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Research article

Initial value problem for fractional differential equations of variable order

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Abstract: This study presented a novel approach to investigating the existence, uniqueness, and stability of solutions for an initial value problem involving fractional differential equations of variable order. In contrast to conventional methods in the literature, which often utilized generalized intervals and piecewise constant functions, we introduced a new fractional operator that is more appropriate for this problem. The existence and uniqueness of the solutions ware demonstrated through Leray-Schauder fixed point theorem and Banach's theorem, with an analysis of the uniform stability of the problem. The strength of our approach lies in its straightforwardness and reliance on fewer restrictive assumptions. The study concluded with an application that features a practical example, accompanied by visual illustrations.

Keywords: existence and uniqueness of solutions; initial value problem; fractional variable order; uniform stability; fixed point theorem

1. Introduction

Fractional calculus and fractional differential equations have garnered significant attention due to the extensive applications of fractional derivative operators in mathematical modelling. These operators often provide a more accurate representation of many real-world processes compared to classical differential equations, particularly when modeling fractional versions of phenomena in nature and biology. For a comprehensive treatment of the subject, refer to the relevant works. Mohammadi et al. [1] presented a model of hearing loss in children caused by mumps, utilizing the Caputo-Fabrizio fractional-order derivative, which retains the system's historical memory. Chávez-Vázquez et al. [2] designed and developed a fractional control strategy for trajectory tracking tasks of the Stanford robot powered by induction motors. This strategy involves fractional proportional–integral controllers for the actuators, utilizing the Atangana–Baleanu integral, in combination with a fractional integral sliding-mode control law generalized to an arbitrary order using the Caputo–Fabrizio derivative and the Atangana–Baleanu integral. In their study, Khan et al. [3] explored the existence, stability, and computational analysis of a waterborne disease model using the fractal-fractional version of the derivative. The existence results were derived through techniques involving convergent iterative sequences. Dehingia et al. [4] presented an epidemiological model describing the within-host transmission dynamics of severe acute respiratory syndrome coronavirus 2 (SARS-CoV-2) influenced by the fractional-order derivative effect. The study first discusses the existence, nonnegativity, and boundedness of the model's solution. Hussain et al. [5] designed a new stochastic mathematical model for the spread of COVID-19, incorporating environmental white noise. The model was examined mathematically for the existence of solutions, disease persistence, and extinction. Aydogan et al. [6] studied the mathematical model of rabies using the Caputo–Fabrizio fractional derivative. The fractional differential equations were solved using the Laplace-Adomian decomposition method. Ahmad et al. [7] aimed to investigate the existence and Ulam-Hyers stability (U-HS) of solutions for a nonlinear neutral stochastic fractional differential system. Hussain et al. [8] developed and analyzed a novel stochastic mathematical model for the spread of COVID-19 with white noise. The model was examined for solution existence and other factors, including disease persistence and extinction. Tuan et al. [9] investigated a mathematical model for the transmission of COVID-19 using the fractional-order Caputo derivative. They calculated the feasibility region, equilibrium points, and R_0, and proved the existence of a unique solution using fixed-point theory. Approximate solutions for System (1) were provided using the Adams-Bashforth scheme. Khan et al. [10] presented a new mathematical model for tuberculosis (TB) in fractal-fractional settings, which describes the status of the disease in China based on a case study. Baleanu et al. [11] extended the model of epidemic childhood diseases using the Caputo-Fabrizio fractional derivative. The fractional differential equations were solved using the Laplace-Adomian decomposition method, and the equilibrium points and conditions for local asymptotic stability of the disease-free equilibrium point were determined. Baleanu et al. [12] examined the mathematical model that describes hematopoiesis (the Mackey–Glass model) using the Caputo operator. The system displayed more complex behavior due to its parameters and the delayed production rate of blood cells. Baleanu et al. [13] investigated the existence of solutions for a novel fractional multi-term boundary value problem, modeling each edge of the graph representation of the Glucose molecule, based on a new labeling method for vertices of arbitrary graphs. Baleanu et al. [11] presented a fractional-order epidemic model for childhood diseases using the Caputo–Fabrizio fractional derivative. The fractional differential equations were solved using the Laplace-Adomian decomposition method. In [14], a new symmetric fractional-order discrete system was introduced. The dynamics and symmetry of the proposed model were analyzed under two initial conditions, focusing on a comparison between commensurate and incommensurate order maps, which highlights their impact on symmetry-breaking bifurcations. Xu et al. [15], they explored the impact of system initialization on the performance of iterative learning control (ILC) for fractional-order systems and investigated strategies to enhance system convergence. Agarwal et al. [16] studied the following constant fractional order problem

$$\begin{cases} \mathbb{D}^{\chi}_{0^+}\mu(\zeta) = \eta(\zeta, \mu(\zeta)), \ \zeta \in [0, \infty), \ \chi \in (1, 2], \\ \mu(0) = 0, \ \mu \text{ bounded on } [0, \infty), \end{cases}$$

where $\mathbb{D}^{\chi}_{0^+}$ stands for the Riemann-Liouville fractional derivative of order χ, and η is a given continuous function. In all these contributions, the fractional operators of constant order were considered and the conclusions were reached using the appropriate hypotheses.

Recently, the concept and formal definition of variable-order fractional operators have emerged. Unlike fixed-order fractional operators, variable-order differentiation and integration allow the order to vary continuously based on the dependent or independent variables involved. This approach offers greater flexibility than traditional fractional-order methods and serves as a natural progression in the mathematical framework [17–19]. These operators have proven effective in capturing the complexity of real-world phenomena across multiple disciplines, including biology, mechanics, control systems, and transport processes. Their capacity to derive evolutionary governing equations has made them a focus of intense research. As a result, many studies have explored their application in the modeling of engineering and physical systems, as shown in some of the foundational literature on the subject [20–22]. However, there is relatively little research that studies the problems for nonlinear differential equations of variable fractional order (see [23–25]), and it is essential to highlight that the study extensively employs the concept of a piecewise constant function (PWCF), which plays a pivotal role. For this reason, the interval of existence $[0,\rho]$ is partitioned as follows: $M := \{I_1 = [0,\rho_1], I_2 = (\rho_1,\rho_2], I_3 = (\rho_2,\rho_3], \cdots, I_\sigma = (\rho_{\sigma-1},\rho]\}$, where σ represents a given natural number. Moreover, the PWCF $q(t) : [0,\rho] \to (0,1]$ with respect to P is expressed as

$$q(t) = \sum_{k=1}^{\sigma} q_k I_k(t),\ t \in [0,\rho],$$

where $0 < q_k \leq 1,\ k = 1,2,\cdots,\sigma$ are constants. Here, $\rho_0 = 0$ and $\rho_\sigma = \rho$, meaning that $I_k = 1$ for $t \in [\rho_{k-1},\rho_k]$, and $I_k = 0$ elsewhere. Most of the results referenced are derived using this method, which initially divides the interval of existence into subintervals, then defines the differential and integral operators relative to these subintervals. This approach enabled researchers to transform fractional problems with variable order into their corresponding conventional fractional problems of constant order.

In this study, we propose an innovative method that eliminates the need for the piecewise constant function and partitioning of the existence interval. The cornerstone of our approach is the development of a more flexible operator that requires no extra steps. We apply this novel method to the following initial value problem (IVP) with variable order

$$\begin{cases} \mathbb{D}_{0^+}^{\chi(\zeta)}\mu(\zeta) = \eta(\zeta,\mu(\zeta)),\ \zeta \in A := [0,L] \\ \mu(0) = 0, \end{cases} \tag{1.1}$$

where $0 < L < +\infty$ and $\chi : A \to (0,1)$, $\eta : A \times \mathbb{R} \to \mathbb{R}$ are continuous functions and $\mathbb{D}_{0^+}^{\chi(\zeta)}$ is the Riemann-Liouville fractional derivative of variable-order $\chi(\zeta)$.

The main purpose of this paper is to propose new criteria on the uniqueness and existence for solutions of IVP (1.1). Further, we study the uniform stability criterion of the obtained solutions. Additionally, we also conducted an approximate numerical study of IVP (1.1). An example is given at the end to illustrate the theoretical results.

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts that are used throughout this paper.

Note that the set $\mathbb{E} = C(A,\mathbb{R})$ is a Banach space of continuous functions μ from A into $\mathbb{R}$, such that, $\mu(0) = 0$ with a norm defined as

$$\|\mu\| = sup\{|\mu(\zeta)|/\zeta \in A\}.$$

Definition 2.1. *[26–28] Let $\chi : A \to (0,1)$ be a continuous function, and the left Riemann Liouville fractional integral of variable order $\chi(\zeta)$ for function $\mu(\zeta)$ is defined by*

$$\mathbb{I}_{0^+}^{\chi(\zeta)}\mu(\zeta) = \int_0^\zeta \frac{(\zeta-\varpi)^{\chi(\varpi)-1}}{\Gamma(\chi(\varpi))}\mu(\varpi)d\varpi,\ \zeta > 0, \tag{2.1}$$

where $\Gamma(\cdot)$ is the gamma function.

Definition 2.2. *[26–28] Let $\chi : A \to (1,2)$ be a continuous function, and the left Riemann Liouville fractional derivative of variable order $\chi(\zeta)$ for function $\mu(\zeta)$ is defined by*

$$\begin{aligned} \mathbb{D}_{0^+}^{\chi(\zeta)}\mu(\zeta) &= \Big(\frac{d}{dt}\Big)\mathbb{I}_{0^+}^{1-\chi(\zeta)}\mu(\zeta) \\ &= \Big(\frac{d}{dt}\Big)\int_0^\zeta \frac{(\zeta-\varpi)^{-\chi(\varpi)}}{\Gamma(1-\chi(\varpi))}\mu(\varpi)d\varpi, \zeta > 0. \end{aligned} \tag{2.2}$$

Remark 2.1. *[29, 30] For general functions $\chi(\zeta)$, $v(\zeta)$, we notice that the semi group property doesn't hold, i.e.*

$$\mathbb{I}_{a^+}^{\chi(\zeta)}\mathbb{I}_{a^+}^{v(\zeta)}\mu(\zeta) \neq \mathbb{I}_{a^+}^{\chi(\zeta)+v(\zeta)}\mu(\zeta).$$

Lemma 2.1. *Let $\chi : A \to (0,1)$ be a continuous function, then for $y \in C_\sigma(A,\mathbb{R}) = \{y(\zeta) \in C(A,\mathbb{R}),\ \zeta^\sigma y(\zeta) \in C(A,\mathbb{R}),\ (0 \leq \sigma \leq \min_{\zeta\in A}|\chi(\zeta)|)\}$, the variable order fractional integral $\mathbb{I}_{0^+}^{\chi(\zeta)}y(\zeta)$ exists $\forall\zeta \in A$.*

Lemma 2.2. *Let $\chi \in C(A,(0,1))$ be a continuous function, then $\mathbb{I}_{0^+}^{\chi(\zeta)}y(\zeta) \in C(A,\mathbb{R})$ for $y \in C(A,\mathbb{R})$.*

Theorem 2.1. *[31] Let $\Im$ be a banach space and $\wp : \Im \to \Im$ be a mapping such that, $\wp^n$ is a contraction, for some $n \in N$. Then, $\wp$ has a unique fixed point in $\Im$.*

Theorem 2.2. *[32] Let $\mathfrak{I}$ be a banach space with $k \subset \mathfrak{I}$ closed and convex. Assume $\mathbb{U}$ is a relatively open subset of k with $0 \in \mathbb{U}$ and $C : \bar{\mathbb{U}} \to k$ is a compact map. Then, either*

(1) *C has a fixed point in $\bar{\mathbb{U}}$;*

or

(2) *there is a point $\mu \in \partial\mathbb{U}$ and $\lambda \in (0, 1)$ with $\mu = \lambda C(\mu)$.*

3. Main results

3.1. Existence criteria

Let us introduce the following hypothesis:

(HY) The function $\zeta^{\sigma}\eta$ is a continuous function on $A \times \mathbb{R}$ and there exist constants $0 \leq \sigma < \min_{\zeta\in A} \mid \chi(\zeta) \mid$, $p > 0$, such that:
$\zeta^{\sigma} \mid \eta(\zeta, \mu(\zeta)) - \eta(\zeta, y(\zeta)) \mid\leq p \mid \mu(\zeta) - y(\zeta) \mid, \quad \forall \mu, y \in \mathbb{R}, \zeta \in A$.

Remark 3.1. **(1)** *The function $\Gamma(1\text{-}\chi(\zeta))$ is continuous as a composition of two continuous functions, and we can let $M_\Gamma = \max_{\zeta\in[0,L]} \mid \frac{1}{\Gamma(1-\chi(\zeta))} \mid$.*

(2) *By the continuity of the function $\chi(\zeta)$, we let $L^{-\chi(\zeta)} \leq 1$, if $1 \leq L < \infty$, $\quad L^{-\chi(\zeta)} \leq L^{-\chi^*}$, if $0 \leq L \leq 1$, where $\chi^* = \max_{\zeta\in[0,L]} \mid \chi(\zeta) \mid$. We conclude that $L^{-\chi(\zeta)} \leq \max(1, L^{-\chi^*}) = L^*$.*

We will need the following lemma to solve IVP (1.1).

Lemma 3.1. *The function $\mu \in C(A, \mathbb{R})$ forms a solution of IVP (1.1), if and only, if μ solves the integral equation*

$$\int_0^{\zeta} \frac{(\zeta - \varpi)^{-\chi(\varpi)}}{\Gamma(1 - \chi(\varpi))}\mu(\varpi)\ d\varpi = \int_0^{\zeta} \eta(\varpi, \mu(\varpi))\ d\varpi, \quad (3.1)$$

and μ fulfills the initial condition $\mu(0) = 0$.

Proof. By the definition of fractional derivative of variable order defined by (2.2), the IVP (1.1) can be written in the form:

$$(\frac{d}{dt})\int_0^{\zeta} \frac{(\zeta - \varpi)^{-\chi(\varpi)}}{\Gamma(1 - \chi(\varpi))}\mu(\varpi)\ d\varpi = \eta(\zeta, \mu(\zeta)).$$

Then,

$$\int_0^{\zeta} \frac{(\zeta - \varpi)^{-\chi(\varpi)}}{\Gamma(1 - \chi(\varpi))}\mu(\varpi)\ d\varpi = \int_0^{\zeta} \eta(\varpi, \mu(\varpi))\ d\varpi + c_1. \quad (3.2)$$

Evaluating Eq (3.2) at $\zeta = 0$, gives us $c_1 = 0$.

Thus,

$$\int_0^{\zeta} \frac{(\zeta - \varpi)^{-\chi(\varpi)}}{\Gamma(1 - \chi(\varpi))}\mu(\varpi)\ d\varpi = \int_0^{\zeta} \eta(\varpi, \mu(\varpi))\ d\varpi.$$

Conversely, by derivation of both sides of the Eq (3.1), we have

$$(\frac{d}{dt})\int_0^{\zeta} \frac{(\zeta - \varpi)^{-\chi(\varpi)}}{\Gamma(1 - \chi(\varpi))}\mu(\varpi)\ d\varpi = \eta(\zeta, \mu(\zeta)),$$

which means that μ is a solution of IVP (1.1). □

The first result is based on Theorem 2.2.

Theorem 3.1. *Consider (HY). Then, the IVP (1.1) has at least one solution on $\mathbb{E}$.*

Proof. We construct the following operator
$C : \mathbb{E} \to \mathbb{E}$ as follows,

$$C_\mu(\zeta) = \mu(\zeta) + \int_0^{\zeta} \eta(\varpi, \mu(\varpi))\ d\varpi - \int_0^{\zeta} \frac{(\zeta - \varpi)^{-\chi(\varpi)}}{\Gamma(1 - \chi(\varpi))}\mu(\varpi)\ d\varpi.$$

Set $E_r = \{\mu \in \mathbb{E}, \|\mu\| < r, r > 0\}$. Clearly, E_r is nonempty, bounded, closed, and convex subset of $\mathbb{E}$.

Now, we will prove that the operator C satisfies the hypotheses of Theorem 2.2.

Step 01: C is continuous.

To prove that C is continuous, we presume that the sequence $(\mu_n)_{n\in\mathbb{N}}$ converges to μ in $\mathbb{E}$, and we show that $C\mu_n$ converges to $C\mu$ in $\mathbb{E}$.

Then, we have

$$\begin{aligned}
&\mid C\mu_n(\zeta) - C\mu(\zeta) \mid \\
&\leq \mid \mu_n(\zeta) - \mu(\zeta) \mid + \int_0^{\zeta} \varpi^{-\sigma}\varpi^{\sigma} \mid \eta(\varpi, \mu_n(\varpi)) - \eta(\varpi, \mu(\varpi)) \mid d\varpi \\
&\quad + \int_0^{\zeta} \frac{(\zeta - \varpi)^{-\chi(\varpi)}}{\Gamma(1 - \chi(\varpi))} \mid \mu(\varpi) - \mu_n(\varpi) \mid d\varpi \\
&\leq \|\mu_n - \mu\| + p \|\mu_n - \mu\| \int_0^{\zeta} \varpi^{-\sigma} d\varpi \\
&\quad + M_\Gamma \|\mu_n - \mu\| \int_0^{\zeta} (\zeta - \varpi)^{-\chi(\varpi)} d\varpi \\
&\leq \|\mu_n - \mu\| + p \|\mu_n - \mu\| \frac{\zeta^{-\sigma+1}}{-\sigma + 1} \\
&\quad + M_\Gamma L^* \|\mu_n - \mu\| \int_0^{\zeta} \big(\frac{\zeta - \varpi}{L}\big)^{-\chi^*} d\varpi \\
&\leq \|\mu_n - \mu\| + p\frac{L^{-\sigma+1}}{-\sigma + 1} \|\mu_n - \mu\| + \frac{M_\Gamma L^*}{L^{-\chi^*}} \frac{(\zeta)^{1-\chi^*}}{(1 - \chi^*)} \|\mu_n - \mu\| \\
&\leq \|\mu_n - \mu\| + p\frac{L^{-\sigma+1}}{-\sigma + 1} \|\mu_n - \mu\| + \frac{M_\Gamma L L^*}{1 - \chi^*} \|\mu_n - \mu\| \\
&\leq \big(1 + \frac{M_\Gamma L L^*}{1 - \chi^*} + p\frac{L^{-\sigma+1}}{-\sigma + 1}\big) \|\mu_n - \mu\|,
\end{aligned}$$

which implies that

$$\| C\mu_n(\zeta) - C\mu(\zeta) \| \to 0, as \ n \to \infty.$$

The above relation shows that the operator C is continuous on $\mathbb{E}$.

Step 02: C maps bounded sets into bounded sets in $\mathbb{E}$. Let $\eta^* = \sup_{\zeta \in A} \mid \eta(\zeta, 0) \mid$, then, for $\mu \in E_r$, we have

$$\begin{aligned}
&\mid C\mu(\zeta) \mid \\
&\leq \mid \mu(\zeta) \mid + \mid \int_0^{\zeta} \eta(\varpi, \mu(\varpi)) \ d\varpi \mid \\
&\quad + \mid \int_0^{\zeta} \frac{(\zeta - \varpi)^{-\chi(\varpi)}}{\Gamma(1-\chi(\varpi))} \mu(\varpi) \ d\varpi \mid \\
&\leq \mid \mu(\zeta) \mid + \int_0^{\zeta} \mid \eta(\varpi, \mu(\varpi)) - \eta(\varpi, 0) + \eta(\varpi, 0) \mid d\varpi \\
&\quad + \int_0^{\zeta} \frac{(\zeta - \varpi)^{-\chi(\varpi)}}{\Gamma(1-\chi(\varpi))} \mid \mu(\varpi) \mid \ d\varpi \\
&\leq \mid \mu(\zeta) \mid + \int_0^{\zeta} \varpi^{-\sigma} \varpi^{\sigma} \mid \eta(\varpi, \mu(\varpi)) - \eta(\varpi, 0) \mid d\varpi \\
&\quad + \int_0^{\zeta} \mid \eta(\varpi, 0) \mid d\varpi + M_\Gamma \int_0^{\zeta} (\zeta - \varpi)^{-\chi(\varpi)} \mid \mu(\varpi) \mid d\varpi \\
&\leq \mid \mu(\zeta) \mid + \int_0^{\zeta} p \mid \mu(\varpi) \mid \varpi^{-\sigma} d\varpi + \int_0^{\zeta} \eta^* d\varpi \\
&\quad + M_\Gamma L^* \int_0^{\zeta} \Big(\frac{\zeta - \varpi}{L}\Big)^{-\chi(\varpi)} \mid \mu(\varpi) \mid d\varpi \\
&\leq \| \mu \| + p \| \mu \| \frac{\zeta^{-\sigma+1}}{-\sigma+1} + \eta^* \zeta + \frac{M_\Gamma L^*}{L^{-\chi^*}} \| \mu \| \int_0^{\zeta} (\zeta - \varpi)^{-\chi^*} d\varpi \\
&\leq \| \mu \| + p \| \mu \| \frac{L^{-\sigma+1}}{-\sigma+1} + \eta^* L + \frac{M_\Gamma L^*}{L^{-\chi^*}} \frac{\zeta^{1-\chi^*}}{(1-\chi^*)} \| \mu \| \\
&\leq \| \mu \| + p \| \mu \| \frac{L^{-\sigma+1}}{-\sigma+1} + \eta^* L + \frac{M_\Gamma L^* L}{(1-\chi^*)} \| \mu \| \\
&\leq \Big[1 + \frac{M_\Gamma L^* L}{1-\chi^*} + p \frac{L^{-\sigma+1}}{-\sigma+1}\Big] \| \mu \| + \eta^* L,
\end{aligned}$$

which implies that

$$\| C\mu \| \leq \Big[1 + \frac{M_\Gamma L^* L}{1-\chi^*} + p \frac{L^{-\sigma+1}}{-\sigma+1}\Big] r + \eta^* L.$$

Hence, $C(E_r)$ is uniformly bounded.

Step 03: C maps bounded sets into equicontinuous sets in $\mathbb{E}$.

First, we can remark that the function $w_\chi(\varpi) = \left(\frac{\zeta_1 - \varpi}{L}\right)^{-\chi(\varpi)} - \left(\frac{\zeta_2 - \varpi}{L}\right)^{-\chi(\varpi)}$ is decreasing with respect to its exponent $-\chi(\varpi)$, for $0 < \frac{\zeta_1 - \varpi}{L} < \frac{\zeta_2 - \varpi}{L} < 1$.

Then, for $\zeta_1, \zeta_2 \in A$, $\zeta_1 < \zeta_2$, and $\mu \in E_r$, we have

$$\begin{aligned}
&\mid C\mu(\zeta_2) - C\mu(\zeta_1) \mid \\
&\leq \mid \mu(\zeta_2) - \mu(\zeta_1) \mid + \mid \int_0^{\zeta_2} \eta(\varpi, \mu(\varpi)) \ d\varpi \\
&\quad - \int_0^{\zeta_1} \eta(\varpi, \mu(\varpi)) \ d\varpi \mid + \mid \int_0^{\zeta_1} \frac{(\zeta_1 - \varpi)^{-\chi(\varpi)}}{\Gamma(1-\chi(\varpi))} \mu(\varpi) \ d\varpi \\
&\quad - \int_0^{\zeta_2} \frac{(\zeta_2 - \varpi)^{-\chi(\varpi)}}{\Gamma(1-\chi(\varpi))} \mu(\varpi) \ d\varpi \mid \\
&\leq \mid \mu(\zeta_2) - \mu(\zeta_1) \mid + \mid \int_{\zeta_1}^{\zeta_2} \eta(\varpi, \mu(\varpi)) \ d\varpi \mid \\
&\quad + \mid \int_0^{\zeta_1} \frac{(\zeta_2 - \varpi)^{-\chi(\varpi)}}{\Gamma(1-\chi(\varpi))} \mu(\varpi) - \frac{(\zeta_1 - \varpi)^{-\chi(\varpi)}}{\Gamma(1-\chi(\varpi))} \mu(\varpi) \ d\varpi \mid \\
&\quad + \mid \int_{\zeta_1}^{\zeta_2} \frac{(\zeta_2 - \varpi)^{-\chi(\varpi)}}{\Gamma(1-\chi(\varpi))} \mu(\varpi) d\varpi \mid \\
&\leq \mid \mu(\zeta_2) - \mu(\zeta_1) \mid + \int_{\zeta_1}^{\zeta_2} \varpi^{-\sigma} \varpi^{\sigma} \mid \eta(\varpi, \mu(\varpi)) - \eta(\varpi, 0) \mid d\varpi \\
&\quad + \int_{\zeta_1}^{\zeta_2} \mid \eta(\varpi, 0) \mid d\varpi + \int_0^{\zeta_1} \mid \frac{1}{\Gamma(1-\chi(\varpi))} \| (\zeta_2 - \varpi)^{-\chi(\varpi)} \\
&\quad - (\zeta_1 - \varpi)^{-\chi(\varpi)} \| \mu(\varpi) \mid d\varpi \\
&\quad + \int_{\zeta_1}^{\zeta_2} \frac{(\zeta_2 - \varpi)^{-\chi(\varpi)}}{\Gamma(1-\chi(\varpi))} \mid \mu(\varpi) \mid d\varpi \\
&\leq \mid \mu(\zeta_2) - \mu(\zeta_1) \mid + \int_{\zeta_1}^{\zeta_2} p \mid \mu(\varpi) \mid \varpi^{-\sigma} d\varpi + \int_{\zeta_1}^{\zeta_2} \eta^* d\varpi \\
&\quad + M_\Gamma \| \mu \| \int_0^{\zeta_1} \Big[(\zeta_1 - \varpi)^{-\chi(\varpi)} - (\zeta_2 - \varpi)^{-\chi(\varpi)}\Big] d\varpi \\
&\quad + M_\Gamma \| \mu \| L^* \int_{\zeta_1}^{\zeta_2} \Big(\frac{\zeta_2 - \varpi}{L}\Big)^{-\chi^*} d\varpi \\
&\leq \mid \mu(\zeta_2) - \mu(\zeta_1) \mid + \int_{\zeta_1}^{\zeta_2} p \mid \mu(\varpi) \mid \varpi^{-\sigma} d\varpi + \int_{\zeta_1}^{\zeta_2} \eta^* d\varpi \\
&\quad + M_\Gamma \| \mu \| \int_0^{\zeta_1} L^{-\chi(\varpi)} \Big[\Big(\frac{\zeta_1 - \varpi}{L}\Big)^{-\chi(\varpi)} - \Big(\frac{\zeta_2 - \varpi}{L}\Big)^{-\chi(\varpi)}\Big] d\varpi \\
&\quad + M_\Gamma \| \mu \| L^* \int_{\zeta_1}^{\zeta_2} \Big(\frac{\zeta_2 - \varpi}{L}\Big)^{-\chi^*} d\varpi \\
&\leq \mid \mu(\zeta_2) - \mu(\zeta_1) \mid + \int_{\zeta_1}^{\zeta_2} p \mid \mu(\varpi) \mid \varpi^{-\sigma} d\varpi + \int_{\zeta_1}^{\zeta_2} \eta^* d\varpi \\
&\quad + M_\Gamma \| \mu \| \int_0^{\zeta_1} L^* \Big[\Big(\frac{\zeta_1 - \varpi}{L}\Big)^{-\chi^*} - \Big(\frac{\zeta_2 - \varpi}{L}\Big)^{-\chi^*}\Big] d\varpi \\
&\quad + M_\Gamma \| \mu \| L^* \int_{\zeta_1}^{\zeta_2} \Big(\frac{\zeta_2 - \varpi}{L}\Big)^{-\chi^*} d\varpi \\
&\leq \mid \mu(\zeta_2) - \mu(\zeta_1) \mid + p \| \mu \| \frac{(\zeta_2^{-\sigma+1} - \zeta_1^{-\sigma+1})}{-\sigma+1} + \eta^*(\zeta_2 - \zeta_1) \\
&\quad + \Big[(\zeta_1)^{1-\chi^*} - (\zeta_2)^{1-\chi^*} + 2(\zeta_2 - \zeta_1)^{1-\chi^*}\Big] \frac{M_\Gamma \| \mu \| L^*}{L^{-\chi^*}(1-\chi^*)}
\end{aligned}$$

$$\leq \mid \mu(\zeta_2) - \mu(\zeta_1) \mid + \frac{p \parallel \mu \parallel}{-\sigma + 1}\Big[\zeta_2^{-\sigma+1} - \zeta_1^{-\sigma+1}\Big] + \eta^*(\zeta_2 - \zeta_1)$$
$$+ \Big[(\zeta_1)^{1-\chi^*} - (\zeta_2)^{1-\chi^*} + 2(\zeta_2 - \zeta_1)^{1-\chi^*}\Big]\frac{M_\Gamma \parallel \mu \parallel L^*}{L^{-\chi^*}(1-\chi^*)}.$$

Hence, $\mid C_\mu(\zeta_2) - C_\mu(\zeta_1) \mid \to 0$ as $\zeta_2 \to \zeta_1$. It implies that $C(E_r)$ is equicontinuous.

Consequently the operator C is compact.

Step 04: A priori bounds.

We now show there exists an open set $\mathbb{U} \subseteq \mathbb{E}$ with $\mu \neq \lambda C\mu$, for $\lambda \in (0,1)$ and $\mu \in \partial\mathbb{U}$.

Let $\mu \in \mathbb{E}$ and $\zeta \in A$, such that, for some $0 < \lambda < 1$, we have

$$\mu(\zeta) = \lambda C\mu(\zeta),$$

and

$$\begin{aligned}
\mid \lambda C\mu(\zeta) \mid =& \lambda \mid \mu(\zeta) + \int_0^\zeta \eta(\varpi,\mu(\varpi))\ d\varpi \\
&- \int_0^\zeta \frac{(\zeta-\varpi)^{-\chi(\varpi)}}{\Gamma(1-\chi(\varpi))}\mu(\varpi)\ d\varpi \mid \\
\leq& \lambda\Big[\mid \mu(\zeta) \mid + \int_0^\zeta \mid \eta(\varpi,\mu(\varpi)) \mid\ d\varpi \\
&- \int_0^\zeta \frac{(\zeta-\varpi)^{-\chi(\varpi)}}{\Gamma(1-\chi(\varpi))} \mid \mu(\varpi) \mid\ d\varpi\Big] \\
\leq& \Big[\mid \mu(\zeta) \mid + \int_0^\zeta \mid \eta(\varpi,\mu(\varpi)) \mid\ d\varpi \\
&- \int_0^\zeta \frac{(\zeta-\varpi)^{-\chi(\varpi)}}{\Gamma(1-\chi(\varpi))} \mid \mu(\varpi) \mid\ d\varpi\Big] \\
\leq& \Big[1 + \frac{M_\Gamma L^* L}{1-\chi^*} + p\frac{L^{-\sigma+1}}{-\sigma+1}\Big] \parallel \mu \parallel + \eta^* L.
\end{aligned}$$

Thus,

$$\parallel \lambda C\mu(\zeta) \parallel \leq \Big(\Big[1 + \frac{M_\Gamma L^* L}{1-\chi^*} + p\frac{L^{-\sigma+1}}{-\sigma+1}\Big] \parallel \mu \parallel + \eta^* L\Big) := \bar{M}.$$

Let

$$\mathbb{U} = \{\mu \in \mathbb{E} : \parallel \mu \parallel < \bar{M} + 1\}.$$

By our choice of $\mathbb{U}$, there is no $\mu \in \partial\mathbb{U}$ such that $\mu = \lambda C\mu$, for $\lambda \in (0,1)$. As a consequence of Theorem 2.2, we deduce that C has a fixed point μ in $\mathbb{E}$ which is solution to IVP(1.1). □

3.2. Results of uniqueness

The next result is based on Theorem 2.1.

Theorem 3.2. *Let (HY) be satisfied, then the ITVP (1.1) has a unique solution on* $\mathbb{E}$.

Proof. For $\mu(\zeta)$, $\mu^*(\zeta) \in$ E, we may write

$$\begin{aligned}
&\mid C_\mu(\zeta) - C_{\mu^*}(\zeta) \mid \\
&\leq \mid \mu(\zeta) - \mu^*(\zeta) \mid + \int_0^\zeta \varpi^{-\sigma}\varpi^{\sigma} \mid \eta(\varpi,\mu(\varpi)) - \eta(\varpi,\mu^*(\varpi)) \mid d\varpi \\
&\quad + \int_0^\zeta \frac{(\zeta-\varpi)^{-\chi(\varpi)}}{\Gamma(1-\chi(\varpi))} \mid \mu^*(\varpi) - \mu(\varpi) \mid d\varpi \\
&\leq \parallel \mu - \mu^* \parallel + p \parallel \mu - \mu^* \parallel \int_0^\zeta \varpi^{-\sigma} d\varpi \\
&\quad + M_\Gamma \parallel \mu - \mu^* \parallel \int_0^\zeta (\zeta-\varpi)^{-\chi(\varpi)} d\varpi \\
&\leq \parallel \mu - \mu^* \parallel + p \parallel \mu - \mu^* \parallel \frac{\zeta^{-\sigma+1}}{-\sigma+1} \\
&\quad + M_\Gamma L^* \parallel \mu - \mu^* \parallel \int_0^\zeta \Big(\frac{\zeta-\varpi}{L}\Big)^{-\chi^*} d\varpi \\
&\leq \parallel \mu - \mu^* \parallel + p\frac{L^{-\sigma+1}}{-\sigma+1} \parallel \mu - \mu^* \parallel \\
&\quad + \frac{M_\Gamma L^*}{L^{-\chi^*}}\frac{(\zeta)^{1-\chi^*}}{(1-\chi^*)} \parallel \mu - \mu^* \parallel \\
&\leq \parallel \mu - \mu^* \parallel + p\frac{L^{-\sigma+1}}{-\sigma+1} \parallel \mu - \mu^* \parallel + \frac{M_\Gamma L L^*}{1-\chi^*} \parallel \mu - \mu^* \parallel \\
&\leq \Big(1 + \frac{M_\Gamma L L^*}{1-\chi^*} + p\frac{L^{-\sigma+1}}{-\sigma+1}\Big) \parallel \mu - \mu^* \parallel .
\end{aligned}$$

We put $\sigma = 1 + \frac{M_\Gamma L L^*}{1-\chi^*} + p\frac{L^{-\sigma+1}}{-\sigma+1}$, and we have

$$\parallel C\mu - C\mu^* \parallel \leq \sigma \parallel \mu - \mu^* \parallel .$$

By induction, we can prove that

$$\parallel C^n\mu - C^n\mu^* \parallel \leq \frac{\sigma^n}{n!} \parallel \mu - \mu^* \parallel,$$

such that $C^n = C \circ C \circ C \circ C \circ \cdots \circ C$ n times.

We have $\lim_{n\to\infty} \frac{\sigma^n}{n!} = 0$, so it tends to zero as n tends to infinity. Then, for n sufficiently large, we get $\frac{\sigma^n}{n!} < 1$.

Accordingly to Theorem 2.1, the operator C has a unique fixed point which is the unique solution of the IVP (1.1). □

3.3. Study of the uniform stability

Definition 3.1. *The solution of the equation of IVP 1.1 is uniformly stable, if for* $\forall\epsilon > 0$ *there* $\exists\psi(\epsilon) > 0$ *such that for any solution* $\mu(\zeta), \mu'(\zeta)$ *corresponding to the initial conditions of IVP 1.1, such that* $\mid \mu^*(\zeta) - \mu'_*(\zeta) \mid < \psi(\epsilon)$, *where* $\mu^* = \max_{\zeta\in A}\mu(\zeta)$; $\mu'_* = \min_{\zeta\in A}\mu'(\zeta)$. *one has* $\parallel \mu - \mu' \parallel \leq \epsilon$.

Theorem 3.3. *Assume (HY) is satisfied, and if*

$$\Big[\frac{M_\Gamma L L^*}{1-\chi^*} + p\frac{L^{-\sigma+1}}{-\sigma+1}\Big] < 1,$$

then the solution of IVP (1.1) is uniformly stable.

Proof.

$$
\begin{aligned}
&| \mu(\zeta) - \mu'(\zeta) | \\
&\le | \mu(\zeta) - \mu'(\zeta) | + \int_0^\zeta \varpi^{-\sigma}\varpi^{\sigma} | \eta(\varpi,\mu(\varpi)) \\
&\quad - \eta(\varpi,\mu'(\varpi)) | d\varpi + \int_0^\zeta \frac{(\zeta-\varpi)^{-\chi(\varpi)}}{\Gamma(1-\chi(\varpi))} | \mu(\zeta) - \mu'(\zeta) | d\varpi \\
&\le | \mu(\zeta) - \mu'(\zeta) | + \int_0^\zeta p | \mu(\zeta) - \mu'(\zeta) | \varpi^{-\sigma} d\varpi \\
&\quad + \| \mu - \mu' \| \frac{M_\Gamma L^*}{L^{-\chi^*}} \int_0^\zeta (\zeta-\varpi)^{-\chi^*} d\varpi \\
&\le | \mu(\zeta) - \mu'(\zeta) | + p \| \mu - \mu' \| \frac{\zeta^{-\sigma+1}}{-\sigma+1} + \| \mu - \mu' \| \frac{M_\Gamma L L^*}{1-\chi^*} \\
&\le | \mu(\zeta) - \mu'(\zeta) | + \Big(\frac{M_\Gamma L L^*}{1-\chi^*} + p\frac{L^{-\sigma+1}}{-\sigma+1} \Big) \| \mu - \mu' \| .
\end{aligned}
$$

Then,

$$
\| \mu - \mu' \| \le \Big(1 - \Big[\frac{M_\Gamma L L^*}{1-\chi^*} + p\frac{L^{-\sigma+1}}{-\sigma+1} \Big] \Big)^{-1} | \mu^*(\zeta) - \mu'_*(\zeta) | .
$$

Then, $\| \mu - \mu' \| < \epsilon$, which completes the proof. □

3.4. Numerical approximation

First, before starting calculation of the approximate solution, it is worth noting that we will take a value of $\chi(\zeta)$ as a constant, because when replacing a value of ζ in $[0, L], \chi(\zeta)$ become a constant. By the definition of fractional derivative of variable order defined by (2.2) with $0 < \chi < 1$, so $-1 < \chi < 0$, and we have:

$$
\begin{aligned}
\mathbb{D}^{-\chi}\eta(\zeta) &= \Big(\frac{d}{dt}\Big)^0 \mathbb{I}^{0+\chi}\eta(\zeta) \\
&= \Big(\frac{d}{dt}\Big)^0 \int_0^\zeta \frac{(\zeta-\varpi)^{0+\chi-1}}{\Gamma(0+\chi)}\eta(\varpi)d\varpi \\
&= \int_0^\zeta \frac{(\zeta-\varpi)^{\chi-1}}{\Gamma(\chi)}\eta(\varpi)d\varpi.
\end{aligned}
$$

Let $\eta(\zeta) \in C(A), A = [0, L]$, $\Delta\zeta = \frac{L}{n}, n \in \mathbb{N}$, and $\zeta_m = m\Delta\zeta, m = 0, 1, 2, \cdots, n-1$.

Let $\eta(\zeta)$ be approximated by a certain function $\check{\eta}(\zeta)$.

Naturally, we come up with polynomial interpolation of $\eta(\zeta)$ on $[0, L]$.

For $\zeta = \zeta_n$, we have $\chi(\zeta_n) = \chi_n$=constant, so:

$$
\begin{aligned}
\mathbb{D}^{-\chi_n}\eta(\zeta)|_{\zeta=\zeta_n} =& \frac{1}{\Gamma(\chi_n)}\Big[\int_{\zeta_0}^{\zeta_1} (\zeta_n-\varpi)^{\chi_n-1}\eta(\varpi)d\varpi \\
&+ \int_{\zeta_1}^{\zeta_2} (\zeta_n-\varpi)^{\chi_n-1}\eta(\varpi)d\varpi \\
&+ \int_{\zeta_2}^{\zeta_3} (\zeta_n-\varpi)^{\chi_n-1}\eta(\varpi)d\varpi \\
&+ \cdots + \int_{\zeta_{n-1}}^{\zeta_n} (\zeta_n-\varpi)^{\chi_n-1}\eta(\varpi)d\varpi \Big] \\
=& \frac{1}{\Gamma(\chi_n)} \sum_{m=0}^{n-1} \int_{\zeta_m}^{\zeta_{m+1}} (\zeta_n-\varpi)^{\chi_n-1}\eta(\varpi)d\varpi.
\end{aligned}
$$

On each sub-interval $[\zeta_m, \zeta_{m+1}]$, $m = 0, 1, 2, \cdots, n-1$, the function $\eta(\zeta)$ is approximated by a constant.

$$
\eta(\zeta)|_{[\zeta_m,\zeta_{m+1})} \approx \check{\eta}(\zeta)|_{[\zeta_m,\zeta_{m+1})} = \eta(\zeta_m),
$$

$$
\mathbb{D}^{-\chi_m}\eta(\zeta)|_{\zeta=\zeta_m} \approx \frac{1}{\Gamma(\chi_m)} \sum_{m=0}^{n-1} \int_{\zeta_m}^{\zeta_{m+1}} (\zeta_n-\varpi)^{\chi_m-1}\eta(\varpi)d\varpi,
$$

with $\chi_m = \chi(\zeta_m)$= constant.

$$
\mathbb{D}^{-\chi_m}\eta(\zeta)|_{\zeta=\zeta_m} = \frac{1}{\Gamma(\chi_m)} \sum_{m=0}^{n-1} \eta(\zeta_m)\Big[\frac{(\zeta_n-\varpi)^{\chi_m}}{-\chi_m}\Big]_{\zeta_m}^{\zeta_{m+1}}.
$$

For $n = 1$,

$$
\begin{aligned}
\mathbb{D}^{-\chi_1}\eta(\zeta)|_{\zeta=\zeta_1} &= \frac{-1}{\chi_1\Gamma(\chi_1)}\eta(\zeta_0)\Big[(\zeta_1-\zeta_1)^{\chi_1} - (\zeta_1-\zeta_0)^{\chi_1}\Big] \\
&= \frac{h^{\chi_1}}{\Gamma(\chi_1+1)}\eta(\zeta_0),
\end{aligned}
$$

since $\zeta_1 - \zeta_0 = h$ and $\Gamma(\chi_1 + 1) = \chi_1\Gamma(\chi_1)$.

If $n = 2$,

$$
\begin{aligned}
\mathbb{D}^{-\chi_2}\eta(\zeta)|_{\zeta=\zeta_2} &= \frac{-1}{\chi_2\Gamma(\chi_2)} \sum_{m=0}^{1} \eta(\zeta_m)[(\zeta_2-\zeta_{m+1})^{\chi_2} - (\zeta_2-\zeta_m)^{\chi_2}] \\
&= \frac{-1}{\chi_2\Gamma(\chi_2)}[\eta(\zeta_0)(h^{\chi_2} - (2h)^{\chi_2}) + \eta(\zeta_1)(0 - h^{\chi_2})] \\
&= \frac{-1}{\chi_2\Gamma(\chi_2)}[h^{\chi_2}(1-2^{\chi_2})\eta(\zeta_0) - h^{\chi_2}\eta(\zeta_1)] \\
&= \frac{h^{\chi_2}}{\chi_2\Gamma(\chi_2)}[\eta(\zeta_1) + (2^{\chi_2}-1)\eta(\zeta_0)].
\end{aligned}
$$

Similarly,
$\mathbb{D}^{-\chi_3}\eta(\zeta)|_{\zeta=\zeta_3} = \frac{h^{\chi_3}}{\chi_3\Gamma(\chi_3)}[\eta(\zeta_2) + (2^{\chi_3}-1)\eta(\zeta_1) + (3^{\chi_3}-2^{\chi_3})\eta(\zeta_0)]$.
Following the above process, we have:

$$
\begin{aligned}
\mathbb{D}^{-\chi_n}\eta(\zeta)|_{\zeta=\zeta_n} =& \frac{h^{\chi_n}}{\Gamma(\chi_n+1)}[\eta(\zeta_{n-1}) + (2^{\chi_n}-1)\eta(\zeta_{n-2}) \\
&+ (3^{\chi_n}-2^{\chi_n})\eta(\zeta_{n-3}) + \cdots + (n^{\chi_n} - (n-1)^{\chi_n})\eta(\zeta_0)],
\end{aligned}
$$

so

$$\mathbb{D}^{-\chi(\zeta)}\eta(\zeta) = \frac{h^{\chi(\zeta)}}{\Gamma(\chi(\zeta)+1)}\sum_{m=1}^{n-1}[(m+1)^{\chi(\zeta)} - m^{\chi(\zeta)}]\eta_{n-m-1}.$$

Converting IVP (1.1) to the volterra integral equation, we get

$$\mu(\zeta) = \mu(0) + \mathbb{D}^{-\chi(\zeta)}\eta(\zeta,\mu(\zeta)),$$

so, we have

$$\mu(\zeta) = \mu(0) + \frac{h^{\chi(\zeta)}}{\Gamma(\chi(\zeta)+1)}\sum_{m=0}^{n-1}[(m+1)^{\chi(\zeta)} - m^{\chi(\zeta)}]\eta_{n-m-1}.$$

3.5. Examples

Example 3.1. *Let the following IVP*

$$\begin{cases} D^{\chi(\zeta)}\mu(\zeta) = \eta(\zeta,\mu(\zeta)), \quad \zeta \in A = [0,\frac{1}{4}], \\ \mu(0) = 0, \end{cases} \tag{3.3}$$

where $\chi(\zeta) = \frac{\exp(\zeta)}{2} - \log(\zeta+1)$*, and*

$$\eta(\zeta,\mu) = \sqrt{\frac{\exp(\zeta)}{2} - \log(\zeta+1)} + \Big[\frac{4\exp(2\zeta)}{\exp(\exp(\zeta+1)) - \exp(-\zeta)} + 1\Big]\mu.$$

Clearly, $\chi(\zeta)$ *is a continuous function with* $0 < \chi(\zeta) < \frac{1}{2} = \chi^* < 1$, $0 \le \sigma < \min_{\zeta\in A} \mid \chi(\zeta) \mid$*, and we get* $\sigma = 0$*.*

$\eta(\zeta,\mu)$ *is a continuous function on* $A \times \mathbb{R}$*, and*

$$\begin{aligned}
&\zeta^{\sigma} \mid \eta(\zeta,\mu) - \eta(\zeta,y) \mid \\
=&\zeta^{\sigma}\Big|\sqrt{\frac{\exp(\zeta)}{2} - \log(\zeta+1)} \\
&+ \Big[\frac{4\exp(2\zeta)}{\exp(\exp(\zeta+1)) - \exp(-\zeta)} + 1\Big]\mu \\
&- \sqrt{\frac{\exp(\zeta)}{2} - \log(\zeta+1)} \\
&- \Big[\frac{4\exp(2\zeta)}{\exp(\exp(\zeta+1)) - \exp(-\zeta)} + 1\Big]y\Big| \\
=&\zeta^{\sigma}\Big|\Big[\frac{4\exp(2\zeta)}{\exp(\exp(\zeta+1)) - \exp(-\zeta)} + 1\Big]\big(\mu - y\big)\Big| \\
\le&\zeta^{\sigma}\Big[\frac{4\exp(2\zeta)}{\exp(\exp(\zeta+1)) - \exp(-\zeta)} + 1\Big]\Big|\big(\mu - y\big)\Big| \\
\le&\Big(\frac{4}{\exp(\exp(1)) - 1} + 1\Big)\Big|\mu - y\Big|,
\end{aligned}$$

so (HY) *satisfied with* $p = \frac{4}{\exp(\exp(1))-1} + 1$*, in addition to*

$$\begin{aligned}
\Big[\frac{M_{\Gamma}LL^*}{1-\chi^*} + p\frac{L^{-\sigma+1}}{-\sigma+1}\Big] &= \Big[\frac{\frac{1}{\sqrt{\pi}}\frac{1}{4}}{\frac{1}{2}} + (\frac{4}{\exp(\exp(1)) - 1} + 1)\frac{1}{4}\Big] \\
&= \Big[\frac{1}{2\sqrt{\pi}} + 1.2826\frac{1}{4}\Big] \\
&= 0.2821 + 0.3206 \\
&= 0.6027 < 1.
\end{aligned}$$

According Theorem 3.2, the IVP (3.3) *has a unique solution; Theorem 3.3, the IVP* (3.3) *is uniform stable.*

Example 3.2. *Let the following IVP*

$$\begin{cases} D^{\chi(\zeta)}\mu(\zeta) = \eta(\zeta,\mu(\zeta)), \quad \zeta \in A = [0,1], \\ \mu(0) = 0, \end{cases} \tag{3.4}$$

where $\chi(\zeta) = \frac{\zeta}{2}$*, and* $\eta(\zeta,\mu) = \sqrt{(\zeta+1)} + \frac{\exp(-\zeta)}{(4\exp(2\zeta)+1)(1+\mu)}$*. Clearly,* $\chi(\zeta)$ *is a continuous function on* $[0,1]$ *with* $0 < \chi(\zeta) < \frac{1}{2} = \chi^* < 1$, $0 \le \sigma < \min_{\zeta\in A} \mid \chi(\zeta) \mid$*. We get* $\sigma = 0$*.* $\eta(\zeta,\mu)$ *is a continuous function, and*

$$\begin{aligned}
&\zeta^{\sigma} \mid \eta(\zeta,\mu) - \eta(\zeta,y) \mid \\
=&\zeta^{\sigma}\Big|\sqrt{(\zeta+1)} + \frac{\exp(-\zeta)}{(4\exp(2\zeta)+1)(1+\mu)} \\
&- \sqrt{(\zeta+1)} - \frac{\exp(-\zeta)}{(4\exp(2\zeta)+1)(1+y)}\Big| \\
=&\zeta^{\sigma}\Big|\frac{\exp(-\zeta)}{(4\exp(2\zeta)+1)}\Big(\frac{1}{1+\mu} - \frac{1}{1+y}\Big)\Big| \\
\le&\zeta^{\sigma}\frac{\exp(-\zeta) \mid \mu - y \mid}{(4\exp(2\zeta)+1)(1+\mu)(1+y)} \\
\le&\zeta^{\sigma}\frac{\exp(-\zeta)}{(4\exp(2\zeta)+1)} \mid \mu - y \mid \\
\le&\frac{\exp(-1)}{(4\exp(2)+1)} \mid \mu - y \mid,
\end{aligned}$$

so (HY) satisfied with $p = \frac{\exp(-1)}{(4\exp(2)+1)}$*.*

By Theorem 3.2, the IVP (3.4) *has a unique solution.*

3.5.1. Numerical results

Now, we present our solution $\mu(\zeta)$ for $\chi(\zeta) = \frac{\zeta}{2}$ with $\zeta \in [0,1]$ and $\mu_i(\zeta)$ for $\chi(\zeta_i) = \frac{\zeta_i}{2}$ where ζ_i is fixed. In Figure 1, we plot the solution μ depending on ζ, and the Figure 2 presents a plot of $\mu_i(\zeta)$ for different $\chi(\zeta)$.

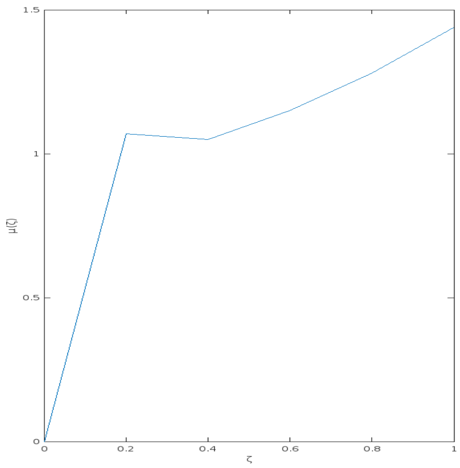

Figure 1. The solution $\mu(\zeta)$ in $[0, 1]$ with $\chi(\zeta) = \frac{\zeta}{2}$.

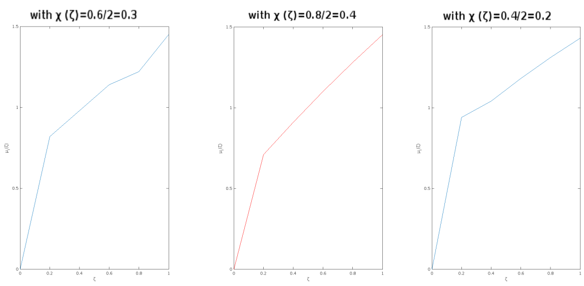

Figure 2. The solution $\mu_i(\zeta)$ in $\zeta \in [0, 1]$ for different $\chi(\zeta)$.

4. Conclusions and perspectives

The paper explores the existence, uniqueness, and stability of solutions for an initial value problem involving fractional differential equations with variable orders. Our methodology differs significantly from earlier approaches, offering a more straightforward and accessible alternative. The findings present a practical solution that minimizes the need for complex computations. This study also opens the door for future research by extending the results using other fractional order operators, such as the (k, ψ)-Hilfer operator and modified Atangana–Baleanu–Caputo operator.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

All authors declare no conflicts of interest in this paper

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