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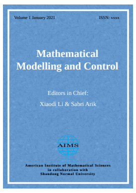


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*Research article*

# Global existence of positive and negative solutions for IFDEs via Lyapunov-Razumikhin method

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**Abstract:** This paper considers the global existence of positive and negative solutions for impulsive functional differential equations (IFDEs). First, we introduce the concept of $\varepsilon$-unstability to IFDEs and establish some sufficient conditions to guarantee the $\varepsilon$-unstability via Lyapunov-Razumikhin method. Based on the obtained results, we present some sufficient conditions for the global existence of positive and negative solutions of IFDEs. An example is also given to demonstrate the effectiveness of the results.

**Keywords:** impulsive functional differential equations (IFDEs); global existence; Lyapunov-Razumikhin method; positive solution; negative solution

## 1. Introduction

As is well known, impulsive differential equations can display numerous practical problems arising in mechanics, electrical engineering, medicine, biology, ecology, etc. The mathematical theory of impulsive differential equations have been extensively studied in many classical problems in the past several decades. Such as stability, persistence, synchronization and control problems. Many important and interesting results have been achieved. The reader can refer to some papers and books by Bainov and Simeonov [1,2], Lakshmikantham et al. [3–5], K. Gopalsamy and Zhang [6], and [7–15] among others.

The method of Lyapunov-Razumikhin functions has been widely applied to dynamical analysis of various delay differential equations, especially in stability of IFDEs. Many important results can be seen in [16–26] and the references therein. In 1892, Lyapunov presented the idea of this method originated with for the ordinary differential equations. And Razumikhin developed it to delay differential equations in 1956. A manifest advantage of this method is that it can exhibit the dynamics of systems and does not require the knowledge of solutions of systems. And then, one may naturally ask that whether it can be applied to the existence problems of positive and negative solutions of systems. In other words, we can establish some Lyapunov-Razumikhin type conditions to guarantee the existence of positive and negative for IFDEs. Several results of existence of positive solutions for IFDEs can be seen in [27–31]. Although Lyapunov-Razumikhin methods have been developed to stability and control problem of impulsive dynamical systems in past years, to the best of author's knowledge, so far there is almost no result of Lyapunov-Razumikhin type on the existence of positive and negative solutions for IFDEs and the aim of this paper is to close this gap.

In this paper, we shall develop the Lyapunov-Razumikhin method to study the existence problems of positive and negative solutions for IFDEs. In order to do this, we first introduce the concept of $\varepsilon$-unstability to IFDEs and establish some sufficient conditions to guarantee the $\varepsilon$-unstability via Lyapunov-Razumikhin method. Based on the obtained results, we present some sufficient conditions for the global existence of positive and negative solutions of IFDEs. An example is given to demonstrate the effectiveness of the results.

## 2. Preliminaries

**Notations.** Let $\mathbb{R}$ denote the set of real numbers, $\mathbb{R}_+$ the set of positive real numbers, $\mathbb{Z}_+$ the set of positive integers and $\mathbb{R}^n$ the $n$-dimensional real space equipped with the Euclidean norm $|\bullet|$. $\mathbb{K} = \{a \in C(\mathbb{R}_+, \mathbb{R}_+)|\ a(0) = 0$ and $a(s) > 0$ for $s > 0$, $a$ is strictly increasing in $s$ and tend to infinite as $s$ tend to infinite $\}$. $C(S, V) = \{\varphi : S \to V$ is continuous$\}$ and $PC(S, V) = \{\varphi : S \to V$ is continuous everywhere except at finite number of points $t$, at which $\varphi(t^+)$, $\varphi(t)$ exist and $\varphi(t^+) = \varphi(t)\}$. In particular, let $PC_r = PC([-r, 0], \mathbb{R})$. For $\varphi \in PC_r$, the norm of $\varphi$ is defined by $\|\varphi\|_r = \max_{-r\leq\theta\leq 0} |\varphi(\theta)|$.

Consider the following IFDEs:

$$\begin{cases} x'(t) = f(t, x_t), & t \in [t_{k-1}, t_k), \\ \Delta x|_{t=t_k} = x(t_k) - x(t_k^-) = I_k(t_k, x(t_k^-)), & k \in \mathbb{Z}_+, \\ x_{t_0} = \phi(s), & -r \leq s \leq 0, \end{cases} \tag{2.1}$$

where $\phi \in PC_r$, the impulse times $t_k$ satisfy $0 \leq t_0 < t_1 < \ldots < t_k \to \infty$ as $k \to \infty$ and $x^{'}$ denotes the right-hand derivative of $x$. For each $t \geq t_0$, $x_t, x_{t^-} \in PC_r$ are defined by $x_t(s) = x(t + s)$, $x_{t^-}(s) = x(t^- + s)$, $s \in [-r, 0]$.

In this paper, we make the following assumptions:

- $(H_1)$ $f : [t_{k-1}, t_k) \times PC_r \to \mathbb{R}, k \in \mathbb{Z}_+$, is continuous and $f(t, 0) = 0$. For any $\varphi \in PC_r, k \in \mathbb{Z}_+$, the limit $\lim_{(t,\theta)\to(t_k^-,\varphi)} f(t, \theta) = f(t_k^-, \varphi)$ exists.
- $(H_2)$ $f(t, \varphi)$ is Lipschitzian in $\varphi$ in each compact set in $PC_r$.
- $(H_3)$ $I_k(t, x) : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}, k \in \mathbb{Z}_+$, is continuous and $I_k(t, 0) = 0$. For any $\rho > 0$, there exists a $\rho_1 \in (0, \rho)$ such that $x \in S(\rho_1)$ implies that $x + I_k(t_k, x) \in S(\rho)$, where $S(\rho) = \{x : |x| < \rho, x \in \mathbb{R}\}$.
- $(H_4)$ For $\varphi \in PC_r, \|\phi\|_{r0} \doteq \min_{-r\leq\theta\leq 0} |\varphi(\theta)| > 0$ holds.
- $(H_5)$ For any $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$, $x[x + I_k(t, x)] \geq 0$ holds for all $k \in \mathbb{Z}_+$.

**Remark 2.1.** Under the assumptions $(H_1) - (H_3)$, the initial value problem (2.1) exists with a unique solution which can be written in the form $x(t, t_0, \phi)$, see [12,7] for detailed information. Assumptions $(H_4)$ and $(H_5)$ are given for later use.

**Definition 2.1.** The function $V : [-r, \infty) \times PC_r \to \mathbb{R}_+$ belongs to class $\nu_0$ if
$(H_1)$ $V$ is continuous on each of the sets $[t_{k-1}, t_k) \times PC_r$ and $\lim_{(t,\varphi_1)\to(t_k^-,\varphi_2)} V(t, \varphi_1) = V(t_k^-, \varphi_2)$ exists;
$(H_2)$ $V(t, x)$ is locally Lipschitzian in $x$ and $V(t, 0) \equiv 0$.

**Definition 2.2.** Let $V \in \nu_0$, for any $(t, \psi) \in [t_{k-1}, t_k) \times PC_r$, the right-upper Dini derivative of $V(t, x)$ along the solution of system (2.1) is defined by

$$D^+V(t, \psi(0)) = \limsup_{h\to 0^+} \frac{1}{h}\{V(t+h, \psi(0)+hf(t, \psi))-V(t, \psi(0))\}.$$

**Definition 2.3.** Assume that $x(t) = x(t, t_0, \phi)$ be the solution of system (2.1) through $(t_0, \phi)$. Then the trivial solution of system (2.1) is said to be

1. $\varepsilon$-unstable, if for any $\varepsilon > 0$ and $t_0 \geq 0$, there exists a $\delta = \delta(t_0, \varepsilon) > 0$ such that $\|\phi\|_{r0} \geq \delta$ implies $|x(t)| \geq \varepsilon, t \geq t_0$.
2. Uniformly $\varepsilon$-unstable, if $\delta$ is independent of $t_0$.

## 3. $\varepsilon$-unstability results

In this section, we shall establish some sufficient conditions to guarantee the $\varepsilon$-unstability of the trivial solution of system (2.1) by using Lyapunov-Razumikhin method and some analysis techniques.

**Theorem 3.1.** *Assume that $(H_1) - (H_4)$ hold. If there exist some functions $w_1, w_2 \in \mathbb{K}, c \in C(\mathbb{R}_+, \mathbb{R}_+), p \in PC(\mathbb{R}_+, \mathbb{R}_+), V \in \nu_0$, and some constants $q > 1, \sigma > 0$, $\beta_k \in [0, 1)$, $k \in \mathbb{Z}_+$ such that*

*(i)* $w_1(|x|) \leq V(t, x) \leq w_2(|x|)$, $(t, x) \in [t_0, \infty) \times \mathbb{R}$;

*(ii)* $D^+V(t, \psi(0)) \geq -p(t)c(V(t, \psi(0)))$, *for all* $t \in [t_{k-1}, t_k), k \in \mathbb{Z}_+$ *whenever* $qV(t + \theta, \psi(\theta)) \geq V(t, \psi(0))$, *for* $\theta \in [-r, 0]$ *and* $\psi \in PC_r$;

*(iii)* $V(t_k, \psi(0) + I_k(t_k, \psi)) \geq q(1 - \beta_k)V(t_k^-, \psi(0))$, *for all* $(t_k, \psi) \in \mathbb{R}_+ \times PC_r$, *and* $\prod_{k=1}^{m}(1 - \beta_k) \geq \sigma$ *for all* $m \in \mathbb{Z}_+$;

*(iv)* $\inf_{s>0} \int_s^{qs} \frac{du}{c(u)} - \int_{t_{k-1}}^{t_k} p(s)ds > 0$ *for all* $k \in \mathbb{Z}_+$.

*Then the trivial solution of system* (2.1) *is uniformly $\varepsilon$-unstable.*

**Proof.** Let $x(t) = x(t, t_0, \phi)$ be a solution of system (2.1) through $(t_0, \phi)$. For any $\varepsilon > 0$, choose $\beta = \beta(\varepsilon) > 0$ and $\delta = \delta(\varepsilon) > 0$ such that

$$w_2(\varepsilon) \leq \sigma w_1(\beta) \leq w_1(\beta) < q w_1(\beta) \leq w_1(\delta). \tag{3.1}$$

For the sake of brevity, we denote $V(t) = V(t, x(t))$. Next we shall show that for any $\phi \in PC_r$, $\|\phi\|_{r0} \geq \delta$ implies

$$V(t) \geq \prod_{k=0}^{m-1}(1-\beta_k)w_1(\beta),\ t \in [t_{m-1}, t_m),\ m \in \mathbb{Z}_+.$$

where $\beta_0 = 0$. Frist, we show that $V(t) \geq w_1(\beta), t \in [t_0, t_1)$. Suppose not, then there exist some $t \in [t_0, t_1)$ such that $V(t) < w_1(\beta)$. Let

$$\bar{t} = \inf\{t \in [t_0, t_1),\ V(t) < w_1(\beta)\}.$$

It follows from (3.1) that $V(t_0) \geq w_1(\delta) > w_1(\beta)$. So it is obvious that $\bar{t} > t_0,\ V(\bar{t}) = w_1(\beta)$ and

$$V(t) \geq w_1(\beta), t \in [t_0, \bar{t}].$$

Considering (3.1) again, it holds:

$$V(t) \geq w_1(\beta), t \in [t_0 - r, \bar{t}]. \tag{3.2}$$

Since $V(t_0) \geq q w_1(\beta)$, we further define

$$\underline{t} = \sup\{t \in [t_0, \bar{t}],\ V(t) \geq q w_1(\beta)\}.$$

Obviously, $\underline{t} < \bar{t}$, $V(\underline{t}) = q w_1(\beta)$ and together with (3.2) yields

$$w_1(\beta) \leq V(t) \leq q w_1(\beta), t \in [\underline{t}, \bar{t}].$$

Thus it can be obtained that

$$qV(t+\theta) \geq q w_1(\beta) \geq V(t), \theta \in [-r, 0],\ t \in [\underline{t}, \bar{t}],$$

which implies that $D^+V(t, \psi(0)) \geq -p(t)c(V(t, \psi(0)))$ for $t \in [\underline{t}, \bar{t}]$.

Hence, we get

$$\begin{aligned} \inf_{s>0} \int_s^{qs} \frac{du}{c(u)} &\leq \int_{w_1(\beta)}^{q w_1(\beta)} \frac{du}{c(u)} \\ &= \int_{V(\bar{t})}^{V(\underline{t})} \frac{du}{c(u)} \\ &\leq \int_{\underline{t}}^{\bar{t}} p(u)du \\ &\leq \int_{t_0}^{t_1} p(u)du, \end{aligned}$$

which is a contradiction with condition ($iv$). So we obtain $V(t) \geq w_1(\beta),\ t \in [t_0, t_1)$.

Meanwhile, we note that

$$V(t_1) \geq q(1-\beta_1)V(t_1^-) \geq q(1-\beta_1)w_1(\beta).$$

Now we suppose that

$$\begin{cases} V(t) \geq \prod_{k=0}^{m-1}(1-\beta_k)w_1(\beta), t \in [t_{m-1}, t_m) \\ V(t_m) \geq q\prod_{k=0}^{m}(1-\beta_k)w_1(\beta) \end{cases} \tag{3.3}$$

for

$$1 \leq m \leq N, N \in \mathbb{Z}_+.$$

Next we show that

$$V(t) \geq \prod_{k=0}^{N}(1-\beta_k)w_1(\beta),\ t \in [t_N, t_{N+1}). \tag{3.4}$$

For the sake of brevity, we define

$$\mathbb{B} = \prod_{k=0}^{N-1}(1-\beta_k)w_1(\beta).$$

Thus, it follows from (3.3) that

$$\begin{cases} V(t) \geq \mathbb{B},\ t \in [t_0 - r, t_N), \\ V(t_N) \geq q(1-\beta_N)\mathbb{B}. \end{cases} \tag{3.5}$$

Now we only need prove that

$$V(t) \geq (1-\beta_N)\mathbb{B},\ t \in [t_N, t_{N+1}).$$

Suppose not, then there exist some $t \in [t_N, t_{N+1})$ such that $V(t) < (1-\beta_N)\mathbb{B}$. Let

$$t^\star = \inf\{t \in [t_N, t_{N+1}),\ V(t) < (1-\beta_N)\mathbb{B}\},$$

then $t^\star > t_N,\ V(t^\star) = (1-\beta_N)\mathbb{B}$ and

$$V(t) \geq (1-\beta_N)\mathbb{B},\ t \in [t_N, t^\star],$$

which together with (3.5) yields

$$V(t) \geq (1-\beta_N)\mathbb{B},\ t \in [t_0 - r, t^\star]. \tag{3.6}$$

Note that $V(t_N) \geq q(1-\beta_N)\mathbb{B}$, we further define

$$t^* = \sup\{t \in [t_N, t^\star],\ V(t) \geq q(1-\beta_N)\mathbb{B}\},$$

then $t^* < t^\star$, $V(t^*) = q(1-\beta_N)\mathbb{B}$ and

$$(1-\beta_N)\mathbb{B} \le V(t) \le q(1-\beta_N)\mathbb{B},\ t \in [t^*, t^\star].$$

It follows from (3.6) and above inequality that

$$qV(t+\theta) \ge q(1-\beta_N)\mathbb{B} \ge V(t), \theta \in [-r, 0],\ t \in [t^*, t^\star],$$

which implies that $D^+V(t,\psi(0)) \ge -p(t)c(V(t,\psi(0)))$ for $t \in [t^*, t^\star]$.
Hence, we get

$$\begin{aligned} \inf_{s>0}\int_s^{qs}\frac{du}{c(u)} &\le \int_{(1-\beta_N)\mathbb{B}}^{q(1-\beta_N)\mathbb{B}}\frac{du}{c(u)} \\ &= \int_{V(t^\star)}^{V(t^*)}\frac{du}{c(u)} \\ &\le \int_{t^*}^{t^\star} p(u)du \\ &\le \int_{t_N}^{t_{N+1}} p(u)du, \end{aligned}$$

which is a contradiction. So we have proven (3.4) holds.
By the method of induction, in general, we get

$$V(t) \ge \prod_{k=0}^{m}(1-\beta_k)w_1(\beta),\ t \in [t_m, t_{m+1}), m \in \mathbb{Z}_+,$$

which implies that

$$w_2(|x(t)|) \ge V(t) \ge \prod_{k=0}^{m}(1-\beta_k)w_1(\beta) \ge \sigma w_1(\beta) \ge w_2(\varepsilon),$$

for

$$t \ge t_0,$$

i.e.,

$$|x(t)| \ge \epsilon,\ \ t \ge t_0.$$

Thus the trivial solution of system (2.1) is uniformly $\varepsilon$-unstable. □

**Corollary 3.1.** *Assume that* $(H_1)-(H_4)$ *hold. If there exist some functions* $w_1, w_2 \in \mathbb{K}, V \in \nu_0$, *and some constants* $p > 0, q > 1, \sigma > 0,\ \beta_k \in [0, 1),\ k \in \mathbb{Z}_+$ *such that*

(*i*) $w_1(|x|) \le V(t, x) \le w_2(|x|),\ (t, x) \in [t_0, \infty) \times \mathbb{R}$;

*(ii)* $D^+V(t,\psi(0)) \ge -pV(t,\psi(0))$, *for all* $t \in [t_{k-1}, t_k), k \in \mathbb{Z}_+$ *whenever* $qV(t+\theta,\psi(\theta)) \ge V(t,\psi(0))$, *for* $\theta \in [-r, 0]$ *and* $\psi \in PC_r$;

*(iii)* $V(t_k, \psi(0) + I_k(t_k, \psi)) \ge q(1-\beta_k)V(t_k^-, \psi(0))$, *for all* $(t_k, \psi) \in \mathbb{R}_+ \times PC_r$, *and* $\prod_{k=1}^{m}(1-\beta_k) \ge \sigma$ *for all* $m \in \mathbb{Z}_+$;

*(iv)* $t_k - t_{k-1} < \frac{\ln q}{p}$, *for all* $k \in \mathbb{Z}_+$. *Then the trivial solution of system* (2.1) *is uniformly* $\varepsilon$*-unstable*

**Remark 3.1.** Theorem 3.1 presents some sufficient conditions from the view of impulsive control to ensure the uniform $\varepsilon$-unstability. In fact, the $\varepsilon$-unstability can also be derived from the view of impulsive perturbation. Next we shall give the main result and its proof is similar to Theorem 3.1 and omitted here.

**Theorem 3.2.** *Assume that* $(H_1)-(H_4)$ *hold. If there exist some functions* $w_1, w_2 \in \mathbb{K}, V \in \nu_0$, *and some constants* $\sigma > 0,\ \beta_k \in [0, 1),\ k \in \mathbb{Z}_+$ *such that*

(*i*) $w_1(|x|) \le V(t, x) \le w_2(|x|),\ (t, x) \in [t_0, \infty) \times \mathbb{R}$;

*(ii)* $D^+V(t,\psi(0)) \ge 0$, *for all* $t \in [t_{k-1}, t_k), k \in \mathbb{Z}_+$ *whenever* $V(t+\theta,\psi(\theta)) \ge V(t,\psi(0))$, *for* $\theta \in [-r, 0]$ *and* $\psi \in PC_r$;

*(iii)* $V(t_k, \psi(0) + I_k(t_k, \psi)) \ge (1-\beta_k)V(t_k^-, \psi(0))$, *for all* $(t_k, \psi) \in \mathbb{R}_+ \times PC_r$, *and* $\prod_{k=1}^{m}(1-\beta_k) \ge \sigma$ *for all* $m \in \mathbb{Z}_+$.

*Then the trivial solution of system* (2.1) *is uniformly* $\varepsilon$*-unstable.*

From Theorem 3.2, we can obtain the uniform $\varepsilon$-unstability result for system (2.1) without impulsive effect.

**Corollary 3.2.** *Assume that* $(H_1)-(H_4)$ *hold. If there exist some functions* $w_1, w_2 \in \mathbb{K}$ *such that*

(*i*) $w_1(|x|) \le V(t, x) \le w_2(|x|),\ (t, x) \in [t_0, \infty) \times \mathbb{R}$;

*(ii)* $D^+V(t,\psi(0)) \ge 0$, *for all* $t \ge t_0$, *whenever* $V(t+\theta,\psi(\theta)) \ge V(t,\psi(0))$, *for* $\theta \in [-r, 0]$.

*Then the trivial solution of system* (2.1) *without impulsive effect is uniformly* $\varepsilon$*-unstable.*

## 4. Global existence of positive and negative solutions

Based on the obtained results in Section 3, we next shall study the global existence of positive and negative solutions for system (2.1).

**Theorem 4.1.** *Assume that* $(H_1)-(H_5)$ *hold. If there exist some functions* $w_1, w_2 \in \mathbb{K}, c \in C(\mathbb{R}_+, \mathbb{R}_+), p \in$

*$PC(\mathbb{R}_+, \mathbb{R}_+), V \in \nu_0$, and some constants $q > 1, \sigma > 0$, $\beta_k \in [0, 1)$, $k \in \mathbb{Z}_+$ such that*

*(i)* $w_1(|x|) \le V(t, x) \le w_2(|x|)$, $(t, x) \in [t_0, \infty) \times \mathbb{R}$;

*(ii)* $D^+V(t, x(t)) \ge -p(t)c(V(t, x(t)))$, *for all* $t \in [t_{k-1}, t_k), k \in \mathbb{Z}_+$ *whenever* $qV(t + \theta, x(t + \theta)) \ge V(t, x(t))$, *for* $\theta \in [-r, 0]$;

*(iii)* $V(t_k, x(t_k^-) + I_k(t_k, x(t_k^-))) \ge q(1 - \beta_k)V(t_k^-, x(t_k^-))$, *and* $\prod_{k=1}^m(1 - \beta_k) \ge \sigma$ *for all* $m \in \mathbb{Z}_+$;

*(iv)*

$$\inf_{s>0} \int_s^{qs} \frac{du}{c(u)} - \int_{t_{k-1}}^{t_k} p(s)ds > 0 \text{ for all } k \in \mathbb{Z}_+,$$

*where* $x(t) = x(t, t_0, \phi)$ *is a solution of system* (2.1) *with* $\|\phi\|_{r0} > 0$ *and* $\phi(0) > 0$.

*Then* $x$ *is a globally positive solution of system* (2.1).

**Proof.** From Theorem 3.1, we know that the trivial solution of system (2.1) is uniformly $\varepsilon$-unstable. So for any $\varepsilon > 0$, one may choose $\delta = w_1^{-1}\left(\frac{q}{\sigma}w_2(\epsilon)\right)$ such that $\|\phi\|_{r0} \ge \delta$ implies $|x(t)| \ge \epsilon, t \ge t_0$. Note that $w_1, w_2 \in \mathbb{K}$ and

$$\lim_{s\to 0} w_1^{-1}\left(\frac{q}{\sigma}w_2(s)\right) = 0.$$

Thus we analyze it from another point of view. Since $x(t) = x(t, t_0, \phi)$ is a solution of system (2.1) with $\|\phi\|_{r0} > 0$, we define

$$\delta_\phi = \|\phi\|_{r0} \quad \text{and} \quad \varepsilon_\phi = w_2^{-1}\left(\frac{\sigma}{q}w_1(\delta_\phi)\right).$$

Obviously, for $\varepsilon_\phi > 0$, we have $|x(t)| \ge \varepsilon_\phi$, $t \ge t_0$. Then note that the continuity of $x(t)$ on $[t_0, t_1)$ and $\phi(0) > 0$, we get $x(t) > 0, t \in [t_0, t_1)$. From $(H_5)$, it is clear that $x(t_1^-) > 0$ and $x(t_1) > 0$. Similarly, we get $x(t) > 0, t \in [t_1, t_2)$ in view of the continuity of $x(t)$ on $[t_1, t_2)$. In this way, we can deduce that $x(t) > 0, t \ge t_0$. Thus the proof is complete. □

**Remark 4.1.** It should be noted that in the proof of Theorem 4.1, we are interested in the the existence of positive constant $\varepsilon$ rather than its concrete value. Moreover, one may find that assumption $(H_5)$ plays an important role in guaranteeing the global existence of positive (negative) solutions.

**Corollary 4.1.** *Under the conditions in Theorem 4.1, assume that* $x(t) = x(t, t_0, \phi)$ *is a solution of system* (2.1) *with* $\|\phi\|_{r0} > 0$ *and* $\phi(0) < 0$. *Then the solution* $x$ *is a globally negative solution of system* (2.1).

**Theorem 4.2.** *Assume that* $(H_1) - (H_5)$ *hold. If there exist some functions* $w_1, w_2 \in \mathbb{K}, V \in \nu_0$, *and some constants* $\sigma > 0$, $\beta_k \in [0, 1)$, $k \in \mathbb{Z}_+$ *such that*

*(i)* $w_1(|x|) \le V(t, x) \le w_2(|x|)$, $(t, x) \in [t_0, \infty) \times \mathbb{R}$;

*(ii)* $D^+V(t, x(t)) \ge 0$, *for all* $t \in [t_{k-1}, t_k), k \in \mathbb{Z}_+$ *whenever* $V(t + \theta, x(t + \theta)) \ge V(t, x(t))$, *for* $\theta \in [-r, 0]$;

*(iii)* $V(t_k, x(t_k^-) + I_k(t_k, x(t_k^-))) \ge (1 - \beta_k)V(t_k^-, x(t_k^-))$, *and* $\prod_{k=1}^m(1 - \beta_k) \ge \sigma$ *for all* $m \in \mathbb{Z}_+$,

*where* $x(t) = x(t, t_0, \phi)$ *is a solution of system* (2.1) *with* $\|\phi\|_{r0} > 0$ *and* $\phi(0) > 0$.

*Then* $x$ *is a globally positive solution of system* (2.1).

**Corollary 4.2.** *Under the conditions in Theorem 4.2, assume that* $x(t) = x(t, t_0, \phi)$ *is a solution of system* (2.1) *with* $\|\phi\|_{r0} > 0$ *and* $\phi(0) < 0$. *Then the solution* $x$ *is a global negative solution of system* (2.1).

**Remark 4.2.** The proofs of Theorem 4.2 and Corollary 4.2 is similar to Theorem 4.1 and omitted here.

**Example.** Consider the following IFDE:

$$\begin{cases} x'(t) = a(t)x(t) + b(t)\int_{-r}^0 |x(t + u)|\text{sign}(x(t))du, \ t \ge 0, t \ne t_k, \\ \Delta x(t_k) = I_k(x(t_k^-)), \ k \in \mathbb{Z}_+, \\ x(s) = \varphi(s), \ s \in [-r, 0], \end{cases} \tag{4.1}$$

where $\varphi \in PC_r, a \in C(\mathbb{R}_+, \mathbb{R}), b \in C(\mathbb{R}_+, \mathbb{R}_+)$. $r > 0$ is a constant. Here we consider the following two cases:

**(I)** $I_k(s) = (\lambda - 1 - \lambda\beta_k)s, \lambda > 1, \beta_k \in [0, 1), k \in \mathbb{Z}_+$;
**(II)** $I_k(s) = -\beta_k s, \beta_k \in [0, 1), k \in \mathbb{Z}_+$.

**Property 1. Case (I).** *Assume that there exist some constants* $q \in (1, \lambda], p > 0$, *and* $\sigma > 0$ *such that*

$(P_1)$ $-a(t) - \frac{r}{q}b(t) \le p, \ t \ge 0$;

$(P_2)$ $\prod_{k=1}^m(1 - \beta_k) \ge \sigma, m \in \mathbb{Z}_+$;

$(P_3)$ $t_k - t_{k-1} < \frac{\ln q}{p}, k \in \mathbb{Z}_+$;

($P_4$) $\min_{-r\le\theta\le 0}|\varphi(\theta)| > 0$.

*Then* $x(t) = x(t,\varphi)$ *is a globally positive (negative) solution of system* (4.1) *if* $\varphi(0) > 0(< 0)$.

**Property 2. Case (II).** *Assume that there exists a constant* $\sigma > 0$ *such that*

($Q_1$) $a(t) + rb(t) \ge 0,\ t \ge 0$;

($Q_2$) $\prod_{k=1}^{m}(1-\beta_k) \ge \sigma$, $m \in \mathbb{Z}_+$;

($Q_3$) $\min_{-r\le\theta\le 0}|\varphi(\theta)| > 0$.

*Then* $x(t) = x(t,\varphi)$ *is a globally positive (negative) solution of system* (3.1) *if* $\varphi(0) > 0(< 0)$.

**Remark 4.3.** Let $V(t) = |x(t)|$, then the results in Properties 1 and 2 can be easily obtained by Theorem 4.1 and 4.2. From among, one may observe that Properties 1 and 2 present the global existence of positive (negative) solutions of system (3.1) from the point of view of the impulsive control and impulsive perturbation, respectively.

## Conflict of interest

The author declares there is no conflict of interests.

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