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*Research article*

# Domain perturbation for biharmonic Steklov problems: spectral stability, boundary homogenization and degeneration

**Bauyrzhan Derbissaly**[1] **and Pier Domenico Lamberti**[2,*]

[1] Institute of Mathematics and Mathematical Modeling, 125 Pushkin, Almaty, 050010, Kazakhstan

[2] Dipartimento di Tecnica e Gestione dei Sistemi Industriali, Università degli Studi di Padova, 3 Stradella S. Nicola, Vicenza, 36100, Italy

* **Correspondence:** Email: lamberti@math.unipd.it.

**Abstract:** In this paper, we investigate the spectral stability of two biharmonic Steklov problems under domain perturbation. We provide optimal conditions on the boundary perturbations ensuring the stability of both eigenvalues and eigenfunctions. To highlight the optimality of those conditions, we present alternative assumptions on the boundary perturbations that lead to either a degeneration of the spectrum or to the appearance of a strange term in the limiting problem. In particular, these phenomena are discussed for a boundary homogenization problem exhibiting a trichotomy in the asymptotic behaviour.

**Keywords:** Steklov boundary conditions; multi-parameter eigenvalue problems; biharmonic Steklov eigenvalues; domain perturbations; spectral stability

## 1. Introduction

Let $\Omega$ be a sufficiently regular, bounded domain (i.e., open connected set) in $\mathbb{R}^N$ with $N \geq 2$. For fixed $\sigma \in (-1/(N-1), 1)$ and $\mu > 0$ we consider the following Steklov problem for the biharmonic operator

$$(\mathrm{DBS})_\mu : \begin{cases} \Delta^2 u = 0, & \text{in } \Omega, \\ (1-\sigma)\, u_{\nu\nu} + \sigma \Delta u = \lambda u_\nu, & \text{on } \partial\Omega, \\ (1-\sigma)\, \mathrm{div}_{\partial\Omega} \left( D^2 u \cdot \nu \right)_{\partial\Omega} + (\Delta u)_\nu = \mu u, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

in the unknowns $u$ (the eigenfunction) and $\lambda$ (the eigenvalue). We call this problem $(\mathrm{DBS})_\mu$ where the acronym $(\mathrm{DBS})_\mu$ stands for Dirichlet Biharmonic Steklov (depending on $\mu$). This is motivated by the well-known (DBS) problem (1.5) below, see also Remark 1.1.

Here $\operatorname{div}_{\partial\Omega} F := \operatorname{div} F - (\nabla F \cdot \nu)\nu$ denotes the tangential divergence of a vector field $F$ and $F_{\partial\Omega} := F - (F \cdot \nu)\nu$ denotes the tangential component of $F$.

As second problem, for fixed $\sigma \in (-1/(N-1), 1)$ and $\rho > 0$ we consider the Steklov problem for the biharmonic operator

$$(\mathrm{NBS})_\rho : \begin{cases} \Delta^2 v = 0, & \text{in } \Omega, \\ (\sigma - 1)\, v_{\nu\nu} - \sigma \Delta v = \rho v_\nu, & \text{on } \partial\Omega, \\ (\sigma - 1) \operatorname{div}_{\partial\Omega} \left(D^2 v \cdot \nu\right)_{\partial\Omega} - (\Delta v)_\nu = \gamma v, & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

in the unknowns $v$ (the eigenfunction) and $\gamma$ (the eigenvalue). We call this problem $(\mathrm{NBS})_\rho$ where the acronym $(\mathrm{NBS})_\rho$ stands for Neumann Biharmonic Steklov (depending on $\rho$). This is motivated by the well-known (NBS) problem (1.6) below, see also Remark 1.1.

Note that the weak formulation of problem (1.1) reads as

$$\int_\Omega (1 - \sigma)\, D^2 u : D^2 \varphi + \sigma \Delta u \Delta \varphi dx + \mu \int_{\partial\Omega} u \varphi d\sigma = \lambda \int_{\partial\Omega} u_\nu \varphi_\nu d\sigma, \ \forall \varphi \in H^2(\Omega), \tag{1.3}$$

in the unknowns $u \in H^2(\Omega)$, $\lambda \in \mathbb{R}$, where

$$D^2 u : D^2 \varphi = \sum_{i,j=1}^{N} \frac{\partial^2 u}{\partial x_i \partial x_j} \frac{\partial^2 \varphi}{\partial x_i \partial x_j}$$

denotes the Frobenius product of the Hessians matrices and $H^2(\Omega)$ denotes the standard Sobolev space of functions in $L^2(\Omega)$ with all weak derivatives of the first and second order in $L^2(\Omega)$.

The weak formulation of problem (1.2) is

$$\int_\Omega (1 - \sigma)\, D^2 v : D^2 \varphi + \sigma \Delta v \Delta \varphi dx + \rho \int_{\partial\Omega} v_\nu \varphi_\nu d\sigma = \gamma \int_{\partial\Omega} v \varphi d\sigma, \ \forall \varphi \in H^2(\Omega), \tag{1.4}$$

in the unknowns $v \in H^2(\Omega)$, $\gamma \in \mathbb{R}$.

In [30] it was proved that, in the limit $\mu \to +\infty$, problem (1.1) reduces to the well-known Dirichlet Biharmonic Steklov problem (DBS), namely

$$(\mathrm{DBS}) : \begin{cases} \Delta^2 u = 0, & \text{in } \Omega, \\ (1 - \sigma)\, u_{\nu\nu} + \sigma \Delta u = \lambda u_\nu, & \text{on } \partial\Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.5}$$

which has been considered by many authors for $\sigma = 1$ (see, e.g., [5–7, 9, 20–22, 24, 35]); for the case $\sigma \neq 1$ we refer to [12, 13], see also [22] for $\sigma = 0$.

In a similar fashion, for $\rho = +\infty$ in problem (1.2) we obtain the known (NBS)-Neumann Biharmonic Steklov problem

$$(\mathrm{NBS}) : \begin{cases} \Delta^2 v = 0, & \text{in } \Omega, \\ v_\nu = 0, & \text{on } \partial\Omega, \\ (\sigma - 1) \operatorname{div}_{\partial\Omega} \left(D^2 v \cdot \nu\right)_{\partial\Omega} - (\Delta v)_\nu = \gamma v, & \text{on } \partial\Omega, \end{cases} \tag{1.6}$$

see [30]. Problem (NBS) has been discussed in [29, 31, 32] for $\sigma = 1$. We point out that (1.2) with $\sigma = \rho = 0$ has been introduced in [14] as the natural fourth order generalization of the classical Steklov problem for the Laplacian.

We refer to the extensive monograph [26] for an introduction to the theory of poly-harmonic operators.

We note that problems (1.1) and (1.2) were introduced in [30] in order to characterize the trace spaces of functions in $H^2(\Omega)$ when $\Omega$ is a bounded Lipschitz domain in $\mathbb{R}^N$ and to provide a spectral representation of the solutions of the Dirichlet and Neumann problems for the biharmonic operator.

From a physical standpoint, it is known that in the two-dimensional case $N = 2$, Steklov boundary conditions for the biharmonic operator naturally appear in the analysis of the vibrations of an elastic plate the mass of which is concentrated at the boundary. In these models, $\sigma$ represents the Poisson coefficient of the elastic material. We refer to [14] for a thorough discussion about the physical motivation of biharmonic Steklov problems, in particular of the $(\mathrm{NBS})_\rho$ problem with $\rho = 0$. Among other papers devoted to the study of biharmonic Steklov problems, we quote [25] where existence and nonexistence results for positive solutions to a linearly perturbed critical growth biharmonic problem under Steklov boundary conditions were determined. Moreover, in [31], a Weyl-type asymptotic formula for the counting function of biharmonic Steklov eigenvalues was established, and in [32], two Weyl-type asymptotic formulas with sharp remainder estimates for the counting functions of two classes of biharmonic Steklov eigenvalue problems on smooth bounded domains in a Riemannian manifold were established. We refer to [19,27,28,36] for further results concerning biharmonic Steklov problems.

It turns out that the spectra of both $(\mathrm{DBS})_\mu$ and $(\mathrm{NBS})_\rho$ problems are discrete and consist of sequences of nonnegative eigenvalues of finite multiplicity.

In this article, we study the stability of the eigenvalues and eigenfunctions of problems (1.1) and (1.2) with respect to domain perturbation. Namely, we consider a family $\{\Omega_\varepsilon\}_{0<\varepsilon\leq\varepsilon_0}$ of domains converging to a fixed domain $\Omega$ in a suitable sense. We denote the eigenvalues of problem (1.1) in $\Omega_\varepsilon$ and $\Omega$ by $\{\lambda_j^\varepsilon\}_{j=1}^\infty$ and $\{\lambda_j\}_{j=1}^\infty$, respectively. Also we denote the eigenvalues of problem (1.2) in $\Omega_\varepsilon$ and $\Omega$ by $\{\gamma_j^\varepsilon\}_{j=1}^\infty$ and $\{\gamma_j\}_{j=1}^\infty$, respectively. (Note that we follow the standard convention whereby eigenvalues are repeated in the respective sequences according to their multiplicity.) Then we provide conditions on $\Omega_\varepsilon$ that guarantee that for any $j \geq 1$, $\lambda_j^\varepsilon \to \lambda_j$ and $\gamma_j^\varepsilon \to \gamma_j$ as $\varepsilon \to 0$ and that an analogous convergence result holds for the corresponding eigenfunctions. This is achieved by studying the spectral convergence of the appropriate resolvents $T_{D,\varepsilon}$ and $T_{N,\varepsilon}$ in $\Omega_\varepsilon$ to the resolvent operators $T_D$ and $T_N$ in $\Omega$. See Section 2.4 for the precise definitions of these operators. The spectral convergence of these operators is achieved by demonstrating their compact convergence, a well-known concept of convergence that ensures the convergence of eigenvalues and eigenfunctions. However, since the underlying function spaces depend on $\varepsilon$, we cannot directly apply the standard notion of compact convergence. Instead, we use appropriate 'connecting systems' that allow the transition from the varying Hilbert spaces defined on $\Omega_\varepsilon$ to the fixed limiting Hilbert space defined on $\Omega$. This approach involves several concepts and results originating from the works of Stummel [37] and Vainikko [38], which have been further developed in [1, 17]. Specifically, we employ the notion of $E$-compact convergence. These concepts are elaborated in detail in Section 2.2.

To establish the $E$-compact convergence of the operators in question, we examine domains $\Omega_\varepsilon$ and $\Omega$ that belong to uniform classes of domains with $C^{0,1}$ boundaries, as defined in Definition 2.6. We require that the boundaries of $\Omega_\varepsilon$ converge to the boundary of $\Omega$ in the sense specified by condition (2.7). This condition enables the construction, detailed in Section 2.5, of a family of linear continuous operators $E_\varepsilon : H^2(\Omega) \to H^2(\Omega_\varepsilon)$. This connecting system allows the treatment of operators defined on the different spaces $H^2(\Omega_\varepsilon)$ as a well-defined sequence with a well-defined limit in $H^2(\Omega)$.

In particular, the family of operators $\{E_\varepsilon\}$ makes possible the definition of the notion of $E$-convergence for a sequence $u_\varepsilon \in H^2(\Omega_\varepsilon)$ to a function $u \in H^2(\Omega)$, characterized by the condition

$$\|u_\varepsilon - E_\varepsilon u\|_{H^2(\Omega_\varepsilon)} \to 0 \text{ as } \varepsilon \to 0.$$

The operator $E_\varepsilon$ is constructed as a usual extension operator $E_\varepsilon : H^2(\Omega) \to H^2(\Omega_\varepsilon)$, which maps any function $u \in H^2(\Omega)$ to $E_\varepsilon u \in H^2(\Omega_\varepsilon)$ and coincides with $u$ over $\Omega$ (for more details about extension operator, see [16], Section 6). This in particular implies that

$$\|u_\varepsilon - u\|_{H^2(\Omega_\varepsilon\cap\Omega)} \to 0, \text{ and } \|\nabla u_\varepsilon - E_\varepsilon \nabla u\|_{L^2(\partial\Omega_\varepsilon)} \to 0. \tag{1.7}$$

One of the primary results of the paper is Theorem 3.1, wherein it is proved that condition (2.7) implies the $E$-compact convergence of $T_{D,\varepsilon}$ to $T_D$ and $T_{N,\varepsilon}$ to $T_N$ as $\varepsilon \to 0$. Based on the general Theorem 2.5, this implies the spectral convergence of $T_{D,\varepsilon}$ to $T_D$ and $T_{N,\varepsilon}$ to $T_N$ as $\varepsilon \to 0$, consequently implying the convergence of the eigenvalues and the $E$-convergence of the eigenfunctions as per Theorem 2.5. In particular, the eigenfunctions converge in the sense of (1.7).

In Section 4, we delve into the optimality analysis of condition (2.7). We explore the scenario where the domain takes the form $\Omega = W \times (-1, 0)$, with $W$ being a cuboid or a bounded domain in $\mathbb{R}^{N-1}$ of class $C^{0,1}$. We assume that the perturbed domain $\Omega_\varepsilon$ is defined as follows:

$$\Omega_\varepsilon = \{(\overline{x}, x_N) : \overline{x} \in W, -1 < x_N < g_\varepsilon(\overline{x})\}$$

where $g_\varepsilon(\overline{x}) = \varepsilon^\alpha b(\overline{x}/\varepsilon)$ for any $\overline{x} \in W$ and the function $b : \mathbb{R}^{N-1} \to [0, +\infty)$ is a non-constant $Y$-periodic function where $Y = \left(-\frac{1}{2}, \frac{1}{2}\right)^{N-1}$ is the unit cell in $\mathbb{R}^{N-1}$. We denote by $\Gamma_\varepsilon$ and $\Gamma$ the sets

$$\Gamma_\varepsilon := \{(\overline{x}, g_\varepsilon(\overline{x})) : \overline{x} \in W\} \text{ and } \Gamma := W \times \{0\}.$$

It appears that condition (2.7) is satisfied for $\alpha > 1$, thereby resulting in the spectral convergence of $T_{D,\varepsilon}$ to $T_D$ and $T_{N,\varepsilon}$ to $T_N$ as $\varepsilon \to 0$. However, when $0 < \alpha < 1$, condition (2.7) is not met, leading to a degeneration phenomenon, see Theorems 4.6 and 4.9. In the case of $\alpha = 1$, we encounter a strange term in the boundary conditions, see the weak formulations (4.10) of the limiting problems and the corresponding classical formulations in Remark 4.3. We note that the appearance of 'strange weights' in the limting problems is a classical theme in homogenization theory, we refer to [10] for an analogous phenomenon in a different context.

Our analysis is in the spirit of the analogous study performed in [23] for the classical second-order Steklov problem and in [22] for the fourth order (DBS) problem (1.5) with $\sigma = 0$ and $\sigma = 1$. (Note that in [22] these problems are called (DBS) when $\sigma = 1$ and (MDBS) when $\sigma = 0$). We find it interesting that the critical threshold for $\alpha$ is $\alpha = 1$ in [23] and $\alpha = 3/2$ in [22] (note that $\alpha = 3/2$ is proved to be optimal only for the (MDBS) problem). In fact, condition (2.7) is the same condition used in [23]. Comparing our analysis with the results in [22, 23] shows that the behavior of problems (1.1) and (1.2) under domain perturbation is consistent with that of the classical second-order Steklov problem, and differs from the behavior of the (DBS) problem (1.5). This highlights the critical nature of the (DBS) problem (1.5), which, in fact, arises as the limit of problem (1.1) as $\mu \to \infty$. Understanding the deep reason of this discontinuity in the limit, possibly analyzing the role of the size of $\mu$ in the asymptotic behavior of this problem, as well as finding the critical threshold for the (NBS) problem are certainly very interesting problems but are outside the scopes of the present paper.

We refer to [8, 11] for analogous results for the second order Steklov problem and to [2, 3, 20] for earlier works concerning poly-harmonic operators.

We remark that the assumption $\mu, \rho > 0$ can be relaxed. Namely, one could easily consider the $(\text{NBS})_\rho$ problem for any $\rho > -\eta_1$ where $\eta_1$ is the first positive eigenvalue of the (DBS) problem. In particular, the case $\rho = 0$ can be included in the analysis. As for the problem $(\text{DBS})_\mu$, the case $\mu = 0$ is more delicate since for $\mu = 0$ constant functions would annul both sides of Eq (1.3). Nevertheless, by using $H^2(\Omega)/\mathbb{R}$ as the energy space for Eq (1.3) would allow to cast a well-posed problem with discrete spectrum also for $\mu = 0$. We note that in Section 4, we consider a simplified setting -imposing Dirichlet boundary conditions on some part of the boundary- and this allows to directly remove the constant functions from the analysis and to consider all cases $\mu, \rho \geq 0$.

Finally, we would like to highlight an important connection between the Steklov problems under consideration in this paper and the corresponding Robin problems for the biharmonic operator. Recall that, for a fixed $\lambda > 0$, the classical second order Steklov problem

$$\begin{cases} \Delta u = 0, & \text{in } \Omega, \\ u_\nu = \lambda u, & \text{in } \partial\Omega, \end{cases}$$

can be viewed as a Robin-type boundary value problem with negative parameter. Indeed, the boundary condition can be written as $u_\nu - \lambda u = 0$ and $-\lambda$ plays the role of the Robin parameter. The same point of view can be adopted for the fourth-order Steklov problem. More precisely, we restate verbatim the general Robin problem for the biharmonic operator introduced in [13]

$$\begin{cases} \Delta^2 u - \alpha \Delta u = f, & \text{in } \Omega, \\ (1-\sigma)\, u_{\nu\nu} + \sigma \Delta u = -\beta u_\nu, & \text{on } \partial\Omega, \\ \alpha u_\nu - (1-\sigma)\, \mathrm{div}_{\partial\Omega} \left( D^2 u \cdot \nu \right)_{\partial\Omega} - (\Delta u)_\nu = -\mu u, & \text{on } \partial\Omega, \end{cases} \tag{1.8}$$

where $\alpha$ is a real parameter and $f$ is the datum. From this formulation, we see that problem (1.1) can be obtained by setting $f = 0$, $\alpha = 0$ and $\beta = -\lambda$ in (1.8), while problem (1.2) can be obtained by setting $f = 0$, $\alpha = 0$, $\mu = -\gamma$ and $\beta = \rho$ in (1.8). In other words, the boundary condition involving the spectral parameter in our problems can be obtained by changing the sign in one of the two Robin parameters in (1.8). We emphasize that the Robin eigenvalue problem is not considered in the present paper and should not be confused with the Steklov eigenvalue problem, since in the Robin case the eigenvalue appears in the interior equation in (1.8), rather than in the boundary conditions.

The paper is organized as follows. In Section 2, we delve into preliminary concepts, focusing on the notion of $E$-convergence and our chosen functional framework. Section 3 presents the primary stability theorem of the paper, assuming the validity of condition (2.7). Section 4 is dedicated to examining the optimality of condition (2.7).

**Remark 1.1.** *Concerning the terminology, the use of the term Dirichlet for problem* (1.5) *in the case* $\sigma = 1$ *dates back at least to [35]. In Chapter 5 of [29], entitled 'Eigenvalues in the boundary* *conditions', problem* (1.5) *with* $\sigma = 1$ *is discussed in the section 'Dirichlet eigenvalues', while problem* (1.6) *is analyzed in the section 'A Stekloff problem for the Neumann condition'. The expression 'Dirichlet Biharmonic Steklov' and the corresponding acronym (DBS) can be found in [6] and have also been used in [5, 7, 20, 22]. Problems* (1.1) *and* (1.2) *are denoted by* $(\mathrm{BS})_\mu$ *and* $(\mathrm{BS})_\rho$*, respectively, in [30], but this notation does not seem sufficiently informative to clearly distinguish between the two problems. For this reason, keeping in mind the corresponding limiting problems (DBS) and (NBS), we propose to refer to problem* (1.1) *as* $(\mathrm{DBS})_\mu$ *and to problem* (1.2) *as* $(\mathrm{NBS})_\rho$*. We mention that in [13]* (1.5) *is also referred to as the Navier–Steklov problem, while* (1.6) *is called the Kuttler–Sigillito problem.*

## 2. Preliminaries and notations

### *2.1. Notations*

We denote by $H^k(\Omega)$ the standard Sobolev space of functions in $L^2(\Omega)$ with all $k$ order weak derivatives in $L^2(\Omega)$. Throughout this paper, we consider $L^2(\Omega)$ as a space of real-valued functions. We denote by $H^k_0(\Omega)$ the closure of $C^\infty_c(\Omega)$ in $H^k(\Omega)$. The space $C^\infty_c(\Omega)$ consists of all functions in $C^\infty(\Omega)$ with compact support in $\Omega$.

By $(\cdot,\cdot)_{\partial\Omega}$ we denote the standard scalar product of $L^2(\partial\Omega)$, namely

$$(u,\varphi)_{\partial\Omega} := \int_{\partial\Omega} u\varphi d\sigma,\ \forall u,\varphi \in L^2(\partial\Omega).$$

For any $\sigma \in (-1/(N-1), 1)$, $\mu, \rho > 0$ and $u, \varphi \in H^2(\Omega)$ we set

$$\begin{aligned}
Q_{\sigma,\Omega}(u,\varphi) &= (1-\sigma)\int_\Omega D^2u : D^2\varphi dx + \sigma\int_\Omega \Delta u\Delta\varphi dx,\\
Q_{D,\Omega}(u,\varphi) &= Q_{\sigma,\Omega}(u,\varphi) + \mu\int_{\partial\Omega} u\varphi d\sigma + \int_{\partial\Omega} u_\nu\varphi_\nu d\sigma,\\
Q_{N,\Omega}(u,\varphi) &= Q_{\sigma,\Omega}(u,\varphi) + \int_{\partial\Omega} u\varphi d\sigma + \rho\int_{\partial\Omega} u_\nu\varphi_\nu d\sigma.
\end{aligned}$$

### *2.2. A general approach to spectral stability*

The study of the spectral stability of problems (1.1) and (1.2) is reduced to analyzing suitable families $\{B_\varepsilon\}_{0<\varepsilon\le\varepsilon_0}$ of nonnegative compact self-adjoint operators defined in Hilbert spaces $\mathcal{H}_\varepsilon$ associated with the domains $\Omega_\varepsilon$.

In the spirit of [1], we denote by $\mathcal{H}_\varepsilon$ a family of Hilbert spaces for $\varepsilon \in [0, \varepsilon_0]$ and assume the existence of a family of linear operators $E_\varepsilon : \mathcal{H}_0 \to \mathcal{H}_\varepsilon$ such that

$$\|E_\varepsilon u\|_{\mathcal{H}_\varepsilon} \xrightarrow{\varepsilon\to 0} \|u\|_{\mathcal{H}_0},\ \text{for all } u \in \mathcal{H}_0. \tag{2.1}$$

**Definition 2.1.** We say that a family $\{u_\varepsilon\}_{0<\varepsilon\le\varepsilon_0}$, with $u_\varepsilon \in \mathcal{H}_\varepsilon$, $E$-converges to $u \in \mathcal{H}_0$ if $\|u_\varepsilon - E_\varepsilon u\|_{\mathcal{H}_\varepsilon} \to 0$ as $\varepsilon \to 0$. We write this as $u_\varepsilon \xrightarrow{E} u$.

**Definition 2.2.** Let $\{B_\varepsilon \in \mathcal{L}(\mathcal{H}_\varepsilon) : \varepsilon \in (0, \varepsilon_0]\}$ be a family of linear and continuous operators. We say that $\{B_\varepsilon\}_{0<\varepsilon\le\varepsilon_0}$ $E$-converges to $B_0 \in \mathcal{L}(\mathcal{H}_0)$ as $\varepsilon \to 0$ if $B_\varepsilon u_\varepsilon \xrightarrow{E} B_0 u$ whenever $u_\varepsilon \xrightarrow{E} u$. We write this as $B_\varepsilon \xrightarrow{EE} B_0$.

**Definition 2.3.** Let $\{u_\varepsilon\}_{0<\varepsilon\le\varepsilon_0}$ be a family such that $u_\varepsilon \in \mathcal{H}_\varepsilon$. We say that $\{u_\varepsilon\}_{0<\varepsilon\le\varepsilon_0}$ is precompact if for any sequence $\varepsilon_n \to 0$ there exist a subsequence $\{\varepsilon_{n_k}\}$ and $u \in \mathcal{H}_0$ such that $u_{\varepsilon_{n_k}} \xrightarrow{E} u$.

**Definition 2.4.** We say that $\{B_\varepsilon\}_{0<\varepsilon\le\varepsilon_0}$ with $B_\varepsilon \in \mathcal{L}(\mathcal{H}_\varepsilon)$ and $B_\varepsilon$ compact, converges compactly to a compact operator $B_0 \in \mathcal{L}(\mathcal{H}_0)$ if $B_\varepsilon \xrightarrow{EE} B_0$ and for any family $\{u_\varepsilon\}_{0<\varepsilon\le\varepsilon_0}$ such that $u_\varepsilon \in \mathcal{H}_\varepsilon$, $\|u_\varepsilon\|_{\mathcal{H}_\varepsilon} = 1$, we have that $\{B_\varepsilon u_\varepsilon\}_{0<\varepsilon\le\varepsilon_0}$ is precompact in the sense of Definition 2.3. We write this as $B_\varepsilon \xrightarrow{C} B_0$.

**Theorem 2.5.** *Let $\{B_\varepsilon\}_{0<\varepsilon\le\varepsilon_0}$ and $B_0$ be nonnegative compact self-adjoint operators in the Hilbert spaces $\mathcal{H}_\varepsilon$ and $\mathcal{H}_0$, respectively. Assume that their nonzero eigenvalues are given by the sequences $\mu_n(\varepsilon)$ and $\mu_n(0)$, $n \in \mathbb{N}$, respectively. If $B_\varepsilon \xrightarrow{C} B_0$ then we have spectral convergence of $B_\varepsilon$ to $B_0$ as $\varepsilon \to 0$ in the sense that the following statements hold.*

(i) *For every $n \in \mathbb{N}$ we have $\mu_n(\varepsilon) \to \mu_n(0)$ as $\varepsilon \to 0$.*

(ii) *Suppose $u_n(\varepsilon)$, $n \in \mathbb{N}$, is an orthonormal sequence of eigenfunctions associated with the eigenvalues $\mu_n(\varepsilon)$. Then there exist a sequence $\varepsilon_k$, $k \in \mathbb{N}$, converging to zero and an orthonormal sequence of eigenfunctions $u_n(0)$, $n \in \mathbb{N}$, associated with $\mu_n(0)$, $n \in \mathbb{N}$, such that $u_n(\varepsilon_k) \xrightarrow{E} u_n(0)$ as $k \to \infty$.*

(iii) *Let $m$ eigenvalues $\mu_n(0), \dots, \mu_{n+m-1}(0)$ satisfy*

$$\mu_n(0) \neq \mu_{n-1}(0) \text{ and } \mu_{n+m-1}(0) \neq \mu_{n+m}(0),$$

*and let $u_n(0), \dots, u_{n+m-1}(0)$ be the corresponding orthonormal eigenfunctions. Then there exist $m$ orthonormal generalized eigenfunctions $v_n(\varepsilon), \dots, v_{n+m-1}(\varepsilon)$ associated with the eigenvalues $\mu_n(\varepsilon), \dots, \mu_{n+m-1}(\varepsilon)$ such that*

$$v_{n+i}(\varepsilon) \xrightarrow{E} u_{n+i}(0) \text{ for all } i = 0, 1, \dots, m-1,$$

*as $\varepsilon \to 0$.*

We refer to [38, Theorem 6.3], see also [1, Theorem 4.10], [4, Theorem 5.1] and [17, Theorem 3.3] for details of spectral convergence.

### *2.3. Classes of domains*

To study the spectral convergence for the eigenvalue problems (1.1) and (1.2) on varying domains, we consider uniform families of domains with prescribed parameters. This approach utilizes the notion of an atlas from [15] and further detailed in [3, Section 5]. Following [3, 15], given a set $V \subset \mathbb{R}^N$ and a number $\delta > 0$, we define

$$V_\delta := \{x \in V : \ d(x, \partial V) > \delta\}. \tag{2.2}$$

Note that in this paper we denote by $\mathbb{N}$ the set of natural number excluding zero and by $\mathbb{N}_0$ the set $\mathbb{N} \cup \{0\}$.

**Definition 2.6.** [15, Definition 2.4]. Let $\rho > 0$, $s$, $s' \in \mathbb{N}$ with $s' < s$. Let $\{V_j\}_{j=1}^s$ be a family of open cuboids (i.e. rotations of rectangle parallelepipeds in $\mathbb{R}^N$) and $\{r_j\}_{j=1}^s$ be a family of rotations in $\mathbb{R}^N$. We say $\mathcal{A} = \left(\rho,\ s,\ s',\ \{V_j\}_{j=1}^s,\ \{r_j\}_{j=1}^s\right)$ is an atlas in $\mathbb{R}^N$ with parameters $\rho,\ s,\ s',\ \{V_j\}_{j=1}^s,\ \{r_j\}_{j=1}^s$ briefly an

atlas in $\mathbb{R}^N$. Moreover, we say that a bounded domain $\Omega$ in $\mathbb{R}^N$ belongs to the class $C^{k,\gamma}(\mathcal{A})$ with $k \in \mathbb{N}$ and $\gamma \in [0,1]$ if the following conditions are satisfied:
($i$) $\Omega \subset \bigcup_{j=1}^{s} \left(V_j\right)_\rho$ and $\left(V_j\right)_\rho \cap \Omega \neq \emptyset$ where $\left(V_j\right)_\rho$ is meant in the sense given in (2.2);
($ii$) $V_j \cap \partial\Omega \neq \emptyset$ for $j = 1, ..., s'$ and $V_j \cap \partial\Omega = \emptyset$ for $s' + 1 \leq j \leq s$;
($iii$) For $j = 1, ..., s$, we have

$$r_j\left(V_j\right) = \{x \in \mathbb{R}^N : \ a_{ij} < x_i < b_{ij}, \ i = 1, ..., N\},$$

for $j = 1, ..., s'$ we have

$$r_j\left(V_j \cap \Omega\right) = \{x = (\overline{x}, x_N) \in \mathbb{R}^N : \ \overline{x} \in W_j, \ a_{Nj} < x_N < b_{Nj}\},$$

where $\overline{x} = (x_1, ..., x_{N-1})$,

$$W_j = \{x \in \mathbb{R}^{N-1} : \ a_{ij} < x_i < b_{ij}, \ i = 1, ..., N-1\}$$

and the functions $g_j \in C^{k,\gamma}\left(\overline{W_j}\right)$ for any $j \in 1, ..., s'$ with $k \in \mathbb{N} \cup \{0\}$ and $\gamma \in [0,1]$. Moreover, for $j = 1, ..., s'$ we assume that $a_{Nj} + \rho \leq g_j(\overline{x}) \leq b_{Nj} - \rho$, for all $\overline{x} \in \overline{W_j}$.

Finally we say that $\Omega$ is of class $C^{k,\gamma}$ if it is of class $C^{k,\gamma}(\mathcal{A})$ for some atlas $\mathcal{A}$.

Let $\Omega$ be a bounded domain in $\mathbb{R}^N$ of class $C^{0,1}$. It is well known that Hilbert space $H^2(\Omega)$ can be endowed with the scalar product

$$\int_\Omega D^2u : D^2\varphi dx + \int_\Omega u\varphi dx \text{ for all } u, \varphi \in H^2(\Omega).$$

Given the structure of problems (1.1) and (1.2), it is reasonable to replace the classical scalar product in $H^2(\Omega)$ with a different one.

**Lemma 2.7.** *Let $\Omega$ be a bounded domain in $\mathbb{R}^N$ of class $C^{0,1}$. Then the following statements hold:*
($i$) *There exists a positive constant $C(N,\Omega)$ depending only on $N$ and $\Omega$ such that*

$$\int_{\partial\Omega} \left(u^2 + u_\nu^2\right) d\sigma \leq C(N,\Omega) \left( \int_\Omega |D^2u|^2 dx + \int_\Omega u^2 dx \right) \text{ for all } u \in H^2(\Omega);$$

*more precisely, if $\mathcal{A}$ is an atlas as in Definition 2.6 such that $\Omega$ is of class $C^{0,1}(\mathcal{A})$, the dependence of $C(N,\Omega)$ on $\Omega$ occurs through the atlas $\mathcal{A}$ and the $C^{0,1}$ norms of the functions $g_j$ introduced in the same definition.*
($ii$) *There exists a constant $C(N, d(\Omega))$ depending only on $N$ and $d(\Omega)$, where $d(\Omega)$ denotes the diameter of $\Omega$, such that*

$$\int_\Omega u^2 dx \leq C(N, d(\Omega)) \left( \int_\Omega |D^2u|^2 dx + \int_{\partial\Omega} \left(u^2 + u_\nu^2\right) d\sigma \right) \text{ for all } u \in H^2(\Omega);$$

($iii$) *The following scalar product*

$$\int_\Omega D^2u : D^2\varphi dx + \int_{\partial\Omega} (u\varphi + u_\nu\varphi_\nu) \, d\sigma \text{ for all } u, \varphi \in H^2(\Omega)$$

*is equivalent to the original scalar product of $H^2(\Omega)$.*

*Proof.* Statement (i) is a well known result from classical trace theorems, see for example, [34].

Let us prove statement (ii). In [23, Lemma 2.7], it was shown that the following inequality holds:

$$\int_\Omega u^2 dx \leq C(N, d(\Omega)) \left( \int_\Omega |\nabla u|^2 dx + \int_{\partial\Omega} u^2 d\sigma \right) \text{ for all } u \in H^1(\Omega). \tag{2.3}$$

By using the divergence formula, the Hölder-Young and the Cauchy-Schwarz inequalities, from (2.3) we obtain for any $0 < \epsilon < 1$

$$\begin{aligned} \int_\Omega u^2 dx &\leq C(N, d(\Omega)) \left( \int_{\partial\Omega} u u_\nu d\sigma - \int_\Omega u \Delta u dx + \int_{\partial\Omega} u^2 d\sigma \right) \\ &\leq C(N, d(\Omega)) \left( \frac{3}{2} \int_{\partial\Omega} u^2 d\sigma + \frac{1}{2} \int_{\partial\Omega} u_\nu^2 d\sigma + \epsilon \int_\Omega u^2 dx + \frac{1}{4\epsilon} \int_\Omega |\Delta u|^2 dx \right) \\ &\leq C(N, d(\Omega)) \left( 3 \int_{\partial\Omega} u^2 d\sigma + \int_{\partial\Omega} u_\nu^2 d\sigma \right) \\ &+ C(N, d(\Omega)) \left( \epsilon \int_\Omega u^2 dx + \frac{C(N)}{4\epsilon} \int_\Omega |D^2 u|^2 dx \right), \end{aligned} \tag{2.4}$$

for all $u \in H^2(\Omega)$. It is easy to see that (2.4) implies the validity of statement (ii). Statement (iii) is an immediate consequence of (i) and (ii). □

**Corollary 2.8.** *The quadratic form $Q_{\sigma,\Omega}$ is coercive in $H^2(\Omega)$ and the norms given by $\left(Q_{\sigma,\Omega}(u,u) + \|u\|^2_{L^2(\Omega)}\right)^{\frac{1}{2}}$ and $\left(Q_{\sigma,\Omega}(u,u) + \|u\|^2_{L^2(\partial\Omega)} + \|u_\nu\|^2_{L^2(\partial\Omega)}\right)^{\frac{1}{2}}$ are also equivalent to the standard norm in $H^2(\Omega)$.*

*Proof.* The fact that the quadratic form $\left(Q_{\sigma,\Omega}(u,u) + \|u\|^2_{L^2(\Omega)}\right)^{\frac{1}{2}}$ is equivalent to the standard norm in $H^2(\Omega)$ is proved, e.g., in [18] by using the inequality

$$Q_{\sigma,\Omega}(u,u) \geq C(N,\sigma) \int_\Omega |D^2 u|^2 dx,$$

where $C(N,\sigma)$ is a positive constant independent of $u$. Consequently, by Lemma 2.7, we deduce that the norm $\left(Q_{\sigma,\Omega}(u,u) + \|u\|^2_{L^2(\partial\Omega)} + \|u_\nu\|^2_{L^2(\partial\Omega)}\right)^{\frac{1}{2}}$ is also equivalent to the standard norm in $H^2(\Omega)$. □

### 2.4. *The functional setting*

We consider in $H^2(\Omega)$ the equivalent norms

$$\|u\|_D^2 = Q_{D,\Omega}(u,u), \ \|v\|_N^2 = Q_{N,\Omega}(v,v)$$

which are associated with the scalar products defined by

$$\langle u, \varphi \rangle_{D,\Omega} := Q_{D,\Omega}(u,\varphi), \ \langle v, \varphi \rangle_{N,\Omega} := Q_{N,\Omega}(v,\varphi),$$

for all $u, v, \varphi \in H^2(\Omega)$. Then we define the operators $B_D$ and $B_N$ from $H^2(\Omega)$ to its dual $\left(H^2(\Omega)\right)'$ by setting

$$B_D(u)[\varphi] = \langle u, \varphi \rangle_{D,\Omega}, \ B_N(v)[\varphi] = \langle v, \varphi \rangle_{N,\Omega}, \ \forall\, u, v, \varphi \in H^2(\Omega).$$

By the Riesz Theorem it follows that $B_D$ and $B_N$ are surjective isometries. Then we consider the operators $J_D$ and $J_N$ from $H^2(\Omega)$ to $\left(H^2(\Omega)\right)'$ defined by

$$J_D(u)[\varphi] = \int_{\partial\Omega} u_\nu \varphi_\nu d\sigma, \ \forall u, \varphi \in H^2(\Omega),$$

$$J_N(v)[\varphi] = \int_{\partial\Omega} v\varphi d\sigma, \ \forall v, \varphi \in H^2(\Omega).$$

The operators $J_D$ and $J_N$ are compact since the corresponding trace maps from $H^2(\Omega)$ to $L^2(\partial\Omega)$ are compact.

**Theorem 2.9.** *Let*

$$T_D := B_D^{-1} \circ J_D, \ T_N := B_N^{-1} \circ J_N.$$

*Then $T_D$ and $T_N$ are compact, self-adjoint and nonnegative operators on $H^2(\Omega)$.*

*Proof.* From the compactness of $J_D$ and $J_N$ and the boundedness of $B_D^{-1}$ and $B_N^{-1}$ it follows that $T_D$ and $T_N$ are compact operators from $H^2(\Omega)$ to itself. Moreover, it is easy to see that they are self-adjoint (with respect to the scalar products induced by $B_D$ and $B_N$, respectively). Since $J_D$ and $J_N$ are nonnegative in the sense that $J_D(u)[u] \geq 0$ and $J_N(u)[u] \geq 0$ for all $u \in H^2(\Omega)$, we also infer that $T_D$ and $T_N$ are nonnegative. □

It can be easily demonstrated that $p \neq 0$ is an eigenvalue of $T_D$ if and only if $\lambda = 1/p - 1$ is an eigenvalue of (1.1), with the same eigenfunctions. In a similar way, $q \neq 0$ is an eigenvalue of $T_N$ if and only if $\gamma = 1/q - 1$ is an eigenvalue of (1.2), with the same eigenfunctions.

### *2.5. Domain perturbations and construction of a connecting system*

Let $\mathcal{A}$ be an atlas. Let $\{\Omega_\varepsilon\}_{0<\varepsilon\leq\varepsilon_0}$ be a family of domains of class $C^{0,1}(\mathcal{A})$ which converges to a fixed domain $\Omega$ of class $C^{0,1}(\mathcal{A})$ in a sense which will be specified below. For any $0 < \varepsilon \leq \varepsilon_0$ denote by

$$T_{D,\varepsilon}, T_{N,\varepsilon} : \ H^2(\Omega_\varepsilon) \to H^2(\Omega_\varepsilon)$$

the resolvent operators associated with problems (1.1) and (1.2) in $\Omega_\varepsilon$ according with the definition given in Section 2.4.

Since our ultimate goal is to apply the abstract results from Section 2.2, we need to define a family of operators $E_\varepsilon$, which satisfy condition (2.1). Specifically, in order to study the behaviour of the operators $T_{D,\varepsilon}$ as $\varepsilon \to 0$, we need to introduce linear operators $E_\varepsilon : \ H^2(\Omega) \to H^2(\Omega_\varepsilon)$ such that

$$\|E_\varepsilon u\|_{D,\varepsilon} \to \|u\|_D, \text{ as } \varepsilon \to 0, \text{ for any } u \in H^2(\Omega), \tag{2.5}$$

where $\|u\|^2_{D,\varepsilon} = Q_{D,\Omega_\varepsilon}(u,u)$. Similarly, to study the behaviour of the operators $T_{N,\varepsilon}$ as $\varepsilon \to 0$ we need operators $E_\varepsilon$ as above satisfying (2.5) with $D$ replaced by $N$, that is,

$$\|E_\varepsilon u\|_{N,\varepsilon} \to \|u\|_N, \text{ as } \varepsilon \to 0, \text{ for any } u \in H^2(\Omega), \tag{2.6}$$

where $\|u\|^2_{N,\varepsilon} = Q_{N,\Omega_\varepsilon}(u,u)$. In fact, we shall use the same operator $E_\varepsilon$ for both problems.

In order to do so, we consider a bounded linear extension operator $E: \ H^2(\Omega) \to H^2\left(\mathbb{R}^N\right)$ which maps any function $u \in H^2(\Omega)$ to a function $Eu \in H^2\left(\mathbb{R}^N\right)$ which coincides with $u$ over $\Omega$. Then, for both problems, we consider the operator $E_\varepsilon: \ H^2(\Omega) \to H^2(\Omega_\varepsilon)$ defined by $E_\varepsilon u = Eu_{|\Omega_\varepsilon}$ for all $u \in H^2(\Omega)$. The existence of $E$ is well-known for bounded Lipschitz domains $\Omega$, see, e.g., [16, Chapter 6].

We note that if $u_\varepsilon \in H^2(\Omega_\varepsilon),\ u \in H^2(\Omega)$ are such that $u_\varepsilon \overset{E}{\to} u$ as $\varepsilon \to 0$, then (1.7) holds. Indeed, combining the definition of $E$-convergence with Corollary 2.8, it follows that there exists a positive constant $C$ independent of $\varepsilon$ such that

$$\|u_\varepsilon - u\|_{H^2(\Omega_\varepsilon\cap\Omega)} \le \|u_\varepsilon - E_\varepsilon u\|_{H^2(\Omega_\varepsilon)} \le C\|u_\varepsilon - E_\varepsilon u\|_{D,\varepsilon},$$

as $\varepsilon \to 0$, with a similar inequality in the case of the $\|\cdot\|_{N,\varepsilon}$ norm. To prove the second limit in (1.7), we note that by the classical trace theorem, there exists a positive constant $C$ independent of $\varepsilon$ such that

$$\|\nabla u_\varepsilon - \nabla(E_\varepsilon u)\|_{L^2(\partial\Omega_\varepsilon)} \le C\|u_\varepsilon - E_\varepsilon u\|_{H^2(\Omega_\varepsilon)} = o(1),$$

as $\varepsilon \to 0$. Note that the independence of $C$ from $\varepsilon$ is due to the fact that the domains $\Omega_\varepsilon$ belongs to the class $C^{0,1}(\mathcal{A})$.

In the sequel we shall use the following condition:

$$\lim_{\varepsilon\to 0}\|D^\beta\left(g_{\varepsilon,j} - g_j\right)\|_{L^\infty(W_j)} = 0,\ \forall\ j = 1,\cdots s, \text{and}\ \beta \in \mathbb{N}_0^N,\ \text{with}\ |\beta| \le 1, \tag{2.7}$$

where we have denoted by $g_{\varepsilon,j},\ g_j \in C^{0,1}\left(\overline{W_j}\right)$ the functions corresponding to $\Omega_\varepsilon$ and $\Omega$ respectively, according to Definition 2.6.

**Lemma 2.10.** *Let $\mathcal{A}$ be an atlas. Let $\{\Omega_\varepsilon\}_{0<\varepsilon\le\varepsilon_0}$ be a family of domains of class $C^{0,1}(\mathcal{A})$, $\Omega$ a domain of class $C^{0,1}(\mathcal{A})$ and for any $\varepsilon \in (0,\varepsilon_0]$ let $E_\varepsilon$ be the map defined above. Assume the validity of condition* (2.7)*. Then the family of operators $\{E_\varepsilon\}_{0<\varepsilon\le\varepsilon_0}$ satisfies* (2.5).

*Proof.* In order to prove this lemma, it is enough to show that

$$\int_{\Omega_\varepsilon} |D^2(E_\varepsilon u)|^2 dx \to \int_\Omega |D^2 u|^2 dx \text{ as } \varepsilon \to 0, \tag{2.8}$$

$$\int_{\Omega_\varepsilon} |\Delta(E_\varepsilon u)|^2 dx \to \int_\Omega |\Delta u|^2 dx \text{ as } \varepsilon \to 0, \tag{2.9}$$

$$\int_{\partial\Omega_\varepsilon} |E_\varepsilon u|^2 d\sigma_\varepsilon \to \int_{\partial\Omega} u^2 d\sigma \text{ as } \varepsilon \to 0, \tag{2.10}$$

$$\int_{\partial\Omega_\varepsilon} |(E_\varepsilon u)_{\nu_\varepsilon}|^2 d\sigma_\varepsilon \to \int_{\partial\Omega} u_\nu^2 d\sigma \text{ as } \varepsilon \to 0, \tag{2.11}$$

for all $u \in H^2(\Omega)$.

For simplicity denote $\tilde{u}_\varepsilon := E_\varepsilon u$. Let us proceed to prove (2.9). Note that proving (2.8) will follow by the same argument. We notice that to establish (2.9) it is enough to demonstrate that

$$\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon} \frac{\partial^2\tilde{u}_\varepsilon}{\partial x_k^2}\frac{\partial^2\tilde{u}_\varepsilon}{\partial x_m^2} dx = \int_\Omega \frac{\partial^2 u}{\partial x_k^2}\frac{\partial^2 u}{\partial x_m^2} dx \tag{2.12}$$

for any $k, m \in \{1, ..., N\}$. We observe that

$$\int_{\Omega_\varepsilon} \frac{\partial^2 Eu}{\partial x_k^2} \frac{\partial^2 \tilde{u}_\varepsilon}{\partial x_m^2} dx = \int_{\Omega} \frac{\partial^2 u}{\partial x_k^2} \frac{\partial^2 u}{\partial x_m^2} dx + \int_{\Omega_\varepsilon \setminus \Omega} \frac{\partial^2 Eu}{\partial x_k^2} \frac{\partial^2 Eu}{\partial x_m^2} dx - \int_{\Omega \setminus \Omega_\varepsilon} \frac{\partial^2 u}{\partial x_k^2} \frac{\partial^2 u}{\partial x_m^2} dx \tag{2.13}$$

for any $k, m \in \{1, ..., N\}$. Since $|\Omega_\varepsilon \triangle \Omega| \to 0$ as $\varepsilon \to 0$, by the absolute continuity of the Lebesgue integral, from (2.13) we get (2.12).

For demonstrating the validity of (2.11), we proceed as follows:

$$\begin{aligned}
&\left| \int_{\partial\Omega_\varepsilon} (\tilde{u}_\varepsilon)^2_{\nu_\varepsilon} \, d\sigma_\varepsilon - \int_{\partial\Omega} u_\nu^2 d\sigma \right| \\
&\leq \sum_{j=1}^{s'} \sum_{k,m=1}^{N} \int_{W_j} \left| \frac{\partial \tilde{u}_\varepsilon}{\partial x_k}(\overline{x}, g_{\varepsilon,j}(\overline{x})) \frac{\partial \tilde{u}_\varepsilon}{\partial x_m}(\overline{x}, g_{\varepsilon,j}(\overline{x})) \left( W_{\varepsilon,k,m,j} - W_{k,m,j} \right)(\overline{x}) \right| d\overline{x} \\
&+ \sum_{j=1}^{s'} \sum_{k,m=1}^{N} \int_{W_j} \left| \left( \frac{\partial \tilde{u}_\varepsilon}{\partial x_k}(\overline{x}, g_{\varepsilon,j}(\overline{x})) - \frac{\partial u}{\partial x_k}(\overline{x}, g_j(\overline{x})) \right) \frac{\partial \tilde{u}_\varepsilon}{\partial x_m}(\overline{x}, g_{\varepsilon,j}(\overline{x})) W_{k,m,j}(\overline{x}) \right| d\overline{x} \\
&+ \sum_{j=1}^{s'} \sum_{k,m=1}^{N} \int_{W_j} \left| \left( \frac{\partial \tilde{u}_\varepsilon}{\partial x_m}(\overline{x}, g_{\varepsilon,j}(\overline{x}) - \frac{\partial u}{\partial x_m}(\overline{x}, g_j(\overline{x})) \right) \frac{\partial u}{\partial x_k}(\overline{x}, g_j(\overline{x})) W_{k,m,j}(\overline{x}) \right| d\overline{x} \\
&\leq \sum_{j=1}^{s'} \sum_{k,m=1}^{N} ||W_{\varepsilon,k,m,j} - W_{k,m,j}||_{L^\infty(W_j)} \int_{W_j} \left| \frac{\partial \tilde{u}_\varepsilon}{\partial x_k}(\overline{x}, g_{\varepsilon,j}(\overline{x})) \frac{\partial \tilde{u}_\varepsilon}{\partial x_m}(\overline{x}, g_{\varepsilon,j}(\overline{x})) \right| d\overline{x} \\
&+ \sum_{j=1}^{s'} \sum_{k,m=1}^{N} \left( \int_{W_j} \left| \frac{\partial \tilde{u}_\varepsilon}{\partial x_k}(\overline{x}, g_{\varepsilon,j}(\overline{x})) - \frac{\partial u}{\partial x_k}(\overline{x}, g_j(\overline{x})) \right|^2 d\overline{x} \right)^{\frac{1}{2}} \\
&\times \left( \int_{W_j} \left| \frac{\partial \tilde{u}_\varepsilon}{\partial x_m}(\overline{x}, g_{\varepsilon,j}(\overline{x})) W_{k,m,j}(\overline{x}) \right|^2 d\overline{x} \right)^{\frac{1}{2}} \\
&+ \sum_{j=1}^{s'} \sum_{k,m=1}^{N} \left( \int_{W_j} \left| \frac{\partial \tilde{u}_\varepsilon}{\partial x_m}(\overline{x}, g_{\varepsilon,j}(\overline{x}) - \frac{\partial u}{\partial x_m}(\overline{x}, g_j(\overline{x})) \right|^2 d\overline{x} \right)^{\frac{1}{2}} \\
&\times \left( \int_{W_j} \left| \frac{\partial u}{\partial x_k}(\overline{x}, g_j(\overline{x})) W_{k,m,j}(\overline{x}) \right|^2 d\overline{x} \right)^{\frac{1}{2}},
\end{aligned} \tag{2.14}$$

where

$$\begin{aligned}
W_{\varepsilon,k,m,j}(\overline{x}) &= \nu_{\varepsilon,k}(\overline{x}) \nu_{\varepsilon,m}(\overline{x}) \sqrt{1 + |\nabla_{\overline{x}} g_{\varepsilon,j}(\overline{x})|^2}, \\
W_{k,m,j}(\overline{x}) &= \nu_k(\overline{x}) \nu_m(\overline{x}) \sqrt{1 + |\nabla_{\overline{x}} g_j(\overline{x})|^2}.
\end{aligned} \tag{2.15}$$

We note that

$$\begin{aligned}
&\int_{W_j} \left| \frac{\partial \tilde{u}_\varepsilon}{\partial x_k}(\overline{x}, g_{\varepsilon,j}(\overline{x})) - \frac{\partial \tilde{u}_\varepsilon}{\partial x_k}(\overline{x}, g_j(\overline{x})) \right|^2 d\overline{x} \\
&\leq \int_{W_j} \left| g_{\varepsilon,j}(\overline{x}) - g_j(\overline{x}) \right| \left| \int_{g_j(\overline{x})}^{g_{\varepsilon,j}(\overline{x})} \left| \frac{\partial^2 \tilde{u}_\varepsilon}{\partial x_N \partial x_k}(\overline{x}, x_N) \right|^2 dx_N \right| d\overline{x} \\
&\leq ||g_{\varepsilon,j} - g_j||_{L^\infty(W_j)} \int_{\Omega_\varepsilon \triangle \Omega} \left| D^2 \tilde{u}_\varepsilon \right|^2 dx \leq ||g_{\varepsilon,j} - g_j||_{L^\infty(W_j)} ||\tilde{u}_\varepsilon||^2_{D,\varepsilon} \to 0,
\end{aligned} \tag{2.16}$$

as $\varepsilon \to 0$. Indeed, since the extension operator $E_\varepsilon$ is bounded, using (2.7) we deduce that (2.16) is true.

Using condition (2.7), the fact that $E_\varepsilon$ is a bounded operator and combining (2.16) with (2.14) we obtain (2.11).

It remains to prove (2.10). In a similar way to (2.16), as $\varepsilon \to 0$, we obtain

$$\begin{aligned}
&\int_{W_j} \left|\tilde{u}_\varepsilon(\overline{x}, g_{\varepsilon,j}(\overline{x})) - \tilde{u}_\varepsilon(\overline{x}, g_j(\overline{x}))\right|^2 d\overline{x} \\
&\leq \int_{W_j} \left|g_{\varepsilon,j}(\overline{x}) - g_j(\overline{x})\right| \left|\int_{g_j(\overline{x})}^{g_{\varepsilon,j}(\overline{x})} \left|\frac{\partial \tilde{u}_\varepsilon}{\partial x_N}(\overline{x}, x_N)\right|^2 dx_N\right| d\overline{x} \qquad (2.17)\\
&\leq \|g_{\varepsilon,j} - g_j\|_{L^\infty(W_j)} \int_{\Omega_\varepsilon \triangle \Omega} |\nabla \tilde{u}_\varepsilon|^2\, dx \leq \|g_{\varepsilon,j} - g_j\|_{L^\infty(W_j)} \|\tilde{u}_\varepsilon\|^2_{D,\varepsilon} \to 0.
\end{aligned}$$

For simplicity, we set $w_{\varepsilon,j}(\bar{x}) := \sqrt{1 + |\nabla_{\bar{x}} g_{\varepsilon,j}(\bar{x})|^2}$, $w_j(\bar{x}) := \sqrt{1 + |\nabla_{\bar{x}} g_j(\bar{x})|^2}$. Then, we proceed as follows:

$$\begin{aligned}
\left|\int_{\partial\Omega_\varepsilon} \tilde{u}_\varepsilon^2 d\sigma_\varepsilon - \int_{\partial\Omega} u^2 d\sigma\right| &\leq \sum_{j=1}^{s'} \int_{W_j} \left|\tilde{u}_\varepsilon^2(\overline{x}, g_{\varepsilon,j}(\overline{x}))\left(w_{\varepsilon,j} - w_j\right)\right| d\overline{x} \\
&+ \sum_{j=1}^{s'} \int_{W_j} \left|\left(\tilde{u}_\varepsilon(\overline{x}, g_{\varepsilon,j}(\overline{x})) - u(\overline{x}, g_j(\overline{x}))\right) \tilde{u}_\varepsilon(\overline{x}, g_{\varepsilon,j}(\overline{x})) w_j\right| d\overline{x} \\
&+ \sum_{j=1}^{s'} \int_{W_j} \left|\left(\tilde{u}_\varepsilon(\overline{x}, g_{\varepsilon,j}(\overline{x})) - u(\overline{x}, g_j(\overline{x}))\right) u(\overline{x}, g_j(\overline{x})) w_j\right| d\overline{x} \qquad (2.18)\\
&\leq \sum_{j=1}^{s'} \left\|w_{\varepsilon,j} - w_j\right\|_{L^\infty(W_j)} \int_{W_j} \left|\tilde{u}_\varepsilon^2(\overline{x}, g_{\varepsilon,j}(\overline{x}))\right| d\overline{x} \\
&+ \sum_{j=1}^{s'} \left(\int_{W_j} \left|\tilde{u}_\varepsilon(\overline{x}, g_{\varepsilon,j}(\overline{x})) - u(\overline{x}, g_j(\overline{x}))\right|^2 d\overline{x}\right)^{\frac{1}{2}} \left(\int_{W_j} \left|\tilde{u}_\varepsilon(\overline{x}, g_{\varepsilon,j}(\overline{x})) w_j\right|^2 d\overline{x}\right)^{\frac{1}{2}} \\
&+ \sum_{j=1}^{s'} \left(\int_{W_j} \left|\tilde{u}_\varepsilon(\overline{x}, g_{\varepsilon,j}(\overline{x})) - u(\overline{x}, g_j(\overline{x}))\right|^2 d\overline{x}\right)^{\frac{1}{2}} \left(\int_{W_j} \left|u(\overline{x}, g_j(\overline{x})) w_j\right|^2 d\overline{x}\right)^{\frac{1}{2}}.
\end{aligned}$$

Arguing as in the proof of (2.11), combining (2.17) and (2.18) we deduce the validity of (2.10). This completes the proof of the lemma. □

## 3. The stability result

In this section, we establish results concerning the spectral convergence for problems (1.1) and (1.2). One of the primary assumptions is condition (2.7), which, as demonstrated in Lemma 2.10, ensures the validity of properties (2.8)–(2.11).

**Theorem 3.1.** *Let $\mathcal{A}$ be an atlas. Let $\{\Omega_\varepsilon\}_{0<\varepsilon\leq\varepsilon_0}$ be a family of domains of class $C^{0,1}(\mathcal{A})$, $\Omega$ a domain of class $C^{0,1}(\mathcal{A})$. Assume the validity of condition* (2.7)*. Then the spectra of problems* (1.1) *and* (1.2) *behave continuously at $\varepsilon = 0$ in the sense of Theorem 2.5.*

*Proof.* The proof of Theorem 3.1 follows directly from Theorem 2.5 and Lemmas 3.2 and 3.3 presented below. □

Now we prove the following lemma, which shows that condition (2.7) yields the $EE$-convergence of $T_{D,\varepsilon}$ and $T_{N,\varepsilon}$.

**Lemma 3.2.** *Let $\mathcal{A}$ be an atlas. Let $\{\Omega_\varepsilon\}_{0<\varepsilon\leq\varepsilon_0}$ be a family of domains of class $C^{0,1}(\mathcal{A})$, $\Omega$ a domain of class $C^{0,1}(\mathcal{A})$. Assume the validity of condition* (2.7). *Let $w_\varepsilon \in H^2(\Omega_\varepsilon)$ with $0 < \varepsilon \leq \varepsilon_0$, and $w \in H^2(\Omega)$ be such that $w_\varepsilon \xrightarrow{E} w$. If we set $u_\varepsilon := T_{D,\varepsilon} w_\varepsilon$ and $u := T_D w$, then $u_\varepsilon \xrightarrow{E} u$ as $\varepsilon \to 0$. Also, if we set $v_\varepsilon := T_{N,\varepsilon} w_\varepsilon$ and $v := T_N w$, then $v_\varepsilon \xrightarrow{E} v$ as $\varepsilon \to 0$. In particular this implies that $T_{D,\varepsilon} \xrightarrow{EE} T_D$ and $T_{N,\varepsilon} \xrightarrow{EE} T_N$ as $\varepsilon \to 0$ in the sense of Definition 2.2.*

*Proof.* We divide the proof of the lemma into several steps. We note that in this proof we need to extend not only the trial functions but also the functions $u_\varepsilon$. With respect to this we observe that, since the domains $\Omega_\varepsilon$ belong to the same class $C^{0,1}(\mathcal{A})$, for any $\varepsilon$ there exists a linear continuous extension operator from $H^2(\Omega_\varepsilon)$ to $H^2(\mathbb{R}^N)$ such that the operator norm $\|T_\varepsilon\|$ is uniformly bounded with respect to $\varepsilon$; see, e.g., [16, Chapter 6]. For the sake of simplicity we shall denote the extended function $T_\varepsilon u_\varepsilon$ by the same symbol $u_\varepsilon$.

**Step 1.** In this step we prove that $\|u_\varepsilon\|_{D,\varepsilon}$ is uniformly bounded with respect to $\varepsilon \in (0, \varepsilon_0]$.

First of all we observe that by the weak formulation of the equation $T_{D,\varepsilon} w_\varepsilon = u_\varepsilon$

$$
\begin{aligned}
\|u_\varepsilon\|^2_{D,\varepsilon} = \int_{\partial\Omega_\varepsilon} (w_\varepsilon)_{\nu_\varepsilon} (u_\varepsilon)_{\nu_\varepsilon} d\sigma_\varepsilon &\leq \left( \int_{\partial\Omega_\varepsilon} (w_\varepsilon)^2_{\nu_\varepsilon} d\sigma_\varepsilon \right)^{\frac{1}{2}} \left( \int_{\partial\Omega_\varepsilon} (u_\varepsilon)^2_{\nu_\varepsilon} d\sigma_\varepsilon \right)^{\frac{1}{2}} \\
&\leq \|w_\varepsilon\|_{D,\varepsilon} \|u_\varepsilon\|_{D,\varepsilon},
\end{aligned}
\tag{3.1}
$$

from which it follows that $\|u_\varepsilon\|_{D,\varepsilon} \leq \|w_\varepsilon\|_{D,\varepsilon}$. Now we observe that $\|w_\varepsilon\|_{D,\varepsilon}$ is uniformly bounded since

$$
\|w_\varepsilon\|_{D,\varepsilon} \leq \|w_\varepsilon - E_\varepsilon w\|_{D,\varepsilon} + \|E_\varepsilon w\|_{D,\varepsilon} = O(1) \text{ as } \varepsilon \to 0 \tag{3.2}
$$

as one can deduce by Definition 2.1, (2.5) and Lemma 2.10.

Thus, there exists a sequence $\varepsilon_n \downarrow 0$ and $\tilde{u} \in H^2(\Omega)$ such that $u_{\varepsilon_n} \rightharpoonup \tilde{u}$ in $H^2(\Omega)$ as $n \to \infty$. For simplicity in the sequel we only write $u_\varepsilon \rightharpoonup \tilde{u}$ as $\varepsilon \to 0$ for denoting the convergence along the sequence $\{\varepsilon_n\}$ or a subsequence of its. The subsequent steps aim to demonstrate that $\tilde{u} = u$. This will be achieved by taking the limit in the following identity:

$$
Q_{D,\Omega_\varepsilon}(u_\varepsilon, E_\varepsilon\varphi) = \int_{\partial\Omega_\varepsilon} (w_\varepsilon)_{\nu_\varepsilon} (E_\varepsilon\varphi)_{\nu_\varepsilon} d\sigma_\varepsilon \text{ for any } \varphi \in H^2(\Omega). \tag{3.3}
$$

**Step 2.** In this step, we pass to the limit in the left-hand side of (3.3).

We split the left hand side of (3.3) in the following way

$$
\begin{aligned}
Q_{D,\Omega_\varepsilon}(u_\varepsilon, E_\varepsilon\varphi) =& Q_{\sigma,\Omega}(u_\varepsilon, E_\varepsilon\varphi) + Q_{\sigma,\Omega_\varepsilon\setminus\Omega}(u_\varepsilon, E_\varepsilon\varphi) - Q_{\sigma,\Omega\setminus\Omega_\varepsilon}(u_\varepsilon, E_\varepsilon\varphi) \\
&+ \mu \int_{\partial\Omega_\varepsilon} u_\varepsilon E_\varepsilon\varphi d\sigma_\varepsilon + \int_{\partial\Omega_\varepsilon} (u_\varepsilon)_{\nu_\varepsilon} (E_\varepsilon\varphi)_{\nu_\varepsilon} d\sigma_\varepsilon.
\end{aligned}
\tag{3.4}
$$

By using the fact that $u_\varepsilon \rightharpoonup \tilde{u}$ in $H^2(\Omega)$, $E_\varepsilon\varphi = \varphi$ in $\Omega$, $|\Omega_\varepsilon \triangle \Omega| \to 0$ as $\varepsilon \to 0$ and by the absolute continuity of the Lebesgue integral, it is easy to deduce that

$$Q_{\sigma,\Omega}(u_\varepsilon, E_\varepsilon\varphi) = Q_{\sigma,\Omega}(\tilde{u}, \varphi) + o(1) \text{ as } \varepsilon \to 0, \tag{3.5}$$

and

$$Q_{\sigma,\Omega_\varepsilon\setminus\Omega}(u_\varepsilon, E_\varepsilon\varphi) = o(1),\ Q_{\sigma,\Omega\setminus\Omega_\varepsilon}(u_\varepsilon, \varphi) = o(1) \text{ as } \varepsilon \to 0. \tag{3.6}$$

Combining (3.5) and (3.6) we obtain

$$\begin{aligned} Q_{\sigma,\Omega}(u_\varepsilon, E_\varepsilon\varphi) + Q_{\sigma,\Omega_\varepsilon\setminus\Omega}(u_\varepsilon, E_\varepsilon\varphi) - Q_{\sigma,\Omega\setminus\Omega_\varepsilon}(u_\varepsilon, E_\varepsilon\varphi) \\ = Q_{\sigma,\Omega}(\tilde{u}, \varphi) + o(1) \text{ as } \varepsilon \to 0. \end{aligned} \tag{3.7}$$

Now, let us examine the fourth term on the right-hand side of (3.4). We proceed as follows:

$$\begin{aligned}
&\Big|\int_{\partial\Omega_\varepsilon} (u_\varepsilon)_{\nu_\varepsilon}(E_\varepsilon\varphi)_{\nu_\varepsilon} d\sigma_\varepsilon - \int_{\partial\Omega} \tilde{u}_\nu \varphi_\nu d\sigma\Big| \\
&\le \sum_{j=1}^{s'}\sum_{n,k=1}^{N}\int_{W_j}\Big|(\partial_{x_n}u_\varepsilon(\bar{x}, g_{\varepsilon,j}(\bar{x})) - \partial_{x_n}u_\varepsilon(\bar{x}, g_j(\bar{x})))\,\partial_{x_k}(E_\varepsilon\varphi)(\bar{x}, g_{\varepsilon,j}(\bar{x}))\,W_{\varepsilon,n,k,j}\Big|d\bar{x} \\
&+ \sum_{j=1}^{s'}\sum_{n,k=1}^{N}\int_{W_j}\Big|(\partial_{x_n}u_\varepsilon(\bar{x}, g_j(\bar{x})) - \partial_{x_n}\tilde{u}(\bar{x}, g_j(\bar{x})))\,\partial_{x_k}(E_\varepsilon\varphi)(\bar{x}, g_{\varepsilon,j}(\bar{x}))W_{\varepsilon,n,k,j}\Big|d\bar{x} \\
&+ \sum_{j=1}^{s'}\sum_{n,k=1}^{N}\int_{W_j}\Big|(\partial_{x_k}(E_\varepsilon\varphi)(\bar{x}, g_{\varepsilon,j}(\bar{x})) - \partial_{x_k}(E_\varepsilon\varphi)(\bar{x}, g_j(\bar{x})))\partial_{x_n}\tilde{u}(\bar{x}, g_j(\bar{x}))\,W_{\varepsilon,n,k,j}\Big|d\bar{x} \\
&+ \sum_{j=1}^{s'}\sum_{n,k=1}^{N}\int_{W_j}\Big|(W_{\varepsilon,n,k,j}(\bar{x}) - W_{n,k,j}(\bar{x}))\partial_{x_n}\tilde{u}(\bar{x}, g_j(\bar{x}))\partial_{x_k}(E_\varepsilon\varphi)(\bar{x}, g_j(\bar{x}))\Big|d\bar{x} \\
&\le \sum_{j=1}^{s'}\sum_{n,k=1}^{N}\|W_{\varepsilon,n,k,j} - W_{n,k,j}\|_{L^\infty(W_j)}\int_{W_j}\Big|\partial_{x_n}\tilde{u}(\bar{x}, g_j(\bar{x}))\partial_{x_k}(E_\varepsilon\varphi)(\bar{x}, g_j(\bar{x}))\Big|d\bar{x} \\
&+ \sum_{j=1}^{s'}\sum_{n,k=1}^{N}\|\partial_{x_n}u_\varepsilon(\cdot, g_{\varepsilon,j}) - \partial_{x_n}u_\varepsilon(\cdot, g_j)\|_{L^2(W_j)}\|\partial_{x_k}(E_\varepsilon\varphi)(\cdot, g_{\varepsilon,j})W_{\varepsilon,n,k,j}\|_{L^2(W_j)} \\
&+ \sum_{j=1}^{s'}\sum_{n,k=1}^{N}\|\partial_{x_n}u_\varepsilon(\cdot, g_j) - \partial_{x_n}\tilde{u}(\cdot, g_j)\|_{L^2(W_j)}\|\partial_{x_k}(E_\varepsilon\varphi)(\cdot, g_{\varepsilon,j})\,W_{\varepsilon,n,k,j}\|_{L^2(W_j)} \\
&+ \sum_{j=1}^{s'}\sum_{n,k=1}^{N}\|\partial_{x_k}(E_\varepsilon\varphi)(\cdot, g_{\varepsilon,j}) - \partial_{x_k}(E_\varepsilon\varphi)(\cdot, g_j)\|_{L^2(W_j)}\,\|\partial_{x_n}\tilde{u}(\cdot, g_j)W_{\varepsilon,n,k,j}\|_{L^2(W_j)}.
\end{aligned} \tag{3.8}$$

where $W_{\varepsilon,n,k,j}$ is defined in (2.15). We note that since $\|u_\varepsilon\|_{D,\varepsilon} = O(1)$, for the function $u_\varepsilon$ estimates (2.16) and (2.17) hold true. Hence, using this fact and condition (2.7), the compactness of the trace map and the boundedness of the operator $E_\varepsilon$, from (3.8) we deduce that

$$\left|\int_{\partial\Omega_\varepsilon}(u_\varepsilon)_{\nu_\varepsilon}(E_\varepsilon\varphi)_{\nu_\varepsilon}\,d\sigma_\varepsilon - \int_{\partial\Omega}\tilde{u}_\nu\varphi_\nu d\sigma\right| = o(1) \text{ as } \varepsilon \to 0. \tag{3.9}$$

Let us consider the third term on the right-hand side of (3.4). We proceed in the following way:

$$\begin{aligned}
&\left|\int_{\partial\Omega_\varepsilon} u_\varepsilon E_\varepsilon\varphi d\sigma_\varepsilon - \int_{\partial\Omega} \tilde{u}\varphi d\sigma\right| \\
&\le \sum_{j=1}^{s'} \int_{W_j} \left|\left(u_\varepsilon(\bar{x}, g_{\varepsilon,j}(\bar{x})) - u_\varepsilon(\bar{x}, g_j(\bar{x}))\right)(E_\varepsilon\varphi)(\bar{x}, g_{\varepsilon,j}(\bar{x}))w_{\varepsilon,j}\right| d\bar{x} \\
&+ \sum_{j=1}^{s'} \int_{W_j} \left|\left(u_\varepsilon(\bar{x}, g_j(\bar{x})) - \tilde{u}(\bar{x}, g_j(\bar{x}))\right)(E_\varepsilon\varphi)(\bar{x}, g_{\varepsilon,j}(\bar{x}))w_{\varepsilon,j}\right| d\bar{x} \\
&+ \sum_{j=1}^{s'} \int_{W_j} \left|\left((E_\varepsilon\varphi)(\bar{x}, g_{\varepsilon,j}(\bar{x})) - (E_\varepsilon\varphi)(\bar{x}, g_j(\bar{x}))\right)\tilde{u}(\bar{x}, g_j(\bar{x}))w_{\varepsilon,j}\right| d\bar{x} \\
&+ \sum_{j=1}^{s'} \int_{W_j} \left|\tilde{u}(\bar{x}, g_j(\bar{x}))(E_\varepsilon\varphi)(\bar{x}, g_j(\bar{x}))\left(w_{\varepsilon,j} - w_j\right)\right| d\bar{x} \qquad (3.10)\\
&\le \sum_{j=1}^{s'} \left\|w_{\varepsilon,j} - w_j\right\|_{L^\infty(W_j)} \int_{W_j} \left|\tilde{u}(\bar{x}, g_j(\bar{x}))(E_\varepsilon\varphi)(\bar{x}, g_j(\bar{x}))\right| d\bar{x} \\
&+ \sum_{j=1}^{s'} \left(\int_{W_j} \left|u_\varepsilon(\bar{x}, g_{\varepsilon,j}(\bar{x})) - u_\varepsilon(\bar{x}, g_j(\bar{x}))\right|^2 d\bar{x}\right)^{\frac{1}{2}} \left(\int_{W_j} \left|(E_\varepsilon\varphi)(\bar{x}, g_{\varepsilon,j}(\bar{x}))w_{\varepsilon,j}\right|^2 d\bar{x}\right)^{\frac{1}{2}} \\
&+ \sum_{j=1}^{s'} \left(\int_{W_j} \left|u_\varepsilon(\bar{x}, g_j(\bar{x})) - \tilde{u}(\bar{x}, g_j(\bar{x}))\right|^2 d\bar{x}\right)^{\frac{1}{2}} \left(\int_{W_j} \left|(E_\varepsilon\varphi)(\bar{x}, g_{\varepsilon,j}(\bar{x}))w_{\varepsilon,j}\right|^2 d\bar{x}\right)^{\frac{1}{2}} \\
&+ \sum_{j=1}^{s'} \left(\int_{W_j} \left|(E_\varepsilon\varphi)(\bar{x}, g_{\varepsilon,j}(\bar{x})) - (E_\varepsilon\varphi)(\bar{x}, g_j(\bar{x}))\right|^2 d\bar{x}\right)^{\frac{1}{2}} \left(\int_{W_j} \left|\tilde{u}(\bar{x}, g_j(\bar{x}))w_{\varepsilon,j}\right|^2 d\bar{x}\right)^{\frac{1}{2}}.
\end{aligned}$$

By using (2.17) twice (with $\tilde{u}_\varepsilon$ replaced by $u_\varepsilon$ or by $E_\varepsilon\varphi$), by condition (2.7), and by the compactness of the trace map, from (3.10) we deduce

$$\left|\int_{\partial\Omega_\varepsilon} u_\varepsilon E_\varepsilon\varphi d\sigma_\varepsilon - \int_{\partial\Omega} \tilde{u}\varphi d\sigma\right| = o(1) \text{ as } \varepsilon \to 0. \tag{3.11}$$

By combining (3.4), (3.7), (3.9) and (3.11) we obtain

$$Q_{D,\Omega_\varepsilon}(u_\varepsilon, E_\varepsilon\varphi) \to Q_{\sigma,\Omega}(\tilde{u}, \varphi) + \mu \int_{\partial\Omega} \tilde{u}\varphi d\sigma + \int_{\partial\Omega} \tilde{u}_\nu \varphi_\nu d\sigma = Q_{D,\Omega}(\tilde{u}, \varphi). \tag{3.12}$$

**Step 3.** The next purpose is to pass to the limit in the right-hand side of (3.3).
We claim that

$$\left|\int_{\partial\Omega_\varepsilon} (w_\varepsilon)_{\nu_\varepsilon} (E_\varepsilon\varphi)_{\nu_\varepsilon} d\sigma_\varepsilon - \int_{\partial\Omega} w_\nu \varphi_\nu d\sigma\right| = o(1) \text{ as } \varepsilon \to 0. \tag{3.13}$$

By arguing as in the proof of Lemma 2.10, using condition (2.7) and the fact that $w_\varepsilon$ $E$-converges to $w$ as $\varepsilon \to 0$ we obtain

$$\begin{aligned}&\left|\int_{\partial\Omega_\varepsilon}(w_\varepsilon)_{\nu_\varepsilon}(E_\varepsilon\varphi)_{\nu_\varepsilon}d\sigma_\varepsilon - \int_{\partial\Omega} w_\nu\varphi_\nu d\sigma\right| \le \left|\int_{\partial\Omega_\varepsilon}\left((w_\varepsilon)_{\nu_\varepsilon} - (E_\varepsilon w)_{\nu_\varepsilon}\right)(E_\varepsilon\varphi)_{\nu_\varepsilon}d\sigma_\varepsilon\right|\\ &+\left|\int_{\partial\Omega_\varepsilon}(E_\varepsilon w)_{\nu_\varepsilon}(E_\varepsilon\varphi)_{\nu_\varepsilon}d\sigma_\varepsilon - \int_{\partial\Omega} w_\nu\varphi_\nu d\sigma\right|\\ &\le \|w_\varepsilon - E_\varepsilon w\|_{D,\varepsilon}\|E_\varepsilon\varphi\|_{D,\varepsilon} + o(1) = o(1)\end{aligned}$$

as $\varepsilon \to 0$, which proves the claim (3.13).

**Step 4.** In this step we complete the proof of the lemma for problem (1.1).

By combining (3.3), (3.12) and (3.13) we deduce that

$$Q_{D,\Omega}(\tilde{u},\varphi) = \int_{\partial\Omega} w_\nu\varphi_\nu d\sigma \text{ for any } \varphi \in H^2(\Omega)\,. \tag{3.14}$$

Additionally, the function $u$ also satisfies the same variational problem. Therefore, by the uniqueness of the solution, we conclude that $\tilde{u} = u$. In particular, this implies that the weak limit $\tilde{u}$ is independent of the choice of the sequence $\varepsilon_n \downarrow 0$.

It remains to prove that $u_\varepsilon \overset{E}{\to} u$ as $\varepsilon \to 0$.

Let us consider

$$\|u_\varepsilon - E_\varepsilon u\|^2_{D,\varepsilon} = \|u_\varepsilon\|^2_{D,\varepsilon} - 2Q_{D,\Omega_\varepsilon}(u_\varepsilon, E_\varepsilon u) + \|E_\varepsilon u\|^2_{D,\varepsilon}. \tag{3.15}$$

By (3.12) with $\varphi$ replaced by $u$ we deduce that

$$Q_{D,\Omega_\varepsilon}(u_\varepsilon, E_\varepsilon u) \to Q_{D,\Omega}(u,u) = \|u\|^2_D. \tag{3.16}$$

Moreover, by Lemma 2.10 we have

$$\|E_\varepsilon u\|^2_{D,\varepsilon} \to \|u\|^2_D. \tag{3.17}$$

It remains to prove that $\|u_\varepsilon\|^2_{D,\varepsilon} \to \|u\|^2_D$. For this we have to show that

$$Q_{D,\Omega_\varepsilon}(u_\varepsilon, u_\varepsilon) = \int_{\partial\Omega_\varepsilon}(w_\varepsilon)_{\nu_\varepsilon}(u_\varepsilon)_{\nu_\varepsilon}d\sigma_\varepsilon \to \int_{\partial\Omega} w_\nu u_\nu d\sigma = \|u\|^2_D. \tag{3.18}$$

To do so, we argue as above and note that

$$\begin{aligned}&\left|\int_{\partial\Omega_\varepsilon}(w_\varepsilon)_{\nu_\varepsilon}(u_\varepsilon)_{\nu_\varepsilon}d\sigma_\varepsilon - \int_{\partial\Omega} w_\nu u_\nu d\sigma\right|\\ &\le \left|\int_{\partial\Omega_\varepsilon}\left((w_\varepsilon)_{\nu_\varepsilon} - (E_\varepsilon w)_{\nu_\varepsilon}\right)(u_\varepsilon)_{\nu_\varepsilon}d\sigma_\varepsilon\right|\\ &+\left|\int_{\partial\Omega_\varepsilon}(E_\varepsilon w)_{\nu_\varepsilon}(u_\varepsilon)_{\nu_\varepsilon}d\sigma_\varepsilon - \int_{\partial\Omega} w_\nu u_\nu d\sigma\right|\\ &\le \|w_\varepsilon - E_\varepsilon w\|_{D,\varepsilon}\|u_\varepsilon\|_{D,\varepsilon} + o(1) = o(1)\end{aligned} \tag{3.19}$$

as $\varepsilon \to 0$, exploiting (3.8) with $\varphi$ replaced by $w$ and the fact that $\|u_\varepsilon\|_{D,\varepsilon} = O(1)$ as $\varepsilon \to 0$ as we have shown in (3.1). This proves (3.18).

Combining (3.15)–(3.18) proves that $u_\varepsilon \overset{E}{\to} u$ as $\varepsilon \to 0$. In particular this implies that $T_{D,\varepsilon} \overset{EE}{\longrightarrow} T_D$ as $\varepsilon \to 0$ in the sense of Definition 2.2.

**Step 5.** In this last step we give a quick proof of the lemma for problem (1.2).
We observe that

$$\begin{aligned}\|v_\varepsilon\|^2_{N,\varepsilon} = \int_{\partial\Omega_\varepsilon} w_\varepsilon v_\varepsilon d\sigma_\varepsilon &\le \left(\int_{\partial\Omega_\varepsilon} w_\varepsilon^2 d\sigma_\varepsilon\right)^{\frac{1}{2}} \left(\int_{\partial\Omega_\varepsilon} v_\varepsilon^2 d\sigma_\varepsilon\right)^{\frac{1}{2}} \\ &\le \|w_\varepsilon\|_{N,\varepsilon}\|v_\varepsilon\|_{N,\varepsilon},\end{aligned} \tag{3.20}$$

from which it follows that $\|v_\varepsilon\|_{N,\varepsilon} \le \|w_\varepsilon\|_{N,\varepsilon}$. Thus, there exists a sequence $\varepsilon_n \downarrow 0$ and $\tilde{v} \in H^2(\Omega)$ such that $v_{\varepsilon_n} \rightharpoonup \tilde{v}$ in $H^2(\Omega)$ as $n \to \infty$. Next, we have to show that $\tilde{v} = v$. This will be achieved by taking the limit in the identity

$$Q_{N,\Omega_\varepsilon}(v_\varepsilon, E_\varepsilon\varphi) = \int_{\partial\Omega_\varepsilon} w_\varepsilon E_\varepsilon \varphi d\sigma_\varepsilon \text{ for any } \varphi \in H^2(\Omega). \tag{3.21}$$

We split the left hand side of (3.21) in the following way

$$\begin{aligned}Q_{N,\Omega_\varepsilon}(v_\varepsilon, E_\varepsilon\varphi) =& Q_{\sigma,\Omega}(v_\varepsilon, E_\varepsilon\varphi) + Q_{\sigma,\Omega_\varepsilon\setminus\Omega}(v_\varepsilon, E_\varepsilon\varphi) - Q_{\sigma,\Omega\setminus\Omega_\varepsilon}(v_\varepsilon, E_\varepsilon\varphi) \\ &+ \int_{\partial\Omega_\varepsilon} v_\varepsilon E_\varepsilon\varphi d\sigma_\varepsilon + \rho\int_{\partial\Omega_\varepsilon} (v_\varepsilon)_{\nu_\varepsilon}(E_\varepsilon\varphi)_{\nu_\varepsilon} d\sigma_\varepsilon.\end{aligned} \tag{3.22}$$

In a similar way as in (3.7), (3.9) and (3.11) from (3.22) we get

$$Q_{N,\Omega}(\tilde{v}, \varphi) = \int_{\partial\Omega} w\varphi d\sigma \text{ for any } \varphi \in H^2(\Omega).$$

It is known that the function $v$ also satisfies the same variational problem. Therefore, by the uniqueness of the solution, we conclude that $\tilde{v} = v$, and this implies that the weak limit $\tilde{v}$ is independent of the choice of the sequence $\varepsilon_n \downarrow 0$.

To prove $v_\varepsilon \overset{E}{\to} v$ as $\varepsilon \to 0$, we follow the same steps as in proving (3.16)–(3.18) and pass to the limit in

$$\|v_\varepsilon - E_\varepsilon v\|^2_{N,\varepsilon} = \|v_\varepsilon\|^2_{N,\varepsilon} - 2Q_{N,\Omega_\varepsilon}(v_\varepsilon, E_\varepsilon v) + \|E_\varepsilon v\|^2_{N,\varepsilon}.$$

□

**Lemma 3.3.** *Let $\mathcal{A}$ be an atlas. Let $\{\Omega_\varepsilon\}_{0<\varepsilon\le\varepsilon_0}$ be a family of domains of class $C^{0,1}(\mathcal{A})$, $\Omega$ a domain of class $C^{0,1}(\mathcal{A})$. Assume the validity of condition (2.7). Let $w_\varepsilon \in H^2(\Omega_\varepsilon)$ with $0 < \varepsilon \le \varepsilon_0$ be such that $\|w_\varepsilon\|_{D,\varepsilon} = 1$. Then $\{T_{D,\varepsilon}w_\varepsilon\}_{0<\varepsilon\le\varepsilon_0}$ and $\{T_{N,\varepsilon}w_\varepsilon\}_{0<\varepsilon\le\varepsilon_0}$ are precompact in the sense of Definition 2.3. In particular we have that $T_{D,\varepsilon} \overset{C}{\to} T_D$ and $T_{N,\varepsilon} \overset{C}{\to} T_N$ as $\varepsilon \to 0$.*

*Proof.* Since $\|w_\varepsilon\|_{D,\varepsilon} = 1$, one can deduce that possibly passing to a subsequence converging to zero, $w_\varepsilon \rightharpoonup w$ weakly in $H^2(\Omega)$ and $(w_\varepsilon)_\nu \to w_\nu$ strongly in $L^2(\partial\Omega)$ as $\varepsilon \to 0$ where it is understood that $w_\varepsilon$ is extended to the whole $\mathbb{R}^N$ with $H^2$ norm uniformly bounded with respect to $\varepsilon$. In accordance with the notations of Lemma 3.2, let us define $u_\varepsilon := T_{D,\varepsilon}w_\varepsilon$. Using the boundedness of $\|w_\varepsilon\|_{D,\varepsilon}$ and looking at (3.1) we deduce that $\|u_\varepsilon\|_{D,\varepsilon} = O(1)$. Then there exists $\tilde{u} \in H^2(\Omega)$ such that, possibly passing to a subsequence, $u_\varepsilon \rightharpoonup \tilde{u}$ in $H^2(\Omega)$. In order to go further, it is useful to observe that

$$\begin{aligned}
&\int_{\partial\Omega_\varepsilon} (u_\varepsilon)_{\nu_\varepsilon}((w_\varepsilon)_{\nu_\varepsilon} - (E_\varepsilon w)_{\nu_\varepsilon}) d\sigma_\varepsilon, \ \int_{\partial\Omega_\varepsilon} (E_\varepsilon\varphi)_{\nu_\varepsilon}((w_\varepsilon)_{\nu_\varepsilon} - (E_\varepsilon w)_{\nu_\varepsilon}) d\sigma_\varepsilon \\
&\quad \le C \left| \int_{\partial\Omega_\varepsilon} \left|\nabla w_\varepsilon - \nabla E_\varepsilon w\right|^2 d\sigma_\varepsilon \right|^{\frac{1}{2}} \\
&\quad \le C \left| \sum_{j=1}^{s} \int_{W_j} \left|\nabla w_\varepsilon(\bar{x}, g_{\varepsilon,j}(\bar{x})) - \nabla(E_\varepsilon w)(\bar{x}, g_{\varepsilon,j}(\bar{x}))\right|^2 d\bar{x} \right|^{\frac{1}{2}} \\
&\quad \le C \sum_{j=1}^{s} \left| \int_{W} \left|\nabla w_\varepsilon(\bar{x}, g_{\varepsilon,j}(\bar{x})) - \nabla w_\varepsilon(\bar{x}, 0)\right|^2 d\bar{x} \right|^{\frac{1}{2}} \\
&\quad + C \sum_{j=1}^{s} \left| \int_{W} \left|\nabla w_\varepsilon(\bar{x}, 0) - \nabla w(\bar{x}, 0)\right|^2 d\bar{x} \right|^{\frac{1}{2}} \\
&\quad + C \sum_{j=1}^{s} \left| \int_{W} \left|\nabla (E_\varepsilon w)(\bar{x}, g_{\varepsilon,j}(\bar{x})) - \nabla w(\bar{x}, 0)\right|^2 d\bar{x} \right|^{\frac{1}{2}} = o(1)
\end{aligned} \tag{3.23}$$

as $\varepsilon \to 0$. Indeed, the first terms and the third in the right-hand side of (3.23) go to zero by the analog of (2.16) while the second one converge to zero by the compactness of the trace operator.

Arguing as in the proof of Lemma 3.2 and using (3.23) we establish that $\tilde{u}$ solves the variational problem (3.14) hence $\tilde{u} = u := T_D w$.

We claim that $u_\varepsilon \overset{E}{\to} u$. This will be deduced by (3.15). In fact, following the same steps as in the proof of Lemma 3.2 one can confirm that (3.16) and (3.17) hold. The remaining task is demonstrate (3.18). This is done as for (3.18) by using triangle inequality as in (3.19) and estimating the two corresponding terms as follows. For one term, we use (3.23). For the other term $\int_{\partial\Omega_\varepsilon} (E_\varepsilon\omega)_{\nu_\varepsilon}(u_\varepsilon)_{\nu_\varepsilon} d\sigma_\varepsilon - \int_{\partial\Omega} w_\nu u_\nu d\sigma$ nothing changes and it is easily seen to be $o(1)$ as $\varepsilon \to 0$.

It follows that $u_\varepsilon \overset{E}{\to} u$ as claimed. This combined with Lemma 3.2 implies that $T_{D,\varepsilon} \overset{C}{\to} T_D$ as $\varepsilon \to 0$.

To prove $T_{N,\varepsilon} \overset{C}{\to} T_N$ as $\varepsilon \to 0$, we can argue as above, since as it is shown in (3.20), we have $\|v_\varepsilon\|_{N,\varepsilon} = O(1)$. □

## 4. Optimality of condition (2.7): stability, degeneration and strange terms

We now discuss the optimality of condition (2.7). We inquire whether spectral stability still holds if we remove condition (2.7) from the statement of Theorem 3.1. To address this question, we assume specific conditions on the domain $\Omega$ and its perturbations $\Omega_\varepsilon$. We assume that $\Omega$ takes the form $\Omega = W \times (-1, 0)$, where $W$ is a cuboid or a bounded domain in $\mathbb{R}^{N-1}$ of class $C^{0,1}$. Then we set

$$\Omega_0 = \Omega, \ \text{ and } \ \Omega_\varepsilon = \{(\overline{x}, x_N) : \ \overline{x} \in W, \ -1 < x_N < g_\varepsilon(\overline{x})\} \tag{4.1}$$

for every $\varepsilon \in (0, \varepsilon_0]$ where $\varepsilon_0$ is a fixed positive constant (that does not play any significant role), $g_\varepsilon(\overline{x}) = \varepsilon^\alpha b(\overline{x}/\varepsilon)$ for any $\overline{x} \in W$ and the function $b : \mathbb{R}^{N-1} \to [0, +\infty)$ satisfies:

$$b \in C^{1,1}(\mathbb{R}^{N-1}),\ b \text{ is a } Y\text{-periodic function,} \tag{4.2}$$

where $Y = \left(-\frac{1}{2}, \frac{1}{2}\right)^{N-1}$ is the unit cell in $\mathbb{R}^{N-1}$. We also assume that

$$\mathcal{H}^{N-1}\left(\{\overline{x} \in \mathbb{R}^{N-1} :\ |\nabla_{\overline{x}} b(\overline{x})| = 0\}\right) = 0, \tag{4.3}$$

where we have denoted by $\mathcal{H}^{N-1}$ the $(N-1)$-dimensional Lebesgue measure (this assumption is actually relevant only for the degeneration results in Section 4.3).

Moreover in order to simplify our analysis we shall impose Steklov boundary conditions only on the relevant part of the boundary of $\Omega_\varepsilon$ and impose Dirichlet boundary conditions on the remaining part. Namely, we set

$$\Gamma_\varepsilon := \{(\overline{x}, g_\varepsilon(\overline{x})) :\ \overline{x} \in W\},\ \Sigma_\varepsilon := \partial\Omega_\varepsilon \backslash \Gamma_\varepsilon.$$

The corresponding limiting subsets of $\partial\Omega$ are defined by

$$\Gamma_0 := W \times \{0\},\ \Sigma_0 := \partial\Omega_0 \backslash \Gamma_0$$

and are also denoted by $\Gamma$ and $\Sigma$ respectively. We emphasize that imposing Dirichlet boundary conditions on $\Sigma_\varepsilon$ does not affect the limiting behavior of the Steklov boundary conditions imposed on $\Gamma_\varepsilon$. This simplification may be considered as the result of a localization process as done in [22]. In principle, even imposing Steklov boundary conditions on $\Sigma_\varepsilon$ would not interfere but the analysis would be more technical.

In this section we consider the following Steklov problems

$$\begin{cases} \Delta^2 u = 0, & \text{in } \Omega_\varepsilon, \\ (1-\sigma)\, u_{\nu\nu} + \sigma \Delta u = \lambda u_\nu, & \text{on } \Gamma_\varepsilon, \\ (1-\sigma)\, \mathrm{div}_{\Gamma_\varepsilon}\left(D^2 u \cdot \nu\right)_{\Gamma_\varepsilon} + (\Delta u)_\nu = \mu u, & \text{on } \Gamma_\varepsilon, \\ u = u_\nu = 0 & \text{on } \Sigma_\varepsilon \end{cases} \tag{4.4}$$

and

$$\begin{cases} \Delta^2 v = 0, & \text{in } \Omega_\varepsilon, \\ (\sigma-1)\, v_{\nu\nu} - \sigma \Delta v = \rho v_\nu, & \text{on } \Gamma_\varepsilon, \\ (\sigma-1)\, \mathrm{div}_{\Gamma_\varepsilon}\left(D^2 v \cdot \nu\right)_{\Gamma_\varepsilon} - (\Delta v)_\nu = \gamma v, & \text{on } \Gamma_\varepsilon, \\ v = v_\nu = 0 & \text{on } \Sigma_\varepsilon. \end{cases} \tag{4.5}$$

Then we study the asymptotic behavior of the eigenvalues of (4.4) and (4.5) as $\varepsilon \to 0$. Note that the energy space associated with problems (4.4) and (4.5) is the space

$$H^2_{0,\Sigma_\varepsilon}(\Omega_\varepsilon) := \{u \in H^2(\Omega_\varepsilon) :\ u = u_\nu = 0 \text{ on } \Sigma_\varepsilon\}.$$

The energy space corresponding to limiting problems defined in $\Omega$ is defined in the same way and will be denoted by $H^2_{0,\Sigma}(\Omega)$.

The weak formulations of problems (4.4) and (4.5) can be written in the same way as those of problems (1.1) and (1.2). Namely,

$$\begin{aligned} &Q_{\sigma,\Omega_\varepsilon}(u,\varphi) + \mu \int_{\Gamma_\varepsilon} u\varphi d\sigma = \lambda \int_{\Gamma_\varepsilon} u_\nu \varphi_\nu d\sigma, \\ &Q_{\sigma,\Omega_\varepsilon}(v,\varphi) + \rho \int_{\Gamma_\varepsilon} v_\nu \varphi_\nu d\sigma = \gamma \int_{\Gamma_\varepsilon} v\varphi d\sigma, \end{aligned} \tag{4.6}$$

for all trial functions $\varphi \in H^2_{0,\Sigma}(\Omega_\varepsilon)$, in the unknowns $u, v \in H^2_{0,\Sigma}(\Omega_\varepsilon)$, $\lambda, \gamma \in \mathbb{R}$. As in Section 2.4, problems (4.4) and (4.5) are associated with nonnegative self-adjoint compact operators. Therefore, both problems admit a discrete real spectrum and the corresponding eigenfunctions form an orthonormal basis of the appropriate Hilbert spaces.

In order to better write the appropriate quadratic forms we set

$$C_{i,j}(\varepsilon) = \begin{cases} \frac{\partial_i g_\varepsilon \partial_j g_\varepsilon}{\sqrt{1+|\nabla_{\overline{x}} g_\varepsilon|^2}}, & \text{if } i,j \in \{1,\cdots N-1\}, \\ \frac{\partial_i g_\varepsilon}{\sqrt{1+|\nabla_{\overline{x}} g_\varepsilon|^2}}, & \text{if } i \in \{1,\cdots N-1\},\ j = N, \\ \frac{\partial_j g_\varepsilon}{\sqrt{1+|\nabla_{\overline{x}} g_\varepsilon|^2}}, & \text{if } i = N,\ j \in \{1,\cdots N-1\}, \\ \frac{1}{\sqrt{1+|\nabla_{\overline{x}} g_\varepsilon|^2}}, & \text{if } i = j = N, \end{cases}$$

and

$$C_{i,j} = \begin{cases} \int_Y \frac{\partial_i b \partial_j b}{\sqrt{1+|\nabla_{\overline{x}} b|^2}} dy, & \text{if } i,j \in \{1,\cdots N-1\}, \\ \int_Y \frac{\partial_i b}{\sqrt{1+|\nabla_{\overline{x}} b|^2}} dy, & \text{if } i \in \{1,\cdots N-1\},\ j = N, \\ \int_Y \frac{\partial_j b}{\sqrt{1+|\nabla_{\overline{x}} b|^2}} dy, & \text{if } i = N,\ j \in \{1,\cdots N-1\}, \\ \int_Y \frac{1}{\sqrt{1+|\nabla_{\overline{x}} b|^2}} dy, & \text{if } i = j = N. \end{cases}$$

Moreover we shall need the following lemma. Note that apart from (4.9) statements (i)-(ii) are taken from Proposition 3.4 (ii)-(iii) in [23]. In any case (4.8) and (4.9) can be deduced from [33].

**Lemma 4.1.** *Let $g_\varepsilon(\overline{x}) = \varepsilon^\alpha b(\overline{x}/\varepsilon)$ for all $\varepsilon > 0$ with $b$ satisfying* (4.2) *and* (4.3)*. Then we have:*
$(i)$ *If $\alpha = 1$, then*

$$\lim_{\varepsilon\to 0} \|g_\varepsilon\|_{L^\infty(W)} = 0, \ \left\|\frac{\partial g_\varepsilon}{\partial x_k}\right\|_{L^\infty(W)} = O(1) \text{ as } \varepsilon \to 0 \text{ for any } k = \{1,...,N\}, \tag{4.7}$$

$$\text{and } \sqrt{1+|\nabla_{\overline{x}} g_\varepsilon(\overline{x})|^2} \to C_b := \int_Y \sqrt{1+|\nabla_{\overline{x}} b(\overline{y})|^2} d\overline{y}, \tag{4.8}$$

$$C_{i,j}(\varepsilon) \to C_{i,j} \tag{4.9}$$

*weakly in $L^p(W)$ for all $p \in [1,\infty[$, and weakly* in $L^\infty(W)$ as $\varepsilon \to 0$.*
$(ii)$ *If $0 < \alpha < 1$, then*

$$\begin{aligned} &\lim_{\varepsilon\to 0} \|g_\varepsilon\|_{L^\infty(W)} = 0, \\ &\lim_{\varepsilon\to 0} \mathcal{H}^{N-1}\left(\left\{\overline{x} \in W : \ \sqrt{1+|\nabla_{\overline{x}} g_\varepsilon(\overline{x})|^2} \le t\right\}\right) = 0 \text{ for any } t > 0. \end{aligned}$$

The analysis of the asymptotic behavior of problems (4.4) and (4.5) depends on the exponent $\alpha$ appearing in the definition of $g_\varepsilon$. Namely, there are three cases determined by critical threshold $\alpha = 1$.

*4.1. Case $\alpha > 1$: stability*

If $\alpha > 1$ condition (2.7) is satisfied. Thanks to this condition, we may proceed exactly as in the proof of Theorem 3.1 with the advantage that here we have to deal with an atlas with only one chart. In particular we have spectral stability and the eigenvalues and eigenfunctions of problems (4.4) and (4.5) converge in the sense of Theorem 2.5 to those of the same problem defined in $\Omega$.

*4.2. Case $\alpha = 1$: strange term*

As it will be clear in the sequel it turns out the limiting problems obtained by letting $\varepsilon \to 0$ in (4.6) can be formulated as follows:

$$\begin{aligned} Q_{\sigma,\Omega}(u,\varphi) + \mu C_b \int_\Gamma u\varphi d\sigma &= \lambda \int_\Gamma \sum_{i,j=1}^N C_{i,j}\partial_i u \partial_j \varphi d\sigma, \\ Q_{\sigma,\Omega}(v,\varphi) + \rho \int_\Gamma \sum_{i,j=1}^N C_{i,j}\partial_i v \partial_j \varphi d\sigma &= \gamma C_b \int_\Gamma v\varphi d\sigma \end{aligned} \tag{4.10}$$

for all trial functions $\varphi \in H^2_{0,\Sigma}(\Omega)$, in the unknowns $u, v \in H^2_{0,\Sigma}(\Omega)$, $\lambda, \gamma \in \mathbb{R}$. Here $C_b$ is as defined in (4.8).

In order reduce the study of problems (4.10) to the study of eigenvalue problems for compact self-adjoint operators, we consider the following quadratic forms

$$\widetilde{Q}_{D,\Omega}(u,\varphi) = Q_{\sigma,\Omega}(u,\varphi) + \mu C_b \int_\Gamma u\varphi d\sigma + \int_\Gamma \sum_{i,j=1}^N C_{i,j}\partial_i u \partial_j \varphi d\sigma,$$

$$\widetilde{Q}_{N,\Omega}(u,\varphi) = Q_{\sigma,\Omega}(u,\varphi) + C_b \int_\Gamma u\varphi d\sigma + \rho \int_\Gamma \sum_{i,j=1}^N C_{i,j}\partial_i u \partial_j \varphi d\sigma.$$

The corresponding norms in $H^2_{0,\Sigma}(\Omega)$ are defined by

$$\|u\|^2_{\widetilde{D}} = \widetilde{Q}_{D,\Omega}(u,u),\ \|v\|^2_{\widetilde{N}} = \widetilde{Q}_{N,\Omega}(v,v),$$

and are equivalent to the standard $H^2$ norm.

In the spirit of Section 2.4 and with a slight abuse of notations we define the operators $B_{D,\varepsilon},\ B_{N,\varepsilon}$ : $H^2_{0,\Sigma_\varepsilon}(\Omega_\varepsilon) \to \left(H^2_{0,\Sigma_\varepsilon}(\Omega_\varepsilon)\right)'$ by setting

$$B_{D,\varepsilon}(u_\varepsilon)[\psi] = Q_{D,\Omega_\varepsilon}(u_\varepsilon,\psi),\ B_{N,\varepsilon}(v_\varepsilon)[\psi] \ = Q_{N,\Omega_\varepsilon}(v_\varepsilon,\psi),$$

for all $u_\varepsilon, v_\varepsilon, \psi \in H^2_{0,\Sigma_\varepsilon}(\Omega_\varepsilon)$.

Next, we consider the operators $J_{D,\varepsilon},\ J_{N,\varepsilon}:\ H^2_{0,\Sigma_\varepsilon}(\Omega_\varepsilon) \to \left(H^2_{0,\Sigma_\varepsilon}(\Omega_\varepsilon)\right)'$ defined by

$$J_{D,\varepsilon}(u_\varepsilon)[\psi] = \int_{\Gamma_\varepsilon} (u_\varepsilon)_{\nu_\varepsilon}\psi_{\nu_\varepsilon} d\sigma_\varepsilon,\ J_{N,\varepsilon}(v_\varepsilon)[\psi] = \int_{\Gamma_\varepsilon} v_\varepsilon \psi_\varepsilon d\sigma_\varepsilon,$$

for all $u_\varepsilon, v_\varepsilon, \psi \in H^2_{0,\Sigma_\varepsilon}(\Omega_\varepsilon)$. We also define the operators

$$T_{D,\varepsilon} = B^{-1}_{D,\varepsilon} \circ J_{D,\varepsilon},\ T_{N,\varepsilon} = B^{-1}_{N,\varepsilon} \circ J_{N,\varepsilon}.$$

From the compactness of $J_{D,\varepsilon}$, $J_{N,\varepsilon}$ and the boundedness of $B_{D,\varepsilon}$, $B_{N,\varepsilon}$ it follows that $T_{D,\varepsilon}$ and $T_{N,\varepsilon}$ are compact and self-adjoint operators from $H^2_{0,\Sigma_\varepsilon}(\Omega_\varepsilon)$ to itself.

Finally, we define the following operators: $\widetilde{B}_D,\ \widetilde{B}_N,\ \widetilde{J}_D,\ \widetilde{J}_N:\ H^2_{0,\Sigma}(\Omega) \to \left(H^2_{0,\Sigma}(\Omega)\right)'$ by setting

$$\widetilde{B}_D(u)[\varphi] = \widetilde{Q}_{D,\Omega}(u,\varphi),\ \widetilde{B}_N(v)[\varphi] = \widetilde{Q}_{N,\Omega}(u,\varphi)$$

and

$$\widetilde{J}_D(u)[\varphi] = \int_\Gamma \sum_{i,j=1}^N C_{i,j} \frac{\partial u}{\partial x_i}\frac{\partial \varphi}{\partial x_j} d\sigma,\ \widetilde{J}_N(v)[\varphi] = C_b \int_\Gamma v\varphi d\sigma,$$

for all $u, v, \varphi \in H^2_{0,\Sigma}(\Omega)$. Then

$$\widetilde{T}_D = \widetilde{B}_D^{-1} \circ \widetilde{J}_D,\ \widetilde{T}_N = \widetilde{B}_N^{-1} \circ \widetilde{J}_N.$$

Also, from the compactness of $\widetilde{J}_D, \widetilde{J}_N$ and the boundedness of $\widetilde{B}_D, \widetilde{B}_N$ it follows that $\widetilde{T}_D, \widetilde{T}_N$ are compact self-adjoint operators from $H^2_{0,\Sigma}(\Omega)$ to itself. As it happens for the operators $T_{D,\varepsilon}$ and $T_{N,\varepsilon}$ (see Section 2.4), it can be proved that $p \neq 0$ is an eigenvalue of $\widetilde{T}_D$ if and only if $\lambda = 1/p - 1$ is an eigenvalue of the first equation in (4.10), with the same eigenfunctions. Also, in a similar way we can demonstrate that $q \neq 0$ is an eigenvalue of $\widetilde{T}_N$ if and only if $\gamma = 1/q - 1$ is an eigenvalue of the second equation in (4.10), with the same eigenfunctions. This implies that in order to study the asymptotic behavior of problems (4.4) and (4.5) as $\varepsilon \to 0$ it suffices to study the asymptotic behavior of the eigenvalues and eigenfunctions of the operators $T_{D,\varepsilon}$, $T_{N,\varepsilon}$, respectively.

In order to apply Theorem 2.5 we need a connecting system. As in Section 2.5 we consider an extension operator $E:\ H^2(\Omega) \to H^2(W \times (-1,+\infty))$, which maps any function $u \in H^2(\Omega)$ to $Eu$, where $Eu:\ W \times (-1,+\infty) \to \mathbb{R}$ coincides with $u$ over $\Omega$. By looking at the definitions of such extension operator available in the literature we can assume that

$$(Eu)_{|\Omega_\varepsilon} \in H^2_{0,\Sigma_\varepsilon}(\Omega_\varepsilon),$$

for all $u \in H^2_{0,\Sigma}(\Omega)$, see for example [16, Chapter 6]. Then we consider the operator $E_\varepsilon:\ H^2_{0,\Sigma}(\Omega) \to H^2_{0,\Sigma_\varepsilon}(\Omega_\varepsilon)$ defined by

$$E_\varepsilon u = Eu_{|\Omega_\varepsilon}$$

for all $u \in H^2_{0,\Sigma}(\Omega)$.

By using the operator $E_\varepsilon$, according to Definitions 2.1–2.4, the following notions of convergence are well-defined:

$$u_\varepsilon \xrightarrow{E} u,\ B_\varepsilon \xrightarrow{EE} B_0,\ B_\varepsilon \xrightarrow{C} B_0,$$

where $\mathcal{H}_\varepsilon = H^2_{0,\Sigma_\varepsilon}(\Omega_\varepsilon)$, $\mathcal{H} = H^2_{0,\Sigma}(\Omega)$. Indeed, by using Lemma 4.1 one can prove that

$$\|E_\varepsilon u\|_{D,\varepsilon} \to \|u\|_{\tilde{D}},\ \|E_\varepsilon u\|_{N,\varepsilon} \to \|u\|_{\tilde{N}} \text{ as } \varepsilon \to 0, \tag{4.11}$$

for all $u \in H^2_{0,\Sigma}(\Omega)$

Then we can prove the following theorem.

**Theorem 4.2.** *If $g_\varepsilon(\overline{x}) = \varepsilon b(\overline{x}/\varepsilon)$ and $b$ satisfies* (4.2)*, then $T_{D,\varepsilon} \xrightarrow{C} \widetilde{T}_D$ and $T_{N,\varepsilon} \xrightarrow{C} \widetilde{T}_N$ as $\varepsilon \to 0$. In particular, the eigenvalues and eigenfunctions of the problems in* (4.6) *converge to the eigenvalues and eigenfunctions of the respective problems in* (4.10) *in the sense of Theorem 2.5.*

*Proof.* First, we observe that $\partial\Omega_\varepsilon$ are Lipschitz domains with Lipschitz constants uniformly bounded with respect to $\varepsilon \in (0, \varepsilon_0]$, hence Lemma 2.7 and Corollary 2.8 still hold true.

We begin by proving that $T_{D,\varepsilon} \overset{C}{\to} \widetilde{T}_D$. Let $w_\varepsilon \in H^2_{0,\Sigma_\varepsilon}(\Omega_\varepsilon)$ with $0 < \varepsilon \le \varepsilon_0$, and $w \in H^2_{0,\Sigma}(\Omega)$ be such that $w_\varepsilon \overset{E}{\to} w$. We define $u_\varepsilon := T_{D,\varepsilon} w_\varepsilon$. We note that $u_\varepsilon \in H^2_{0,\Sigma_\varepsilon}(\Omega_\varepsilon)$ is uniformly bounded in norm as $\varepsilon \to 0$. This follows by the fact that, as in (3.1), we have $\|u_\varepsilon\|_{D,\varepsilon} \le \|w_\varepsilon\|_{D,\varepsilon}$ and by boundness of $\|w_\varepsilon\|_{D,\varepsilon}$ as $\varepsilon \to 0$.

In particular, $\{(u_\varepsilon)_{|\Omega}\}_{\varepsilon>0}$ is bounded in $H^2_{0,\Sigma}(\Omega)$ as $\varepsilon \to 0$. Consequently, along a sequence, there exists $u \in H^2_{0,\Sigma}(\Omega)$ such that $(u_\varepsilon)_{|\Omega} \rightharpoonup u$ in $H^2_{0,\Sigma}(\Omega)$ as $\varepsilon \to 0$.

In order to show that $u$ satisfies the appropriate limiting problem, we fix $\varphi$ we analyze the following equality

$$
\begin{aligned}
Q_{D,\Omega_\varepsilon}(u_\varepsilon, E_\varepsilon\varphi) &= Q_{\sigma,\Omega}(u_\varepsilon, E_\varepsilon\varphi) + Q_{\sigma,\Omega_\varepsilon\setminus\Omega}(u_\varepsilon, E_\varepsilon\varphi) \\
&\quad + \mu\int_{\Gamma_\varepsilon} u_\varepsilon E_\varepsilon\varphi d\sigma_\varepsilon + \int_{\Gamma_\varepsilon} (u_\varepsilon)_{\nu_\varepsilon}(E_\varepsilon\varphi)_{\nu_\varepsilon} d\sigma_\varepsilon = \int_{\Gamma_\varepsilon} (w_\varepsilon)_{\nu_\varepsilon}(E_\varepsilon\varphi)_{\nu_\varepsilon} d\sigma_\varepsilon,
\end{aligned}
\tag{4.12}
$$

where $E_\varepsilon : H^2_{0,\Sigma}(\Omega) \to H^2(\Omega_\varepsilon)$ is the extension operator defined above. Since $u_\varepsilon \rightharpoonup u$ in $H^2(\Omega)$, $E_\varepsilon\varphi = \varphi$ on $\Omega$, $|\Omega_\varepsilon\setminus\Omega| \to 0$ as $\varepsilon \to 0$. By the absolute continuity of the Lebesgue integral, it follows that

$$
Q_{\sigma,\Omega}(u_\varepsilon, E_\varepsilon\varphi) + Q_{\sigma,\Omega_\varepsilon\setminus\Omega}(u_\varepsilon, E_\varepsilon\varphi) = Q_{\sigma,\Omega}(u, \varphi) + o(1) \text{ as } \varepsilon \to 0. \tag{4.13}
$$

We observe that

$$
\begin{aligned}
&\int_W |u_\varepsilon(\bar{x}, g_\varepsilon(\bar{x})) - u_\varepsilon(\bar{x}, 0)|^2 \, d\bar{x} \\
&\le \int_W |g_\varepsilon(\bar{x})| \left| \int_0^{g_\varepsilon(\bar{x})} \left| \frac{\partial u_\varepsilon}{\partial x_N}(\bar{x}, x_N) \right|^2 dx_N \right| d\bar{x} \\
&\le \varepsilon \|b\|_{L^\infty(\mathbb{R}^{N-1})} \int_{\Omega_\varepsilon\setminus\Omega} \left| \frac{\partial u_\varepsilon}{\partial x_N} \right|^2 dx \le \varepsilon \|b\|_{L^\infty(\mathbb{R}^{N-1})} \|u_\varepsilon\|^2_{D,\varepsilon} \to 0,
\end{aligned}
\tag{4.14}
$$

as $\varepsilon \to 0$, and similarly

$$
\begin{aligned}
&\int_W \left| \frac{\partial u_\varepsilon}{\partial x_i}(\bar{x}, g_\varepsilon(\bar{x})) - \frac{\partial u_\varepsilon}{\partial x_i}(\bar{x}, 0) \right|^2 d\bar{x} \\
&\le \int_W |g_\varepsilon(\bar{x})| \left| \int_0^{g_\varepsilon(\bar{x})} \left| \frac{\partial^2 u_\varepsilon}{\partial x_N \partial x_i}(\bar{x}, x_N) \right|^2 dx_N \right| d\bar{x} \\
&\le \varepsilon \|b\|_{L^\infty(\mathbb{R}^{N-1})} \int_{\Omega_\varepsilon\setminus\Omega} \left| \frac{\partial^2 u_\varepsilon}{\partial x_N \partial x_i} \right|^2 dx \le \varepsilon \|b\|_{L^\infty(\mathbb{R}^{N-1})} \|u_\varepsilon\|^2_{D,\varepsilon} \to 0,
\end{aligned}
\tag{4.15}
$$

as $\varepsilon \to 0$. We also note that

$$
\begin{aligned}
&\int_{\Gamma_\varepsilon} u_\varepsilon E_\varepsilon\varphi d\sigma_\varepsilon - C_b \int_{\Gamma} u\varphi d\sigma \\
&= \int_W (u_\varepsilon(\bar{x}, g_\varepsilon(\bar{x})) - u_\varepsilon(\bar{x}, 0)) (E_\varepsilon\varphi)(\bar{x}, g_\varepsilon(\bar{x})) \sqrt{1 + |\nabla_{\bar{x}} g_\varepsilon(\bar{x})|^2} d\bar{x}
\end{aligned}
$$

$$+\int_W (u_\varepsilon(\overline{x},0)-u(\overline{x},0))\,(E_\varepsilon\varphi)(\overline{x},g_\varepsilon(\overline{x}))\,\sqrt{1+|\nabla_{\overline{x}}g_\varepsilon(\overline{x})|^2}d\overline{x}$$
$$+\int_W ((E_\varepsilon\varphi)(\overline{x},g_\varepsilon(\overline{x}))-\varphi(\overline{x},0))\,u(\overline{x},0)\,\sqrt{1+|\nabla_{\overline{x}}g_\varepsilon(\overline{x})|^2}d\overline{x}$$
$$+\int_W \left(\sqrt{1+|\nabla_{\overline{x}}g_\varepsilon(\overline{x})|^2}-C_b\right)u(\overline{x},0)\varphi(\overline{x},0)d\overline{x}. \tag{4.16}$$

By using (4.14), for the first and the third term in the right-hand side of (4.16) we have

$$\begin{aligned}&\int_W (u_\varepsilon(\overline{x},g_\varepsilon(\overline{x}))-u_\varepsilon(\overline{x},0))\,(E_\varepsilon\varphi)(\overline{x},g_\varepsilon(\overline{x}))\,\sqrt{1+|\nabla_{\overline{x}}g_\varepsilon(\overline{x})|^2}d\overline{x}\\ &\leq\left(\int_W |u_\varepsilon(\overline{x},g_\varepsilon(\overline{x}))-u_\varepsilon(\overline{x},0)|^2\,d\overline{x}\right)^{\frac{1}{2}}\\ &\times\left(\int_W \left|(E_\varepsilon\varphi)(\overline{x},g_\varepsilon(\overline{x}))\,\sqrt{1+|\nabla_{\overline{x}}g_\varepsilon(\overline{x})|^2}\right|^2 d\overline{x}\right)^{\frac{1}{2}}=o(1)\end{aligned} \tag{4.17}$$

and

$$\begin{aligned}&\int_W ((E_\varepsilon\varphi)(\overline{x},g_\varepsilon(\overline{x}))-\varphi(\overline{x},0))\,u(\overline{x},0)\,\sqrt{1+|\nabla_{\overline{x}}g_\varepsilon(\overline{x})|^2}d\overline{x}\\ &\leq\left(\int_W |(E_\varepsilon\varphi)(\overline{x},g_\varepsilon(\overline{x}))-\varphi(\overline{x},0)|^2\,d\overline{x}\right)^{\frac{1}{2}}\\ &\times\left(\int_W \left|u(\overline{x},0)\,\sqrt{1+|\nabla_{\overline{x}}g_\varepsilon(\overline{x})|^2}\right|^2 d\overline{x}\right)^{\frac{1}{2}}=o(1).\end{aligned} \tag{4.18}$$

Similarly by the compactness of the trace operator we have

$$\begin{aligned}&\int_W (u_\varepsilon(\overline{x},0)-u(\overline{x},0))\,(E_\varepsilon\varphi)(\overline{x},g_\varepsilon(\overline{x}))\,\sqrt{1+|\nabla_{\overline{x}}g_\varepsilon(\overline{x})|^2}d\overline{x}\\ &\leq\left(\int_W |u_\varepsilon(\overline{x},0)-u(\overline{x},0)|^2\,d\overline{x}\right)^{\frac{1}{2}}\\ &\times\left(\int_W \left|(E_\varepsilon\varphi)(\overline{x},g_\varepsilon(\overline{x}))\,\sqrt{1+|\nabla_{\overline{x}}g_\varepsilon(\overline{x})|^2}\right|^2 d\overline{x}\right)^{\frac{1}{2}}=o(1).\end{aligned} \tag{4.19}$$

By Lemma 4.1 we have

$$\int_W \left(\sqrt{1+|\nabla_{\overline{x}}g_\varepsilon(\overline{x})|^2}-C_b\right)u(\overline{x},0)\varphi(\overline{x},0)d\overline{x}=o(1). \tag{4.20}$$

Finally, by combining (4.16)–(4.20) we get

$$\int_{\Gamma_\varepsilon} u_\varepsilon E_\varepsilon\varphi d\sigma_\varepsilon = C_b\int_\Gamma u\varphi d\sigma + o(1). \tag{4.21}$$

We claim that

$$\int_{\Gamma_\varepsilon}(u_\varepsilon)_{\nu_\varepsilon}(E_\varepsilon\varphi)_{\nu_\varepsilon}d\sigma_\varepsilon=\int_\Gamma\sum_{i,j=1}^N C_{i,j}\frac{\partial u}{\partial x_i}\frac{\partial\varphi}{\partial x_j}d\sigma+o(1)\text{ as }\varepsilon\to 0. \tag{4.22}$$

First of all we note that

$$
\begin{aligned}
&\int_{\Gamma_\varepsilon}(u_\varepsilon)_{\nu_\varepsilon}(E_\varepsilon\varphi)_{\nu_\varepsilon}d\sigma_\varepsilon-\int_\Gamma\sum_{i,j=1}^N C_{i,j}\frac{\partial u}{\partial x_i}\frac{\partial\varphi}{\partial x_j}d\sigma\\
&=\sum_{i,j=1}^N\int_{\Gamma_\varepsilon}\frac{\partial u_\varepsilon}{\partial x_i}\nu_{\varepsilon,i}\frac{\partial(E_\varepsilon\varphi)}{\partial x_j}\nu_{\varepsilon,j}d\sigma_\varepsilon-\int_\Gamma\sum_{i,j=1}^N C_{i,j}\frac{\partial u}{\partial x_i}\frac{\partial\varphi}{\partial x_j}d\sigma\\
&=\sum_{i,j=1}^N\int_W\left(\frac{\partial u_\varepsilon}{\partial x_i}(\overline{x},g_\varepsilon(\overline{x}))-\frac{\partial u_\varepsilon}{\partial x_i}(\overline{x},0)\right)\frac{\partial(E_\varepsilon\varphi)}{\partial x_j}(\overline{x},g_\varepsilon(\overline{x}))C_{i,j}(\varepsilon)d\overline{x}\\
&+\sum_{i,j=1}^N\int_W\left(\frac{\partial u_\varepsilon}{\partial x_i}(\overline{x},0)-\frac{\partial u}{\partial x_i}(\overline{x},0)\right)\frac{\partial(E_\varepsilon\varphi)}{\partial x_j}(\overline{x},g_\varepsilon(\overline{x}))C_{i,j}(\varepsilon)d\overline{x}\\
&+\sum_{i,j=1}^N\int_W\left(\frac{\partial(E_\varepsilon\varphi)}{\partial x_j}(\overline{x},g_\varepsilon(\overline{x}))-\frac{\partial\varphi}{\partial x_j}(\overline{x},0)\right)\frac{\partial u}{\partial x_i}(\overline{x},0)C_{i,j}(\varepsilon)d\overline{x}\\
&+\sum_{i,j=1}^N\int_W\frac{\partial\varphi}{\partial x_j}(\overline{x},0)\frac{\partial u}{\partial x_i}(\overline{x},0)\left(C_{i,j}(\varepsilon)-C_{i,j}\right)d\overline{x}.
\end{aligned}
\tag{4.23}
$$

By exploiting (4.15) for the first and the third term in the right-hand side of (4.23) we have

$$
\begin{aligned}
&\sum_{i,j=1}^N\int_W\left(\frac{\partial u_\varepsilon}{\partial x_i}(\overline{x},g_\varepsilon(\overline{x}))-\frac{\partial u_\varepsilon}{\partial x_i}(\overline{x},0)\right)\frac{\partial(E_\varepsilon\varphi)}{\partial x_j}(\overline{x},g_\varepsilon(\overline{x}))C_{i,j}(\varepsilon)d\overline{x}\\
&\le\left(\sum_{i,j=1}^N\int_W\left|\frac{\partial u_\varepsilon}{\partial x_i}(\overline{x},g_\varepsilon(\overline{x}))-\frac{\partial u}{\partial x_i}(\overline{x},0)\right|^2d\overline{x}\right)^{\frac12}\times\left(\int_W\left|\frac{\partial(E_\varepsilon\varphi)}{\partial x_j}(\overline{x},g_\varepsilon(\overline{x}))C_{i,j}(\varepsilon)\right|^2d\overline{x}\right)^{\frac12}=o(1),
\end{aligned}
\tag{4.24}
$$

and

$$
\begin{aligned}
&\sum_{i,j=1}^N\int_W\left(\frac{\partial(E_\varepsilon\varphi)}{\partial x_j}(\overline{x},g_\varepsilon(\overline{x}))-\frac{\partial\varphi}{\partial x_j}(\overline{x},0)\right)\frac{\partial u}{\partial x_i}(\overline{x},0)C_{i,j}(\varepsilon)d\overline{x}\\
&\le\left(\sum_{i,j=1}^N\int_W\left|\frac{\partial(E_\varepsilon\varphi)}{\partial x_j}(\overline{x},g_\varepsilon(\overline{x}))-\frac{\partial\varphi}{\partial x_j}(\overline{x},0)\right|^2d\overline{x}\right)^{\frac12}\times\left(\int_W\left|\frac{\partial u}{\partial x_i}(\overline{x},0)C_{i,j}(\varepsilon)\right|^2d\overline{x}\right)^{\frac12}=o(1).
\end{aligned}
\tag{4.25}
$$

Now, similarly by the compactness of the trace operator we have

$$
\begin{aligned}
&\sum_{i,j=1}^N\int_W\left(\frac{\partial u_\varepsilon}{\partial x_i}(\overline{x},0)-\frac{\partial u}{\partial x_i}(\overline{x},0)\right)\frac{\partial(E_\varepsilon\varphi)}{\partial x_j}(\overline{x},g_\varepsilon(\overline{x}))C_{i,j}(\varepsilon)d\overline{x}\\
&\le\left(\sum_{i,j=1}^N\int_W\left|\frac{\partial u_\varepsilon}{\partial x_i}(\overline{x},0)-\frac{\partial u}{\partial x_i}(\overline{x},0)\right|^2d\overline{x}\right)^{\frac12}\\
&\times\left(\int_W\left|\frac{\partial(E_\varepsilon\varphi)}{\partial x_j}(\overline{x},g_\varepsilon(\overline{x}))C_{i,j}(\varepsilon)\right|^2d\overline{x}\right)^{\frac12}=o(1).
\end{aligned}
\tag{4.26}
$$

Next, by Lemma 4.1 we have

$$\sum_{i,j=1}^{N} \int_W \frac{\partial \varphi}{\partial x_j}(\overline{x},0)\frac{\partial u}{\partial x_i}(\overline{x},0)\left(C_{i,j}(\varepsilon) - C_{i,j}\right) d\overline{x} = o(1). \tag{4.27}$$

Combining (4.23)–(4.27) we prove the claim (4.22).

By passing to the limit in (4.12) as $\varepsilon \to 0$ and using $E$-convergence of $\Omega_\varepsilon$ to $\Omega$ we deduce that

$$\begin{aligned} Q_{\sigma,\Omega}(u,\varphi) + \mu C_b \int_\Gamma u\varphi d\sigma + \sum_{i,j=1}^{N} \int_W \frac{\partial u}{\partial x_i}(\overline{x},0)\frac{\partial \varphi}{\partial x_j}(\overline{x},0)C_{i,j}d\overline{x} \\ = \sum_{i,j=1}^{N} \int_W \frac{\partial w}{\partial x_i}(\overline{x},0)\frac{\partial \varphi}{\partial x_j}(\overline{x},0)C_{i,j}d\overline{x}. \end{aligned} \tag{4.28}$$

Hence, we conclude that $u$ depends solely on $w$, and is independent of the sequence converging to zero. Namely, we have that $u = \widetilde{T}_D w$.

Next we prove that $u_\varepsilon \xrightarrow{E} u$. We proceed as in the proof of Lemma 3.2 and we note that

$$\|u_\varepsilon - E_\varepsilon u\|^2_{D,\varepsilon} = \|u_\varepsilon\|^2_{D,\varepsilon} - 2Q_{D,\Omega_\varepsilon}(u_\varepsilon, E_\varepsilon u) + \|E_\varepsilon u\|^2_{D,\varepsilon}. \tag{4.29}$$

Then

$$Q_{D,\Omega_\varepsilon}(u_\varepsilon, E_\varepsilon u) = \int_{\partial\Omega_\varepsilon} (w_\varepsilon)_{\nu_\varepsilon}(E_\varepsilon u)_{\nu_\varepsilon} d\sigma_\varepsilon, \tag{4.30}$$

$$\|u_\varepsilon\|^2_{D,\varepsilon} = Q_{D,\Omega_\varepsilon}(u_\varepsilon, u_\varepsilon) = \int_{\partial\Omega_\varepsilon} (w_\varepsilon)_{\nu_\varepsilon}(u_\varepsilon)_{\nu_\varepsilon} d\sigma_\varepsilon, \tag{4.31}$$

$$\|E_\varepsilon u\|^2_{D,\varepsilon} = Q_{D,\Omega_\varepsilon}(E_\varepsilon u, E_\varepsilon u). \tag{4.32}$$

By (4.22) with $\varphi$ replaced by $u$ we get

$$\int_{\Gamma_\varepsilon} (u_\varepsilon)_{\nu_\varepsilon}(E_\varepsilon u)_{\nu_\varepsilon} d\sigma_\varepsilon \to \int_\Gamma \sum_{i,j=1}^{N} C_{i,j}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j} d\sigma = \|u\|^2_{\tilde{D}}. \tag{4.33}$$

In a similar way we can prove that

$$\int_{\Gamma_\varepsilon} (u_\varepsilon)_{\nu_\varepsilon}(w_\varepsilon)_{\nu_\varepsilon} d\sigma_\varepsilon \to \int_\Gamma \sum_{i,j=1}^{N} C_{i,j}\frac{\partial u}{\partial x_i}\frac{\partial w}{\partial x_j} d\sigma = \|u\|^2_{\tilde{D}}. \tag{4.34}$$

Indeed,

$$\begin{aligned} &\int_{\Gamma_\varepsilon} (u_\varepsilon)_{\nu_\varepsilon}(w_\varepsilon)_{\nu_\varepsilon} d\sigma_\varepsilon - \int_\Gamma \sum_{i,j=1}^{N} C_{i,j}\frac{\partial u}{\partial x_i}\frac{\partial w}{\partial x_j} d\sigma \\ &= \int_{\Gamma_\varepsilon} (u_\varepsilon)_{\nu_\varepsilon}(w_\varepsilon)_{\nu_\varepsilon} d\sigma_\varepsilon - \int_{\Gamma_\varepsilon} (u_\varepsilon)_{\nu_\varepsilon}(E_\varepsilon w)_{\nu_\varepsilon} d\sigma_\varepsilon \end{aligned} \tag{4.35}$$

$$+\int_{\Gamma_\varepsilon}(u_\varepsilon)_{\nu_\varepsilon}(E_\varepsilon w)_{\nu_\varepsilon}d\sigma_\varepsilon-\int_\Gamma\sum_{i,j=1}^N C_{i,j}\frac{\partial u}{\partial x_i}\frac{\partial w}{\partial x_j}d\sigma \tag{4.36}$$

and the term in (4.35) goes to zero by the $E$-convergence of $w_\varepsilon$ to $w$. Also the term in (4.36) goes to zero by (4.22). Thus by (4.29), (4.33) and (4.34) and by observing that

$$Q_{D,\Omega_\varepsilon}(E_\varepsilon u, E_\varepsilon u)\to\|u\|^2_{\tilde{D}}, \tag{4.37}$$

we infer that $u_\varepsilon \xrightarrow{E} u$ as $\varepsilon\to 0$.

It remains to prove that if $\|w_\varepsilon\|_{D,\varepsilon}=O(1)$ then possibly passing to a subsequence $u_\varepsilon$ is $E$-converge to some $u$.

By the boundness of $H^2$ norm of $w_\varepsilon$, there exists $w\in H^2(\Omega)$ such that, possibly passing to a subsequence, $w_\varepsilon\rightharpoonup w$ weakly in $H^2(\Omega)$ and $(w_\varepsilon)_\nu\to w_\nu$ strongly in $L^2(\partial\Omega)$ as $\varepsilon\to 0$. Then by the same arguments used above one can prove that $u$ and $w$ satisfies (4.28). In order to prove that $u_\varepsilon\xrightarrow{E} u$ we consider all equations from (4.29) to (4.36) and we deduce that $u_\varepsilon\xrightarrow{E} u$ as $\varepsilon\to 0$ by using the same arguments with one exception. Namely, in this case, in order to prove that the term in (4.35) goes to zero we use the compactness of the trace map as follows. We write

$$\begin{aligned}
&\int_{\Gamma_\varepsilon}(u_\varepsilon)_{\nu_\varepsilon}((w_\varepsilon)_{\nu_\varepsilon}-(E_\varepsilon w)_{\nu_\varepsilon})d\sigma_\varepsilon\le C\int_{\Gamma_\varepsilon}\left|\nabla w_\varepsilon-\nabla E_\varepsilon w\right|^2d\sigma_\varepsilon\\
&\quad\le C\int_W\left|\nabla w_\varepsilon(\bar{x},g_\varepsilon(\bar{x}))-\nabla(E_\varepsilon w)(\bar{x},g_\varepsilon(\bar{x}))\right|^2d\bar{x}\\
&\quad\le C\int_W\left|\nabla w_\varepsilon(\bar{x},g_\varepsilon(\bar{x}))-\nabla w_\varepsilon(\bar{x},0)\right|^2d\bar{x}+C\int_W\left|\nabla w_\varepsilon(\bar{x},0)-\nabla w(\bar{x},0)\right|^2d\bar{x}\\
&\quad+C\int_W\left|\nabla(E_\varepsilon w)(\bar{x},g_\varepsilon(\bar{x}))-\nabla w(\bar{x},0)\right|^2d\bar{x}
\end{aligned} \tag{4.38}$$

and we note that the first term and the third in (4.38) goes to zero by (4.15) while the second one converge to zero by the compactness of the trace operator. Then according to Definition 2.2, we have $T_{D,\varepsilon}\xrightarrow{C}\widetilde{T}_D$ as $\varepsilon\to 0$.

The proof of $T_{N,\varepsilon}\xrightarrow{C}\widetilde{T}_N$ is very similar and can be performed by repeating the argument above. We omit the details. □

**Remark 4.3.** *The classical formulations of problems* (4.10) *can be obtained in standard way by integration by parts. Assuming that $u$ is smooth enough, keeping in mind that $u$ and $\nabla u$ vanish at the boundary of $W$ (recall that $u\in H^2_{0,\Sigma}(\Omega)$), we have*

$$\begin{aligned}
&\int_W\sum_{i,j=1}^N C_{i,j}\frac{\partial u}{\partial x_i}(\bar{x},0)\frac{\partial\varphi}{\partial x_j}(\bar{x},0)d\bar{x}\\
&=-\int_W\sum_{i=1}^N\sum_{j=1}^{N-1}C_{i,j}\frac{\partial^2 u}{\partial x_i x_j}(\bar{x},0)\varphi(\bar{x},0)d\bar{x}+\int_W\sum_{i=1}^N C_{i,N}\frac{\partial u}{\partial x_i}(\bar{x},0)\frac{\partial\varphi}{\partial x_N}(\bar{x},0)d\bar{x}.
\end{aligned} \tag{4.39}$$

*Hence the first equation in* (4.10) *can be written as follows:*

$$\begin{aligned} &Q_{\sigma,\Omega}(u,\varphi)+\mu C_b\int_\Gamma u\varphi d\sigma \\ &=-\lambda\int_\Gamma\sum_{i=1}^N\sum_{j=1}^{N-1}C_{i,j}\frac{\partial^2 u}{\partial x_i x_j}\varphi d\sigma+\lambda\int_\Gamma\sum_{i=1}^N C_{i,N}\frac{\partial u}{\partial x_i}\frac{\partial\varphi}{\partial\nu}d\sigma. \end{aligned} \tag{4.40}$$

*Thus, we can deduce that the classical formulation* (4.40) *is the following*

$$\begin{cases} \Delta^2 u=0, & \text{in } \Omega,\\ u=u_\nu=0, & \text{on } \Sigma,\\ (1-\sigma)\,u_{\nu\nu}+\sigma\Delta u=\lambda\sum_{i=1}^N C_{i,N}\frac{\partial u}{\partial x_i}, & \text{on } \Gamma,\\ (1-\sigma)\,\mathrm{div}_\Gamma\left(D^2u\cdot\nu\right)_\Gamma+(\Delta u)_\nu-\mu C_b u=-\lambda\sum_{i=1}^N\sum_{j=1}^{N-1}C_{i,j}\frac{\partial^2 u}{\partial x_i x_j}, & \text{on } \Gamma. \end{cases}$$

*In the same way the classical formulation of the second equation in* (4.10) *is*

$$\begin{cases} \Delta^2 v=0, & \text{in } \Omega,\\ v=v_\nu=0, & \text{on } \Sigma,\\ (1-\sigma)\,v_{\nu\nu}+\sigma\Delta v+\rho\sum_{i=1}^N C_{i,N}\frac{\partial v}{\partial x_i}=0, & \text{on } \Gamma,\\ (\sigma-1)\,\mathrm{div}_\Gamma\left(D^2v\cdot\nu\right)_\Gamma-(\Delta v)_\nu-\rho\sum_{i=1}^N\sum_{j=1}^{N-1}C_{i,j}\frac{\partial^2 v}{\partial x_i x_j}=\gamma C_b v, & \text{on } \Gamma. \end{cases}$$

### 4.3. *Case* $0<\alpha<1$*: degeneration*

First, let us consider problem (4.4) with $\mu=0$.

If $0<\alpha<1$, by using Theorem 1 in [33] we can easily prove that for any $i,j\in\{1,\cdots N\}$ we have

$$\nu_{\varepsilon,i}\nu_{\varepsilon,j}\rightharpoonup^*\tilde\nu_{i,j}:=\begin{cases}\int_Y\frac{\partial_j b\partial_i b}{|\nabla_{\bar x}b|^2}dy, & \text{if } i,j\in\{1,\cdots N-1\},\\ 0, & \text{otherwise,}\end{cases} \tag{4.41}$$

in $L^\infty(W)$ as $\varepsilon\to 0$. Recall that here we assume the validity of condition (4.3). We denote by $\mathfrak{N}_\varepsilon$ the matrix defined by

$$\mathfrak{N}_\varepsilon=\begin{cases}(\tilde\nu_{i,j})_{i,j=1}^N, & \text{if } \varepsilon=0,\\ (\nu_{\varepsilon,i}\nu_{\varepsilon,j})_{i,j=1}^N, & \text{if } \varepsilon>0.\end{cases}$$

We note that by (4.41) this matrix is symmetric and nonnegative, that is, $\nu\mathfrak{N}_\varepsilon\nu^t\geq 0$ for all $\nu\in\mathbb{R}^N$.

Before proceeding further we need to introduce the following results. Recall that for simplicity we write $\Omega_0=\Omega$, $\Gamma_0=\Gamma$ and $\Sigma_0=\Sigma$.

**Lemma 4.4.** *For any integer $n\geq 0$ there exist $n+1$ functions $\varphi_0,...,\varphi_n$ in $C^\infty(\mathbb{R}^N)$ such that, for any $\varepsilon\geq 0$ their restrictions to $\Omega_\varepsilon$ are linearly independent, belong to $H^2_{0,\Sigma_\varepsilon}(\Omega_\varepsilon)$ and if $u=\sum_{i=0}^n a_i\varphi_i$ and $\nabla u\mathfrak{N}_\varepsilon(\nabla u)^t=0$ on $\Gamma_\varepsilon$ then $a_0=\cdots=a_n=0$.*

*Proof.* The proof can be carried out by following the same steps of the proof of Lemma 8.1 in [23] with minor modifications. Namely, by condition (4.3) we can assume without loss of generality, that $\nu_{\varepsilon,1}\neq 0$ for all $\varepsilon\geq 0$. Then, we can explicitly construct the functions $\varphi_0,...,\varphi_n$ by choosing, for example,

$$\varphi_k(x)=x_1^{k+1}\eta_1(\bar x)\eta_2(x_N)\text{ for all } x=(x_1,...,x_N)\in\mathbb{R}^N,\ k\in\{0,...,n\},$$

where $\eta_1 \in C_c^\infty(W)$ $\eta_1 \geq 0$ in $W$, $\eta_1 \equiv 1$ in $B$ where $B \subset W$ is an open ball of $\mathbb{R}^{N-1}$, and $\eta_2 \in C^\infty(\mathbb{R})$ with $\eta_2(t) = 0$ for any $t \leq -1$ and $\eta_2(t) = 1$ for any $t \geq -\frac{1}{2}$. It is obvious that the restrictions of those functions to $\Omega_\varepsilon$ are linearly independent. Assume now that $a_0, ..., a_n \in \mathbb{R}$ are such that $\nabla u \mathfrak{R}_\varepsilon (\nabla u)^t = 0$ on $\Gamma_\varepsilon$. Then we have

$$\nabla u(x)\mathfrak{R}_\varepsilon(x)(\nabla u(x))^t = \nu_{\varepsilon,1}(x) \sum_{i=0}^n \sum_{j=0}^n a_i a_j (i+1)(j+1) x_1^{i+j},$$

for all $x \in \{(\bar{x}, g_\varepsilon(\bar{x})) : \bar{x} \in B\}$. Then the assumption $\nabla u \mathfrak{R}_\varepsilon (\nabla u)^t = 0$ on $\Gamma_\varepsilon$ and $\nu_{\varepsilon,1} \neq 0$ imply

$$\sum_{i=0}^n \sum_{j=0}^n a_i a_j (i+1)(j+1) x_1^{i+j} = 0. \tag{4.42}$$

It is easy to prove by induction that (4.42) implies that $a_i = 0$ for all $i = 0, ..., n$. Indeed, it is immediate to check that by condition (4.42) $a_0 = 0$ since the constant term in (4.42) is $a_0^2$. Next, assuming by inductive hypothesis that $a_i = 0$ for all $i = 1, ..., k$ with $k < n$, we see that the coefficient of the monomial of degree $2k+2$ in (4.42) is $a_{k+1}^2 (k+2)^2$, hence, $a_{k+1} = 0$. The proof is complete. □

**Lemma 4.5.** *Let* $\{\Omega_\varepsilon\}_{0 \leq \varepsilon \leq \varepsilon_0}$ *be a family of domains as in* (4.1) *with* $g_\varepsilon(\overline{x}) = \varepsilon^\alpha b(\overline{x}/\varepsilon)$*,* $0 < \alpha < 1$*, and* $b$ *satisfying* (4.2) *and* (4.3)*. Let* $\{u_\varepsilon\}_{0 < \varepsilon \leq \varepsilon_0} \subset H^2_{loc}(\mathbb{R}^N)$ *be a family of functions such that*

$$\int_D |D^2 u_\varepsilon|^2 dx + \int_D |\nabla u_\varepsilon|^2 dx + \int_D u_\varepsilon^2 dx = O(1), \text{ as } \varepsilon \to 0,$$

*for some bounded domain* $D$ *satisfying* $D \supset \bigcup_{0 \leq \varepsilon \leq \varepsilon_0} \Omega_\varepsilon$*.*

*Then the following statements hold:*

(*i*) *There exists a subsequence* $\{u_{\varepsilon_k}\}_{k \in \mathbb{N}}$ *with* $\varepsilon_k \to 0$ *and* $u \in H^2(\Omega)$ *such that* $(u_{\varepsilon_k})_{|\Omega} \rightharpoonup u$ *weakly in* $H^2(\Omega)$ *as* $k \to \infty$*;*

(*ii*) *If* $\int_{\Gamma_{\varepsilon_k}} (u_{\varepsilon_k})^2_{\nu_{\varepsilon_k}} d\sigma_{\varepsilon_k} = O(1)$ *as* $k \to \infty$ *then* $\nabla u \mathfrak{R} (\nabla u)^t = 0$ *on* $\Gamma$*.*

*Proof.* For simplicity, in the proof of the lemma, we will denote the restrictions $(u_\varepsilon)_{|\Omega_\varepsilon}$ and $(u_\varepsilon)_{|\Omega}$ simply by $u_\varepsilon$.

To begin with, we observe that $\|u_\varepsilon\|_{H^2(\Omega)} = O(1)$ as $\varepsilon \to 0$ being $\Omega \subseteq D$. The proof of (i) then proceeds from the reflexivity of $H^2(\Omega)$. Moreover, by the compactness of the trace map we deduce that

$$u_{\varepsilon_k} \to u, \ \nabla u_{\varepsilon_k} \to \nabla u \text{ strongly in } L^2(\partial\Omega) \text{ as } k \to +\infty. \tag{4.43}$$

For any $t > 0$ we set

$$W_{t,\varepsilon} := \left(\left\{\overline{x} \in W : \ \sqrt{1 + |\nabla_{\overline{x}} g_\varepsilon(\overline{x})|^2} \leq t\right\}\right) = 0, \ W_{t,\varepsilon}^c := W \backslash W_{t,\varepsilon}.$$

For simplicity, we omit the subindex $k$. Then, we have

$$
\begin{aligned}
O(1) &= \int_{\Gamma_\varepsilon} (u_\varepsilon)^2_{\nu_\varepsilon} d\sigma_\varepsilon = \int_{\Gamma_\varepsilon} \left( \sum_{i=1}^{N} \frac{\partial u_\varepsilon}{\partial x_i} \nu_{\varepsilon,i} \right)^2 d\sigma_\varepsilon \\
&= \int_W \left( \sum_{i=1}^{N} \frac{\partial u_\varepsilon}{\partial x_i}(\overline{x}, g_\varepsilon(\overline{x}))\nu_{\varepsilon,i}(\overline{x}) \right)^2 \sqrt{1 + |\nabla_{\overline{x}} g_\varepsilon(\overline{x})|^2} d\overline{x} \\
&= \int_{W_{t,\varepsilon}} \left( \sum_{i=1}^{N} \frac{\partial u_\varepsilon}{\partial x_i}(\overline{x}, g_\varepsilon(\overline{x}))\nu_{\varepsilon,i}(\overline{x}) \right)^2 \sqrt{1 + |\nabla_{\overline{x}} g_\varepsilon(\overline{x})|^2} d\overline{x} \\
&+ \int_{W^c_{t,\varepsilon}} \left( \sum_{i=1}^{N} \frac{\partial u_\varepsilon}{\partial x_i}(\overline{x}, g_\varepsilon(\overline{x}))\nu_{\varepsilon,i}(\overline{x}) \right)^2 \sqrt{1 + |\nabla_{\overline{x}} g_\varepsilon(\overline{x})|^2} d\overline{x}.
\end{aligned}
\tag{4.44}
$$

We can estimate the first term on the right-hand side of (4.44) as follows:

$$
\begin{aligned}
&\int_{W_{t,\varepsilon}} \left( \sum_{i=1}^{N} \frac{\partial u_\varepsilon}{\partial x_i}(\overline{x}, g_\varepsilon(\overline{x}))\nu_{\varepsilon,i}(\overline{x}) \right)^2 \sqrt{1 + |\nabla_{\overline{x}} g_\varepsilon(\overline{x})|^2} d\overline{x} \\
&\leq \int_{W_{t,\varepsilon}} \left( \sum_{i=1}^{N} \left( \frac{\partial u_\varepsilon}{\partial x_i}(\overline{x}, g_\varepsilon(\overline{x})) \right)^2 \right) \left( \sum_{i=1}^{N} \left| \nu_{\varepsilon,i}(\overline{x}) \right|^2 \right) \sqrt{1 + |\nabla_{\overline{x}} g_\varepsilon(\overline{x})|^2} d\overline{x} \\
&\leq t \sum_{i=1}^{N} \int_{W_{t,\varepsilon}} \left( \frac{\partial u_\varepsilon}{\partial x_i}(\overline{x}, g_\varepsilon(\overline{x})) \right)^2 d\overline{x} \\
&\leq Ct \sum_{i=1}^{N} \int_W \left| \frac{\partial u_\varepsilon}{\partial x_i}(\overline{x}, g_\varepsilon(\overline{x})) - \frac{\partial u_\varepsilon}{\partial x_i}(\overline{x}, 0) \right|^2 d\overline{x} \\
&+ Ct \sum_{i=1}^{N} \int_W \left| \frac{\partial u_\varepsilon}{\partial x_i}(\overline{x}, 0) - \frac{\partial u}{\partial x_i}(\overline{x}, 0) \right|^2 d\overline{x} \\
&+ Ct \sum_{i=1}^{N} \int_{W_{t,\varepsilon}} \left| \frac{\partial u}{\partial x_i}(\overline{x}, 0) \right|^2 d\overline{x}.
\end{aligned}
\tag{4.45}
$$

The second term on the right-hand side of (4.45) converges to zero due to (4.43), and the third term follows suit as the measure of $W_{t,\varepsilon}$ converges to zero, as per Proposition 4.1 (ii). As for the first term, we have (omitting the summation symbol for simplicity):

$$
\begin{aligned}
&\int_W \left| \frac{\partial u_\varepsilon}{\partial x_i}(\overline{x}, g_\varepsilon(\overline{x})) - \frac{\partial u_\varepsilon}{\partial x_i}(\overline{x}, 0) \right|^2 d\overline{x} \\
&\leq \int_W |g_\varepsilon(\overline{x})| \left| \int_0^{g_\varepsilon(\overline{x})} \left| \frac{\partial^2 u_\varepsilon}{\partial x_N \partial x_i}(\overline{x}, x_N) \right|^2 dx_N \right| d\overline{x} \\
&\leq \varepsilon^\alpha \|b\|_{L^\infty(\mathbb{R}^{N-1})} \int_{\Omega_\varepsilon \setminus \Omega} \left| D^2 u_\varepsilon \right|^2 dx \\
&\leq \varepsilon^\alpha \|b\|_{L^\infty(\mathbb{R}^{N-1})} \|u_\varepsilon\|^2_{H^2(D)} \to 0 \text{ as } \varepsilon \to 0.
\end{aligned}
\tag{4.46}
$$

This demonstrates that the left-hand side of (4.45) tends to zero as $\varepsilon \to 0$ along the prescribed sequence.

Combining this fact with (4.44), we infer the existence of a positive constant $C$ independent of $\varepsilon$ and $t$ such that

$$\begin{aligned} C &\geq \int_{W^c_{t,\varepsilon}} \left( \sum_{i=1}^{N} \frac{\partial u_\varepsilon}{\partial x_i}(\overline{x}, g_\varepsilon(\overline{x}))\nu_{\varepsilon,i}(\overline{x}) \right)^2 \sqrt{1+|\nabla_{\overline{x}} g_\varepsilon(\overline{x})|^2} d\overline{x} + o(1) \\ &> t \int_{W^c_{t,\varepsilon}} \left( \sum_{i=1}^{N} \frac{\partial u_\varepsilon}{\partial x_i}(\overline{x}, g_\varepsilon(\overline{x}))\nu_{\varepsilon,i}(\overline{x}) \right)^2 d\overline{x} + o(1). \end{aligned} \tag{4.47}$$

We claim that

$$\int_{W^c_{t,\varepsilon}} \left( \sum_{i=1}^{N} \frac{\partial u_\varepsilon}{\partial x_i}(\overline{x}, g_\varepsilon(\overline{x}))\nu_{\varepsilon,i}(\overline{x}) \right)^2 d\overline{x} = \int_{W^c_{t,\varepsilon}} \left( \sum_{i=1}^{N} \frac{\partial u}{\partial x_i}(\overline{x}, 0)\nu_{\varepsilon,i}(\overline{x}) \right)^2 d\overline{x} + o(1). \tag{4.48}$$

Indeed, using (4.43) and (4.46), we have

$$\begin{aligned} &\left| \left( \int_{W^c_{t,\varepsilon}} \left( \sum_{i=1}^{N} \frac{\partial u_\varepsilon}{\partial x_i}(\overline{x}, g_\varepsilon(\overline{x}))\nu_{\varepsilon,i} \right)^2 d\overline{x} \right)^{\frac{1}{2}} - \left( \int_{W^c_{t,\varepsilon}} \left( \sum_{i=1}^{N} \frac{\partial u}{\partial x_i}(\overline{x}, 0)\nu_{\varepsilon,i} \right)^2 d\overline{x} \right)^{\frac{1}{2}} \right| \\ &\leq \left( \int_{W^c_{t,\varepsilon}} \left| \sum_{i=1}^{N} \frac{\partial u_\varepsilon}{\partial x_i}(\overline{x}, g_\varepsilon(\overline{x}))\nu_{\varepsilon,i}(\overline{x}) - \sum_{i=1}^{N} \frac{\partial u}{\partial x_i}(\overline{x}, 0)\nu_{\varepsilon,i}(\overline{x}) \right|^2 d\overline{x} \right)^{\frac{1}{2}} \\ &\leq \sum_{i=1}^{N} \left( \int_{W^c_{t,\varepsilon}} \left| \frac{\partial u_\varepsilon}{\partial x_i}(\overline{x}, g_\varepsilon(\overline{x})) - \frac{\partial u}{\partial x_i}(\overline{x}, 0) \right|^2 d\overline{x} \right)^{\frac{1}{2}} \\ &\leq \sum_{i=1}^{N} \left( \int_{W^c_{t,\varepsilon}} \left| \frac{\partial u_\varepsilon}{\partial x_i}(\overline{x}, g_\varepsilon(\overline{x})) - \frac{\partial u_\varepsilon}{\partial x_i}(\overline{x}, 0) \right|^2 d\overline{x} \right)^{\frac{1}{2}} \\ &+ \sum_{i=1}^{N} \left( \int_{W^c_{t,\varepsilon}} \left| \frac{\partial u_\varepsilon}{\partial x_i}(\overline{x}, 0) - \frac{\partial u}{\partial x_i}(\overline{x}, 0) \right|^2 d\overline{x} \right)^{\frac{1}{2}} = o(1). \end{aligned}$$

On the other hand, it is easy to see that

$$\int_{W^c_{t,\varepsilon}} \left( \sum_{i=1}^{N} \frac{\partial u}{\partial x_i}(\overline{x}, 0)\nu_{\varepsilon,i} \right)^2 d\overline{x} = O(1)$$

as $\varepsilon \to 0$. This combined with the previous inequality proves the validity of (4.48). Then, from (4.47) and (4.48), using (4.41) we have

$$\begin{aligned} C &\geq t \left( \int_{W^c_{t,\varepsilon}} \left( \sum_{i=1}^{N} \frac{\partial u}{\partial x_i}(\overline{x}, 0)\nu_{\varepsilon,i}(\overline{x}) \right)^2 d\overline{x} + o(1) \right) + o(1) \\ &= t \left( \sum_{i,j=1}^{N} \int_{W^c_{t,\varepsilon}} \frac{\partial u}{\partial x_i}(\overline{x}, 0) \frac{\partial u}{\partial x_j}(\overline{x}, 0) \tilde{\nu}_{i,j} d\overline{x} + o(1) \right) + o(1). \end{aligned} \tag{4.49}$$

Taking into account

$$\sum_{i,j=1}^{N}\int_{W_{t,\varepsilon}}\frac{\partial u}{\partial x_i}(\overline{x},0)\frac{\partial u}{\partial x_j}(\overline{x},0)\tilde{\nu}_{i,j}d\overline{x}=o(1)\text{ as }\varepsilon\to 0,$$

from (4.49) we obtain

$$C>t\left(\sum_{i,j=1}^{N}\int_{W}\frac{\partial u}{\partial x_i}(\overline{x},0)\frac{\partial u}{\partial x_j}(\overline{x},0)\tilde{\nu}_{i,j}d\overline{x}+o(1)\right)+o(1)$$

as $\varepsilon\to 0$, which implies

$$\int_{W}\sum_{i,j=1}^{N}\frac{\partial u}{\partial x_i}(\overline{x},0)\frac{\partial u}{\partial x_j}(\overline{x},0)\tilde{\nu}_{i,j}d\overline{x}<\frac{C+o(1)}{t}+o(1)\text{ for any }t>0,$$

as $\varepsilon\to 0$. Here the quantities denoted by $o(1)$ do not depend on $t$. Letting $t\to+\infty$, we infer that $\nabla u\mathfrak{R}(\nabla u)^t=0$ on $\Gamma$, because the matrix $\mathfrak{R}$ is nonnegative. □

**Theorem 4.6.** *Let* $\{\Omega_\varepsilon\}_{0\le\varepsilon\le\varepsilon_0}$ *be a family of domains as in* (4.1) *with* $g_\varepsilon(\overline{x})=\varepsilon^\alpha b(\overline{x}/\varepsilon)$*,* $0<\alpha<1$ *and* $b$ *satisfying* (4.2) *and* (4.3). *Then for any* $j\ge 1$ *we have that* $\lambda_j^\varepsilon\to 0$ *as* $\varepsilon\to 0$*, where* $\lambda_j^\varepsilon$ *are the eigenvalues of* (4.4) *with* $\mu=0$.

*Proof.* By the classical Min-Max principle we have

$$\lambda_j^\varepsilon=\inf_{\substack{\Pi\subset H^2_{0,\Sigma_\varepsilon}(\Omega_\varepsilon)\\ \dim\Pi=j+1}}\sup_{\substack{u\in\Pi\\ u_\nu\neq 0}}\frac{Q_{\sigma,\Omega_\varepsilon}(u,u)}{\int_{\Gamma_\varepsilon}u_\nu^2d\sigma}. \tag{4.50}$$

Considering $j+1$ functions as described in Lemma 4.4, denoted by $\varphi_0,...,\varphi_j$, we introduce the following family of subspaces of $H^2_{0,\Sigma_\varepsilon}(\Omega_\varepsilon)$:

$$\Pi_\varepsilon:=\operatorname{span}\{(\varphi_0)_{|\Omega_\varepsilon},...,\left(\varphi_j\right)_{|\Omega_\varepsilon}\}.$$

Note that by Lemma 4.4 the dimension of $\Pi_\varepsilon$ is $j+1$ and $u_\nu$ is not identically zero for all $u\in\Pi_\varepsilon\setminus\{0\}$. By the equality (4.50), we infer

$$\lambda_j^\varepsilon\le\sup_{\substack{u\in\Pi_\varepsilon\\ u_\nu\neq 0}}\frac{Q_{\sigma,\Omega_\varepsilon}(u,u)}{\int_{\Gamma_\varepsilon}u_\nu^2d\sigma_\varepsilon}. \tag{4.51}$$

Let $a_0^\varepsilon,...,a_j^\varepsilon\in\mathbb{R}$ be such that $u_\varepsilon:=\sum_{i=0}^j a_i^\varepsilon\varphi_i$ satisfies

$$\frac{Q_{\sigma,\Omega_\varepsilon}(u_\varepsilon,u_\varepsilon)}{\int_{\Gamma_\varepsilon}(u_\varepsilon)^2_{\nu_\varepsilon}d\sigma_\varepsilon}=\sup_{\substack{u\in\Pi_\varepsilon\\ u_\nu\neq 0}}\frac{Q_{\sigma,\Omega_\varepsilon}(u,u)}{\int_{\Gamma_\varepsilon}u_\nu^2d\sigma_\varepsilon}. \tag{4.52}$$

Without loss of generality, we can assume, up to normalization, that $\sum_{i=0}^j(a_i^\varepsilon)^2=1$.

We claim that

$$\int_{\Gamma_\varepsilon}(u_\varepsilon)^2_{\nu_\varepsilon}d\sigma_\varepsilon\to+\infty\text{ as }\varepsilon\to 0. \tag{4.53}$$

Assume, for contradiction, that there exists a sequence $\varepsilon_k \to 0$ such that

$$\int_{\Gamma_{\varepsilon_k}} (u_{\varepsilon_k})^2_{\nu_{\varepsilon_k}} d\sigma_{\varepsilon_k} = O(1) \text{ as } k \to +\infty.$$

Then, according to Lemma 4.5, we conclude that there exists $u \in H^2_{0,\Sigma}(\Omega)$ such that

$$(u_{\varepsilon_k})_{|\Omega} \rightharpoonup u \text{ weakly in } H^2(\Omega) \tag{4.54}$$

and moreover $\nabla u \mathfrak{R}(\nabla u)^t = 0$ on $\Gamma$. In fact, the weak convergence in (4.54) is strong because the sequence of functions $\{(u_{\varepsilon_k})_{|\Omega}\}$ belong to finite-dimensional space. By possibly passing to a subsequence, we can assume that there exist $a_0, ..., a_j \in \mathbb{R}$ such that

$$a_i^{\varepsilon_k} \to a_i \text{ as } k \to +\infty, \text{ for any } i \in \{0, ..., j\}. \tag{4.55}$$

Clearly, the normalization condition $\sum_{i=0}^{j} (a_i^{\varepsilon_k})^2 = 1$ implies

$$\sum_{i=0}^{j} (a_i)^2 = 1. \tag{4.56}$$

Combining (4.53) and (4.55), we deduce that $u = \sum_{i=0}^{j} a_i \varphi_i$. Since $\nabla u \mathfrak{R}(\nabla u)^t = 0$ on $\Gamma$, by Lemma 4.4 we have that

$$a_0 = \cdots = a_j = 0,$$

which contradicts (4.56). This completes the proof of claim (4.53).

The theorem's proof now immediately follows by combining (4.50) and (4.51)–(4.53). □

Let us consider problem (4.5) with $\rho = 0$.

Before proceeding further we recall the following results from [23].

**Lemma 4.7.** *For any integer $n \geq 0$ there exist $n + 1$ functions $\psi_0, ..., \psi_n$ in $C^\infty(\mathbb{R}^N)$ such that for any $\varepsilon \geq 0$ their restrictions to $\Omega_\varepsilon$ belong to $H^2_{0,\Sigma_\varepsilon}(\Omega_\varepsilon)$ and such that their restrictions to $\Gamma_\varepsilon$ are linearly independent in the sense that if $c_0, ..., c_n \in \mathbb{R}$ are such that*

$$c_0\psi_0(x) + \cdots + c_n\psi_n(x) = 0 \text{ for all } x \in \Gamma_\varepsilon, \tag{4.57}$$

*then $c_0 = \cdots = c_n = 0$.*

*Proof.* The proof is precisely the one provided for Lemma 8.1 in [23]. Here we only point out that in Lemma 8.1 in [23] it is stated that the functions $\psi_1, \ldots, \psi_n$ vanish on $\Sigma_\varepsilon$; in fact, the proof of that lemma shows that also the normal derivatives vanish on $\Sigma_\varepsilon$, hence they belong to $H^2_{0,\Sigma_\varepsilon}(\Omega_\varepsilon)$. □

**Lemma 4.8.** *Let $\{\Omega_\varepsilon\}_{0\leq\varepsilon\leq\varepsilon_0}$ be a family of domains as in (4.1) with $g_\varepsilon(\overline{x}) = \varepsilon^\alpha b(\overline{x}/\varepsilon)$ $0 < \alpha < 1$, $b$ satisfying (4.2) and (4.3). Let $\{v_\varepsilon\}_{0<\varepsilon\leq\varepsilon_0} \subset H^2_{loc}(\mathbb{R}^N)$ be a family of functions such that*

$$\int_D |D^2 v_\varepsilon|^2 dx + \int_D |\nabla v_\varepsilon|^2 dx + \int_D v_\varepsilon^2 dx = O(1), \text{ as } \varepsilon \to 0,$$

*for some bounded domain $D$ satisfying $D \supset \bigcup_{0\leq\varepsilon\leq\varepsilon_0} \Omega_\varepsilon$.*

*Then we have:*

*(i) There exists a subsequence $\{v_{\varepsilon_k}\}_{k\in\mathbb{N}}$ with $\varepsilon_k \to 0$ and $v \in H^2(\Omega)$ such that $(v_{\varepsilon_k})_{|\Omega} \rightharpoonup v$ weakly in $H^2(\Omega)$;*

*(ii) If $\int_{\Gamma_{\varepsilon_k}} v_{\varepsilon_k}^2 d\sigma_{\varepsilon_k} = O(1)$ as $k \to +\infty$ then $v = 0$ on $\Gamma$.*

*Proof.* The proof of this lemma follows the same lines of the proof of Lemma 6.1 in [23]. □

**Theorem 4.9.** *Let* $\{\Omega_\varepsilon\}_{0\le\varepsilon\le\varepsilon_0}$ *be a family of domains like in* (4.1) *with* $g_\varepsilon(\overline{x}) = \varepsilon^\alpha b(\overline{x}/\varepsilon)$ $0 < \alpha < 1$, *b satisfying* (4.2) *and* (4.3). *Then for any* $j \ge 1$ *we have that* $\gamma_j^\varepsilon \to 0$ *as* $\varepsilon \to 0$ *where* $\gamma_j^\varepsilon$ *are the eigenvalues of* (4.5) *with* $\rho = 0$.

*Proof.* By the classical Min-Max Principle, if $\rho = 0$ we have

$$\gamma_j = \min_{\substack{V\subset H^2(\Omega)\\ \dim V=j+1}} \max_{\substack{v\in V\\ v\neq 0}} \frac{Q_{\sigma,\Omega}(v,v)}{\int_\Gamma v^2 d\sigma}. \tag{4.58}$$

Considering $j+1$ functions as described in Lemma 4.7, denoted by $\psi_0, ..., \psi_j$, we introduce the following family of subspaces within $H^2(\Omega_\varepsilon)$:

$$V_\varepsilon := \text{span}\{(\psi_0)_{|\Omega_\varepsilon}, ..., \left(\psi_j\right)_{|\Omega_\varepsilon}\}.$$

By equality (4.58), we infer

$$\gamma_j^\varepsilon \le \max_{\substack{v\in V_\varepsilon\\ v\neq 0}} \frac{Q_{\sigma,\Omega_\varepsilon}(v,v)}{\int_{\Gamma_\varepsilon} v^2 d\sigma_\varepsilon}. \tag{4.59}$$

Let $c_0^\varepsilon, ..., c_j^\varepsilon \in \mathbb{R}$ be such that $v_\varepsilon := \sum_{i=0}^j c_i^\varepsilon \psi_i$ satisfies

$$\frac{Q_{\sigma,\Omega_\varepsilon}(v_\varepsilon,v_\varepsilon)}{\int_{\Gamma_\varepsilon} v_\varepsilon^2 d\sigma_\varepsilon} = \max_{\substack{v\in V_\varepsilon\\ v\neq 0}} \frac{Q_{\sigma,\Omega_\varepsilon}(v,v)}{\int_{\Gamma_\varepsilon} v^2 d\sigma_\varepsilon}. \tag{4.60}$$

Without loss of generality, we can assume, up to normalization, that $\sum_{i=0}^j (c_i^\varepsilon)^2 = 1$.

We claim that

$$\int_{\Gamma_\varepsilon} v_\varepsilon^2 d\sigma_\varepsilon \to +\infty \text{ as } \varepsilon \to 0. \tag{4.61}$$

Assume by contradiction that there exists a sequence $\varepsilon_k \to 0$ such that

$$\int_{\Gamma_\varepsilon} v_\varepsilon^2 d\sigma_\varepsilon = O(1) \text{ as } k \to +\infty.$$

Then, according to Lemma 4.8, we conclude that there exists $v \in H^2(\Omega)$ such that

$$(v_{\varepsilon_k})_{|\Omega} \rightharpoonup v \text{ weakly in } H^2(\Omega) \tag{4.62}$$

and moreover $v = 0$ on $\Gamma$. In fact, the weak convergence in (4.62) is strong because the sequence of functions $\{(v_{\varepsilon_k})_{|\Omega}\}$ belongs to a finite-dimensional space. By possibly passing to a subsequence, we can assume that there exist $c_0, ..., c_j \in \mathbb{R}$ such that

$$c_i^{\varepsilon_k} \to c_i \text{ as } k \to +\infty, \text{ for any } i \in \{0, ..., j\}. \tag{4.63}$$

Clearly, the normalization condition $\sum_{i=0}^j (c_i^{\varepsilon_k})^2 = 1$ implies

$$\sum_{i=0}^j (c_i)^2 = 1. \tag{4.64}$$

Combining (4.62) and (4.63), we deduce that $v = \sum_{i=0}^j c_i\psi_i$. Since $v = 0$ on $\Gamma$, we have that condition (4.57) on $\Gamma$ is satisfied. Hence, we infer $c_0 = \cdots = c_j = 0$, which contradicts (4.64). This completes the proof of claim (4.61).

The theorem's proof now immediately follows by combining (4.58) and (4.59)–(4.61). □

## 5. Conclusions

In this paper, we have studied the spectral stability of the biharmonic Steklov problems $(DBS)_\mu$ and $(NBS)_\rho$ under domain perturbations. Working within a framework of varying Hilbert spaces and employing suitable connecting operators, we have shown that condition (2.7) guarantees the $E$-compact convergence of the associated resolvent operators. As a consequence, the eigenvalues converge, and the corresponding eigenfunctions converge in the sense of Theorem 2.5.

We then investigated the sharpness of condition (2.7) for perturbations of the form

$$g_\varepsilon(x) = \varepsilon^\alpha b(x/\varepsilon),$$

showing that the asymptotic behaviour exhibits a trichotomy. More precisely, for $\alpha > 1$ spectral stability is preserved; for $\alpha = 1$ a nontrivial "strange term" appears in the limit boundary conditions; and for $0 < \alpha < 1$ the spectrum undergoes a degeneration phenomenon.

Our results also highlight several natural open problems. A first question is to better understand the role of $\mu$ in the asymptotic regime. A second issue is to identify the critical threshold for the (NBS) problem, in analogy with what is known for related Steklov-type problems. It would also be of interest to extend the present analysis to more general perturbations and to weaker assumptions on the parameters $\mu$ and $\rho$.

### Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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### Conflict of interest

The authors declare no conflicts of interest.

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