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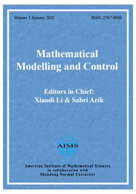

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Research article

On the sum of powers of the A_α-eigenvalues of graphs

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Abstract: Let $A(G)$ and $D(G)$ be the adjacency matrix and the degree diagonal matrix of a graph G, respectively. For any real number $\alpha \in [0, 1]$, Nikiforov recently defined the A_α-matrix of G as $A_\alpha(G) = \alpha D(G) + (1 - \alpha)A(G)$. The graph invariant $S^p_\alpha(G)$ is the sum of the p-th power of the A_α-eigenvalues of G for $\frac{1}{2} < \alpha < 1$, which has a close relation to the α-Estrada index. In this paper, we establish some bounds on $S^p_\alpha(G)$ and characterize the extremal graphs. In particular, we present some bounds on $S^p_\alpha(G)$ in terms of the degree sequences, order and size of G by using majorization techniques. Moreover, we give lower and upper bounds for $S^p_\alpha(G)$ of a bipartite graph and characterize the extremal graphs.

Keywords: A_α-matrix; A_α-eigenvalues; majorization

1. Introduction

Let G be a simple finite undirected connected graph with vertex set $V(G)$ and edge set $E(G)$, where $|V(G)|$ is the order and $|E(G)|$ is the size of G. Let $A(G)$ and $D(G)$ be the adjacency matrix and the degree diagonal matrix of a graph G, respectively. Then $L(G) = D(G) - A(G)$, $Q(G) = D(G) + A(G)$ and $\mathcal{L}(G) = D^{-\frac{1}{2}}(G)L(G)D^{-\frac{1}{2}}(G)$ are called the Laplacian matrix, the signless Laplacian matrix and the normalized Laplacian matrix of the graph G, respectively.

The investigation on the sum of the p-th power of the eigenvalues of graphs is a topic of interest in Mathematical Chemistry. Based on the mathematical methods, scholars get many bounds for the sum of the p-th power of the eigenvalues of graphs. For a non-zero real number p, Zhou [1] introduced the sum of the p-th power of the non-zero Laplacian eigenvalues of G, denoted by $S^p_L(G)$. Since $S^p_L(G)$ has close relation with the Laplacian-energy-like invariant [2], the Laplacian Estrada index [3] and the Kirhhoff index [4], there are considerable results regarding $S^p_L(G)$ in the literature. For related results, one may refer to [1, 5–9] and references therein. For a non-zero real number p, M. Liu and B. Liu [10] defined $S^p_Q(G)$ as the sum of the p-th power of the non-zero signless Laplacian eigenvalues of G, which has close relation with the incidence energy [11] and the signless Laplacian Estrada index [12]. For details on $S^p_Q(G)$, see the papers [13, 14] and the references cited therein. Moreover, Akbari et al. [15, 16] compared between $S^p_L(G)$ and $S^p_Q(G)$ when the parameter p takes different values. For a non-zero real number p, Ş.B. Bozkurt and D. Bozkurt [17] defined $S^p_{\mathcal{L}}(G)$ as the sum of the p-th power of the normalized Laplacian eigenvalues of G, which has close relation with the degree-Kirchhoff index [18] and the general Randić index [19]. For related results, one may refer to [20, 21].

For any real number $\alpha \in [0, 1]$, Nikiforov [22] defined the A_α-matrix of G as

$$A_\alpha(G) = \alpha D(G) + (1 - \alpha)A(G).$$

It is easy to see that $A_\alpha(G)$ is the adjacency matrix $A(G)$

if $\alpha = 0$, and $A_\alpha(G)$ is essentially equivalent to signless Laplacian matrix $Q(G)$ if $\alpha = \frac{1}{2}$. The new matrix $A_\alpha(G)$ not only can underpin a unified theory of $A(G)$ and $Q(G)$, but it also brings many new interesting problems, see for example [22–25]. In particular, $A_\alpha(G)$ is a positive definite matrix for $\frac{1}{2} < \alpha < 1$, which is a hitherto uncharted territory of worth our investigation and exploration, see [22]. Moreover, X. Liu and S. Liu [26] found that A_α-eigenvalues (especially, $\frac{1}{2} < \alpha < 1$) are much more efficient than A-eigenvalues and Q-eigenvalues when we use them to distinguish graphs, by enumerating the A_α-characteristic polynomials for all graphs on at most ten vertices. The A_α-matrix has been an interesting topic in mathematical literature and has been studied extensively, see for example [22, 23, 27–30] and references therein.

Let $\lambda_1(A_\alpha(G)) \geq \lambda_2(A_\alpha(G)) \geq \cdots \geq \lambda_n(A_\alpha(G))$ be the A_α-eigenvalues of a graph G of order n. Motivated by the above work, we define $S_\alpha^p(G)$ as the sum of the p-th power of the A_α-eigenvalues of G, that is,

$$S_\alpha^p(G) = \sum_{i=1}^{n} \lambda_i^p(A_\alpha(G)),$$

where $\frac{1}{2} < \alpha < 1$ and p is a real number. $S_\alpha^p(G)$ can be regard as a generalization of $S_Q^p(G)$ due to the fact that our results are correct for the sum of the p-th power of the non-zero $A_{\frac{1}{2}}$-eigenvalues of G. By using the Maclaurin development, we have

$$E_\alpha(G) = \sum_{i=1}^{n} e^{\lambda_i(A_\alpha(G))} = \sum_{p=0}^{\infty} \frac{S_\alpha^p(G)}{p!},$$

where p is an integer and $E_\alpha(G)$ is called the α-Estrada index defined by Cardoso et al. [31]. Thus the bound for $S_\alpha^p(G)$ can be naturally converted to the bound of the α-Estrada index. In addition, we find that $S_\alpha^p(G)$ is connected with the first general Zagreb index, which is a useful topological index and has important applications in chemistry.

The primary purpose of this paper is to establish the bounds of $S_\alpha^p(G)$. The cases $p = 0$ and $p = 1$ are trivial as $S_\alpha^0(G) = n$ and $S_\alpha^1(G) = 2\alpha m$, where m is the size of G. We will not consider both cases in the following results. The rest of the paper is organized as follows. In Section 2, we recall some useful notions and lemmas used further. In Section 3, some bounds on the $S_\alpha^p(G)$ are presented. In Sections 4 and 5, several bounds for $S_\alpha^p(G)$ related to degree sequences, order and size are given through majorization techniques. In Section 6, lower and upper bounds for $S_\alpha^p(G)$ of a bipartite graph G are obtained, and the extremal graphs characterized.

2. Preliminaries

Let $G - e$ denote the graph that arises from G by deleting the edge $e \in E(G)$. A connected graph is called a c-cyclic graph if it contains n vertices and $n + c - 1$ edges. For $v_i \in V(G)$, $d_G(v_i) = d_i(G)$ denotes the degree of vertex v_i in G. The minimum and the maximum degree of G are denoted by $\delta = \delta(G)$ and $\Delta = \Delta(G)$, respectively. A pendant vertex is a vertex of degree one and a quasi-pendant vertex is a vertex adjacent to a pendant vertex. Li and Zheng [32] defined the first general Zagreb index of G as $Z_p = Z_p(G) = \sum_{v \in V(G)} d^p(v)$, where p is an arbitrary real number except 0 and 1. A subset I of $V(G)$ is called an independent set of a graph G if no two vertices in I are adjacent in G. Given a graph G, the independence number $\theta(G)$ of G is the numbers of vertices of the largest independent set. Denote by K_n, $K_{a,b}$ and $\overline{G}$ the complete graph, the complete bipartite graph and the complement of a graph G, respectively. The join $G_1 \vee G_2$ of two vertex-disjoint graphs G_1 and G_2 is the graph formed from the union of G_1 and G_2 by joining each vertex of G_1 to each vertex of G_2.

Lemma 2.1. **([33])** *Let G be a graph with n vertices. If $e \in E(G)$ and $\frac{1}{2} \leq \alpha \leq 1$, then $\lambda_i(A_\alpha(G)) \geq \lambda_i(A_\alpha(G - e))$ for $1 \leq i \leq n$.*

Lemma 2.2. **([34,35])** *Let G be a graph of order n and size m. Then*

$$Z_2(G) \geq \frac{4m^2}{n} + \frac{1}{2}(\Delta - \delta)^2$$

with equality if and only if G has the property $d_2 = d_3 = \cdots = d_{n-1} = \frac{\Delta+\delta}{2}$, which includes also the regular graphs.

Lemma 2.3. **([36])** *Let $s_1, s_2, \ldots, s_n$ be the singular values of a matrix $M = (m_{ij}) \in M_n$. Then*

$$\sum_{j=1}^{n} s_j^p \leq \sum_{i,j=1}^{n} |m_{ij}|^p \quad \text{for} \quad 0 < p \leq 2,$$

$$\sum_{j=1}^{n} s_j^p \geq \sum_{i,j=1}^{n} |m_{ij}|^p \quad \text{for} \quad p \geq 2.$$

Lemma 2.4. **([37])** *For c-cyclic graphs with n vertices, the minimal degree sequences with respect to the majorization* *order are given by* $(2, 2, \ldots, 2, 1, 1)$*, in case* $c = 0$ *and* $n > 2$*,* $(2, 2, \ldots, 2)$*, in case* $c = 1$ *and* $n > 2$*,* $(\underbrace{3, 3, \ldots, 3}_{2c-2}, 2, 2, \ldots, 2)$*, in case* $2 \le c \le 6$ *and* $n > 2c - 2$*.*

Lemma 2.5. **([38])** *Let* $a(G)$*,* $b(G)$ *and* $m_G(\alpha)$ *be the number of pendant vertices, quasi-pendant vertices of* G *and the multiplicity of* α *as an eigenvalue of* $A_\alpha(G)$*, respectively. Then* $m_G(\alpha) \ge a(G) - b(G)$ *with equality if each internal vertex is a quasi-pendant vertex.*

Lemma 2.6. **([39])** *Let* G *be a graph of order* n *and size* m*. If* $\alpha \in (\frac{1}{2}, 1)$*, then*

$$\lambda_1(A_\alpha(G)) \le \frac{2m}{n-1}(1-\alpha) + \alpha n - 1,$$

the equality holds if and only if $G \cong K_n$*.*

3. Some bounds on $S^p_\alpha(G)$

Theorem 3.1. *Let* $\frac{1}{2} < \alpha < 1$*,* G *be a connected graph with* n *vertices and* $e \in E(G)$*.*

(i) *If* $p > 0$ *and* $p \ne 1$*, then* $S^p_\alpha(G - e) < S^p_\alpha(G)$*.*

(ii) *If* $p < 0$*, then* $S^p_\alpha(G - e) > S^p_\alpha(G)$*.*

Proof. By Perron-Frobenius Theorem, we have $\lambda_1(A_\alpha(G)) > \lambda_1(A_\alpha(G - e))$. By Lemma 2.1, the result follows. □

Corollary 3.1. *Let* $\frac{1}{2} < \alpha < 1$*,* G *be a connected graph of order* n*.*

(i) *If* $p > 0$ *and* $p \ne 1$*, then*

$$S^p_\alpha(G) \le (n-1)^p + (n-1)(\alpha n - 1)^p$$

with equality if and only if $G \cong K_n$*.*

(ii) *If* $p < 0$*, then*

$$S^p_\alpha(G) \ge (n-1)^p + (n-1)(\alpha n - 1)^p$$

with equality if and only if $G \cong K_n$*.*

Proof. From Proposition 36 in [22], it follows that $\lambda_1(A_\alpha(K_n)) = n - 1$ and $\lambda_i(A_\alpha(K_n)) = \alpha n - 1$ for $2 \le i \le n$. By Theorem 3.1, we have the proof. □

Corollary 3.2. *Let* $\frac{1}{2} < \alpha < 1$*,* G *be a connected graph of order* n *with independence number* θ*.*

(i) *If* $p > 0$ *and* $p \ne 1$*, then*

$$S^p_\alpha(G) \le (n-\theta-1)(\alpha n - 1)^p + (\theta - 1)(n-\theta)^p\alpha^p + x_1^p + x_2^p,$$

where x_1 *and* x_2 *are the roots of the equation*

$$x^2 - (\alpha n + n - \theta - 1)x + \alpha\theta + \alpha n^2 - \alpha n - \alpha\theta^2 - \theta n + \theta^2 = 0$$

and equality holds if and only if $G \cong \overline{K_\theta} \vee K_{n-\theta}$*.*

(ii) *If* $p < 0$*, then*

$$S^p_\alpha(G) \ge (n-\theta-1)(\alpha n - 1)^p + (\theta - 1)(n-\theta)^p\alpha^p + x_1^p + x_2^p,$$

where x_1 *and* x_2 *are the roots of the equation*

$$x^2 - (\alpha n + n - \theta - 1)x + \alpha\theta + \alpha n^2 - \alpha n - \alpha\theta^2 - \theta n + \theta^2 = 0$$

and equality holds if and only if $G \cong \overline{K_\theta} \vee K_{n-\theta}$*.*

Proof. Let $\phi_\alpha(G, x)$ be the characteristic polynomial of $A_\alpha(G)$. By direct computation, we have

$$\begin{aligned}\phi_\alpha(\overline{K_\theta} \vee K_{n-\theta}, x) &= (x - \alpha n + 1)^{n-\theta-1}[x - (n-\theta)\alpha]^{\theta-1}[x^2 \\ &\quad -(\alpha n + n - \theta - 1)x + \alpha\theta + \alpha n^2 - \alpha n \\ &\quad -\alpha\theta^2 - \theta n + \theta^2].\end{aligned}$$

Thus

$$S^p_\alpha(\overline{K_\theta} \vee K_{n-\theta}) = (n-\theta-1)(\alpha n-1)^p + (\theta-1)(n-\theta)^p\alpha^p + x_1^p + x_2^p,$$

where x_1 and x_2 are the roots of the equation $x^2 - (\alpha n + n - \theta - 1)x + \alpha\theta + \alpha n^2 - \alpha n - \alpha\theta^2 - \theta n + \theta^2 = 0$. By Theorem 3.1, we have the proof. □

Theorem 3.2. *Let* G *be a connected graph of order* n *and size* m*. If* $\frac{1}{2} < \alpha < 1$ *and* $p \ne 0$ *and* $p \ne 1$*, then*

$$S^p_\alpha(G) \ge \left(\frac{2m}{n}\right)^p + (n-1)\left(\frac{n\det(A_\alpha(G))}{2m}\right)^{\frac{p}{n-1}} \tag{3.1}$$

with equality if and only if $G \cong K_n$*.*

Proof. By the arithmetic-geometric mean inequality, we have

$$S^p_\alpha(G) = \lambda_1^p(A_\alpha(G)) + \sum_{i=2}^{n} \lambda_i^p(A_\alpha(G))$$

$$\geq \lambda_1^p(A_\alpha(G)) + (n-1)\left(\prod_{i=2}^{n} \lambda_i(A_\alpha(G))\right)^{\frac{p}{n-1}}$$
$$= \lambda_1^p(A_\alpha(G)) + (n-1)\left(\frac{\det(A_\alpha(G))}{\lambda_1(A_\alpha(G))}\right)^{\frac{p}{n-1}}.$$

Let $h(x) = x^p + (n-1)\left(\frac{\det(A_\alpha(G))}{x}\right)^{\frac{p}{n-1}}$. Then $h'(x) = p(x^{p-1} - \det(A_\alpha(G))^{\frac{p}{n-1}} x^{-\frac{p}{n-1}-1})$. It is easy to see that $h(x)$ is increasing on $[\det(A_\alpha(G))^{\frac{1}{n}}, +\infty)$ whether $p > 0$ or $p < 0$. From Corollary 19 in [22], it follows that

$$\begin{aligned} \lambda_1(A_\alpha(G)) &\geq \frac{2m}{n} > \frac{2\alpha m}{n} = \frac{\sum_{i=1}^{n} \lambda_i(A_\alpha(G))}{n} \\ &\geq \left(\prod_{i=1}^{n} \lambda_i(A_\alpha(G))\right)^{\frac{1}{n}} = \det(A_\alpha(G))^{\frac{1}{n}}. \end{aligned}$$

Thus

$$\begin{aligned} S_\alpha^p(G) &\geq h(\lambda_1(A_\alpha(G))) \geq h\left(\frac{2m}{n}\right) \\ &= \left(\frac{2m}{n}\right)^p + (n-1)\left(\frac{n\det(A_\alpha(G))}{2m}\right)^{\frac{p}{n-1}} \end{aligned}$$

with equality if and only if $\lambda_1(A_\alpha(G)) = \frac{2m}{n}$ and $\lambda_2(A_\alpha(G)) = \cdots = \lambda_n(A_\alpha(G))$. From Corollary 33 in [22], the diameter of G is 1. Thus, $G \cong K_n$. Conversely, if $G \cong K_n$, then $\lambda_1(A_\alpha(G)) = n-1$, and $\lambda_i(A_\alpha(G)) = \alpha n - 1$ for $2 \leq i \leq n$. It is easy to check that equality holds in (3.1). This completes the proof. □

Theorem 3.3. *Let $\frac{1}{2} < \alpha < 1$, G be a connected graph of order n and size m.*

(i) *If $p < 0$ or $p > 1$, then*

$$S_\alpha^p(G) \geq \left(\frac{Z_2}{n}\right)^{\frac{p}{2}} + \frac{1}{(n-1)^{p-1}}\left(2\alpha m - \sqrt{\frac{Z_2}{n}}\right)^p \quad (3.2)$$

with equality if and only if $G \cong K_n$.

(ii) *If $0 < p < 1$, then*

$$S_\alpha^p(G) \leq \left(\frac{Z_2}{n}\right)^{\frac{p}{2}} + \frac{1}{(n-1)^{p-1}}\left(2\alpha m - \sqrt{\frac{Z_2}{n}}\right)^p \quad (3.3)$$

with equality if and only if $G \cong K_n$.

Proof. Since $p < 0$ or $p > 1$, we know that $f(x) = x^p$ is a strictly convex function. By Jensen's inequality, we have

$$\left(\sum_{i=2}^{n} \frac{1}{n-1}\lambda_i(A_\alpha(G))\right)^p \leq \sum_{i=2}^{n} \frac{1}{n-1}\lambda_i^p(A_\alpha(G)),$$

that is,

$$\sum_{i=2}^{n} \lambda_i^p(A_\alpha(G)) \geq \frac{1}{(n-1)^{p-1}}(2\alpha m - \lambda_1(A_\alpha(G)))^p.$$

Thus

$$\begin{aligned} S_\alpha^p(G) &= \lambda_1^p(A_\alpha(G)) + \sum_{i=2}^{n} \lambda_i^p(A_\alpha(G)) \\ &\geq \lambda_1^p(A_\alpha(G)) + \frac{1}{(n-1)^{p-1}}(2\alpha m - \lambda_1(A_\alpha(G)))^p. \end{aligned}$$

Let $g(x) = x^p + \frac{1}{(n-1)^{p-1}}(2\alpha m - x)^p$. Then $g'(x) = p\left(x^{p-1} - \frac{(2\alpha m - x)^{p-1}}{(n-1)^{p-1}}\right) \geq 0$ for $x \geq \frac{2\alpha m}{n}$. Hence $g(x)$ is increasing on $[\frac{2\alpha m}{n}, +\infty)$. From Lemma 2.2 and Corollary 19 in [22], it follows that $\lambda_1(A_\alpha(G)) \geq \sqrt{\frac{Z_2}{n}} \geq \frac{2\alpha m}{n}$. Thus

$$\begin{aligned} S_\alpha^p(G) &\geq g(\lambda_1(A_\alpha(G))) \geq g\left(\sqrt{\frac{Z_2}{n}}\right) \\ &= \left(\frac{Z_2}{n}\right)^{\frac{p}{2}} + \frac{1}{(n-1)^{p-1}}\left(2\alpha m - \sqrt{\frac{Z_2}{n}}\right)^p \end{aligned}$$

with equality if and only if $\lambda_1(A_\alpha(G)) = \sqrt{\frac{Z_2}{n}}$ and $\lambda_2(A_\alpha(G)) = \cdots = \lambda_n(A_\alpha(G))$. From Corollary 33 in [22], the diameter of G is 1. Thus, $G \cong K_n$. Conversely, if $G \cong K_n$, then $\lambda_1(A_\alpha(G)) = n-1$, and $\lambda_i(A_\alpha(G)) = \alpha n - 1$ for $2 \leq i \leq n$. It is easy to check that equality holds in (3.2).

Now suppose that $0 < p < 1$. Then

$$\left(\sum_{i=2}^{n} \frac{1}{n-1}\lambda_i(A_\alpha(G))\right)^p \geq \sum_{i=2}^{n} \frac{1}{n-1}\lambda_i^p(A_\alpha(G)),$$

with equality if and only if $\lambda_2(A_\alpha(G)) = \cdots = \lambda_n(A_\alpha(G))$, and $g(x)$ is decreasing on $[\frac{2\alpha m}{n}, +\infty)$. By similar arguments as above, the second part of the theorem follows.

Combining the above arguments, we have the proof. □

By Lemma 2.2 and Theorem 3.3, we have

Corollary 3.3. *Let $\frac{1}{2} < \alpha < 1$, G be a connected graph of order n and size m.*

(i) *If $p < 0$ or $p > 1$, then*

$$\begin{aligned} S_\alpha^p(G) \geq & \left(\frac{4m^2}{n^2} + \frac{1}{2n}(\Delta - \delta)^2\right)^{\frac{p}{2}} \\ & + \frac{1}{(n-1)^{p-1}}\left(2\alpha m - \sqrt{\frac{4m^2}{n^2} + \frac{1}{2n}(\Delta - \delta)^2}\right)^p \end{aligned}$$

with equality if and only if $G \cong K_n$.

(ii) *If $0 < p < 1$, then*

$$S_\alpha^p(G) \le \left(\frac{4m^2}{n^2} + \frac{1}{2n}(\Delta - \delta)^2\right)^{\frac{p}{2}} + \frac{1}{(n-1)^{p-1}}\left(2\alpha m - \sqrt{\frac{4m^2}{n^2} + \frac{1}{2n}(\Delta - \delta)^2}\right)^p$$

with equality if and only if $G \cong K_n$.

Theorem 3.4. *Let $\frac{1}{2} < \alpha < 1$, G be a connected graph of order n and size m.*

(i) *If $0 < p \le 2$, then $S_\alpha^p(G) \le \alpha^p Z_p + 2m(1-\alpha)^p$.*

(ii) *If $p > 2$, then $S_\alpha^p(G) \ge \alpha^p Z_p + 2m(1-\alpha)^p$.*

Proof. Since $A_\alpha(G)$ is a real symmetric and positive definite matrix for $\frac{1}{2} < \alpha < 1$, the singular values of $A_\alpha(G)$ are equal to the eigenvalues of $A_\alpha(G)$. By Lemma 2.3, we have the proof. □

4. Bounds for $S_\alpha^p(G)$ related to degree sequences

Suppose $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ are two non-increasing sequences of real numbers, we say x is majorized by y, denoted by $x \preceq y$, if $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i$ and $\sum_{i=1}^j x_i \le \sum_{i=1}^j y_i$ for $j = 1, 2, \ldots, n-1$. For a real-valued function f defined on a set in $\mathbb{R}^n$, if $f(x) \le f(y)$ whenever $x \preceq y$ but $x \ne y$, then f is said to be Schur-convex.

Theorem 4.1. *Let $\frac{1}{2} < \alpha < 1$, G be a connected graph of order n and size m with the degree sequence $d_1 \ge d_2 \ge \cdots \ge d_n$.*

(i) *If $p < 0$ or $p > 1$, then*

$$\frac{(2\alpha m)^p}{n^{p-1}} \le \alpha^p Z_p(G) \le S_\alpha^p(G).$$

(ii) *If $0 < p < 1$, then*

$$S_\alpha^p(G) \le \alpha^p Z_p(G) \le \frac{(2\alpha m)^p}{n^{p-1}}.$$

Proof. Let $x = \frac{2\alpha m}{n}(1, \ldots, 1)$, $y = (\alpha d_1, \ldots, \alpha d_n)$ and $z = (\lambda_1(A_\alpha(G)), \ldots, \lambda_n(A_\alpha(G)))$. It is well known that the spectrum of any symmetric, positive semi-definite matrix majorizes its main diagonal [40], hence $x \preceq y \preceq z$. Since $p < 0$ or $p > 1$, $f(x) = x^p$ is a convex function. From [41], we know that if the real-valued function f defined on an interval in $\mathbb{R}$ is a convex then $\sum_{i=1}^n f(x_i)$ is Schur-convex. Thus $\frac{(2\alpha m)^p}{n^{p-1}} \le \sum_{i=1}^n \alpha^p d_i^p \le S_\alpha^p(G)$.

If $0 < p < 1$, then $g(x) = -x^p$ is a convex function. By similar arguments as above, the second part of the theorem follows. □

By Lemma 2.4 and Theorem 4.1, we have

Corollary 4.1. *Let $\frac{1}{2} < \alpha < 1$, $0 \le c \le 6$ and G be a c-cyclic graph with n vertices.*

(i) *If $p < 0$ or $p > 1$, $c = 0$ and $n > 2$, then*

$$S_\alpha^p(G) \ge (n-2)(2\alpha)^p + 2\alpha^p.$$

If $p < 0$ or $p > 1$, $c = 1$ and $n > 2$, then

$$S_\alpha^p(G) \ge n(2\alpha)^p.$$

If $p < 0$ or $p > 1$, $2 \le c \le 6$ and $n > 2c - 2$, then

$$S_\alpha^p(G) \ge \alpha^p((2c-2)3^p + (n-2c+2)2^p).$$

(ii) *If $0 < p < 1$, $c = 0$ and $n > 2$, then*

$$S_\alpha^p(G) \le (n-2)(2\alpha)^p + 2\alpha^p.$$

If $0 < p < 1$, $c = 1$ and $n > 2$, then $S_\alpha^p(G) \le n(2\alpha)^p$.
If $0 < p < 1$, $2 \le c \le 6$ and $n > 2c - 2$, then

$$S_\alpha^p(G) \le \alpha^p((2c-2)3^p + (n-2c+2)2^p).$$

Theorem 4.2. *Let $\frac{1}{2} < \alpha < 1$, G be a connected graph of order n and size m with the degree sequence $d_1 \ge d_2 \ge \cdots \ge d_n = \delta$.*

(i) *If $p < 0$ or $p > 1$, then*

$$S_\alpha^p(G) \le \sum_{i=1}^{n-1}(\alpha d_i + (1-\alpha)(n-i))^p + \frac{1}{2^p}(2\alpha\delta - (1-\alpha)n(n-1))^p.$$

(ii) *If $0 < p < 1$, then*

$$S_\alpha^p(G) \ge \sum_{i=1}^{n-1}(\alpha d_i + (1-\alpha)(n-i))^p + \frac{1}{2^p}(2\alpha\delta - (1-\alpha)n(n-1))^p.$$

Proof. Let $x = (\lambda_1(A_\alpha(G)), \lambda_2(A_\alpha(G)), \ldots, \lambda_n(A_\alpha(G)))$ and $y = (\alpha d_1 + (1-\alpha)(n-1), \alpha d_2 + (1-\alpha)(n-2), \ldots, \alpha d_{n-1} + (1-\alpha), 2\alpha m - \alpha(2m-\delta) - (1-\alpha)\frac{n(n-1)}{2})$. From Theorem 3.1 in [28], it follows that $\lambda_i(A_\alpha(G)) \le \alpha d_i + (1-\alpha)(n-i)$ for $1 \le i \le n$. Thus $x \preceq y$. Similar to the method used in Theorem 4.1, we have the proof. □

Theorem 4.3. *Let $\frac{1}{2} < \alpha < 1$, G be a connected graph of order n and size m with the degree sequence $\Delta = d_1 \geq d_2 \geq \cdots \geq d_n$.*

(i) *If $p < 0$ or $p > 1$ and $d_2 \geq d_3 \geq \cdots \geq d_k \geq \alpha n - 1$, then*

$$S_\alpha^p(G) \leq \Delta^p + (k-1)(\alpha n - 1)^p + [2\alpha m - \Delta - (k-1)(\alpha n - 1)]^p, \quad (4.1)$$

where $2 \leq k \leq n$. If $p < 0$ or $p > 1$ and $d_2 \leq \alpha n - 1$, then

$$S_\alpha^p(G) \leq \sum_{i=1}^{k} d_i^p + \left(2\alpha m - \sum_{i=1}^{k} d_i\right)^p,$$

where $2 \leq k \leq n$.

(ii) *If $0 < p < 1$ and $d_2 \geq d_3 \geq \cdots \geq d_k \geq \alpha n - 1$, then*

$$S_\alpha^p(G) \geq \Delta^p + (k-1)(\alpha n - 1)^p + [2\alpha m - \Delta - (k-1)(\alpha n - 1)]^p, \quad (4.2)$$

where $2 \leq k \leq n$. If $0 < p < 1$ and $d_2 \leq \alpha n - 1$, then

$$S_\alpha^p(G) \geq \sum_{i=1}^{k} d_i^p + \left(2\alpha m - \sum_{i=1}^{k} d_i\right)^p,$$

where $2 \leq k \leq n$.

Proof. Let $x = (\lambda_1(A_\alpha(G)), \lambda_2(A_\alpha(G)), \ldots, \lambda_n(A_\alpha(G)))$ and $y = (d_1, \alpha n-1, \ldots, \alpha n-1, 2\alpha m - d_1 - (k-1)(\alpha n-1), 0, \ldots, 0)$. From Proposition 10 in [22], it follows that $\lambda_1(A_\alpha(G)) \leq d_1$. By Lemma 2.1, we have $\lambda_i(A_\alpha(G)) \leq \lambda_i(A_\alpha(K_n)) = \alpha n - 1$ for $2 \leq i \leq n$. Thus $x \preceq y$. Similar to the method used in Theorem 4.1, we have the proof. □

Remark 4.1. *It is easy to see that the equality in (4.1) and (4.2) holds if $G \cong K_n$.*

Theorem 4.4. *Let $\frac{1}{2} < \alpha < 1$, G be a connected graph of order n and size m with the degree sequence $\Delta = d_1 \geq d_2 \geq \cdots \geq d_n$.*

(i) *If $p < 0$ or $p > 1$ and $n - 1 > d_1 \geq d_2 \geq d_3 \geq \cdots \geq d_k \geq \alpha(n-2)$, then*

$$S_\alpha^p(G) \leq \Delta^p + \alpha^p(k-1)(n-2)^p + (2\alpha m - \Delta - \alpha(k-1)(n-2))^p,$$

where $2 \leq k \leq n$. If $p < 0$ or $p > 1$, $d_1 < n-1$ and $d_2 \leq \alpha(n-2)$, then

$$S_\alpha^p(G) \leq \sum_{i=1}^{k} d_i^p + \left(2\alpha m - \sum_{i=1}^{k} d_i\right)^p,$$

where $2 \leq k \leq n$.

(ii) *If $0 < p < 1$ and*

$$n - 1 > d_1 \geq d_2 \geq d_3 \geq \cdots \geq d_k \geq \alpha(n-2),$$

then

$$S_\alpha^p(G) \geq \Delta^p + \alpha^p(k-1)(n-2)^p + (2\alpha m - \Delta - \alpha(k-1)(n-2))^p,$$

where $2 \leq k \leq n$. If $0 < p < 1$, $d_1 < n-1$ and $d_2 \leq \alpha(n-2)$, then

$$S_\alpha^p(G) \geq \sum_{i=1}^{k} d_i^p + \left(2\alpha m - \sum_{i=1}^{k} d_i\right)^p,$$

where $2 \leq k \leq n$.

Proof. Let $x = (\lambda_1(A_\alpha(G)), \lambda_2(A_\alpha(G)), \ldots, \lambda_n(A_\alpha(G)))$ and $y = (d_1, \alpha(n-2), \ldots, \alpha(n-2), 2\alpha m - d_1 - \alpha(k-1)(n-2), 0, \ldots, 0)$. From Proposition 10 in [22] and Theorem 3.1 in [27], it follows that $\lambda_1(A_\alpha(G)) \leq d_1$ and $\lambda_i(A_\alpha(G)) \leq \alpha(n-2)$ for $2 \leq i \leq n$. Thus $x \preceq y$. Similar to the method used in Theorem 4.1, we have the proof. □

Theorem 4.5. *Let $\frac{1}{2} < \alpha < 1$, G be a connected graph of order n and size m with the degree sequence $\Delta = d_1 \geq d_2 \geq \cdots \geq d_n$.*

(i) *If $p < 0$ or $p > 1$ and $m_G(\lambda_j)$ is the multiplicity of λ_j as an eigenvalue of $A_\alpha(G)$, then*

$$S_\alpha^p(G) \leq \Delta^p + \sum_{i=2}^{k} d_i^p + m_G(\lambda_j)\lambda_j^p + \left(2\alpha m - \Delta - \sum_{i=2}^{k} d_i - m_G(\lambda_j)\lambda_j\right)^p,$$

where $2 \leq k \leq j \leq n$.

(ii) *If $0 < p < 1$ and $m_G(\lambda_j)$ is the multiplicity of λ_j as an eigenvalue of $A_\alpha(G)$, then*

$$S_\alpha^p(G) \geq \Delta^p + \sum_{i=2}^{k} d_i^p + m_G(\lambda_j)\lambda_j^p + \left(2\alpha m - \Delta - \sum_{i=2}^{k} d_i - m_G(\lambda_j)\lambda_j\right)^p,$$

where $2 \leq k \leq j \leq n$.

Proof. Let $x = (\lambda_1(A_\alpha(G)), \ldots, \lambda_n(A_\alpha(G)))$ and $y = (d_1, d_2, \ldots, d_k, \lambda_j, \ldots, \lambda_j, 2\alpha m - d_1 - \sum_{i=2}^{k} d_k - m_G(\lambda_j)\lambda_j, 0, \ldots, 0)$. From Proposition 10 in [22], it follows that $\lambda_i(A_\alpha(G)) \leq d_i$ for $1 \leq i \leq n$. Then $x \preceq y$. Similar to the method used in Theorem 4.1, we have the proof. □

By Lemma 2.5 and Theorem 4.5, we have

Corollary 4.2. *Let $\frac{1}{2} < \alpha < 1$, G be a connected graph of order n and size m with the degree sequence $\Delta = d_1 \geq d_2 \geq \cdots \geq d_n$, and let a and b be the number of pendant vertices and quasi-pendant vertices of G, respectively.*

(i) *If $p < 0$ or $p > 1$ and $a - b \geq 1$, then*

$$S_\alpha^p(G) \leq \Delta^p + \sum_{i=2}^{n-a+b-1} d_i^p + (a-b)\alpha^p + \left(2\alpha m - \Delta - \sum_{i=2}^{n-a+b-1} d_i - (a-b)\alpha\right)^p.$$

(ii) *If $0 < p < 1$ and $a - b \geq 1$, then*

$$S_\alpha^p(G) \geq \Delta^p + \sum_{i=2}^{n-a+b-1} d_i^p + (a-b)\alpha^p + \left(2\alpha m - \Delta - \sum_{i=2}^{n-a+b-1} d_i - (a-b)\alpha\right)^p.$$

5. Bounds for $S_\alpha^p(G)$ related to order and size

Theorem 5.1. *Let $\frac{1}{2} < \alpha < 1$, G be a connected graph of order n and size m.*

(i) *If $p < 0$ or $p > 1$ and there is c such that $\lambda_1(A_\alpha(G)) \geq c > 0$, then*

$$S_\alpha^p(G) \geq c^p + \frac{(2\alpha m - c)^p}{(n-1)^{p-1}}.$$

(ii) *If $0 < p < 1$ and there is c such that $\lambda_1(A_\alpha(G)) \geq c > 0$, then*

$$S_\alpha^p(G) \leq c^p + \frac{(2\alpha m - c)^p}{(n-1)^{p-1}}.$$

Proof. Let $x = (c, \frac{2\alpha m - c}{n-1}, \ldots, \frac{2\alpha m - c}{n-1})$ and

$$y = (\lambda_1(A_\alpha(G)), \lambda_2(A_\alpha(G)), \ldots, \lambda_n(A_\alpha(G))).$$

Since $\lambda_1(A_\alpha(G)) \geq c > 0$, we have $x \leq y$. Similar to the method used in Theorem 4.1, we have the proof. □

Corollary 5.1. *Let $\frac{1}{2} < \alpha < 1$, G be a connected graph of order n and size m.*

(i) *If $p < 0$ or $p > 1$, then*

$$S_\alpha^p(G) \geq \frac{1}{\alpha^p}(\alpha^2\Delta + (1-\alpha)^2)^p + \frac{(2\alpha^2 m - \alpha^2\Delta - (1-\alpha)^2)^p}{\alpha^p(n-1)^{p-1}}. \quad (5.1)$$

(ii) *If $0 < p < 1$, then*

$$S_\alpha^p(G) \leq \frac{1}{\alpha^p}(\alpha^2\Delta + (1-\alpha)^2)^p + \frac{(2\alpha^2 m - \alpha^2\Delta - (1-\alpha)^2)^p}{\alpha^p(n-1)^{p-1}}. \quad (5.2)$$

Proof. From Corollary 13 in [22], it follows that

$$\lambda_1(A_\alpha(G)) \geq \alpha\Delta + \frac{(1-\alpha)^2}{\alpha}.$$

By Theorem 5.1, we have the proof. □

Corollary 5.2. *Let $\frac{1}{2} < \alpha < 1$, G be a connected graph of order n and size m.*

(i) *If $p < 0$ or $p > 1$, then*

$$S_\alpha^p(G) \geq \left(\frac{2m}{n}\right)^p \left(1 + \frac{(\alpha n - 1)^p}{(n-1)^{p-1}}\right).$$

(ii) *If $0 < p < 1$, then*

$$S_\alpha^p(G) \leq \left(\frac{2m}{n}\right)^p \left(1 + \frac{(\alpha n - 1)^p}{(n-1)^{p-1}}\right).$$

Proof. From Corollary 19 in [22], it follows that

$$\lambda_1(A_\alpha(G)) \geq \frac{2m}{n}.$$

By Theorem 5.1, we have the proof. □

Remark 5.1. *It is easy to see that the equality in Corollary 5.2 holds if $G \cong K_n$.*

Corollary 5.3. *Let $\frac{1}{2} < \alpha < 1$, G be a connected graph of order n and size m with chromatic number χ.*

(i) *If $p < 0$ or $p > 1$, then*

$$S_\alpha^p(G) \geq (\chi - 1)^p + \frac{(2\alpha m - \chi + 1)^p}{(n-1)^{p-1}}. \quad (5.3)$$

(ii) *If $0 < p < 1$, then*

$$S_\alpha^p(G) \leq (\chi - 1)^p + \frac{(2\alpha m - \chi + 1)^p}{(n-1)^{p-1}}. \quad (5.4)$$

The equality holds in (5.3) and (5.4) if and only if $G \cong K_n$.

Proof. It is well known that $\lambda_1(A(G)) \geq \chi - 1$ with equality if and only if G is a complete graph or an odd cycle, see [42]. From Proposition 18 in [22], it follows that

$$\lambda_1(A_\alpha(G)) \geq \lambda_1(A(G)) \geq \chi - 1$$

with equality if and only if G is a complete graph or an odd cycle. By Theorem 5.1, we have the proof. □

Theorem 5.2. *Let $\frac{1}{2} < \alpha < 1$, G be a connected graph of order n and size m.*

(i) *If $p < 0$ or $p > 1$, then*

$$S_\alpha^p(G) \leq \left(\frac{2(1-\alpha)m}{n-1} + \alpha n - 1\right)^p + (n-2)(\alpha n - 1)^p + \left(2\alpha m - \frac{2(1-\alpha)m}{n-1} - (n-1)(\alpha n - 1)\right)^p. \quad (5.5)$$

(ii) *If $0 < p < 1$, then*

$$S_\alpha^p(G) \geq \left(\frac{2(1-\alpha)m}{n-1} + \alpha n - 1\right)^p + (n-2)(\alpha n - 1)^p + \left(2\alpha m - \frac{2(1-\alpha)m}{n-1} - (n-1)(\alpha n - 1)\right)^p. \quad (5.6)$$

The equality holds in (5.5) and (5.6) if and only if $G \cong K_n$.

Proof. Let $x = (\lambda_1(A_\alpha(G)), \ldots, \lambda_n(A_\alpha(G)))$ and $y = (\frac{2m}{n-1}(1-\alpha)+\alpha n-1, \alpha n-1, \ldots, \alpha n-1, 2\alpha m-\frac{2m}{n-1}(1-\alpha)-(n-1)(\alpha n-1))$. By Lemmas 2.1 and 2.6, we have

$$\lambda_1(A_\alpha(G)) \leq \frac{2m}{n-1}(1-\alpha) + \alpha n - 1$$

and

$$\lambda_i(A_\alpha(G)) \leq \lambda_i(A_\alpha(K_n)) = \alpha n - 1$$

for $2 \leq i \leq n$. Thus $x \preceq y$. Similar to the method used in Theorem 4.1, we have the proof. □

6. $S_\alpha^p(G)$ of bipartite graphs

Theorem 6.1. *Let $\frac{1}{2} < \alpha < 1$, G be a connected bipartite graph with n vertices.*

(i) *If $p > 1$, then $S_\alpha^p(G) \leq S_\alpha^p(K_{1,n-1})$ with equality if and only if $G \cong K_{1,n-1}$.*

(ii) *If $0 < p < 1$, then $S_\alpha^p(G) \leq S_\alpha^p(K_{\lceil n/2\rceil,\lfloor n/2\rfloor})$ with equality if and only if $G \cong K_{\lceil n/2\rceil,\lfloor n/2\rfloor}$.*

(iii) *If $p < 0$, then $S_\alpha^p(G) \geq S_\alpha^p(K_{\lceil n/2\rceil,\lfloor n/2\rfloor})$ with equality if and only if $G \cong K_{\lceil n/2\rceil,\lfloor n/2\rfloor}$.*

Proof. Let $G = (X, Y)$ be a connected bipartite graph on n vertices and suppose that $|X| = a$, $|Y| = b$ and $a \geq b \geq 1$, where $a + b = n$. From Proposition 38 in [22], it follows that

$$\begin{aligned} S_\alpha^p(K_{a,b}) &= \frac{1}{2^p}(\alpha n + \sqrt{\alpha^2 n^2 + 4a(n-a)(1-2\alpha)})^p \\ &\quad + \alpha^p a^p + \alpha^p (n-a)^p \\ &\quad + \frac{1}{2^p}(\alpha n - \sqrt{\alpha^2 n^2 + 4a(n-a)(1-2\alpha)})^p. \end{aligned}$$

Let

$$\begin{aligned} f(x) &= \frac{1}{2^p}(\alpha n + \sqrt{\alpha^2 n^2 + 4x(n-x)(1-2\alpha)})^p \\ &\quad + \alpha^p x^p + \alpha^p (n-x)^p \\ &\quad + \frac{1}{2^p}(\alpha n - \sqrt{\alpha^2 n^2 + 4x(n-x)(1-2\alpha)})^p. \end{aligned}$$

Then

$$\begin{aligned} f'(x) &= \frac{p(1-2\alpha)(n-2x)}{2^{p-1}\sqrt{\alpha^2 n^2 + 4x(n-x)(1-2\alpha)}}[(\alpha n \\ &\quad + \sqrt{\alpha^2 n^2 + 4x(n-x)(1-2\alpha)})^{p-1} \\ &\quad - (\alpha n - \sqrt{\alpha^2 n^2 + 4x(n-x)(1-2\alpha)})^{p-1}] \\ &\quad + p\alpha^p (x^{p-1} - (n-x)^{p-1}). \end{aligned}$$

If $p > 1$, then $f(x)$ is decreasing for $1 \leq x \leq \frac{n}{2}$ and increasing for $\frac{n}{2} \leq x \leq n-1$. Hence $f(n/2) \leq f(x) \leq f(1)$. By Theorem 3.1, we have $S_\alpha^p(G) < S_\alpha^p(K_{a,b}) \leq S_\alpha^p(K_{1,n-1})$ for $p > 0$, $p \neq 1$ and $G \neq K_{a,b}$.

If $0 < p < 1$, then $f(x)$ is increasing for $1 \leq x \leq \frac{n}{2}$ and decreasing for $\frac{n}{2} \leq x \leq n-1$. Hence $f(1) \leq f(x) \leq f(n/2)$. By Theorem 3.1, we have $S_\alpha^p(G) < S_\alpha^p(K_{a,b}) \leq S_\alpha^p(K_{\lceil n/2\rceil,\lfloor n/2\rfloor})$ for $p > 0$, $p \neq 1$ and $G \neq K_{a,b}$.

If $p < 0$, then $f(x)$ is decreasing for $1 \leq x \leq \frac{n}{2}$ and increasing for $\frac{n}{2} \leq x \leq n-1$. Hence $f(n/2) \leq f(x) \leq f(1)$. By Theorem 3.1, we have $S_\alpha^p(G) > S_\alpha^p(K_{a,b}) \geq S_\alpha^p(K_{\lceil n/2\rceil,\lfloor n/2\rfloor})$ for $p < 0$ and $G \neq K_{a,b}$.

Combining the above arguments, we have the proof. □

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Conflict of interest

The authors declared that they have no conflicts of interest to this work.

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