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Research article

$\mathcal{H}$-representation method for solving reduced biquaternion matrix equation

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Abstract: In this paper, we study the Hankel and Toeplitz solutions of reduced biquaternion matrix equation (1.1). Using semi-tensor product of matrices, the reduced biquaternion matrix equation (1.1) can be transformed into a general matrix equation of the form $AX = B$. Then, due to the special structure of Hankel matrix and Toeplitz matrix, the independent elements of Hankel matrix or Toeplitz matrix can be extracted by combing the $\mathcal{H}$-representation method of matrix, so as to reduce the elements involved in the operation in the process of solving matrix equation and reduce the complexity of the problem. Finally, by using Moore-Penrose generalized inverse, the necessary and sufficient conditions for the existence of solutions of reduced biquaternion matrix equation (1.1) are given, and the corresponding numerical examples are given.

Keywords: reduced biquaternion matrix equation; semi-tensor product of matrices; $\mathcal{H}$-representation; real representation; Hankel matrix; Toeplitz matrix

1. Introduction

The symbols used in this article are as follows: $\mathbb{R}$ is the set of all real numbers; $\mathbb{R}^m$ is the set of all real column vectors with m-dimensional and $\mathbb{R}^{m\times n}$ is the set of all $m \times n$ dimensional real matrices. $\mathbb{Q}_b$ is the set of all reduced biquaternions and $\mathbb{Q}_b^n$ is the set of all n-dimensional reduced biquaternion vectors. $\mathbb{Q}_b^{m\times n}$ is the set of all $m \times n$ dimensional reduced biquaternion matrices. $\mathbb{H}_r^{n\times n}$ and $\mathbb{T}_r^{n\times n}$ represent the set of all $n \times n$ dimensional real Hankel matrices and real Toeplitz matrices respectively. $\mathbb{H}_b^{n\times n}$ and $\mathbb{T}_b^{n\times n}$ represent the set of all $n \times n$ dimensional reduced biquaternion Hankel matrices and reduced biquaternion Toeplitz matrices respectively. A^{T} is the transpose of A; $A^{\dagger}$ is the Moore-Penrose generalized inverse of A. 0 and I_n represent zero matrix of suitable size and identity matrix of n-dimensional respectively. $A \otimes B$ is the Kronecker product of A and B, V_c and V_r represent the column expansion and row expansion of the matrix.

Efficient methods for solving matrix equations play an important role in promoting the application in engineering mechanics [1], color images [2,3], control and system theory [4], neural network, and so on. Many scholars are devoted to the study of the solutions of matrix equations, [5] gave the necessary and sufficient conditions for Lyapunov matrix equation to have Hermitian solution, and the sensitivity of Lyapunov equation to disturbance was analyzed; the iterative solutions of discrete-time periodic Sylvester matrix equations were discussed in [6] based on jacobian gradient gradient algorithm and accelerated iteration algorithm, and their applications in antilinear periodic systems were introduced; in [7], a method similar to Hermitian splitting and oblique Hermitian splitting was used to solve the continuous time algebraic Riccati matrix equation, and the convergence of the iterative method was analyzed. Because of the importance of matrix equation, whether it is real, complex or quaternion matrix equation, it has attracted extensive attention. In [8], Yuan derived the expression of the least squares Hermitian solution of the quaternion matrix equation $(AXB, CXD) = (E, F)$ with the least norm over the skew field of quaternions. Wang [9] divided the quaternion bisymmetric matrix into four blocks by using the partition idea, and gave the necessary and sufficient conditions for the existence of bisymmetric solutions of the quaternion matrix equation $AXB = C$ by using the relation between the block matrices. Zhang [10] gave the necessary and sufficient conditions for quaternion matrix equation $AX = B$ to have the pure imaginary least squares solution and the real least squares solution, she also used pure imaginary quaternions to represent color images, and applies quaternion matrix equation $AX = B$ to color image restoration.

In recent years, semi-tensor product of matrices proposed by professor Cheng has attracted extensive attention of scholars. Semi-tensor product of matrices relieves the dimension limitation of ordinary matrix multiplication and makes it possible to multiply matrices of any dimension. Semi-tensor product of matrices are not only widely used in finite game theory, graph theory [11] and formation control, fuzzy control [12] and some other engineering fields. Many scholars also connect semi-tensor product of matrices with matrix equation and solve matrix equation by semi-tensor product of matrices method. Wang [13], Ding [14] studied the special solutions of generalized Lyapunov equation and Sylvester equation based on semi-tensor product of matrices and quaternion matrix real vector representation, respectively; by using semi-tensor product of matrices, Zhao [15] gave the necessary and sufficient conditions for the existence of the minimum norm least squares Tridiagonal (Anti-) Hermite solution of the quaternion matrix Stein equation.

A quaternion is a generalization of complex numbers. A quaternion has four components, one real part and three imaginary parts

$$q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}, q_i \in \mathbb{R}(i = 0, 1, 2, 3),$$

and $\mathbf{i}, \mathbf{j}, \mathbf{k}$ obey the rules as follows, $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1, \mathbf{ij} = -\mathbf{ji} = \mathbf{k}, \mathbf{jk} = -\mathbf{kj} = \mathbf{i}, \mathbf{ki} = -\mathbf{ik} = \mathbf{j}$. We know that the multiplication of quaternions does not satisfy the commutative law of multiplication, which has become an obstacle to solve some quaternion problems. The difference between reduced biquaternions and quaternions are that the multiplication of reduced biquaternions satisfies the commutative law. The representation of reduced biquaternions is [16–18]

$$q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}, q_i \in \mathbb{R}(i = 0, 1, 2, 3),$$

where $\mathbf{i}^2 = \mathbf{k}^2 = -1, \mathbf{j}^2 = 1, \mathbf{ij} = \mathbf{ji} = \mathbf{k}, \mathbf{jk} = \mathbf{kj} = \mathbf{i}, \mathbf{ki} = \mathbf{ik} = -\mathbf{j}$. In [19], Davenport gave the matrix representation for the four-dimensional commutative hypercomplex algebras. The matrix representation is determined by the multiplication rules, so for reduced biquaternions, we have:

$$1 \to I_4 \equiv \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, i \to N_i \equiv \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

$$j \to N_j \equiv \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, k \to N_k \equiv \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix},$$

where $N_j^2 = I_4, N_i^2 = N_k^2 = -I_4, N_iN_j = N_jN_i = N_k, N_jN_k = N_kN_j = N_i$, and $N_iN_k = N_kN_i = -N_j$. Therefore, the matrix representation of a reduced biquaternion $q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$ is

$$q \to N_q \equiv \begin{bmatrix} q_0 & -q_1 & q_2 & -q_3 \\ q_1 & q_0 & q_3 & q_2 \\ q_2 & -q_3 & q_0 & -q_1 \\ q_3 & q_2 & q_1 & q_0 \end{bmatrix}.$$

Similarly, for reduced biquaternion matrix $Q = Q_0 + Q_1\mathbf{i} + Q_2\mathbf{j} + Q_3\mathbf{k}$, the matrix representation is

$$Q \to N_Q \equiv \begin{bmatrix} Q_0 & -Q_1 & Q_2 & -Q_3 \\ Q_1 & Q_0 & Q_3 & Q_2 \\ Q_2 & -Q_3 & Q_0 & -Q_1 \\ Q_3 & Q_2 & Q_1 & Q_0 \end{bmatrix}.$$

Define an operator $\Phi_Q = \begin{bmatrix} Q_0 & Q_1 & Q_2 & Q_3 \end{bmatrix}$, obviously $V_c(\Phi_Q) = \begin{bmatrix} V_c(Q_0) \\ V_c(Q_1) \\ V_c(Q_2) \\ V_c(Q_3) \end{bmatrix}$. The unique advantage of reduced biquaternion is that its multiplication is commutative. In addition, reduced biquaternion matrix theory is widely used in digital signal and color image processing. For example, in [20], the complex symmetric multichannel system and symmetric lattice filter system were analyzed by reduced biquaternion, which greatly reduces the complexity of digital signal processing; in [21], the SVD of reduced biquaternion matrix was used to process color images, the SVD method of reduced biquaternion greatly reduces the complexity of reconstructing the original color image; [22] the pure imaginary solutions and real solutions of the reduced biquaternion matrix equation $AX = B$ were solved in the form of $e_1 - e_2$, and the $e_1 - e_2$ representation of the reduced biquaternion was applied to color image restoration. The methods to solve the reduced biquaternion matrix equation mainly include using the $e_1 - e_2$ representation of reduced biquaternion, or with the help of the real representation and complex representation of the reduced biquaternion matrix.

In this paper, we mainly consider the Hankel and Toeplitz solutions of reduced biquaternion matrix equation

$$\sum_{i=1}^{k} A_i X B_i = C \tag{1.1}$$

by using semi-tensor product of matrices, real representation.

Hankel matrix and Toeplitz matrix are collectively referred to as banded matrix. Banded matrix has important applications in many fields, such as image, signal processing, communication system analysis and so on. For example, Toeplitz system arise in a variety of applications in mathematics, scientific computing and engineering, numerical partial and ordinary differential equations; numerical solution of convolution-type integral equations; optimization problems in control theory; signal processing and image restoration [23, 24]. In addition, Hankel matrix has important applications in electric power, microseismic data processing and array signal processing [25]. In this paper, we will solve the special solutions of reduced biquaternion matrix equation (1.1) by combining semi-tensor product of matrices, real representation of reduced biquaternion matrix and $\mathcal{H}$-representation of special matrices, Moore-Penrose generalized inverse. The main questions we considered are as follows.

Problem 1 Given $A_i \in \mathbb{Q}_b^{m\times n}$, $B_i \in \mathbb{Q}_b^{n\times s}$ $(i = 1, 2, \ldots, k)$, $C \in \mathbb{Q}_b^{m\times s}$, find

$$M_h = \left\{ X \middle| X \in \mathbb{H}_b^{n\times n}, \sum_{i=1}^{k} A_i X B_i = C \right\}. \tag{1.2}$$

Problem 2 Given $A_i \in \mathbb{Q}_b^{m\times n}$, $B_i \in \mathbb{Q}_b^{n\times s}$ $(i = 1, 2, \ldots, k)$, $C \in \mathbb{Q}_b^{m\times s}$, find

$$M_t = \left\{ X \middle| X \in \mathbb{T}_b^{n\times n}, \sum_{i=1}^{k} A_i X B_i = C \right\}. \tag{1.3}$$

We will find a sufficient and necessary condition for M_h and M_t to be nonempty. When the reduced biquaternion matrix equation (1.1) has a solution, we will give a general expression for $X \in M_h$ and $X \in M_t$. When the reduced biquaternion matrix equation (1.1) has a unique solution, and we will give a unique expression for $\widehat{X_h} \in \widehat{M_h}$ and $\widehat{X_t} \in \widehat{M_t}$.

This paper is organized as follows. In Section 2, we give some basic knowledge of semi-tensor product of matrices. In Section 3, we first introduce the definition of $\mathcal{H}$-representation [29], and the $\mathcal{H}$-representation methods of Hankel matrix and Toeplitz matrix are given. By using the results to drive the explicit expression for the solutions of Problem 1,2, in Section 4, respectively. Finally, in Section 5, we give a brief summary.

2. Semi-tensor product of matrices

We know that matrix multiplication is limited by dimension. It needs to meet that the dimension of the first matrix column is equal to that of the second matrix row. If the matrix operation is extended to any dimension matrix, the application of matrix method will be greatly expanded. This section first introduces a matrix multiplication between arbitrary dimensions, semi-tensor product of matrices.

Definition 2.1. *[26] Suppose $A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{p\times q}$, $t = lcm(n, p)$ is the least common multiple of n and p. The semi-tensor product of A and B is denoted by*

$$A \ltimes B = (A \otimes I_{t/n})(B \otimes I_{t/p}).$$

Denoted by the symbol $\ltimes$.

From the definition of semi-tensor product of matrices, we can see that semi-tensor product of matrices is the generalization of ordinary matrix multiplication. That is, in the definition, when $n = p$, semi-tensor product of matrices is ordinary matrix multiplication. And when the dimension requirement is satisfied, semi-tensor product of matrices symbol is usually omitted.

The semi-tensor product of matrices has the following properties.

Lemma 2.1. *[26] Suppose A, B, C be real matrices, then (1) (Associative rule)*

$$(A \ltimes B) \ltimes C = A \ltimes (B \ltimes C);$$

(2) (Distributive rule)

$$A \ltimes (B + C) = A \ltimes B + A \ltimes C;$$

$$(B + C) \ltimes A = B \ltimes A + C \ltimes A.$$

Lemma 2.2. *[26] Suppose $X \in \mathbb{R}^t$ is a column of vector, A is an arbitrary matrix, then*

$$X \ltimes A = (I_t \otimes A) \ltimes X.$$

Swap matrix, which exchange the order of factors, are defined as follows

Definition 2.2. *[27] A swap matrix $W_{[m,n]} \in \mathbb{R}^{mn\times mn}$ is defined as*

$$W_{[m,n]} = \delta_{mn}[1, m + 1, \ldots, (n - 1)m + 1, \ldots, m, 2m, \ldots, nm],$$

where $\delta_k[i_1, \ldots, i_s]$ is a shorthand of $[\delta_k^{i_1}, \ldots, \delta_k^{i_s}]$.

The swap matrix has the function of exchanging the order of two vector factors and some other propositions.

Lemma 2.3. *[26] (1) Suppose $X \in \mathbb{R}^m$, $Y \in \mathbb{R}^n$ is two columns, then*

$$W_{[m,n]} \ltimes X \ltimes Y = Y \ltimes X;$$

(2) Suppose $A \in \mathbb{R}^{m\times n}$, then

$$W_{[m,n]} V_r(A) = V_c(A);$$

$$W_{[n,m]} V_c(A) = V_r(A);$$

(3) Suppose $A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{s\times t}$, then

$$A \otimes B = W_{[s,m]} \ltimes B \ltimes W_{[m,t]} \ltimes A = (I_m \otimes B) \ltimes A.$$

Since reduced biquaternons are commutative, we can know that the above conclusion for semi-tensor product of matrices is also true for reduced biquaternons. In addition, Li [28] gave the properties for semi-tensor product of matrices which play an important role in solving matrix equations. For reduced biquaternion matrix, we have the similar conclusions.

Theorem 2.1. *Suppose $A \in \mathbb{Q}_b^{m\times n}$, $X \in \mathbb{Q}_b^{n\times q}$, $Y \in \mathbb{Q}_b^{p\times m}$, (1)*

$$V_c(YA) = A^{\mathrm{T}} \ltimes V_c(Y);$$

$$V_r(AX) = A \ltimes V_r(X).$$

(2)

$$V_c(AX) = (I_q \otimes A) \ltimes V_c(X);$$

$$V_r(YA) = (I_p \otimes A^{\mathrm{T}}) \ltimes V_r(Y).$$

Proof. (1) We proof $V_c(YA) = A^{\mathrm{T}} \ltimes V_c(Y)$.

Let $A = (a_1, \ldots, a_n), a_i \in \mathbb{Q}_b^m (i = 1, \ldots, n)$, $Y = (y_1, \ldots, y_m), y_j \in \mathbb{Q}_b^p (j = 1, \ldots, m)$, then

$$V_c(YA) = V_c(Ya_1, \ldots, Ya_n) = \begin{bmatrix} Ya_1 \\ \vdots \\ Ya_n \end{bmatrix},$$

from the commutability of reduced biquaternion, there have

$$\begin{aligned} Ya_i &= a_{1i}y_1 + \ldots + a_{mi}y_m \\ &= \begin{bmatrix} a_{1i}I_p & \ldots & a_{mi}I_p \end{bmatrix} V_c(Y). \end{aligned}$$

So

$$\begin{aligned} V_c(YA) &= \begin{bmatrix} a_{11}I_p & a_{21}I_p & \ldots & a_{m1}I_p \\ a_{12}I_p & a_{22}I_p & \ldots & a_{m2}I_p \\ \vdots & \vdots & & \vdots \\ a_{1n}I_p & a_{2n}I_p & \ldots & a_{mn}I_p \end{bmatrix} V_c(Y) \\ &= (A^{\mathrm{T}} \otimes I_p)V_c(Y) \\ &= A^{\mathrm{T}} \ltimes V_c(Y). \end{aligned}$$

(2) We proof $V_r(YA) = W_{[n,p]} \ltimes A^{\mathrm{T}} \ltimes W_{[p,m]} \ltimes V_r(Y)$.

Suppose $A \in \mathbb{Q}_b^{m\times n}$, $Y \in \mathbb{Q}_b^{p\times m}$, then by Lemma 2.3 we have

$$\begin{aligned} V_r(YA) &= W_{[n,p]} V_c(YA) \\ &= W_{[n,p]} \ltimes A^{\mathrm{T}} \ltimes V_c(Y) \\ &= W_{[n,p]} \ltimes A^{\mathrm{T}} \ltimes W_{[p,m]} \ltimes V_r(Y) \\ &= (I_p \otimes A^{\mathrm{T}}) \ltimes V_r(Y). \end{aligned}$$

□

3. $\mathcal{H}$-representation

In this article, we are very interested in $\mathcal{H}$-representation of Hankel matrix and Toeplitz matrix. The following part mainly introduces the $\mathcal{H}$-representation of Hankel matrix and Toeplitz matrix. The concept of $\mathcal{H}$-representation is as follows.

Definition 3.1. *[29] Consider a q-dimensional real matrix subspace $\mathbb{X} \subset \mathbb{R}^{n\times n}$ over the field $\mathbb{R}$. Assume that $e_1, e_2, \ldots, e_q$ form the basis of $\mathbb{X}$, and define $H = \begin{bmatrix} V_c(e_1) & V_c(e_2) & \ldots & V_c(e_q) \end{bmatrix}$. For each $X \in \mathbb{X}$, if we express $\Psi(X) = V_c(X)$ in the form of*

$$\Psi(X) = H\widetilde{X},$$

with a $q \times 1$ vector $\widetilde{X} = \begin{bmatrix} x_1 & x_2 & \ldots & x_q \end{bmatrix}^{\mathrm{T}}$. Then $H\widetilde{X}$ is called an $\mathcal{H}$-representation of $\Psi(X)$, and H is called an $\mathcal{H}$-representation matrix of $\Psi(X)$.

Definition 3.2. *Suppose $H_n = (h_{ij})_{n\times n} \in \mathbb{R}^{n\times n}$ satisfies $h_{ij} = h_{i+j-1}, (i, j = 1, \ldots, n)$, that is*

$$H_n = \begin{bmatrix} h_1 & h_2 & h_3 & \cdots & h_n \\ h_2 & h_3 & h_4 & \cdots & h_{n+1} \\ h_3 & h_4 & h_5 & \cdots & h_{n+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ h_n & h_{n+1} & h_{n+2} & \cdots & h_{2n-1} \end{bmatrix},$$

then H_n is called a real Hankel matrix, and the set of all n-dimensional real Hankel matrices is recorded as $\mathbb{H}_r^{n\times n}$.

Definition 3.3. *Suppose $T_n = (t_{ij})_{n\times n} \in \mathbb{R}^{n\times n}$ satisfies $t_{ij} = t_{j-i}, (i, j = 1, \ldots, n)$, that is*

$$T_n = \begin{bmatrix} t_0 & t_1 & t_2 & \cdots & t_{n-1} \\ t_{-1} & t_0 & t_1 & \cdots & t_{n-2} \\ t_{-2} & t_{-1} & t_0 & \cdots & t_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ t_{-n+1} & t_{-n+2} & t_{-n+3} & \cdots & t_0 \end{bmatrix},$$

then T_n is called a real Toeplitz matrix, and the set of all n-dimensional real Toeplitz matrices is recorded as $\mathbb{T}_r^{n\times n}$.

Example 3.1. *Let $\mathbb{X} = \mathbb{H}_r^{n\times n}$, $H_3 = (h_{ij})_{3\times 3} \in \mathbb{X}$, and then $dim(\mathbb{X}) = 5$. If we select a basis of $\mathbb{X}$ as*

$$e_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, e_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix},$$

$$e_4 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, e_5 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

It is easy to compute

$$\Psi(H_3) = \begin{bmatrix} h_1 & h_2 & h_3 & h_2 & h_3 & h_4 & h_3 & h_4 & h_5 \end{bmatrix}^{\mathrm{T}},$$

$$\widetilde{H_3} = \begin{bmatrix} h_1 & h_2 & h_3 & h_4 & h_5 \end{bmatrix}^{\mathrm{T}},$$

and

$$H_h = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Example 3.2. *Let $\mathbb{X} = \mathbb{T}_r^{n\times n}$, $T_3 = (t_{ij})_{3\times 3} \in \mathbb{X}$, and then $dim(\mathbb{X}) = 5$. If we select a basis of $\mathbb{X}$ as*

$$f_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, f_2 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, f_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix},$$

$$f_4 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, f_5 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

It is easy to compute

$$\Psi(T_3) = \begin{bmatrix} t_0 & t_{-1} & t_{-2} & t_1 & t_0 & t_{-1} & t_2 & t_1 & t_0 \end{bmatrix}^{\mathrm{T}},$$

$$\widetilde{T_3} = \begin{bmatrix} t_0 & t_{-1} & t_{-2} & t_1 & t_2 \end{bmatrix}^{\mathrm{T}},$$

and

$$H_t=\begin{bmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\1&0&0&0&0\\0&1&0&0&0\\0&0&0&0&1\\0&0&0&1&0\\1&0&0&0&0\end{bmatrix}.$$

For the general case, if $\mathbb{X}=\mathbb{H}_r^{n\times n}$, we select a standard basis as

$$\{E_1,E_2,\dots,E_{2n-1}\},$$

$$\text{where } E_i=\begin{cases}\begin{bmatrix}J_i&0\\0&0\end{bmatrix}_{n\times n},(i\le n),\\ \begin{bmatrix}0&0\\0&J_{2n-i}\end{bmatrix}_{n\times n},(n<i\le 2n-1),\end{cases}$$

$$J_n=\begin{bmatrix}0&\dots&1\\ \vdots&1&\vdots\\1&\dots&0\end{bmatrix}_{n\times n}.$$

Based on above standard basis, for any $H_n=(h_{ij})_{n\times n}\in\mathbb{X}$, we have

$$\widetilde{H_n}=\begin{bmatrix}h_1&h_2&\dots&h_{2n-1}\end{bmatrix}^{\mathrm{T}},$$

and $H_h=\begin{bmatrix}V_c(E_1)&V_c(E_2)&\dots&V_c(E_{2n-1})\end{bmatrix}\in\mathbb{R}^{n^2\times 2n-1}$.

Similarly if $\mathbb{X}=\mathbb{T}_r^{n\times n}$, we select a standard basis as

$$\{F_1,F_2,\dots,F_{2n-1}\},$$

$$\text{where } F_i=\begin{cases}\begin{bmatrix}0&0\\I_{n-i+1}&0\end{bmatrix}_{n\times n},(i\le n),\\ \begin{bmatrix}0&I_{2n-i}\\0&0\end{bmatrix}_{n\times n},(n<i\le 2n-1).\end{cases}$$

Based on above standard basis, for any $T_n=(t_{ij})_{n\times n}\in\mathbb{X}$, we have

$$\widetilde{T_n}=\begin{bmatrix}t_0&t_{-1}&\dots&t_{-n+1}&t_1&\dots&t_{n-1}\end{bmatrix}^{\mathrm{T}},$$

and $H_t=\begin{bmatrix}V_c(F_1)&V_c(F_2)&\dots&V_c(F_{2n-1})\end{bmatrix}\in\mathbb{R}^{n^2\times 2n-1}$.

The following Theorem is obvious from Definition 3.1.

Theorem 3.1. *For an $n^2\times 1$ vector α_1, if $\Psi^{-1}(\alpha_1)\in\mathbb{H}_r^{n\times n}$, then there exists an $(2n-1)\times 1$ vector β_1, such that $\alpha_1=H_h\beta_1$. For an $n^2\times 1$ vector α_2, if $\Psi^{-1}(\alpha_2)\in\mathbb{T}_r^{n\times n}$, then there exists an $(2n-1)\times 1$ vector β_2, such that $\alpha_2=H_t\beta_2$.*

4. The Solution of Problem 1,2

In order to obtain the solution of the reduced biquaternion matrix equation (1.1), we first give the following Lemma.

Lemma 4.1. *[30] The matrix equation $Ax=b$, with $A\in\mathbb{R}^{m\times n}$ and $b\in\mathbb{R}^m$, has a solution $x\in\mathbb{R}^n$ if and only if*

$$AA^{\dagger}b=b, \tag{4.1}$$

in this case it has the general solution

$$x=A^{\dagger}b+(I_n-A^{\dagger}A)y, \tag{4.2}$$

where $y\in\mathbb{R}^n$ is an arbitrary vector, and it has the unique solution $x=A^{\dagger}b$ for the case when $rank(A)=n$.

In order to obtain the Hankel solution and Toeplitz solution of reduced biquaternion matrix equation (1.1), we first make some signs as follows, for $A_i\in\mathbb{Q}_b^{m\times n}, B_i\in\mathbb{Q}_b^{n\times s}(i=1,2,\dots,k), C\in\mathbb{Q}_b^{m\times s}$, let

$$P=\sum_{i=1}^{k}(B_i^{\mathrm{T}}\otimes A_i)=P_0+P_1\mathbf{i}+P_2\mathbf{j}+P_3\mathbf{k};$$

$$W_1=diag(H_h,H_h,H_h,H_h), W_2=diag(H_t,H_t,H_t,H_t);$$

$$R=N_PW_1, S=N_PW_2;$$

$$V_c(\Phi_C)=\begin{bmatrix}V_c(C_0)\\V_c(C_1)\\V_c(C_2)\\V_c(C_3)\end{bmatrix}.$$

Theorem 4.1. *Suppose $A_i\in\mathbb{Q}_b^{m\times n}, B_i\in\mathbb{Q}_b^{n\times s}(i=1,2,\dots,k)$, $C\in\mathbb{Q}_b^{m\times s}$, the necessary and sufficient condition for the equation (1.1) to have Hankel solution is*

$$RR^{\dagger}V_c(\Phi_C)=V_c(\Phi_C). \tag{4.3}$$

Under the condition of solution, the solution set M_h of the equation (1.1) can be expressed as

$$M_h=\left\{X\middle|V_c(\Phi_X)=W_1R^{\dagger}V_c(\Phi_C)+W_1\left(I_{8n-4}-R^{\dagger}R\right)y\right\}, \tag{4.4}$$

where for any $y\in\mathbb{R}^{8n-4}$. Further, when

$$rank(R)=8n-4, \tag{4.5}$$

the unique solution set of equation (1.1) is

$$\widehat{M_h}=\left\{\widehat{X_h}\middle|V_c(\Phi_{\widehat{X_h}})=W_1R^{\dagger}V_c(\Phi_C)\right\}. \tag{4.6}$$

Proof. By Theorem 2.1, combing with $\mathcal{H}$-representation of Hankel matrix, the reduced biquaternion matrix equation (1.1) is equivalent to

$$\begin{aligned}
&\sum_{i=1}^{k} A_i X B_i = C \\
\Longleftrightarrow &\sum_{i=1}^{k} (I_s \otimes A_i) \ltimes V_c(XB_i) = V_c(C) \\
\Longleftrightarrow &\sum_{i=1}^{k} (I_s \otimes A_i) \ltimes B_i^{\mathrm{T}} \ltimes V_c(X) = V_c(C) \\
\Longleftrightarrow &\sum_{i=1}^{k} (I_s \otimes A_i)(B_i^{\mathrm{T}} \otimes I_n) \ltimes V_c(X) = V_c(C) \\
\Longleftrightarrow &\sum_{i=1}^{k} (B_i^{\mathrm{T}} \otimes A_i) V_c(X) = V_c(C) \\
\Longleftrightarrow & N_P V_c(\Phi_X) = V_c(\Phi_C) \\
\Longleftrightarrow & N_P W_1 \begin{bmatrix} \widetilde{X_0} \\ \widetilde{X_1} \\ \widetilde{X_2} \\ \widetilde{X_3} \end{bmatrix} = V_c(\Phi_C) \\
\Longleftrightarrow & R \begin{bmatrix} \widetilde{X_0} \\ \widetilde{X_1} \\ \widetilde{X_2} \\ \widetilde{X_3} \end{bmatrix} = V_c(\Phi_C).
\end{aligned}$$

By Lemma 4.1, we have

$$\begin{bmatrix} \widetilde{X_0} \\ \widetilde{X_1} \\ \widetilde{X_2} \\ \widetilde{X_3} \end{bmatrix} = R^{\dagger} V_c(\Phi_C) + \left(I_{8n-4} - R^{\dagger} R\right) y,$$

further, we can obtain

$$V_c(\Phi_X) = W_1 \begin{bmatrix} \widetilde{X_0} \\ \widetilde{X_1} \\ \widetilde{X_2} \\ \widetilde{X_3} \end{bmatrix} = W_1 R^{\dagger} V_c(\Phi_C) + W_1 \left(I_{8n-4} - R^{\dagger} R\right) y.$$

Then if $rank(R) = 8n - 4$, the unique solution of equation (1.1) satisfies

$$V_c(\Phi_{\widehat{X_h}}) = W_1 R^{\dagger} V_c(\Phi_C).$$

$\square$

Similar to the conclusion of Hankel solution, we can obtain the following conditions for the existence of Toeplitz solution of reduced biquaternion matrix equation (1.1).

Theorem 4.2. *Suppose $A_i \in \mathbb{Q}_b^{m\times n}, B_i \in \mathbb{Q}_b^{n\times s} (i = 1, 2, \ldots, k), C \in \mathbb{Q}_b^{m\times s}$, the necessary and sufficient condition for the equation* (1.1) *to have Toeplitz solution is*

$$SS^{\dagger} V_c(\Phi_C) = V_c(\Phi_C). \tag{4.7}$$

Under the condition of solution, the solution set M_t of the equation (1.1) *can be expressed as*

$$M_t = \left\{ X \middle| V_c(\Phi_X) = W_2 S^{\dagger} V_c(\Phi_C) + W_2 \left(I_{8n-4} - S^{\dagger} S\right) y \right\}, \tag{4.8}$$

where $\forall y \in \mathbb{R}^{8n-4}$. Further, when

$$rank(S) = 8n - 4, \tag{4.9}$$

the unique solution set of equation (1.1) *is*

$$\widehat{M_t} = \left\{ \widehat{X_t} \middle| V_c(\Phi_{\widehat{X_t}}) = W_2 S^{\dagger} V_c(\Phi_C) \right\}. \tag{4.10}$$

5. Numerical exemplification

We now provide numerical algorithms and examples for finding the solutions of Problem 1,2. Algorithms 5.1 and 5.2 are based on Theorem 4.1 and Theorem 4.2. Examples are based on Algorithms 5.1 and 5.2, respectively.

Algorithm 5.1. *(For Problem 1)*
(1) **Input***: $A_i \in \mathbb{Q}_b^{m\times n}$, $B_i \in \mathbb{Q}_b^{n\times s} (i = 1, 2, \ldots, k)$, $C \in \mathbb{Q}_b^{m\times s}$, and P, H_h.*
(2) *Compute N_P, and W_1, R and $V_c(\Phi_C)$.*
(3) *If both* (4.3) *and* (4.5) *hold, then output the Hankel solution $\dot{X}_h (\dot{X}_h \in \widehat{M_h})$ according to* (4.6).
(4) **Output***: the solution $\dot{X}_h \in \widehat{M_h}$.*

Algorithm 5.2. *(For Problem 2)*
(1) **Input***: $A_i \in \mathbb{Q}_b^{m\times n}$, $B_i \in \mathbb{Q}_b^{n\times s} (i = 1, 2, \ldots, k)$, $C \in \mathbb{Q}_b^{m\times s}$, and P, H_t.*
(2) *Compute N_P, and W_2, S and $V_c(\Phi_C)$.*
(3) *If both* (4.7) *and* (4.9) *hold, then output the Toeplitz solution $\dot{X}_t (\dot{X}_t \in \widehat{M_t})$ according to* (4.10).
(4) **Output***: the solution $\dot{X}_t \in \widehat{M_t}$.*

Example 5.1. *Suppose $m = n = p$, $A_i, B_i \in \mathbb{Q}_b^{n\times n}(i = 1, 2)$ be generated randomly for $n = 2l, l = 1 : 15$. Randomly generated Hankel or Toeplitz reduced biquaternion matrix X_h, X_t for equation* (1.1)*, respectively, calculate $C = A_1X_hB_1 + A_2X_hB_2$, $C = A_1X_tB_1 + A_2X_tB_2$. In this case, $rank(R) = 8n - 4$, $rank(S) = 8n - 4$ is satisfied, at this time, the reduced biquaternion matrix equation* (1.1) *has a unique solution. Substitute $\dot{X}_h$, $\dot{X}_t$, into the reduced biquaternion equation* (1.1)*, its computational solutions can be obtained by using Algorithm 5.1-Algorithm 5.2, respectively. Define $\varepsilon_1 = log_{10}(\left\|V_c(\Phi_{X_h}) - V_c(\Phi_{\dot{X}_h})\right\|)$, $\varepsilon_2 = log_{10}(\left\|V_c(\Phi_{X_t}) - V_c(\Phi_{\dot{X}_t})\right\|)$ as the error obtained by Algorithms 5.1 and 5.2 respectively. As the dimension changes, ε_i $(i = 1, 2)$ is shown in the Figure 5.1 and Figure 5.2.*

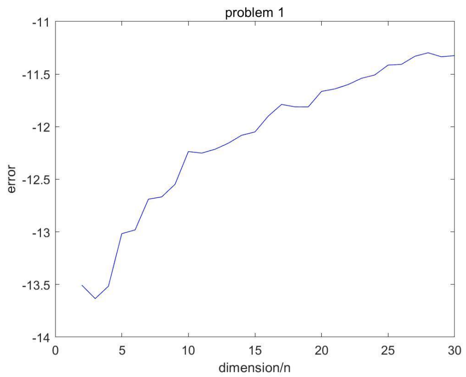

Figure 5.1. Errors in different dimensions for Problem 1.

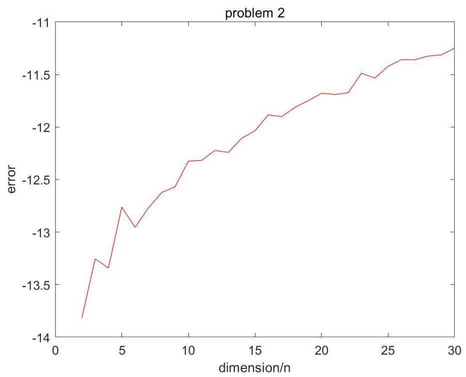

Figure 5.2. Errors in different dimensions for Problem 2.

It can be seen from the figure that the error value obtained by the algorithm is less than -11, which fully proves the effectiveness and rationality of the algorithm.

6. Conclusions

The multiplication of reduced biquaternion satisfies the commutative law, which provides great convenience for some of our quaternion calculations. This paper mainly uses the semi-tensor product of matrices, combines the real representation of reduced biquaternion matrix and the $\mathcal{H}$-representation of matrix to solve the Hankel and Toeplitz solutions of reduced biquaternion matrix equation (1.1), this provides a new method for the solution of reduced biquaternion matrix equation. It is believed that the semi-tensor product of matrices and $\mathcal{H}$-representation of matrices will be more widely used in solving matrix equation problems.

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Conflict of interest

The authors declare that there is no conflict of interest.

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