t, v)}{v^{(p-1)n}} \geq d_3$$

uniformly on $t \in [0, 1]$;

(H5) There exists $d_4 \in \left(0, 2^{\frac{(p-1)n}{q}-1} \kappa_2^{\frac{(1-p)n}{q}}\right)$ (q is as in (2.2)) such that

$$\limsup_{v \to +\infty} \frac{f(t, v)}{v^{(p-1)n}} \leq d_4$$

uniformly on $t \in [0, 1]$.

Theorem 3.1. *Suppose that (H1)–(H3) hold. Then,* (1.1) *has at least one positive radial solution.*

Proof. From (H2), there exists $e_1 > 0$ such that

$$f(t, v) \geq d_1 v^{(p-1)n} - e_1,\ v \in \mathbb{R}^+, t \in [0, 1]. \tag{3.1}$$

Noting that $\frac{1}{(p-1)n} \leq \frac{1}{2}$, we have

$$(d_1 v^{(p-1)n})^{\frac{1}{(p-1)n}} \leq [f(t, v) + e_1]^{\frac{1}{(p-1)n}} \leq f^{\frac{1}{(p-1)n}}(t, v) + e_1^{\frac{1}{(p-1)n}},$$

and thus

$$f^{\frac{1}{(p-1)n}}(t, v) \geq d_1^{\frac{1}{(p-1)n}} v - e_1^{\frac{1}{(p-1)n}},\ v \in \mathbb{R}^+, t \in [0, 1]. \tag{3.2}$$

We now shall prove that the set

$$W_1 = \{v \in P_0 : v = Av + \lambda v_0,\ \lambda \geq 0\}$$

is bounded in P_0, where $v_0 \in P_0$ is concave and decreasing on $[0, 1]$, i.e.,

$$v_0'(t) \leq 0, v_0''(t) \leq 0,\ t \in [0, 1]. \tag{3.3}$$

Indeed, if $v \in W_1$, then we have

$$v(t) \geq (Av)(t),\ t \in [0, 1].$$

This, together with Lemma 2.5, implies that

$$\begin{aligned} v(t) &\geq \int_t^1 \left(\int_0^\tau ns^{n-1} f(s, v(s))ds\right)^{\frac{1}{(p-1)n}} d\tau \\ &\geq \int_t^1 \tau^{\frac{1}{(p-1)n}-1} \int_0^\tau n^{\frac{1}{(p-1)n}} s^{\frac{n-1}{(p-1)n}} f^{\frac{1}{(p-1)n}}(s, v(s))dsd\tau \\ &\geq \int_t^1 \int_0^\tau n^{\frac{1}{(p-1)n}} s^{\frac{n-1}{(p-1)n}} f^{\frac{1}{(p-1)n}}(s, v(s))dsd\tau \\ &= \int_0^1 G(t, s) f^{\frac{1}{(p-1)n}}(s, v(s))ds, \end{aligned} \tag{3.4}$$

where G is defined in (2.1). Multiplying by $\varphi(t)$ on both sides of (3.4) and integrating over $[0, 1]$, from Lemma 2.1 and (3.2), we have

$$\begin{aligned} \int_0^1 v(t)\varphi(t)dt &\geq \int_0^1 \varphi(t) \int_0^1 G(t, s) f^{\frac{1}{(p-1)n}}(s, v(s))dsdt \\ &\geq \kappa_1 \int_0^1 \varphi(s) f^{\frac{1}{(p-1)n}}(s, v(s))ds \\ &\geq \kappa_1 \int_0^1 \varphi(s) \left[d_1^{\frac{1}{(p-1)n}} v(s) - e_1^{\frac{1}{(p-1)n}}\right] ds. \end{aligned}$$

Noting that $\kappa_1 d_1^{\frac{1}{(p-1)n}} > 1$, we have

$$\int_0^1 v(t)\varphi(t)dt \leq \frac{\kappa_1 e_1^{\frac{1}{(p-1)n}} \int_0^1 \varphi(s)ds}{\kappa_1 d_1^{\frac{1}{(p-1)n}} - 1}. \tag{3.5}$$

From (H1), we have

$$(Av)'(t) = -\left(\int_0^t ns^{n-1} f(s, v(s))ds\right)^{\frac{1}{(p-1)n}} \leq 0,\ t \in [0, 1], \tag{3.6}$$

and

$$\begin{aligned} (Av)''(t) = &-\frac{1}{p-1}\left(\int_0^t ns^{n-1} f(s, v(s))ds\right)^{\frac{1}{(p-1)n}-1} t^{n-1} f(t, v(t)) \\ &\leq 0,\ t \in [0, 1]. \end{aligned} \tag{3.7}$$

Note that $v \in W_1$, (3.3), and (3.6)-(3.7) imply that this $v \in P_0$ is concave, decreasing on $[0, 1]$, and reaches its maximum at $t = 0$. Therefore, from Lemma 2.2 and (3.5), we have

$$\begin{aligned} \|v\| \leq &\frac{1}{\kappa_1} \int_0^1 v(t)\varphi(t)dt \\ \leq &\frac{(ne_1)^{\frac{1}{(p-1)n}}}{\left(\kappa_1 d_1^{\frac{1}{(p-1)n}} - 1\right)\left(\frac{n-1}{(p-1)n} + 1\right)\left(\frac{n-1}{(p-1)n} + 2\right)}. \end{aligned}$$

This proves that W_1 is a bounded set in P_0, as required. If we choose a sufficiently large $R_1 > \sup W_1$, then we have

$$v \neq Av + \lambda v_0,\ v \in \partial B_{R_1} \cap P_0, \lambda \geq 0, \tag{3.8}$$

where $B_{R_1} = \{v \in E : \|v\| < R_1\}$. Therefore, Lemma 2.3 implies that

$$i(A, B_{R_1} \cap P_0, P_0) = 0. \tag{3.9}$$

From (H3), there is a sufficiently small $r_1 \in (0, R_1)$ such that

$$f(t, v) \leq d_2 v^{(p-1)n},\ v \in [0, r_1], t \in [0, 1]. \tag{3.10}$$

In what follows, we prove that

$$v \neq \lambda Av,\ v \in \partial B_{r_1} \cap P_0,\ \lambda \in [0, 1], \tag{3.11}$$

where $B_{r_1} = \{v \in E : \|v\| < r_1\}$. Arguing by contradiction, there exist $v_1 \in \partial B_{r_1} \cap P_0$ and $\lambda_1 \in [0, 1]$ such that

$$v_1 = \lambda_1 A v_1. \tag{3.12}$$

Let q be as in (2.2). Then, we have

$$\begin{aligned} v_1^q(t) &\leq \left(\int_t^1 \left(\int_0^\tau ns^{n-1} f(s, v_1(s))ds\right)^{\frac{1}{(p-1)n}} d\tau\right)^q \\ &\leq \int_t^1 (1-t)^{q-1} \left(\int_0^\tau ns^{n-1} f(s, v_1(s))ds\right)^{\frac{q}{(p-1)n}} d\tau \\ &\leq \int_t^1 \tau^{\frac{q}{(p-1)n}-1} \int_0^\tau n^{\frac{q}{(p-1)n}} s^{\frac{q(n-1)}{(p-1)n}} f^{\frac{q}{(p-1)n}}(s, v_1(s))dsd\tau \\ &\leq \int_t^1 \int_0^\tau n^{\frac{q}{(p-1)n}} s^{\frac{q(n-1)}{(p-1)n}} f^{\frac{q}{(p-1)n}}(s, v_1(s))dsd\tau \\ &= \int_0^1 \overline{G}(t, s) f^{\frac{q}{(p-1)n}}(s, v_1(s))ds, \end{aligned} \tag{3.13}$$

and (3.10) enables us to obtain

$$v_1^q(t) \leq d_2^{\frac{q}{(p-1)n}} \int_0^1 \overline{G}(t, s) v_1^q(s)ds. \tag{3.14}$$

Multiplying by $\overline{\varphi}(t)$ on both sides of (3.14) and integrating over $[0,1]$, from Lemma 2.1 we have

$$\begin{aligned}\int_0^1 v_1^q(t)\overline{\varphi}(t)dt &\le \int_0^1\int_0^1 \overline{G}(t,s)d_2^{\frac{q}{(p-1)n}} v_1^q(s)ds\overline{\varphi}(t)dt\\ &\le \kappa_2 d_2^{\frac{q}{(p-1)n}}\int_0^1 v_1^q(s)\overline{\varphi}(s)ds.\end{aligned}$$

Noting that $\kappa_2 d_2^{\frac{q}{(p-1)n}} \in (0,1)$, we have

$$\int_0^1 v_1^q(t)\overline{\varphi}(t)dt = 0. \tag{3.15}$$

(3.6), (3.7), and (3.12), imply that $v_1 \in P_0$ is concave, decreasing on $[0,1]$, and reaches its maximum at $t = 0$. From (3.15) and Lemmas 2.2 and 2.5, we have

$$\begin{aligned}(\kappa_3\|v_1\|)^q &\le \left(\int_0^1 v_1(t)\overline{\varphi}(t)dt\right)^q\\ &\le \int_0^1 v_1^q(t)\overline{\varphi}^q(t)dt\\ &= \int_0^1 v_1^q(t)\left[\frac{\overline{\varphi}(t)}{n^{\frac{q}{(p-1)n}}}\right]^q n^{\frac{q^2}{(p-1)n}}dt\\ &\le \int_0^1 v_1^q(t)\frac{\overline{\varphi}(t)}{n^{\frac{q}{(p-1)n}}}n^{\frac{q^2}{(p-1)n}}dt = 0.\end{aligned} \tag{3.16}$$

Therefore, $\|v_1\| = 0$ contradicts $v_1 \in \partial B_{r_1} \cap P_0$, and thus (3.11) holds, as required. As a result, Lemma 2.4 implies that

$$i(A, B_{r_1} \cap P_0, P_0) = 1. \tag{3.17}$$

Note that $R_1 > r_1$, and from (3.9) and (3.17), we have

$$\begin{aligned}&i(A,(B_{R_1}\backslash\overline{B}_{r_1})\cap P_0, P_0)\\ &= i(A, B_{R_1}\cap P_0, P_0) - i(A, B_{r_1}\cap P_0, P_0) = -1.\end{aligned}$$

Therefore, the operator A has at least one fixed point in $(B_{R_1}\backslash\overline{B}_{r_1})\cap P_0$. This means that (1.1) has at least one positive radial solution. This completes the proof. □

Theorem 3.2. *Suppose that (H1) and (H4)-(H5) hold. Then,* (1.1) *has at least one positive radial solution.*

Proof. From (H4), there is a sufficiently small $r_2 > 0$ such that

$$f(t,v) \ge d_3 v^{(p-1)n},\ v \in [0, r_2],\ t \in [0,1]. \tag{3.18}$$

Next, we prove that

$$v \ne Av + \lambda\widetilde{v}_0,\ v \in \partial B_{r_2}\cap P_0, \lambda \ge 0, \tag{3.19}$$

where $B_{r_2} = \{v \in E : \|v\| < r_2\}$ and $\widetilde{v}_0 \in P_0$ is a given element. If the claim is not satisfied, then there exist $v_2 \in \partial B_{r_2}\cap P_0$ and $\lambda_2 \ge 0$ such that

$$v_2 = Av_2 + \lambda_2\widetilde{v}_0.$$

Combining this, (3.4), and (3.18), we obtain

$$v_2(t) \ge (Av_2)(t) \ge d_3^{\frac{1}{(p-1)n}}\int_0^1 G(t,s)v(s)ds,$$

and

$$\begin{aligned}\int_0^1 v_2(t)\varphi(t)dt &\ge \int_0^1\int_0^1 G(t,s)d_3^{\frac{1}{(p-1)n}}v(s)ds\varphi(t)dt\\ &\ge \kappa_1 d_3^{\frac{1}{(p-1)n}}\int_0^1 v(s)\varphi(s)ds.\end{aligned}$$

Noting that $\kappa_1 d_3^{\frac{1}{(p-1)n}} > 1$, we have

$$\int_0^1 v_2(t)\varphi(t)dt = 0.$$

Noting that $v_2(t), \varphi(t) \ge 0, \varphi(t) \not\equiv 0, t \in [0,1]$, we have

$$v_2(t) \equiv 0,\ \text{and}\ \|v_2\| = 0,$$

and this contradicts $v_2 \in \partial B_{r_2}\cap P_0$. Therefore, (3.19) holds, as required. Lemma 2.3 implies that

$$i(A, B_{r_2}\cap P_0, P_0) = 0. \tag{3.20}$$

From (H5), there exists $e_2 > 0$ such that

$$f(t,v) \le d_4 v^{(p-1)n} + e_2,\ v \in \mathbb{R}^+, t \in [0,1].$$

Let q be as in (2.2). Then,

$$\begin{aligned}f^{\frac{q}{(p-1)n}}(t,v) &\le [d_4 v^{(p-1)n} + e_2]^{\frac{q}{(p-1)n}}\\ &\le 2^{\frac{q}{(p-1)n}-1}\left(d_4^{\frac{q}{(p-1)n}}v^q + e_2^{\frac{q}{(p-1)n}}\right),\ v \in \mathbb{R}^+, t \in [0,1].\end{aligned} \tag{3.21}$$

Now, we shall prove that the set

$$W_2 = \{v \in P_0 : v = \lambda Av,\ \lambda \in [0,1]\}$$

is bounded in P_0. If $v \in W_2$, then by (3.13) and (3.21) we obtain

$$\begin{aligned}v^q(t) &\le \int_0^1 \overline{G}(t,s)f^{\frac{q}{(p-1)n}}(s,v(s))ds\\ &\le 2^{\frac{q}{(p-1)n}-1}\int_0^1 \overline{G}(t,s)\left(d_4^{\frac{q}{(p-1)n}}v^q(s) + e_2^{\frac{q}{(p-1)n}}\right)ds.\end{aligned} \tag{3.22}$$

Multiplying by $\overline{\varphi}(t)$ on both sides of (3.22) and integrating over [0, 1], from Lemma 2.1 we have

$$\begin{aligned}&\int_0^1 v^q(t)\overline{\varphi}(t)dt\\ \leq&2^{\frac{q}{(p-1)n}-1}\int_0^1\int_0^1 \overline{G}(t,s)\left(d_4^{\frac{q}{(p-1)n}}v^q(s)+e_2^{\frac{q}{(p-1)n}}\right)ds\overline{\varphi}(t)dt\\ \leq&2^{\frac{q}{(p-1)n}-1}\kappa_2\int_0^1\left(d_4^{\frac{q}{(p-1)n}}v^q(s)+e_2^{\frac{q}{(p-1)n}}\right)\overline{\varphi}(s)ds,\end{aligned}$$

and

$$\int_0^1 v^q(t)\overline{\varphi}(t)dt\leq\frac{2^{\frac{q}{(p-1)n}-1}e_2^{\frac{q}{(p-1)n}}\kappa_2\int_0^1\overline{\varphi}(s)ds}{1-2^{\frac{q}{(p-1)n}-1}d_4^{\frac{q}{(p-1)n}}\kappa_2}.$$

Note that $v\in W_2$. Then, from (3.6)-(3.7) we obtain $v\in P_0$ is concave and decreasing on [0, 1], and reaches its maximum at $t=0$. Using (3.16), we obtain

$$\begin{aligned}(\kappa_3\|v\|)^q\leq&\int_0^1 v^q(t)\frac{\overline{\varphi}(t)}{n^{\frac{q}{(p-1)n}}}n^{\frac{q^2}{(p-1)n}}dt\\ \leq&\frac{n^{\frac{q^2-q}{(p-1)n}}2^{\frac{q}{(p-1)n}-1}e_2^{\frac{q}{(p-1)n}}\kappa_2\int_0^1\overline{\varphi}(s)ds}{1-2^{\frac{q}{(p-1)n}-1}d_4^{\frac{q}{(p-1)n}}\kappa_2}.\end{aligned}$$

This implies that

$$\|v\|\leq\kappa_3^{-1}\sqrt[q]{\frac{n^{\frac{q^2-q}{(p-1)n}}2^{\frac{q}{(p-1)n}-1}e_2^{\frac{q}{(p-1)n}}\kappa_2^2}{1-2^{\frac{q}{(p-1)n}-1}d_4^{\frac{q}{(p-1)n}}\kappa_2}},$$

and W_2 is bounded in P_0, as required. Hence, we can choose a sufficiently large $R_2>\max\{r_2,\sup W_2\}$ such that

$$v\neq\lambda Av,\ v\in\partial B_{R_2}\cap P_0,\ \lambda\in[0,1],$$

where $B_{R_2}=\{v\in E:\|v\|<R_2\}$. As a result, Lemma 2.4 implies that

$$i(A,B_{R_2}\cap P_0,P_0)=1. \tag{3.23}$$

Then, from (3.20) and (3.23), we have

$$\begin{aligned}&i(A,(B_{R_2}\backslash\overline{B}_{r_2})\cap P_0,P_0)\\ =&i(A,B_{R_2}\cap P_0,P_0)-i(A,B_{r_2}\cap P_0,P_0)\\ =&1.\end{aligned}$$

Therefore, the operator A has at least one fixed point in $(B_{R_2}\backslash\overline{B}_{r_2})\cap P_0$. This means that (1.1) has at least one positive radial solution. This completes the proof. □

Theorem 3.3. *Suppose that (H1) and (H5) hold with* $d_4\in\left(0,\frac{(2q+2)^{\frac{(p-1)n}{q}}}{2n}\right)$*. Then,* (1.1) *has at least one positive radial solution if f satisfies the following conditions:*

(H6) f is nondecreasing about v, i.e., $f(t,u)\geq f(t,v)$ *if* $u\geq v,\ u,v\in\mathbb{R}^+$*, for all* $t\in[0,1]$*;*

(H7) $f(t,0)\not\equiv 0, t\in[0,1]$.

Proof. Note that (H5) implies that (3.21) is still satisfied. If we choose a sufficiently large

$$M\geq\sqrt[q]{\frac{(q+1)2^{\frac{q}{(p-1)n}-1}n^{\frac{q}{(p-1)n}}e_2^{\frac{q}{(p-1)n}}}{q+1-2^{\frac{q}{(p-1)n}-1}n^{\frac{q}{(p-1)n}}d_4^{\frac{q}{(p-1)n}}}}$$

and let $v_0(t)=M(1-t), t\in[0,1]$. Then, from (3.13) and (3.22), we have

$$\begin{aligned}&[(Av_0)(t)]^q\\ \leq&2^{\frac{q}{(p-1)n}-1}\int_0^1(1-t)^{q-1}\overline{G}(t,s)\left(d_4^{\frac{q}{(p-1)n}}v_0^q(s)+e_2^{\frac{q}{(p-1)n}}\right)ds\\ \leq&2^{\frac{q}{(p-1)n}-1}n^{\frac{q}{(p-1)n}}(1-t)^q\int_0^1\left(d_4^{\frac{q}{(p-1)n}}[M(1-s)]^q+e_2^{\frac{q}{(p-1)n}}\right)ds\\ \leq&[M(1-t)]^q,\end{aligned}$$

and thus

$$Av_0\leq v_0.$$

Now we establish a sequence $\{v_n\}_{n=0}^{\infty}$ with

$$v_{n+1}=Av_n. \tag{3.24}$$

Using (H6), we have

$$v_1(t)=(Av_0)(t)\leq v_0(t),\ t\in[0,1],$$

and

$$\begin{aligned}v_2(t)=&(Av_1)(t)\\ =&\int_t^1\left(\int_0^\tau ns^{n-1}f(s,v_1(s))ds\right)^{\frac{1}{(p-1)n}}d\tau\\ \leq&\int_t^1\left(\int_0^\tau ns^{n-1}f(s,v_0(s))ds\right)^{\frac{1}{(p-1)n}}d\tau\\ =&(Av_0)(t)=v_1(t),\ t\in[0,1].\end{aligned}$$

By means of the mathematical induction, for all $n\in\mathbb{N}^+$, we obtain

$$v_{n+1}\leq v_n\leq\cdots\leq v_1\leq v_0\leq M.$$

Therefore, there exists $v^* \in P_0$ such that $\lim_{n\to\infty} v_n = v^*$. Letting $n \to \infty$, by (3.24) we have

$$v^* = Av^*.$$

From (H7), 0 is not a fixed point of A, and thus this v^* is a positive radial solution for (1.1). This completes the proof. □

Now we study the multiplicity of positive solutions for (1.1). From [2, 11], we obtain

$$A(P_0) \subset P_1, \tag{3.25}$$

where

$$P_1 = \left\{ v \in P_0 : \min_{\frac{1}{4}\le t\le\frac{3}{4}} v(t) \ge \frac{1}{4}\|v\| \right\}. \tag{3.26}$$

Theorem 3.4. *Suppose that (H1) and (H5) hold with* $d_4 \in \left(0, \frac{\left[2\left(1+\frac{(p-1)n}{q(n-1)}\right)\left(1+\frac{2(p-1)n}{q(n-1)}\right)\right]^{\frac{(p-1)n}{q}}}{2n}\right)$. *Then,* (1.1) *has at least three positive radial solutions if f satisfies the following conditions:*

(H8) there exist $\widetilde{d} > 0$ *and* $d_5 \in \left(0, \frac{\left[\left(1+\frac{(p-1)n}{q(n-1)}\right)\left(1+\frac{2(p-1)n}{q(n-1)}\right)\right]^{\frac{(p-1)n}{q}}}{n}\right)$ *such that*

$$f(t, v) \leqslant d_5\widetilde{d}^{(p-1)n},\ v \in [0, \widetilde{d}], t \in [0, 1];$$

(H9) there exist $\widetilde{a}$ *with* $\widetilde{a} > \widetilde{d}$ *and* $d_6 > \left(\frac{4(pn-1)}{(p-1)n}\right)^{(p-1)n} \big/ n$ *such that*

$$f(t, v) \geqslant d_6\widetilde{a}^{(p-1)n},\ v \in [\widetilde{a}, 4\widetilde{a}],\ \ t \in [0, 1].$$

Then, (1.1) *has at least three positive solutions.*

Proof. From (H5), we know that (3.21) holds. Then, by (3.22), choosing

$$\widetilde{c} \ge \frac{2^{\frac{q}{(p-1)n}-1} n^{\frac{q}{(p-1)n}} e_2^{\frac{q}{(p-1)n}}}{\left(1+\frac{(p-1)n}{q(n-1)}\right)\left(1+\frac{2(p-1)n}{q(n-1)}\right) - 2^{\frac{q}{(p-1)n}-1} n^{\frac{q}{(p-1)n}} d_4^{\frac{q}{(p-1)n}}}$$

for $\|v\| \le \widetilde{c}$ we have

$$\begin{aligned}
&[(Av)(t)]^q \\
\le& 2^{\frac{q}{(p-1)n}-1}\int_0^1 \overline{G}(t,s)\left(d_4^{\frac{q}{(p-1)n}} v^q(s) + e_2^{\frac{q}{(p-1)n}}\right)ds \\
\le& 2^{\frac{q}{(p-1)n}-1}\int_0^1 n^{\frac{q}{(p-1)n}} s^{\frac{q(n-1)}{(p-1)n}}(1-s)\left(d_4^{\frac{q}{(p-1)n}}\widetilde{c}^q + e_2^{\frac{q}{(p-1)n}}\right)ds \\
\le& \widetilde{c}^q.
\end{aligned}$$

This implies that $A : \overline{P}_{\widetilde{c}} \to \overline{P}_{\widetilde{c}}$.

For $v \in P_1$, define $\widetilde{\beta}(v) = \min_{\frac{1}{4}\le t\le\frac{3}{4}} v(t)$. Then, $\widetilde{\beta}$ is a nonnegative continuous concave functional on P_1, and the following inequality holds:

$$\widetilde{\beta}(v) \leqslant \max_{t\in[0,1]} v(t) = \|v\|,\ v \in P_1.$$

If we let $v(t) \equiv 2.5\,\widetilde{a} > \widetilde{a}$, then this v belongs to $\left\{v \in P\left(\widetilde{\beta}, \widetilde{a}, 4\widetilde{a}\right) : \widetilde{\beta}(v) > \widetilde{a}\right\} := W_3$, i.e., $W_3 \neq \emptyset$. Moreover, if $v \in P\left(\widetilde{\beta}, \widetilde{a}, 4\widetilde{a}\right)$ and $\widetilde{\beta}(v) > \widetilde{a}$, we have

$$4\widetilde{a} \geqslant \|v\| \geqslant \widetilde{\beta}(v) > \widetilde{a}.$$

From (H9) and (3.4), we obtain

$$\begin{aligned}
\widetilde{\beta}(Av)(t) \ge& \min_{\frac{1}{4}\le t\le\frac{3}{4}} \int_0^1 G(t,s) f^{\frac{1}{(p-1)n}}(s, v(s))ds \\
\ge& \min_{\frac{1}{4}\le t\le\frac{3}{4}} (1-t)\int_0^1 n^{\frac{1}{(p-1)n}} s^{\frac{n-1}{(p-1)n}} (d_6\widetilde{a}^{(p-1)n})^{\frac{1}{(p-1)n}} ds \\
\ge& \frac{(p-1)n}{4(pn-1)}(nd_6)^{\frac{1}{(p-1)n}}\widetilde{a} \\
>& \widetilde{a},
\end{aligned}$$

and by this we find $\widetilde{\beta}(Av) > \widetilde{a}$ for $v \in P\left(\widetilde{\beta}, \widetilde{a}, 4\widetilde{a}\right)$.

Next, we claim that $\|Av\| < \widetilde{d}$ for $\|v\| \leqslant \widetilde{d}$. In fact, if $v \in \overline{P}_{\widetilde{d}}$, from (H8) and (3.13) we have

$$\begin{aligned}
[(Av)(t)]^q \le& \int_0^1 \overline{G}(t,s) f^{\frac{q}{(p-1)n}}(s, v(s))ds \\
\le& d_5^{\frac{q}{(p-1)n}}\widetilde{d}^{\,q}\int_0^1 n^{\frac{q}{(p-1)n}} s^{\frac{q(n-1)}{(p-1)n}}(1-s)ds \\
<& \widetilde{d}^{\,q}.
\end{aligned}$$

This shows that $\|Av\| < \widetilde{d}$, and

$$A : \overline{P}_{\widetilde{d}} \to P_{\widetilde{d}} \text{ for } v \in \overline{P}_{\widetilde{d}}.$$

Finally, we show that if $v \in P(\widetilde{\beta}, \widetilde{a}, \widetilde{c})$ and $\|Av\| > 4\widetilde{a}$, then $\widetilde{\beta}(Av) > \widetilde{a}$. To prove this, if $v \in P(\widetilde{\beta}, \widetilde{a}, \widetilde{c})$ and $\|Av\| > 4\widetilde{a}$, then we have

$$\begin{aligned}
&\widetilde{\beta}(Av)(t) \\
=& \min_{\frac{1}{4}\le t\le\frac{3}{4}} \int_t^1 \left(\int_0^\tau ns^{n-1} f(s, v(s))ds\right)^{\frac{1}{(p-1)n}} d\tau \\
\ge& \min_{\frac{1}{4}\le t\le\frac{3}{4}} (1-t)\int_0^1 \left(\int_0^\tau ns^{n-1} f(s, v(s))ds\right)^{\frac{1}{(p-1)n}} d\tau \\
\ge& \frac{1}{4}\|Av\|.
\end{aligned}$$

Consequently, we have

$$\widetilde{\beta}(Av) \geqslant \frac{1}{4}\|Av\| > \frac{1}{4} \times 4\widetilde{a} = \widetilde{a}.$$

As a result, all the conditions of Lemma 2.6 are satisfied by taking $\widetilde{b} = 4\widetilde{a}$. Hence, A has at least three fixed points, i.e., (1.1) has at least three positive solutions v_i ($i = 1, 2, 3$) such that

$$\|v_1\| < \widetilde{d}, \quad \widetilde{a} < \widetilde{\beta}(v_2), \text{ and } \|v_3\| > \widetilde{d} \text{ with } \widetilde{\beta}(v_3) < \widetilde{a}.$$

This completes the proof. □

In what follows, we provide some examples to verify our main theorems. Let $n = 2, p = 3$, and $q = 4$. Then, $\kappa_1 = 0.26$, $\kappa_2 = \frac{1}{3}$, and $\kappa_3 = \frac{1}{6}$.

Example 3.1. *Let $f(t, v) = v^5$, $v \in \mathbb{R}^+$, and $t \in [0, 1]$. Then,*

$$\liminf_{v\to+\infty} \frac{f(t, v)}{v^4} = +\infty, \ \limsup_{v\to 0^+} \frac{f(t, v)}{v^4} = 0$$

uniformly on $t \in [0, 1]$, and thus (H1)–(H3) hold. From Theorem 3.1, (1.1) *has a positive radial solution.*

Example 3.2. *Let $f(t, v) = v^3$, $v \in \mathbb{R}^+$, and $t \in [0, 1]$. Then,*

$$\liminf_{v\to 0^+} \frac{f(t, v)}{v^4} = +\infty, \ \limsup_{v\to+\infty} \frac{f(t, v)}{v^4} = 0$$

uniformly on $t \in [0, 1]$, and thus (H1) and (H4)-(H5) hold. From Theorem 3.2, (1.1) *has a positive radial solution.*

Example 3.3. *Let $f(t, v) = e^t + \delta v^4$, $v \in \mathbb{R}^+, t \in [0, 1]$ and $\delta \in (0, 2.5)(2.5 = \frac{(2q+2)^{\frac{(p-1)n}{q}}}{2n})$. Then, (H1) and (H6)-(H7) hold. Note that*

$$\limsup_{v\to+\infty} \frac{f(t, v)}{v^4} = \delta$$

uniformly on $t \in [0, 1]$, and (H5) is also satisfied. From Theorem 3.3, (1.1) *has a positive radial solution.*

Example 3.4. *Let $\widetilde{a} = 100$, $\widetilde{d} = 1$, and*

$$f(t, v) = \begin{cases} \frac{e^t}{e} + v^4, \ v \in [0, 1], t \in [0, 1], \\ \frac{e^t}{e} + 3.13v^5 - 2.13, \ v \in [1, 100], t \in [0, 1], \\ \frac{e^t}{e} + 3.13 \times 10^{10} - 2.13, \ v \in [100, 400], t \in [0, 1], \\ \frac{e^t}{e} + \frac{7825}{16}v^3 - 2.13, \ v \geq 400, t \in [0, 1]. \end{cases}$$

Then,

$$\limsup_{v\to+\infty} \frac{f(t, v)}{v^4} = \limsup_{v\to+\infty} \frac{\frac{e^t}{e} + \frac{7825}{16}v^3 - 2.13}{v^4} = 0$$

uniformly on $t \in [0, 1]$. Moreover, we obtain that

(i) if $v \in [0, 1], t \in [0, 1]$, then $f(t, v) \leq 2 \in (0, 3)$;

(ii) if $v \in [100, 400], t \in [0, 1]$, then

$$f(t, v) \geq 3.13 \times 10^{10} - 2.13 \approx 31299999997.87$$
$$\geq d_6 \times 10^8, d_6 > 312.5.$$

Consequently, (H1), (H5), and (H8)-(H9) hold. Theorem 3.4 implies that (1.1) *has at least three positive radial solutions.*

4. Conclusions

In Theorems 3.1 and 3.2, we utilize the properties of the operator A, combining some inequality techniques, such as the Jensen inequality, to study problem (1.1). So, our work space only needs to be P_0. This is different from the related works [2, 8, 9, 11], where their work space is P_1 (see (3.26)). Moreover, in Theorem 3.3, when the nonlinearity f grows $(p-1)n$-sublinearly at ∞, we use the monotone bounded principle to obtain at least one solution, and provide an iterative sequence for the solution.

Finally, by means of the Leggett-Williams fixed point theorem and the cone P_1, in Theorem 3.4 we study the multiplicity of positive radial solutions for our problem. The method here is also different from the aforementioned papers.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare that there are no conflicts of interests in this paper. Authors are listed in alphabetical order of their surnames.

References

1. J. Bao, Q. Feng, Necessary and sufficient conditions on global solvability for the p-k-Hessian inequalities, *Canad. Math. Bull.*, **65** (2022), 1004–1019. https://doi.org/10.4153/s0008439522000066

2. M. Feng, Nontrivial p-convex solutions to singular p-Monge-Ampère problems: existence, multiplicity and nonexistence, *Commun. Anal. Mech.*, **16** (2024), 71–93. https://doi.org/10.3934/cam.2024004

3. N. S. Trudinger, X. Wang, Hessian measures II, *Ann. Math.*, **150** (1999), 579–604.

4. Z. Bai, Z. Yang, Existence of k-convex solutions for the k-Hessian equation, *Mediterr. J. Math.*, **20** (2023), 150. https://doi.org/10.1007/s00009-023-02364-8

5. W. Cao, L. Székelyhidi, Very weak solutions to the two-dimensional Monge-Ampère equation, *Sci. China Math.*, **62** (2019), 1041–1056. https://doi.org/10.1007/s11425-018-9516-7

6. G. Du, X. Wang, Monotonicity of solutions for parabolic equations involving nonlocal Monge-Ampère operator, *Adv. Nonlinear Anal.*, **13** (2024), 20230135. https://doi.org/10.1515/anona-2023-0135

7. B. Emamizadeh, Y. Liu, G. Porru, Overdetermined problems for p-Laplace and generalized Monge-Ampère equations, *Complex Var. Elliptic Equ.*, **67** (2022), 807–821. https://doi.org/10.1080/17476933.2020.1843447

8. M. Feng, A class of singular coupled systems of superlinear Monge-Ampère equations, *Acta Math. Appl. Sin. Engl. Ser.*, **38** (2022), 925–942. https://doi.org/10.1007/s10255-022-1024-5

9. M. Feng, Convex solutions of Monge-Ampère equations and systems: existence, uniqueness and asymptotic behavior, *Adv. Nonlinear Anal.*, **10** (2021), 371–399. https://doi.org/10.1515/anona-2020-0139

10. M. Feng, A class of singular k_i-Hessian systems, *Topol. Methods Nonlinear Anal.*, **62** (2023), 341–365. https://doi.org/10.12775/tmna.2022.072

11. M. Feng, Eigenvalue problems for singular p-Monge-Ampère equations, *J. Math. Anal. Appl.*, **528** (2023), 127538. https://doi.org/10.1016/j.jmaa.2023.127538

12. H. Gao, L. Wang, J. Li, Existence of radial solutions for k-Hessian system, *AIMS Math.*, **8** (2023), 26498–26514. https://doi.org/10.3934/math.20231353

13. N. Q. Le, Uniqueness for a system of Monge-Ampère equations, *Methods Appl. Anal.*, **28** (2021), 15–30. https://doi.org/10.4310/MAA.2021.v28.n1.a2

14. J. Shi, F. Jiang, On Neumann problem for the degenerate Monge-Ampère type equations, *Bound. Value Probl.*, **2021** (2021), 11. https://doi.org/10.1186/s13661-021-01486-w

15. M. Sroka, Weak solutions to the quaternionic Monge-Ampère equation, *Anal. PDE*, **13** (2020), 1755–1776. https://doi.org/10.2140/apde.2020.13.1755

16. Z. Yang, Z. Bai, Existence and multiplicity of radial solutions for a k-Hessian system, *J. Math. Anal. Appl.*, **512** (2022), 126159. https://doi.org/10.1016/j.jmaa.2022.126159

17. Z. Yang, Z. Bai, Existence results for the k-Hessian type system with the gradients via $\mathbb{R}^n_+$-monotone matrices, *Nonlinear Anal.*, **240** (2024), 113457. https://doi.org/10.1016/j.na.2023.113457

18. Z. Yu, Z. Bai, The existence of radial solutions for a class of k-Hessian systems with the nonlinear gradient terms, *J. Appl. Math. Comput.*, **70** (2024), 2225–2240. https://doi.org/10.1007/s12190-024-02049-9

19. X. Zhang, M. Feng, The existence and asymptotic behavior of boundary blow-up solutions to the k-Hessian equation, *J. Differ. Equations*, **267** (2019), 4626–4672. https://doi.org/10.1016/j.jde.2019.05.004

20. X. Zhang, M. Feng, Boundary blow-up solutions to singular k-Hessian equations with gradient terms: sufficient and necessary conditions and asymptotic behavior, *J. Differ. Equations*, **375** (2023), 475–513. https://doi.org/10.1016/j.jde.2023.08.022

21. D. Guo, V. Lakshmikantham, *Nonlinear Problems in Abstract Cones*, Academic Press, 1988. https://doi.org/10.1016/C2013-0-10750-7

22. J. Xu, Z. Yang, Positive solutions for a fourth order p-laplacian boundary value problem, *Nonlinear Anal.*, **74** (2011), 2612–2623. https://doi.org/10.1016/j.na.2010.12.016

23. R. W. Leggett, L. R. Williams, Multiple positive fixed points of nonlinear operators on ordered Banach spaces, *Indiana Univ. Math. J.*, **28** (1979), 673–688. https://doi.org/10.1512/iumj.1979.28.28046