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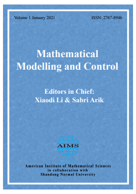

MMC, 2(2): 75–80
DOI:10.3934/mmc.2022009
Received: 04 January 2022
Revised: 11 April 2022
Accepted: 03 June 2022
Published: 27 June 2022

Research article

Approximate solution of initial boundary value problems for ordinary differential equations with fractal derivative

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Abstract: Fractal ordinary differential equations are successfully established by He's fractal derivative in a fractal space, and their variational principles are obtained by semi-inverse transform method.Taylor series method is used to solve the given fractal equations with initial boundary value conditions, and sometimes *Ying Buzu* algorithm play an important role in this process. Examples show the Taylor series method and *Ying Buzu* algorithm are powerful and simple tools.

Keywords: He's fractal derivative; semi-inverse method; *Ying Buzu* algorithm; Taylor series method; variational principle

1. Introduction

The ordinary differential equations (ODEs) arise in many fields like the physics, mechanics, economics and management, etc. There are many methods for solving nonlinear ODEs, for example, the He's frequency formulation [1,2], homotopy perturbation method [3,4], variational iteration method [5,6], and variational-based method [7,8]. In addition, there have been some research results on differential equations recently [9–15]. Each method has its advantages and disadvantages. For example, the Exp-function method can lead to the analytical solutions, but its complex calculation makes those inaccessible who are not familiar some mathematics software. The variational-based methods can obtain a globally valid solution, however, it is extremely difficult to establish a needed variational principle for a complex nonlinear problem, etc.

In my memory, most researchers in the wold pay little attention to Chinese mathematics, especially on ancient Chinese mathematics. So the present author feels strongly necessary to do some work let the world know that China has 5000 years of civilization, not only in social sciences, but also in natural sciences. This paper concerns briefly a famous ancient Chinese algorithm, named *Ying Buzu*. Every student knows Newton's iteration method from a textbook, which is widely used in numerical simulation, what few may know is that its ancient Chinese partner, *Ying Buzu* algorithm, in about second century BC has much advantages over Newton's method [16].

In this paper, we mainly study using Taylor series method [17–22] and *Ying Buzu* algorithm [23] to solve ordinary differential equations with initial boundary value conditions in a fractal space. To the best of our knowledge, the hybrid method has not been studied.

The remainder of this paper is organized as follows. In section 2, we briefly study the initial value problems for ODEs. In section 3, boundary value problems for ODEs with Neumann boundary conditions are studied. In section 4, boundary value problems for ODEs with Dirichlet boundary conditions are considered. In section 5, we conclude this paper and some discussions are also given there.

2. Initial value problems for ODEs

Consider the initial value problem [24]

$$y'' + y = 2x - 1, \;\; y(1) = 1, \;\; y'(1) = 3. \tag{2.1}$$

2.1. Variational principle of fractal Eq.(2.1)

In a fractal space, Eq.(2.1) can be described as follows

$$\frac{d}{dx^{\alpha}}\left(\frac{dy}{dx^{\alpha}}\right) + y = 2x - 1, \;\; y(1) = 1, \;\; \frac{dy}{dx^{\alpha}}(1) = 3, \tag{2.2}$$

where $\frac{dy}{dx^{\alpha}}$ is the fractal derivative [25,26]

$$\frac{dy}{dx^{\alpha}}(x_0) = \Gamma(1+\alpha) \lim_{\substack{x - x_0 \to \Delta x \\ \Delta x \neq 0}} \frac{y(x) - y(x_0)}{x - x_0}. \tag{2.3}$$

The variational principle of Eq (2.2) can be given by semi-inverse method as follows

$$J = \int \left\{ -\frac{1}{2}\left(\frac{dy}{dx^{\alpha}}\right)^2 + \frac{1}{2}y^2 - 2xy + y \right\} dx^{\alpha}. \tag{2.4}$$

2.2. Approximate solution of fractal Eq (2.1)

Using the two-scale transform [27,28]

$$s = x^{\alpha}, \tag{2.5}$$

we can convert Eq (2.2) approximately into the following one

$$\frac{d^2y}{ds^2} + y = 2s - 1, \;\; y(1) = 1, \;\; \frac{dy}{ds}(1) = 3. \tag{2.6}$$

From Eq.(2.6),we have

$$y''(1) = 0, \;\; y^{(3)}(1) = -1, \;\; y^{(4)}(1) = 0. \tag{2.7}$$

The Taylor series solution of Eq (2.6) is

$$\begin{aligned} y(s) &= y(1) + (s-1)y'(1) + \frac{1}{2}(s-1)^2 y''(1) \\ &+ \frac{1}{6}(s-1)^3 y^{(3)}(1) + \frac{1}{24}(s-1)^4 y^{(4)}(1) \\ &= 1 + 3(s-1) - \frac{1}{6}(s-1)^3, \end{aligned} \tag{2.8}$$

which converges to the exact solution, which is

$$y(s) = 2s - 1 + \sin(s-1). \tag{2.9}$$

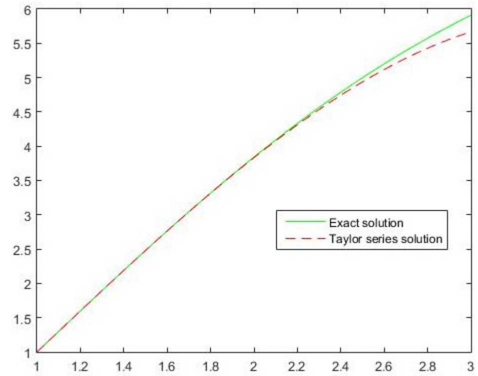

Figure 1. Taylor series solution.

Figure 1 shows the Taylor series solutions, which is very close to the exact solutions.

3. Boundary value problems for ODEs with Neumann boundary conditions

Consider the boundary value problem [24]

$$\frac{d^2\theta}{dx^2} = \epsilon\theta^4, \;\; \theta'(0) = 0, \;\; \theta(1) = 1. \tag{3.1}$$

3.1. Variational principle of fractal Eq (3.1)

In a fractal space, Eq (3.1) can be described as follows

$$\frac{d}{dx^{\alpha}}\left(\frac{d\theta}{dx^{\alpha}}\right) = \epsilon\theta^4, \;\; \frac{d\theta}{dx^{\alpha}}(0) = 0, \;\; \theta(1) = 1, \tag{3.2}$$

where $\frac{d\theta}{dx^{\alpha}}$ is the fractal derivative [25,26]

$$\frac{d\theta}{dx^{\alpha}}(x_0) = \Gamma(1+\alpha) \lim_{\substack{x - x_0 \to \Delta x \\ \Delta x \neq 0}} \frac{\theta(x) - \theta(x_0)}{x - x_0}. \tag{3.3}$$

The variational principle of Eq (3.2) can be given by semi-inverse method as follows

$$J = \int \left\{ -\frac{1}{2}\left(\frac{d\theta}{dx^{\alpha}}\right)^2 - \frac{1}{5}\epsilon\theta^5 \right\} dx^{\alpha}. \tag{3.4}$$

3.2. Approximate solution of fractal Eq (3.1)

Using the two-scale transform

$$s = x^{\alpha}, \tag{3.5}$$

Equation (3.2) can be converted approximately into the following one

$$\frac{d^2\theta}{ds^2} = \epsilon\theta^4, \quad \frac{d\theta}{ds}(0) = 0, \quad \theta(1) = 1. \tag{3.6}$$

For simplicity, we assume

$$\epsilon = 1, \quad \theta(0) = c, \tag{3.7}$$

from (3.6), we have

$$\theta''(0) = c^4, \tag{3.8}$$

$$\theta^{(3)}(0) = 4c^3, \tag{3.9}$$

$$\theta^{(4)}(0) = 12c^2. \tag{3.10}$$

The 4th-order Taylor series solution is

$$\theta(s) = \theta(0) + s\theta'(0) + \frac{1}{2}s^2\theta''(0) + \frac{1}{6}s^3\theta^{(3)}(0) + \frac{1}{24}s^4\theta^{(4)}(0)$$

$$= c + \frac{1}{2}c^4 s^2 + \frac{2}{3}c^3 s^3 + \frac{1}{2}c^2 s^4, \tag{3.11}$$

incorporating the boundary condition $\theta(1) = 1$, we have

$$c + \frac{1}{2}c^4 + \frac{2}{3}c^3 + \frac{1}{2}c^2 = 1. \tag{3.12}$$

We use the *Ying Buzu* algorithm [23] to solve c and write (3.12) in the form

$$R(c) = c + \frac{1}{2}c^4 + \frac{2}{3}c^3 + \frac{1}{2}c^2 - 1. \tag{3.13}$$

Assume the two initial solutions are

$$\theta_1(0) = 0.6, \quad \theta_2(0) = 0.8, \tag{3.14}$$

we obtain the following residuals

$$R_1(0.6) = -0.0112, \quad R_2(0.8) = 0.666133, \tag{3.15}$$

the inital guess $\theta(0)$ can be updated as

$$\theta_3(0) = \frac{R_2\theta_1 - R_1\theta_2}{R_2 - R_1} = 0.603307, \tag{3.16}$$

the shooting process using (3.16) results in

$$\theta_3(1) = 0.997931, \tag{3.17}$$

which deviates the exact value of $\theta(1) = 1$ with a relative error of 0.2%.

The 4th-order Taylor series solution is

$$\theta(s) = 0.603307 + 0.066241s^2 + 0.146394s^3 + 0.18199s^4. \tag{3.18}$$

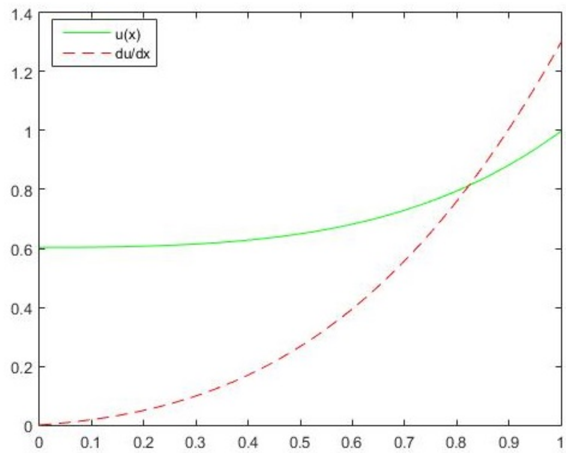

Figure 2. Taylor series solution based on the *Ying Buzu* algorithm.

Figure 2 shows the Taylor series solutions, which approximately meet the requirement of the boundary condition.

4. Boundary value problems for ODEs with Dirichlet boundary conditions

Consider the boundary value problem [29]

$$u''(t) + tu(t) = t^3 + 2, \quad u(0) = 0, \quad u(1) = 1. \tag{4.1}$$

4.1. Variational principle of fractal Eq (4.1)

In a fractal space, Equation (4.1) can be described as follows

$$\frac{d}{dt^{\alpha}}\left(\frac{du}{dt^{\alpha}}\right) + tu(t) = t^3 + 2, \quad u(0) = 0, \quad u(1) = 1, \tag{4.2}$$

where $\frac{du}{dt^{\alpha}}$ is the fractal derivative [25,26]

$$\frac{du}{dt^{\alpha}}(t_0) = \Gamma(1+\alpha) \lim_{\substack{t-t_0 \to \Delta t \\ \Delta t \neq 0}} \frac{u(t) - u(t_0)}{t - t_0}. \tag{4.3}$$

The variational principle of Eq (4.2) can be given by semi-inverse method as follows

$$J = \int \left\{ -\frac{1}{2}(\frac{du}{dt^\alpha})^2 + \frac{1}{2}tu(t)^2 - (t^3+2)u(t) \right\} dt^\alpha. \quad (4.4)$$

4.2. Approximate solution of fractal Eq (4.1)

Using the two-scale transform

$$s = t^\alpha, \quad (4.5)$$

Equation (4.2) can be converted approximately into the following one

$$\frac{d^2u}{ds^2} + su(s) = s^3 + 2, \ \ u(0) = 0, \ \ u(1) = 1. \quad (4.6)$$

We assume

$$u'(0) = \alpha, \quad (4.7)$$

from (4.6), we have

$$u''(0) = 2, \quad (4.8)$$

$$u^{(3)}(0) = 0, \quad (4.9)$$

$$u^{(4)}(0) = -2\alpha. \quad (4.10)$$

The 4th-order Taylor series solution is

$$u(s) = u(0) + su'(0) + \frac{1}{2}s^2u''(0) + \frac{1}{6}s^3u^{(3)}(0) + \frac{1}{24}s^4u^{(4)}(0)$$

$$= \alpha s + s^2 - \frac{1}{12}\alpha s^4. \quad (4.11)$$

by using the boundary condition $u(1) = 1$, we have

$$\alpha + 1 - \frac{1}{12}\alpha = 1, \quad (4.12)$$

and $\alpha = 0$. The 4th-order Taylor series solution is

$$u(s) = s^2. \quad (4.13)$$

which is also the exact solution of Eq (4.1).

5. Conclusions

Fractal ordinary differential equations are successfully established by He's fractal derivative in a fractal space, and their variational principles are obtained by semi-inverse transform method.The two-scale transform method and Taylor series method are adopted to solve the fractal ODEs with initial boundary value conditions.The examples show the Taylor series method is simple and effective and the ancient Chinese *Ying Buzu* algorithm is a simple and straightforward tool to two-point boundary value problems. In the future, we will study how to extend this hybrid method to PDEs.

Conflict of interest

The author has no conflicts of interest to declare that are relevant to the content of this article.

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