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# Regularity for mixed-order nonlinear fractional equations with degenerate coefficients[†]

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**Abstract:** We consider a class of nonlinear integro-differential equations whose leading operator is modeled on a superposition of $(-\Delta_p)^s$ and $(-\Delta_p)^t$, where $0 < s < t < 1 < p < \infty$, weighted via two possibly degenerate coefficients $a(\cdot,\cdot) \geq 0$ and $b(\cdot,\cdot) \geq 0$, respectively. We prove local boundedness and Hölder regularity of its weak solutions under natural assumptions on the coefficients $a(\cdot,\cdot)$, $b(\cdot,\cdot)$ and the powers $s, t, p$. Moreover, when $a(\cdot,\cdot) \equiv 1$, we also prove a Harnack inequality for weak solutions.

**Keywords:** nonlinear nonlocal operator; mixed order; degenerate coefficient; regularity; Harnack inequality

## 1. Introduction

This paper is concerned with regularity results for a class of nonlinear nonlocal equations whose differentiability order shows a drastic change depending on the point. More precisely, with $\Omega \subset \mathbb{R}^n$ being a bounded domain, we consider the following equation

$$\mathcal{L}u = 0 \quad \text{in} \ \Omega, \tag{1.1}$$

whose leading operator $\mathcal{L}(\cdot)$ is given by

$$\mathcal{L}u(x) := \text{P.V.} \int_{\mathbb{R}^n} |u(x)-u(y)|^{p-2}(u(x)-u(y)) \left[a(x,y)K_{sp}(x,y) + b(x,y)K_{tp}(x,y)\right] dy, \qquad x \in \mathbb{R}^n.$$

Here, $K_{sp}, K_{tp} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ are measurable kernels of orders $(s, p)$ and $(t, p)$, respectively, for some $0 < s < t < 1 < p < \infty$, and $a, b : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ are nonnegative functions. We refer to Section 1.1 below for the detailed assumptions.

There have been several researches on various kinds of anisotropic nonlocal equations, see for instance [10, 25, 29] for the linear case and [9, 11] for the nonlinear case. The Eq (1.1) is modeled on the following example, which is the case where $K_{sp}(x, y) \equiv |x - y|^{-n-sp}$ and $K_{tp}(x, y) \equiv |x - y|^{-n-tp}$:

$$\mathrm{P.V.} \int_{\mathbb{R}^n} |u(x) - u(y)|^{p-2}(u(x) - u(y)) \left( \frac{a(x, y)}{|x - y|^{n+sp}} + \frac{b(x, y)}{|x - y|^{n+tp}} \right) dy = 0 \quad \text{in } \Omega. \tag{1.2}$$

It is straightforward to see that (1.2) is the Euler–Lagrange equation of the functional

$$v \mapsto \iint_{C_\Omega} \left( a(x, y) \frac{|v(x) - v(y)|^p}{|x - y|^{n+sp}} + b(x, y) \frac{|v(x) - v(y)|^p}{|x - y|^{n+tp}} \right) dx\, dy, \tag{1.3}$$

where

$$C_\Omega := (\mathbb{R}^n \times \mathbb{R}^n) \setminus ((\mathbb{R}^n \setminus \Omega) \times (\mathbb{R}^n \setminus \Omega)). \tag{1.4}$$

A main point in (1.1) is that the (possibly degenerate) coefficients $a(\cdot, \cdot)$ and $b(\cdot, \cdot)$ make the associated integro-differential operator $\mathcal{L}(\cdot)$ switch between two different fractional elliptic phases. In this respect, our problem is closely related to the following local equation

$$\operatorname{div} \left( a(x)|Du|^{p-2}Du + b(x)|Du|^{q-2}Du \right) = 0,$$

where $1 < p < q < \infty$ and $a(\cdot), b(\cdot) \geq 0$ satisfy $\nu \leq a(x) + b(x) \leq L$ for some constants $0 < \nu \leq L < \infty$. This is called a double phase problem with two modulating coefficients. Since the pioneering works of Colombo and Mingione [15, 16], the regularity theory for local double phase problems with a single modulating coefficient (i.e., the case $a(\cdot) \equiv 1$ in the equation above) has been extensively developed; see for instance [2,3,17,19] and the references therein. More general classes of problems, including the double phase type, have subsequently been investigated in [27, 30]. In particular, [30] studies various regularity results for equations of the form above, including the case in which $a(\cdot)$ may be degenerate, meaning that $a(\cdot)$ can vanish. A key idea in [30] is the following observation: when $a(x)$ is close to zero, the equation can be regarded as a $q$-Laplace equation with a lower-order perturbation of $p$-Laplace type, whereas when $a(x)$ stays away from zero, the equation behaves genuinely as a double phase problem with a single modulating coefficient.

We now turn to regularity results for nonlinear nonlocal equations. The De Giorgi–Nash–Moser theory for the fractional $p$-Laplacian was first investigated by Di Castro, Kuusi and Palatucci [21, 22]; they proved local regularity and Harnack inequalities for weak solutions to fractional $p$-Laplacian type equations involving measurable kernels. Cozzi [18] extended such results to minimizers of non-differentiable functionals with lower-order dependencies, via a slightly different approach using fractional De Giorgi classes. We also refer to [4,33] and references therein for similar results concerned with more general equations.

Very recently, the results and techniques in [18, 21, 22] were further developed and extended to nonlocal problems with nonstandard growth and/or differentiability conditions. Similarly to the case of local problems, one can consider several typical examples, such as Orlicz growth [5, 13], variable powers [12, 34], and double phase [9, 20, 24]. These papers are concerned with local boundedness and Hölder continuity of weak solutions. Moreover, in the case of Orlicz growth, Harnack inequalities are proved in [6, 14]. We also mention the paper [35] concerned with regularity results and Harnack inequalities for nonlinear nonlocal equations with kernels of general differentiability order.

Nonlocal double phase problems with one modulating coefficient, whose prototype is

$$\text{P.V.} \int_{\mathbb{R}^n} \left( \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{n+sp}} + b(x,y)\frac{|u(x)-u(y)|^{q-2}(u(x)-u(y))}{|x-y|^{n+tq}} \right) dy = 0 \tag{1.5}$$

with $s, t \in (0,1)$ and $1 < p \leq q < \infty$, were first considered by De Filippis and Palatucci [20]. More precisely, they proved Hölder continuity of viscosity solutions for nonhomogeneous equations with bounded source terms. We also refer to [7,24,36] for various results for (1.5). The papers [7,20,24,36] consider solutions bounded in $\mathbb{R}^n$ and are under the assumption that $t \leq s$, which implies that the second term in (1.5) plays a role as a lower order term. Under these settings, $b(\cdot,\cdot)$ is imposed to be bounded and possibly discontinuous.

On the contrary, in the paper [9], joint with Byun, the second and third authors of this paper proved the local boundedness and Hölder continuity of weak solutions to (1.5) in the case $s \leq t$, under natural assumptions on the powers and the modulating coefficients. Specifically, it was proved that if

$$b \in C^{0,\alpha}(\mathbb{R}^n \times \mathbb{R}^n) \ \text{ for some } \ \alpha \in (0,1] \quad \text{and} \quad tq \leq sp + \alpha,$$

then every locally bounded weak solution to (1.5) is locally Hölder continuous. We emphasize that when $s \leq t$, the second term in (1.2) or (1.5) has a higher order and, as in the local case, the interplay between the Hölder regularity of $b(\cdot,\cdot)$ and the growth/differentiability condition of the problem plays a key role in the analysis. Moreover, in light of the Lavrentiev type phenomena considered in [1], the above assumption is essentially sharp.

In this paper, we prove the local boundedness and Hölder continuity of weak solutions to the general Eq (1.1). Moreover, in the case of one modulating coefficient (i.e. when $a(\cdot,\cdot) \equiv 1$), we further prove a Harnack inequality for weak solutions to (1.1). To our knowledge, each of our regularity results is the first one for (1.1); moreover, our Harnack inequality given in Theorem 1.4 below is the first one for nonlocal equations of double phase type.

### *1.1. Assumptions and main results*

We say that a function $F : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is symmetric if $F(x,y) = F(y,x)$ for every $x, y \in \mathbb{R}^n$.

The kernels $K_{sp}, K_{tp} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ are measurable, symmetric functions that satisfy

$$\frac{\nu}{|x-y|^{n+sp}} \leq K_{sp}(x,y) \leq \frac{L}{|x-y|^{n+sp}}, \qquad \frac{\nu}{|x-y|^{n+tp}} \leq K_{tp}(x,y) \leq \frac{L}{|x-y|^{n+tp}} \tag{1.6}$$

for a.e. $x, y \in \mathbb{R}^n$ with $x \neq y$, where $0 < \nu \leq 1 \leq L < \infty$ and

$$0 < s < t < 1 < p < \infty. \tag{1.7}$$

The modulating coefficients $a, b : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ are nonnegative, measurable and symmetric functions that satisfy the bound

$$\nu \leq a(x,y) + b(x,y) \leq L \tag{1.8}$$

for a.e. $x, y \in \mathbb{R}^n$.

In addition, for Hölder regularity and Harnack inequality, we further assume the Hölder continuity of $b(\cdot,\cdot)$ as follows: There exist two numbers $[b]_\alpha \geq 0$ and $\alpha \in (0,1]$ such that

$$|b(x_1,y_1) - b(x_2,y_2)| \leq [b]_\alpha(|x_1 - x_2| + |y_1 - y_2|)^\alpha \tag{1.9}$$

for every $(x_1,y_1),(x_2,y_2) \in \mathbb{R}^n \times \mathbb{R}^n$, and

$$(t-s)p \leq \alpha. \tag{1.10}$$

**Remark 1.1.** For any ball $B_r \subset \Omega$, we set

$$b^+_{B_r} := \sup_{x,y\in B_r} b(x,y), \qquad b^-_{B_r} := \inf_{x,y\in B_r} b(x,y).$$

Moreover, when (1.9) is in force, we choose a small radius $R_0 \equiv R_0(\nu,\alpha,[b]_\alpha) \in (0,1)$ satisfying

$$[b]_\alpha(2R_0)^\alpha \leq \frac{\nu}{8}. \tag{1.11}$$

We observe that, when $r \leq R_0$,

$$b^+_{B_r} \leq \frac{\nu}{4} \implies \frac{3\nu}{4} \leq a(\cdot,\cdot) \leq L \text{ in } B_r \times B_r, \tag{1.12}$$

$$b^+_{B_r} > \frac{\nu}{4} \implies b^+_{B_r} \overset{(1.9)}{\leq} b^-_{B_r} + [b]_\alpha(2r)^\alpha \overset{(1.11)}{\leq} 2b^-_{B_r}. \tag{1.13}$$

Accordingly, if we set

$$G_{B_r}(\tau) := \sup_{x,y\in B_r}\left(a(x,y)\frac{\tau^p}{r^{sp}} + b(x,y)\frac{\tau^p}{r^{tp}}\right), \qquad g_{B_r}(\tau) := \frac{G_{B_r}(\tau)}{\tau} \qquad (\tau \geq 0), \tag{1.14}$$

then we have

$$b^+_{B_r} \leq \frac{\nu}{4} \implies G_{B_r}(\tau) \approx \frac{\tau^p}{r^{sp}} + b^+_{B_r}\frac{\tau^p}{r^{tp}} \approx \frac{\tau^p}{r^{sp}} + b^-_{B_r}\frac{\tau^p}{r^{tp}}, \tag{1.15}$$

$$b^+_{B_r} > \frac{\nu}{4} \implies G_{B_r}(\tau) \approx \frac{\tau^p}{r^{tp}}. \tag{1.16}$$

We next introduce the nonlocal tail, which is one of the essential tools in analyzing local regularity for fractional equations. For $f : \mathbb{R}^n \to \mathbb{R}$, $x_0 \in \mathbb{R}^n$ and $r, \rho > 0$, we denote

$$T(f; x_0, r, \rho) := \sup_{x\in B_\rho(x_0)} \int_{\mathbb{R}^n\setminus B_r(x_0)} \left(a(x,y)\frac{|f(y)|^{p-1}}{|y-x_0|^{n+sp}} + b(x,y)\frac{|f(y)|^{p-1}}{|y-x_0|^{n+tp}}\right) dy. \tag{1.17}$$

We will omit the point $x_0$ if it is clear from context.

With the definitions of weak solutions and relevant function spaces to be introduced in the next section, here we state our main results.

**Theorem 1.2** (Local boundedness). *Let* $u \in \mathcal{A}(\Omega) \cap \mathcal{T}(\mathbb{R}^n)$ *be a weak subsolution to* (1.1) *under assumptions* (1.6)–(1.8), *and let* $B_r \Subset \Omega$ *be a ball with* $r \leq 1$.

*(1) We have*

$$\operatorname*{ess\,sup}_{B_{r/2}} u \leq c\, r^{-(t-s)/\beta} \left( \fint_{B_r} u_+^p \, dx \right)^{1/p} + [r^{sp} T(u_+; r/2, r)]^{1/(p-1)} \tag{1.18}$$

*for a constant* $c \equiv c(n, s, t, p, \nu, L) > 0$, *where*

$$\beta := \min\{sp/n, p-1\}. \tag{1.19}$$

*(2) Assume further that* $b(\cdot, \cdot)$ *satisfies* (1.9) *and* (1.10). *Then we have*

$$\begin{aligned} \operatorname*{ess\,sup}_{B_{r/2}} u &\leq c \left( \fint_{B_r} u_+^p \, dx \right)^{1/p} + \left[ \frac{1}{G_{B_r}(1)} T(u_+; r/2, r) \right]^{1/(p-1)} \\ &= c \left( \fint_{B_r} u_+^p \, dx \right)^{1/p} + g_{B_r}^{-1}(T(u_+; r/2, r)) \end{aligned} \tag{1.20}$$

*whenever* $r \leq R_0$, *where* $R_0$ *is given in* (1.11) *and* $c \equiv c(n, s, t, p, \nu, L, \alpha, [b]_\alpha) > 0$.

*Consequently, if* $u \in \mathcal{A}(\Omega) \cap \mathcal{T}(\mathbb{R}^n)$ *is a weak solution to* (1.1), *then* $u \in L^\infty_{\mathrm{loc}}(\Omega)$ *and estimate* (1.18) *or* (1.20) *holds with* $|u|$ *in each case.*

**Theorem 1.3** (Hölder regularity). *Let* $u \in \mathcal{A}(\Omega) \cap \mathcal{T}(\mathbb{R}^n)$ *be a weak solution to* (1.1) *under assumptions* (1.6)–(1.10). *Then there exists an exponent* $\gamma \equiv \gamma(n, s, t, p, \nu, L, \alpha, [b]_\alpha) \in (0, 1)$ *such that* $u \in C^{0,\gamma}_{\mathrm{loc}}(\Omega)$. *Moreover, for any ball* $B_r \Subset \Omega$ *with* $r \leq R_0$, *where* $R_0$ *is given in* (1.11), *we have*

$$[u]_{C^{0,\gamma}(B_{r/2})} \leq c r^{-\gamma} \left[ \left( \fint_{B_r} |u|^p \, dx \right)^{1/p} + g_{B_r}^{-1}(T(u; r/2, r)) \right] \tag{1.21}$$

*for a constant* $c \equiv c(n, s, t, p, \nu, L, \alpha, [b]_\alpha) > 0$.

**Theorem 1.4** (Harnack inequality). *Let* $u \in \mathcal{A}(\Omega) \cap \mathcal{T}(\mathbb{R}^n)$ *be a weak solution to* (1.1) *under assumptions* (1.6)–(1.10) *with* $a(\cdot, \cdot) \equiv 1$. *If* $u$ *is nonnegative in a ball* $B_{10r} \Subset \Omega$ *with* $10r \leq R_0$, *where* $R_0$ *is given in* (1.11), *then we have*

$$\sup_{B_r} u \leq c \inf_{B_r} u + c g_{B_r}^{-1}(T(u_-; 10r, 10r)) \tag{1.22}$$

*for a constant* $c \equiv c(n, s, t, p, \nu, L, \alpha, [b]_\alpha) > 0$, *where* $u_- := \max\{-u, 0\}$.

**Remark 1.5.** In Theorem 1.4, the additional assumption $a(\cdot, \cdot) \equiv 1$ is concerned with the tail estimate given in Lemma 5.4 below. Indeed, even if $a(\cdot, \cdot) \not\equiv 1$, we can prove an estimate similar to (1.22) which involves nonlocal tails of $u$ instead of $u_-$. However, in view of the natural Harnack inequalities obtained in [18, 21, 28], we confine ourselves to the case $a(\cdot, \cdot) \equiv 1$ in Theorem 1.4. We also remark that Theorem 1.4 continues to hold under the slightly weaker assumption $\inf_{x,y \in \mathbb{R}^n} a(x, y) > 0$.

### 1.2. Approach

Our proofs of Theorems 1.2 and 1.3 are based on the Moser type approach in [22] which employs a logarithmic estimate; the extension of this approach to the double phase setting was first presented in [9]. Moreover, for the Harnack inequality in Theorem 1.4, we additionally modify and develop the techniques used in [21] adapted to our problem. We stress that, in obtaining both regularity results and Harnack inequalities for (1.1), these processes exhibit several difficulties which did not appear in [9]. First, while we only had to consider the case $sp \leq n$ in [9], here we also treat the case $sp > n$ as well; note that in the latter case we automatically obtain the local boundedness of weak solutions via the fractional Sobolev–Morrey embedding. However, in both cases, we have to obtain explicit and precise local sup-estimates which extend the one in [22, Theorem 1.1]. Second, since we are dealing with weak solutions which are not necessarily bounded in $\mathbb{R}^n$, we need a delicate analysis to handle the nonlocal tails in proceeding towards the proof of our Harnack inequality. More precisely, with the same spirit as in the case of fractional $p$-Laplacian [18,21], we aim to prove a Harnack inequality involving the tail of $u_-$ only, which reduces to the classical Harnack inequality when considering solutions nonnegative in the whole $\mathbb{R}^n$. This is indeed the most problematic issue in the case of anisotropic nonlocal problems. In order to address this issue, we establish a weak Harnack type estimate (Lemma 5.3) and a tail estimate (Lemma 5.4), and then combine them with the local sup-estimate (1.20). All of these estimates encode the long-range interactions of solutions in an optimal way.

In our proofs, we utilize assumptions (1.6)–(1.10) to deal with the two modulating coefficients $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ in (1.1). Specifically, with $B_r \subset \Omega$ being a fixed ball, we divide cases as follows:

(i) If $b^+_{B_r} \leq \nu/4$, then (1.12) implies that $a(x,y)$ satisfies uniform ellipticity in the sense that the first term $a(x,y)K_{sp}(x,y) \approx K_{sp}(x,y)$ in $B_r \times B_r$, while the second term $b(x,y)K_{tp}(x,y)$ has a higher order with $b(x,y)$ being possibly degenerate in $B_r \times B_r$. As a consequence, the Eq (1.1) features a nonlocal double phase structure analogous to (1.5) with $p = q$.

(ii) If $b^+_{B_r} > \nu/4$, then (1.13) implies that $b(x,y)$ satisfies uniform ellipticity in $B_r \times B_r$ with ellipticity ratio $2b^-_{B_r}/b^-_{B_r} = 2$. In other words, the term $b(x,y)K_{tp}(x,y) \approx K_{tp}(x,y)$ becomes a dominating term and $a(x,y)K_{sp}(x,y)$ is regarded as a lower order term in $B_r \times B_r$. In this case, the Eq (1.1) becomes a $(t,p)$-Laplacian type equation with an $(s,p)$-Laplacian type lower-order term.

The remaining part of this paper is organized as follows. In the next section, we introduce basic notations and function spaces which will be used throughout this paper. In Section 3, we employ a Caccioppoli type estimate to prove Theorem 1.2. After obtaining a logarithmic lemma and an expansion of positivity lemma in Section 4, we finally prove Theorems 1.3 and 1.4 in Section 5.

## 2. Preliminaries

### 2.1. Notation and function spaces

We denote by $c$ a generic positive constant, whose specific value may vary from line to line. We denote its dependencies in parentheses when needed, using the abbreviation

$$\mathtt{data} := (n, s, t, p, \nu, L, \alpha, [b]_\alpha),$$

where $\alpha$ and $[b]_\alpha$ are given in (1.9). For numbers $A, B > 0$, we write $A \lesssim B$ if $A \leq cB$ holds for a constant $c > 1$ depending only on data. Moreover, we write $A \approx B$ if $A \lesssim B$ and $B \lesssim A$.

As usual, $B_r(x_0) := \{x \in \mathbb{R}^n : |x - x_0| < r\}$ is the open ball in $\mathbb{R}^n$ with center $x_0 \in \mathbb{R}^n$ and radius $r > 0$. We omit the center when it is clear in the context. Moreover, unless otherwise stated, different balls in the same context are concentric.

For a real-valued function $f$, we write $f_\pm := \max\{\pm f, 0\}$. If $f$ is integrable over a measurable set $U$ with $0 < |U| < \infty$, we denote its integral average over $U$ by

$$(f)_U := \fint_U f\,dx := \frac{1}{|U|}\int_U f\,dx.$$

For any open set $U \subseteq \mathbb{R}^n$, $s \in (0,1)$ and $p \geq 1$, the fractional Sobolev space $W^{s,p}(U)$ is the set of all functions $f \in L^p(U)$ satisfying

$$\|f\|_{W^{s,p}(U)} := \left(\int_U |f|^p\,dx\right)^{1/p} + \left(\int_U\int_U \frac{|f(x) - f(y)|^p}{|x-y|^{n+sp}}\,dx\,dy\right)^{1/p} < \infty.$$

We always assume that $s$, $t$, and $p$ satisfy (1.7) and that $K_{sp}, K_{tp}, a, b : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ satisfy (1.6) and (1.8). Recalling (1.4), we define the admissible set $\mathcal{A}(\Omega)$ of (1.3) by saying that $f \in \mathcal{A}(\Omega)$ if and only if

$$f|_\Omega \in L^p(\Omega) \ \text{ and } \ \iint_{C_\Omega}\left(a(x,y)\frac{|f(x)-f(y)|^p}{|x-y|^{n+sp}} + b(x,y)\frac{|f(x)-f(y)|^p}{|x-y|^{n+tp}}\right)dx\,dy < \infty,$$

where $C_\Omega$ is defined in (1.4). In particular, by (1.8) and Lemma 2.1 below, we have

$$f \in \mathcal{A}(\Omega) \implies f|_\Omega \in W^{s,p}(\Omega).$$

Accordingly, we say that a function $u \in \mathcal{A}(\Omega)$ is a weak solution to (1.1) if

$$\begin{aligned}\iint_{C_\Omega} & a(x,y)|u(x)-u(y)|^{p-2}(u(x)-u(y))(\varphi(x)-\varphi(y))K_{sp}(x,y)\,dx\,dy \\ &+ \iint_{C_\Omega} b(x,y)|u(x)-u(y)|^{p-2}(u(x)-u(y))(\varphi(x)-\varphi(y))K_{tp}(x,y)\,dx\,dy = 0\end{aligned} \tag{2.1}$$

holds for every $\varphi \in \mathcal{A}(\Omega)$ with $\varphi = 0$ a.e. in $\mathbb{R}^n \setminus \Omega$. In a similar way, we say that $u \in \mathcal{A}(\Omega)$ is a weak subsolution (resp. supersolution) if (2.1) with "=" replaced by "≤ (resp. ≥)" holds for every $\varphi \in \mathcal{A}(\Omega)$ satisfying $\varphi \geq 0$ a.e. in $\mathbb{R}^n$ and $\varphi = 0$ a.e. in $\mathbb{R}^n \setminus \Omega$. The existence of weak solutions to (1.1) (coupled with a suitable Dirichlet boundary condition) can be proved via direct methods in the calculus of variations, see [9, Section 3].

Recalling the definition of nonlocal tail in (1.17), we also consider the tail space

$$\mathcal{T}(\mathbb{R}^n) := \left\{f \in L^{p-1}_{\mathrm{loc}}(\mathbb{R}^n) : T(f; x_0, r, \rho) \text{ is finite for any } x_0 \in \mathbb{R}^n, r > 0, \rho > 0\right\}.$$

It is not difficult to see that our tail space includes the standard one:

$$L^{p-1}_{sp}(\mathbb{R}^n) := \left\{f \in L^{p-1}_{\mathrm{loc}}(\mathbb{R}^n) : \int_{\mathbb{R}^n}\frac{|f(x)|^{p-1}}{(1+|x|)^{n+sp}}\,dx < \infty\right\} \subset \mathcal{T}(\mathbb{R}^n),$$

see for instance [9, Section 2.1]. In particular, if $f \in L^q(\mathbb{R}^n)$ for some $q > p-1$, or if $f \in L^{p-1}(B_R) \cap L^\infty(\mathbb{R}^n \setminus B_R)$ for a ball $B_R$, then $f \in \mathcal{T}(\mathbb{R}^n)$.

### *2.2. Technical results*

We recall several inequalities concerning fractional Sobolev functions. For more on fractional Sobolev spaces, we refer to [23].

**Lemma 2.1** ( [9, Lemma 2.2])**.** *Assume that $s$, $t$ and $p$ satisfy* (1.7). *If $f \in W^{t,p}(U)$ for a bounded open set $U \subset \mathbb{R}^n$, then*

$$\left(\int_U\int_U \frac{|f(x)-f(y)|^p}{|x-y|^{n+sp}}\,dx\,dy\right)^{1/p} \le c(\mathrm{diam}(U))^{t-s}\left(\int_U\int_U \frac{|f(x)-f(y)|^p}{|x-y|^{n+tp}}\,dx\,dy\right)^{1/p}$$

*holds for a constant $c \equiv c(n,s,t,p) > 0$.*

**Lemma 2.2** ( [34, Lemma 2.5])**.** *Let $s \in (0,1)$ and $p \ge 1$; denote the $s$-fractional Sobolev conjugate of $p$ by*

$$p_s^* := \begin{cases} np/(n-sp) & \text{when } \; sp < n, \\ \text{any number in } (p,\infty) & \text{when } \; sp \ge n. \end{cases}$$

*Then we have*

$$\left(\fint_{B_r} |f-(f)_{B_r}|^{p_s^*}\,dx\right)^{p/p_s^*} \le c r^{sp} \fint_{B_r}\int_{B_r} \frac{|f(x)-f(y)|^p}{|x-y|^{n+sp}}\,dx\,dy$$

*for any $f \in W^{s,p}(B_r)$, where $c \equiv c(n,s,p) > 0$.*

The following lemma can be proved in the same way as in [8, Lemma 2.2] and [9, Lemma 2.4].

**Lemma 2.3.** *Assume that $s$, $t$ and $p$ satisfy* (1.7). *Then for any $f \in L^p(B_r)$ and any $a_0, b_0 \ge 0$, we have*

$$\begin{aligned} &\fint_{B_r}\left(a_0\frac{|f|^p}{r^{sp}} + b_0\frac{|f|^p}{r^{tp}}\right)dx \\ &\le c\left(\frac{|\mathrm{supp}\, f|}{|B_r|}\right)^{sp/n} \fint_{B_r}\int_{B_r}\left(a_0\frac{|f(x)-f(y)|^p}{|x-y|^{n+sp}} + b_0\frac{|f(x)-f(y)|^p}{|x-y|^{n+tp}}\right)dx\,dy \\ &\quad + c\left(\frac{|\mathrm{supp}\, f|}{|B_r|}\right)^{p-1}\fint_{B_r}\left(a_0\frac{|f|^p}{r^{sp}} + b_0\frac{|f|^p}{r^{tp}}\right)dx \end{aligned}$$

*whenever the right-hand side is finite, where $c \equiv c(n,s,t,p) > 0$.*

*Proof.* Using Hölder's inequality and Lemma 2.2, we have

$$\begin{aligned} \fint_{B_r}\frac{|f|^p}{r^{sp}}\,dx &\le c\left(\frac{|\mathrm{supp} f|}{|B_r|}\right)^{sp/n}\left(\fint_{B_r}\left|\frac{f-(f)_{B_r}}{r^s}\right|^{p_s^*}dx\right)^{p/p_s^*} + c\frac{|(f)_{B_r}|^p}{r^{sp}} \\ &\le c\left(\frac{|\mathrm{supp} f|}{|B_r|}\right)^{sp/n}\fint_{B_r}\int_{B_r}\frac{|f(x)-f(y)|^p}{|x-y|^{n+sp}}\,dx\,dy + c\left(\frac{|\mathrm{supp} f|}{|B_r|}\right)^{p-1}\fint_{B_r}\frac{|f|^p}{r^{sp}}\,dx, \end{aligned}$$

whenever the right-hand side is finite; the same is true when $s$ is replaced by $t$. □

Finally, we recall standard iteration lemmas.

**Lemma 2.4** ( [26, Lemma 7.1]). *Let* $\{y_i\}_{i=0}^{\infty}$ *be a sequence of nonnegative numbers satisfying*

$$y_{i+1} \le b_1 b_2^i y_i^{1+\beta}, \qquad i = 0, 1, 2, \dots$$

*for some constants* $b_1, \beta > 0$ *and* $b_2 > 1$. *If*

$$y_0 \le b_1^{-1/\beta} b_2^{-1/\beta^2},$$

*then* $y_i \to 0$ *as* $i \to \infty$.

**Lemma 2.5** ( [26, Lemma 6.1]). *Let* $Z : [r_0, r_1] \to \mathbb{R}$ *be a nonnegative and bounded function; let* $\vartheta \in (0, 1)$ *and* $C_1, C_2, \chi > 0$ *be numbers. Assume that*

$$Z(\varrho_0) \le \vartheta Z(\varrho_1) + \frac{C_1}{(\varrho_1 - \varrho_0)^{\chi}} + C_2$$

*holds for every choice of* $\varrho_0$ *and* $\varrho_1$ *such that* $r_0 \le \varrho_0 < \varrho_1 \le r_1$. *Then the following inequality holds with* $c \equiv c(\vartheta, \chi) > 0$*:*

$$Z(r_0) \le c\left[\frac{C_1}{(r_1 - r_0)^{\chi}} + C_2\right].$$

## 3. Proof of Theorem 1.2

The following Caccioppoli type estimate can be proved in a standard way, see for instance [9, Lemma 4.2].

**Lemma 3.1.** *Let* $u \in \mathcal{A}(\Omega) \cap \mathcal{T}(\mathbb{R}^n)$ *be a weak solution to* (1.1) *under assumptions* (1.6)–(1.8). *There exists a constant* $c \equiv c(n, s, t, p, \nu, L) > 0$ *such that*

$$\begin{aligned} &\int_{B_\rho}\int_{B_\rho}\left(a(x,y)\frac{|w_\pm(x) - w_\pm(y)|^p}{|x-y|^{n+sp}} + b(x,y)\frac{|w_\pm(x) - w_\pm(y)|^p}{|x-y|^{n+tp}}\right)dx\,dy \\ &\le \frac{c}{(r-\rho)^p}\int_{B_r}\int_{B_r}\left(a(x,y)\frac{(w_\pm(x))^p}{|x-y|^{n+(s-1)p}} + b(x,y)\frac{(w_\pm(x))^p}{|x-y|^{n+(t-1)p}}\right)dx\,dy \\ &\quad + c\left(\frac{r}{r-\rho}\right)^{n+tp} T(w_\pm; r, r)\int_{B_r} w_\pm\,dx \end{aligned} \tag{3.1}$$

*holds whenever* $B_\rho \subset B_r \Subset \Omega$ *are concentric balls, where* $w_\pm := (u-k)_\pm$ *for any* $k \in \mathbb{R}$. *Moreover, estimate* (3.1) *continues to hold for* $w_+$ *(resp.* $w_-$*) when* $u$ *is merely a weak subsolution (resp. supersolution) to* (1.1).

We now prove Theorem 1.2.

*Proof of Theorem 1.2.* Let $B_r \equiv B_r(x_0) \Subset \Omega$ be a fixed ball with $r \le 1$. For $r/2 \le \rho < \sigma \le r$ and $k > 0$, we denote

$$A^+(k, \rho) := \{x \in B_\rho : u(x) \ge k\}.$$

Note that for $0 < h < k$ and $x \in A^+(k, \rho) \subset A^+(h, \rho)$,

$$(u(x) - h)_+ = u(x) - h \ge k - h,$$

$$(u(x)-h)_+ = u(x)-h \geq u(x)-k = (u(x)-k)_+.$$

Thus, we have

$$|A^+(k,\rho)| \leq \int_{A^+(k,\rho)} \frac{(u-h)_+^p}{(k-h)^p}\,dx \leq \frac{1}{(k-h)^p}\int_{A^+(h,\sigma)} (u-h)_+^p\,dx \tag{3.2}$$

and

$$\int_{B_\sigma} (u-k)_+\,dx \leq \int_{B_\sigma} (u-h)_+\left(\frac{(u-h)_+}{k-h}\right)^{p-1} dx \leq \frac{1}{(k-h)^{p-1}}\int_{B_\sigma} (u-h)_+^p\,dx. \tag{3.3}$$

*Step 1: Proof of* (1.18).

We first prove (1.18) in the case that $b(\cdot,\cdot)$ is merely measurable. Applying Lemma 2.3 with $f \equiv (u-k)_+$, $a_0 = 1$ and $b_0 = 0$, we get

$$\begin{aligned} &\fint_{B_\rho} (u-k)_+^p\,dx \\ &\leq c\left(\frac{|A^+(k,\rho)|}{|B_\rho|}\right)^{\beta}\left(\rho^{sp}\fint_{B_\rho}\int_{B_\rho} \frac{|(u(x)-k)_+-(u(y)-k)_+|^p}{|x-y|^{n+sp}}\,dx\,dy + \fint_{B_\sigma}(u-k)_+^p\,dx\right), \end{aligned} \tag{3.4}$$

where $\beta$ is given in (1.19). Observe that, since $r \leq 1$,

$$1 \leq \frac{1}{\nu}(a(x,y)+b(x,y)) \leq \frac{1}{\nu}\left(a(x,y)+b(x,y)\frac{2^{(t-s)p}}{|x-y|^{(t-s)p}}\right) \qquad \text{for any } x,y \in B_r.$$

Therefore, an application of Lemma 3.1 gives

$$\begin{aligned} &\fint_{B_\rho}\int_{B_\rho} \frac{|(u(x)-k)_+-(u(y)-k)_+|^p}{|x-y|^{n+sp}}\,dx\,dy \\ &\leq \frac{c}{(\sigma-\rho)^p}\fint_{B_\sigma}(u(x)-h)_+^p\int_{B_\sigma}\left(\frac{1}{|x-y|^{n+(s-1)p}}+\frac{1}{|x-y|^{n+(t-1)p}}\right)dy\,dx \\ &\quad + c\left(\frac{\sigma}{\sigma-\rho}\right)^{n+tp} T((u-k)_+;\sigma,\sigma)\fint_{B_\sigma}(u-k)_+\,dx \\ &\leq \frac{cr^{(1-t)p}}{(\sigma-\rho)^p}\fint_{B_\sigma}(u-h)_+^p\,dx + \frac{cr^{n+tp}}{(\sigma-\rho)^{n+tp}}T((u-k)_+;r/2,r)\fint_{B_\sigma}(u-k)_+\,dx. \end{aligned}$$

Combining this estimate together with (3.2)–(3.4), we arrive at

$$\begin{aligned} \fint_{B_\rho}(u-k)_+^p\,dx &\leq c\left(\fint_{B_\sigma}\left[\frac{(u-h)_+}{k-h}\right]^p dx\right)^{\beta}\fint_{B_\sigma}(u-h)_+^p\,dx \\ &\quad \cdot\left[\frac{r^{p+(s-t)p}}{(\sigma-\rho)^p}+\frac{r^{n+tp}}{(\sigma-\rho)^{n+tp}}\frac{r^{sp}}{(k-h)^{p-1}}T((u-k)_+;r/2,r)+1\right]. \end{aligned} \tag{3.5}$$

Now, for $i = 0,1,2,\ldots$ and $k_0 > 0$ with

$$k_0 \geq \left[r^{sp}T(u_+;r/2,r)\right]^{1/(p-1)}, \tag{3.6}$$

define

$$\sigma_i := \frac{r}{2}(1+2^{-i}) \quad \text{and} \quad k_i := 2k_0(1-2^{-i-1}).$$

We choose $k = k_{i+1}$, $h = k_i$, $\rho = \sigma_{i+1}$, and $\sigma = \sigma_i$ in (3.5). Then, dividing both sides of the resulting inequality by $k_0^p$, we arrive at

$$\fint_{B_{\sigma_{i+1}}} \left[\frac{(u-k_{i+1})_+}{k_0}\right]^p dx \le c2^{i(\beta p+n+tp+p-1)} r^{(s-t)p} \left( \fint_{B_{\sigma_i}} \left[\frac{(u-k_i)_+}{k_0}\right]^p dx \right)^{1+\beta}.$$

Setting

$$y_i := \frac{1}{|B_r|} \int_{A^+(k_i,\sigma_i)} \left[\frac{(u-k_i)_+}{k_0}\right]^p dx, \quad i = 0,1,2,\ldots,$$

the above inequality becomes

$$y_{i+1} \le \tilde{c} r^{(s-t)p} 2^{i(\beta p+n+tp+p-1)} y_i^{1+\beta}$$

for some $\tilde{c} \equiv \tilde{c}(n,s,t,p,\nu,L) > 0$. Therefore, if

$$y_0 = \frac{1}{k_0^p} \fint_{B_r} (u-k_0)_+^p \, dx \le \tilde{c}^{-1/\beta} r^{(t-s)p/\beta} 2^{-(\beta p+n+tp+p-1)/\beta^2}, \tag{3.7}$$

we have that $y_i \to 0$ as $i \to \infty$ by Lemma 2.4, which together with the fact that $\lim_{i\to\infty} \sigma_i = r/2$ implies

$$u \le 2k_0 \quad \text{a.e. in } B_{r/2}.$$

Note that the conditions (3.6) and (3.7) of $k_0$ are satisfied if we choose $k_0 > 0$ such that

$$k_0 = cr^{-(t-s)/\beta} \left( \fint_{B_r} u_+^p \, dx \right)^{1/p} + [r^{sp} T(u_+; r/2, r)]^{1/(p-1)}$$

for some large constant $c \equiv c(n,s,t,p,\nu,L) > 0$. Therefore, we obtain (1.18).

We now assume that $b(\cdot,\cdot)$ satisfies (1.9) and (1.10). To prove (1.20), we distinguish two cases.

*Step 2: Proof of* (1.20) *in the case* (1.13).

Assume that $r \le R_0$. We apply Lemma 2.3 with $f \equiv (u-k)_+$, $a_0 = 0$ and $b_0 = 1$, which gives

$$\begin{aligned} \fint_{B_\rho} (u-k)_+^p \, dx \le{}& c\rho^{tp} \left( \frac{|A^+(k,\rho)|}{|B_\rho|} \right)^{sp/n} \fint_{B_\rho} \int_{B_\rho} \frac{|(u(x)-k)_+ - (u(y)-k)_+|^p}{|x-y|^{n+tp}} \, dx\, dy \\ &+ c \left( \frac{|A^+(k,\rho)|}{|B_\rho|} \right)^{p-1} \fint_{B_\sigma} (u-k)_+^p \, dx. \end{aligned} \tag{3.8}$$

Since $b(x,y) \ge \nu/4$ for $x,y \in B_r$ by (1.13), an application of Lemma 3.1 gives

$$\begin{aligned} &\fint_{B_\rho} \int_{B_\rho} \frac{|(u(x)-k)_+ - (u(y)-k)_+|^p}{|x-y|^{n+tp}} \, dx\, dy \\ &\le \frac{cr^{(1-t)p}}{(\sigma-\rho)^p} \fint_{B_\sigma} (u-h)_+^p \, dx + \frac{cr^{n+tp}}{(\sigma-\rho)^{n+tp}} T((u-k)_+; r/2, r) \fint_{B_\sigma} (u-k)_+ \, dx. \end{aligned}$$

Combining this estimate together with (3.2), (3.3), (3.8), and recalling $G_{B_r}(1) \approx r^{-tp}$ from (1.15), we deduce

$$\begin{aligned}\fint_{B_\rho}(u-k)_+^p\,dx \le c\left(\fint_{B_\sigma}\left[\frac{(u-h)_+}{k-h}\right]^p dx\right)^{sp/n}&\fint_{B_\sigma}(u-h)_+^p\,dx\\ &\cdot\left[\frac{r^p}{(\sigma-\rho)^p}+\frac{r^{n+tp}}{(\sigma-\rho)^{n+tp}}\frac{1}{(k-h)^{p-1}G_{B_r}(1)}T((u-k)_+;r/2,r)\right]\\ &+\frac{c}{(k-h)^{p(p-1)}}\left(\fint_{B_\sigma}(u-h)_+^p\,dx\right)^p.\end{aligned}\tag{3.9}$$

Note that the estimate (3.9) is similar to (3.5), except that the terms $r^{p+(s-t)p}$ and $r^{sp}$ are replaced by $r^p$ and $1/G_{B_r}(1)$, respectively. The remaining part of the proof is exactly the same as in *Step 1*, hence one can obtain the estimate (1.20).

*Step 3: Proof of* (1.20) *in the case* (1.12).

Here we recall the function $G_{B_r}(\cdot)$ defined in (1.14); in the following, we simply denote $G = G_{B_r}$. Observe that

$$\int_{B_\sigma}(u-k)_+\,dx \le \frac{k-h}{G(k-h)}\int_{B_\sigma}G((u-h)_+)\,dx \le \frac{c}{(k-h)^{p-1}G(1)}\int_{B_\sigma}G((u-h)_+)\,dx.\tag{3.10}$$

We use Lemma 2.3 with $f \equiv (u-k)_+$, $a_0 = 1$ and $b_0 = b^-_{B_r}$ to get

$$\begin{aligned}&\fint_{B_\rho}G((u-k)_+)\,dx \le c\fint_{B_\rho}\frac{(u-k)_+^p}{r^{sp}}+b^-_{B_r}\frac{(u-k)_+^p}{r^{tp}}\,dx\\ &\le c\left(\frac{|A^+(k,\rho)|}{|B_\rho|}\right)^{\frac{sp}{n}}\fint_{B_\rho}\int_{B_\rho}\frac{|(u(x)-k)_+-(u(y)-k)_+|^p}{|x-y|^{n+sp}}+b^-_{B_r}\frac{|(u(x)-k)_+-(u(y)-k)_+|^p}{|x-y|^{n+tp}}\,dx\,dy\\ &\quad+c\left(\frac{|A^+(k,\rho)|}{|B_\rho|}\right)^{p-1}\fint_{B_\rho}G((u-k)_+)\,dx.\end{aligned}\tag{3.11}$$

Since $a(x,y) \ge 3\nu/4$ for $x, y \in B_r$ by (1.12), an application of Lemma 3.1 gives

$$\begin{aligned}&\fint_{B_\rho}\int_{B_\rho}\left(\frac{|(u(x)-k)_+-(u(y)-k)_+|^p}{|x-y|^{n+tp}}+b^-_{B_r}\frac{|(u(x)-k)_+-(u(y)-k)_+|^p}{|x-y|^{n+tp}}\right)dx\,dy\\ &\le\frac{c}{(\sigma-\rho)^p}\fint_{B_\sigma}(u(x)-h)_+^p\int_{B_\sigma}\left(\frac{a(x,y)}{|x-y|^{n+(s-1)p}}+\frac{b(x,y)}{|x-y|^{n+(t-1)p}}\right)dy\,dx\\ &\quad+c\left(\frac{\sigma}{\sigma-\rho}\right)^{n+tp}T\left((u-k)_+;\sigma,\sigma\right)\fint_{B_\sigma}(u-k)_+\,dx\\ &\le\frac{c}{(\sigma-\rho)^p}\fint_{B_\sigma}(u(x)-h)_+^p\int_{B_\sigma}\left(\frac{1}{|x-y|^{n+(s-1)p}}+\frac{b^+_{B_r}}{|x-y|^{n+(t-1)p}}\right)dy\,dx\\ &\quad+c\left(\frac{\sigma}{\sigma-\rho}\right)^{n+tp}T\left((u-k)_+;\sigma,\sigma\right)\fint_{B_\sigma}(u-k)_+\,dx\\ &\le\frac{cr^p}{(\sigma-\rho)^p}\fint_{B_\sigma}G((u-h)_+)\,dx+c\left(\frac{r}{\sigma-\rho}\right)^{n+tp}T((u-k)_+;r/2,r)\fint_{B_\sigma}(u-k)_+\,dx.\end{aligned}$$

Combining this estimate together with (3.2), (3.10), (3.11), and recalling $G_{B_r}(1) \approx r^{-sp} + b^+_{B_r} r^{-tp}$ from (1.16), we arrive at

$$\begin{aligned}\fint_{B_\rho} G((u-k)_+)\,dx \le{}& c\left(\fint_{B_\sigma}\left[\frac{(u-h)_+}{k-h}\right]^p dx\right)^{sp/n}\fint_{B_\sigma} G((u-h)_+)\,dx\\ &\cdot\left[\frac{r^p}{(\sigma-\rho)^p}+\frac{r^{n+tp}}{(\sigma-\rho)^{n+tp}}\frac{1}{(k-h)^{p-1}G(1)}T((u-k)_+;r/2,r)\right]\\ &+c\left(\fint_{B_\sigma}\left[\frac{(u-h)_+}{k-h}\right]^p dx\right)^{p-1}\fint_{B_\sigma} G((u-h)_+)\,dx.\end{aligned}$$

Dividing both sides of the above inequality by $r^{-sp} + b^+_{B_r} r^{-tp}$, we obtain the estimate (3.5) with $r^{p+(s-t)p}$ and $r^{sp}$ replaced by $r^p$ and $1/G(1)$, respectively. Hence we can conclude with the desired estimate (1.20), and the proof is complete. □

## 4. Expansion of positivity

Throughout this section, we assume that the modulating coefficient $b(\cdot,\cdot)$ satisfies (1.9) with (1.10). We start this section with the following logarithmic type estimate, whose proof is analogous to those of [9, Corollary 5.2] and [22, Corollary 3.2].

**Lemma 4.1.** *Let $u \in \mathcal{A}(\Omega) \cap \mathcal{T}(\mathbb{R}^n)$ be a weak supersolution to* (1.1) *under assumptions* (1.6)–(1.10)*, which is nonnegative in a ball $B_R \equiv B_R(x_0)$ with $R \le R_0$. For any $d, \zeta > 0$ and $\xi > 1$, define*

$$v := \min\{(\log(\zeta + d) - \log(u + d))_+, \log\xi\}.$$

*Then for any $r \in (0, R/2]$ and $d > 0$, we have*

$$\fint_{B_r} |v - (v)_{B_r}|\,dx \le c + cd^{1-p}\frac{1}{G_{B_{2r}}(1)}T(u_-; R, 2r) \tag{4.1}$$

*for a constant $c \equiv c(\mathtt{data}) > 0$.*

*Proof.* In the case $b^+_{B_R} \le \nu/4$, since $\nu/4 \le a(x,y) \le L$, the estimate is proved in [9, Lemma 5.1] with $q = p$. Note that, in the setting of the present paper, [9, (5.10)] can be replaced by

$$\begin{aligned}&\int_{B_{3r/2}}\int_{\mathbb{R}^n\setminus B_R}\frac{1}{g_{B_{2r}}(u(x)+d)}\left(a(x,y)\frac{(u_-(y))^{p-1}}{|x-y|^{n+sp}}+b(x,y)\frac{(u_-(y))^{p-1}}{|x-y|^{n+tp}}\right)dy\,dx\\ &\le \frac{cr^n}{g_{B_{2r}}(d)}\sup_{x\in B_{2r}}\int_{\mathbb{R}^n\setminus B_R}\left(a(x,y)\frac{(u_-(y))^{p-1}}{|y-x_0|^{n+sp}}+b(x,y)\frac{(u_-(y))^{p-1}}{|y-x_0|^{n+tp}}\right)dy\\ &= \frac{cr^n d^{1-p}}{G_{B_{2r}}(1)}T(u_-;R,2r).\end{aligned}$$

We now consider the case $b^+_{B_R} > \nu/4$. For brevity, we denote

$$\Phi_p(\tau) := |\tau|^{p-2}\tau, \qquad \tau \in \mathbb{R}. \tag{4.2}$$

Choose a cut-off function $\phi \in C_0^\infty(B_{3r/2})$ such that $0 \le \phi \le 1$, $\phi \equiv 1$ in $B_r$ and $|D\phi| \le 4/r$. Testing (2.1) with $\varphi \equiv (u+d)^{1-p}\phi^p$, we have

$$\begin{aligned}
0 \le & \int_{B_{2r}}\int_{B_{2r}} \left[a(x,y)K_{sp}(x,y) + b(x,y)K_{tp}(x,y)\right] \Phi_p(u(x)-u(y))(\varphi(x)-\varphi(y))\,dx\,dy \\
& + 2\int_{\mathbb{R}^n\setminus B_{2r}}\int_{B_{2r}} \left[a(x,y)K_{sp}(x,y) + b(x,y)K_{tp}(x,y)\right] \Phi_p(u(x)-u(y))(\varphi(x)-\varphi(y))\,dx\,dy \\
=: & \, I_1 + I_2.
\end{aligned}$$

Estimating in the same way as in [22, Lemma 1.3], we have

$$\begin{aligned}
I_1 \le & -\frac{1}{c}\int_{B_{2r}}\int_{B_{2r}} \left[a(x,y)K_{sp}(x,y) + b(x,y)K_{tp}(x,y)\right] \left|\log\left(\frac{u(x)+d}{u(y)+d}\right)\right|^p \phi^p(y)\,dx\,dy \\
& + c\int_{B_{2r}}\int_{B_{2r}} \left[a(x,y)K_{sp}(x,y) + b(x,y)K_{tp}(x,y)\right] |\phi(x)-\phi(y)|^p\,dx\,dy \\
\le & -\frac{1}{c}\int_{B_r}\int_{B_r} \frac{|\log(u(x)+d)-\log(u(y)+d)|^p}{|x-y|^{n+tp}}\,dx\,dy + cr^{n-tp}
\end{aligned}$$

and

$$\begin{aligned}
I_2 \le & \, c\int_{\mathbb{R}^n\setminus B_{2r}}\int_{B_{2r}} \left[a(x,y)K_{sp}(x,y) + b(x,y)K_{tp}(x,y)\right] \phi^p(x)\,dx\,dy \\
& + cd^{1-p}\int_{\mathbb{R}^n\setminus B_R}\int_{B_{2r}} \left[a(x,y)K_{sp}(x,y) + b(x,y)K_{tp}(x,y)\right] u_-^{p-1}(y)\,dx\,dy \\
\le & \, cr^{n-tp} + cd^{1-p}r^nT(u_-;R,2r).
\end{aligned}$$

Combining the above three displays, we get

$$\int_{B_r}\int_{B_r} \frac{|\log(u(x)+d)-\log(u(y)+d)|^p}{|x-y|^{n+tp}}\,dx\,dy \le cr^{n-tp} + cd^{1-p}r^nT(u_-;R,2r).$$

In turn, an application of the fractional Poincaré inequality gives

$$\begin{aligned}
\fint_{B_r} |v-(v)_{B_r}|\,dx \le & \fint_{B_r} |v-(v)_{B_r}|^p\,dx + 1 \\
& \le c + cd^{1-p}r^{tp}T(u_-;R,2r) \le c + \frac{cd^{1-p}}{G_{B_{2r}}(1)}T(u_-;R,2r),
\end{aligned}$$

which is (4.1). The proof is complete. □

Using the above lemma, we prove the following result concerning expansion of positivity.

**Lemma 4.2.** *Let* $u \in \mathcal{A}(\Omega)\cap\mathcal{T}(\mathbb{R}^n)$ *be a weak supersolution to* (1.1) *under assumptions* (1.6)–(1.10)*, which is nonnegative in a ball* $B_R \equiv B_R(x_0) \Subset \Omega$ *with* $R \le R_0$*. Let* $k > 0$ *and assume that there exists* $\sigma \in (0,1]$ *satisfying*

$$|B_{2r}\cap\{u \ge k\}| \ge \sigma|B_{2r}| \quad \textit{for some} \quad r \in (0,R/4]. \tag{4.3}$$

*Then there exists a constant* $\delta \equiv \delta(\mathtt{data},\sigma) \in (0,1/4)$ *such that if*

$$d := \left[\frac{1}{G_{B_{2r}}(1)}T(u_-;R,2r)\right]^{1/(p-1)} = g_{B_{2r}}^{-1}(T(u_-;R,2r)) \le \delta k, \tag{4.4}$$

*then*

$$\operatorname*{ess\,inf}_{B_r} u \geq \delta k. \tag{4.5}$$

*Proof.* We divide the proof into two steps.

*Step 1: A density estimate.*

We first show that

$$\frac{|B_{2r} \cap \{u \leq 2\delta k\}|}{|B_{2r}|} \leq \frac{\bar{c}}{\sigma \log(1/3\delta)} \tag{4.6}$$

holds for any $\delta \in (0, 1/4)$, where $\bar{c} \equiv \bar{c}(\mathtt{data}) > 0$. Using Lemma 4.1 with the choice

$$v := \left[\min\left\{\log\frac{1}{3\delta}, \log\frac{k+d}{u+d}\right\}\right]_+,$$

we have

$$\fint_{B_{2r}} |v - (v)_{B_{2r}}|\, dx \leq c(\mathtt{data}). \tag{4.7}$$

Now, since assumption (4.3) is equivalent to

$$\frac{|B_{2r} \cap \{v = 0\}|}{|B_{2r}|} \geq \sigma,$$

a direct modification gives

$$\log\frac{1}{3\delta} = \frac{1}{|B_{2r} \cap \{v = 0\}|}\int_{B_{2r}\cap\{v=0\}} \left(\log\frac{1}{3\delta} - v\right) dx \leq \frac{1}{\sigma}\left[\log\frac{1}{3\delta} - (v)_{B_{2r}}\right].$$

Integrating this inequality over $B_{2r} \cap \{v = \log(1/3\delta)\}$ and using (4.7), we find

$$\left|B_{2r} \cap \left\{v = \log\frac{1}{3\delta}\right\}\right| \log\frac{1}{3\delta} \leq \frac{1}{\sigma}\int_{B_{2r}} |v - (v)_{B_{2r}}|\, dx \leq \frac{\bar{c}}{\sigma}|B_{2r}|.$$

Recalling the definition of $v$, we obtain

$$|B_{2r} \cap \{u \leq 2\delta k\}| \leq \left|B_{2r} \cap \left\{v = \log\frac{1}{3\delta}\right\}\right| \leq \frac{\bar{c}}{\sigma \log(1/3\delta)}|B_{2r}|,$$

which is (4.6). We also note that $\delta$ is still free; we will fix its value in the next step.

*Step 2: A pointwise bound.*

We now prove (4.5). For each $j \in \mathbb{N} \cup \{0\}$, we set

$$\rho_j := (1 + 2^{-j})r, \qquad \tilde{\rho}_j := \frac{\rho_j + \rho_{j+1}}{2}, \qquad B_j := B_{\rho_j}(x_0)$$

and

$$\ell_j := (1 + 2^{-j})\delta k, \qquad w_j := (\ell_j - u)_+, \qquad A_j := \frac{|B_j \cap \{u < \ell_j\}|}{|B_j|}.$$

Then for any $j$, we have

$$\rho_j, \tilde{\rho}_j \in (r, 2r), \qquad d \leq \delta k \leq \ell_j \leq 2\delta k, \qquad \ell_j - \ell_{j+1} = 2^{-j-1}\delta k \geq 2^{-j-2}\ell_j \tag{4.8}$$

and

$$w_j \geq (\ell_j - \ell_{j+1})\chi_{\{u<\ell_{j+1}\}} \geq 2^{-j-2}\ell_j\chi_{\{u<\ell_{j+1}\}}. \tag{4.9}$$

Moreover, (4.6) reads as

$$A_0 = \frac{|B_0 \cap \{u < \ell_0\}|}{|B_0|} \leq \frac{\bar{c}}{\sigma \log(1/3\delta)}. \tag{4.10}$$

To proceed, we consider the following two cases separately: $b^+_{B_{2r}} \leq \nu/4$ and $b^+_{B_{2r}} > \nu/4$.

**Case 1: $b^+_{B_{2r}} \leq \nu/4$.**

In this case, we have $a^-_{B_{2r}} \geq 3\nu/4$. Recalling (1.14), we simply denote $b^+_{B_{2r}} \equiv b^+$, $G_{B_{2r}} = G$ and $g_{B_{2r}} = g$. Using (1.15), (4.8), (4.9) and Lemma 2.3 with $a_0 = 1$ and $b_0 = b^-_{B_{2r}}$, we have

$$\begin{aligned}
A_{j+1}G(2^{-j-2}\ell_j) &\leq A_{j+1}G(\ell_j - \ell_{j+1}) = \frac{1}{|B_{j+1}|}\int_{B_{j+1}\cap\{u<\ell_{j+1}\}} G(\ell_j - \ell_{j+1})\,dx \\
&\leq c\fint_{B_{j+1}}\left(\frac{w_j^p}{r^{sp}} + b^-\frac{w_j^p}{r^{tp}}\right)dx \\
&\leq cA_j^{\beta}\left[\fint_{B_{j+1}}\int_{B_{j+1}}\left(\frac{|w_j(x)-w_j(y)|^p}{|x-y|^{n+sp}} + b^-\frac{|w_j(x)-w_j(y)|^p}{|x-y|^{n+tp}}\right)dx\,dy + \fint_{B_{j+1}} G(w_j)\,dx\right],
\end{aligned} \tag{4.11}$$

where $\beta$ is given in (1.19). We then estimate the right-hand side of (4.11). It is straightforward to see that

$$\fint_{B_j} G(w_j)\,dx \leq \frac{1}{|B_j|}\int_{B_j\cap\{u<\ell_j\}} G(\ell_j)\,dx = G(\ell_j)A_j. \tag{4.12}$$

Next, we apply (3.1) to $w_j$ and $B_j$, which gives

$$\begin{aligned}
&\fint_{B_{j+1}}\int_{B_{j+1}}\left(\frac{|w_j(x)-w_j(y)|^p}{|x-y|^{n+sp}} + b^-\frac{|w_j(x)-w_j(y)|^p}{|x-y|^{n+tp}}\right)dx\,dy \\
&\leq c\fint_{B_{j+1}}\int_{B_{j+1}}\left(a(x,y)\frac{|w_j(x)-w_j(y)|^p}{|x-y|^{n+sp}} + b(x,y)\frac{|w_j(x)-w_j(y)|^p}{|x-y|^{n+tp}}\right)dx\,dy \\
&\leq c\,2^{jp}r^{-p}\fint_{B_j}\int_{B_j}\left(a(x,y)\frac{(w_j(x))^p}{|x-y|^{n+(s-1)p}} + b(x,y)\frac{(w_j(x))^p}{|x-y|^{n+(t-1)p}}\right)dx\,dy \\
&\quad + c2^{j(n+tp)}T(w_j;r_j,r_j)\fint_{B_j} w_j\,dx.
\end{aligned} \tag{4.13}$$

The first term in the right-hand side of (4.13) is estimated as

$$\begin{aligned}
&2^{jp}r^{-p}\fint_{B_j}\int_{B_j}\left(a(x,y)\frac{(w_j(x))^p}{|x-y|^{n+(s-1)p}} + b(x,y)\frac{(w_j(x))^p}{|x-y|^{n+(t-1)p}}\right)dx\,dy \\
&\leq c\,2^{jp}r^{-p}\ell_j^p\frac{1}{|B_j|}\int_{B_j\cap\{u<\ell_j\}}\int_{B_j}\frac{1}{|x-y|^{n+(s-1)p}}\,dy\,dx \\
&\quad + c\,2^{jp}r^{-p}b^+\ell_j^p\frac{1}{|B_j|}\int_{B_j\cap\{u<\ell_j\}}\int_{B_j}\frac{1}{|x-y|^{n+(t-1)p}}\,dy\,dx \\
&\leq c\,2^{jp}\frac{|B_j\cap\{u<\ell_j\}|}{|B_j|}\left(r^{-sp}\ell_j^p + b^+r^{-tp}\ell_j^p\right) \\
&\leq c\,2^{jp}G(\ell_j)A_j.
\end{aligned} \tag{4.14}$$

We then estimate the tail term. We use the facts that $w_j = (\ell_j - u)_+ \le \ell_j + u_-$ and that $u \ge 0$ in $B_R$ in order to get

$$\begin{aligned} T(w_j; r_j, r_j) &= \sup_{x\in B_j} \int_{\mathbb{R}^n \setminus B_j} \left( a(x,y) \frac{(w_j(y))^{p-1}}{|y-x_0|^{n+sp}} + b(x,y) \frac{(w_j(y))^{p-1}}{|y-x_0|^{n+tp}} \right) dy \\ &\le c \sup_{x\in B_{2r}} \int_{\mathbb{R}^n \setminus B_j} \left( a(x,y) \frac{\ell_j^{p-1}}{|y-x_0|^{n+sp}} + b(x,y) \frac{\ell_j^{p-1}}{|y-x_0|^{n+tp}} \right) dy \\ &\quad + c \sup_{x\in B_{2r}} \int_{\mathbb{R}^n \setminus B_R} \left( a(x,y) \frac{(u_-(y))^{p-1}}{|y-x_0|^{n+sp}} + b(x,y) \frac{(u_-(y))^{p-1}}{|y-x_0|^{n+tp}} \right) dy. \end{aligned}$$

For the first term in the right-hand side, we further observe that (1.9) and (1.10) imply

$$\begin{aligned} b(x,y) &\le b(x,y) - b(x,x_0) + b^+ \\ &\le |b(x,y) - b(x,x_0)|^{(t-s)p/\alpha} (2L)^{1-(t-s)p/\alpha} + b^+ \\ &\le c|y-x_0|^{(t-s)p} + b^+ \end{aligned} \tag{4.15}$$

for any $x \in B_{2r}$ and $y \in \mathbb{R}^n$, which along with (4.4) and $(4.8)_2$ leads to

$$\begin{aligned} T(w_j; r_j, r_j) &\le c\ell_j^{p-1} \int_{\mathbb{R}^n \setminus B_j} \left( \frac{1}{|y-x_0|^{n+sp}} + b^+ \frac{1}{|y-x_0|^{n+tp}} \right) dy + cT(u_-; R, 2r) \\ &\le c\Big(g(\ell_j) + g(d)\Big) \\ &\le cg(\ell_j). \end{aligned} \tag{4.16}$$

Also, by the definitions of $w_j$ and $A_j$, we directly have

$$\fint_{B_j} w_j \, dx \le \ell_j A_j. \tag{4.17}$$

Combining all the estimates (4.11)–(4.14), (4.16) and (4.17), we obtain

$$A_{j+1} \le c_0 2^{j(n+tp+2p)} A_j^{1+\beta}$$

for a constant $c_0 \equiv c_0(\texttt{data}) > 0$. By choosing

$$\delta = \frac{1}{4} \exp\left( -\frac{\bar{c}}{\sigma} c_0^{1/\beta} 2^{(n+tp+2p)\beta^2} \right) \in \left(0, \frac{1}{4}\right)$$

in (4.10), we have

$$A_0 \le \frac{\bar{c}}{\sigma \log(1/3\delta)} \le c_0^{-1/\beta} 2^{-(n+tp+2p)/\beta^2}.$$

Therefore, we can apply Lemma 2.4 in order to conclude that $A_j \to 0$, from which (4.5) follows.

**Case 2:** $b^+_{B_{2r}} > \nu/4$.

In this case, observe that $b^-_{B_{2r}} \geq b^+_{B_{2r}} - [b]_\alpha (2R_0)^\alpha \geq \nu/8$ from (1.11). Using (4.8), (4.9) and Lemma 2.3 with $a_0 = 0$ and $b_0 = 1$, we have

$$\begin{aligned} A_{j+1}(2^{-j-2}\ell_j)^p \leq A_{j+1}(\ell_j - \ell_{j+1})^p &= \frac{1}{|B_{j+1}|}\int_{B_{j+1}\cap\{u<\ell_j\}} (\ell_j - \ell_{j+1})^p \,dx \\ &\leq cr^{tp} A_j^\beta \left( ⨍_{B_{j+1}} \int_{B_{j+1}} \frac{|w_j(x) - w_j(y)|^p}{|x-y|^{n+tp}} \,dx\,dy + ⨍_{B_j} w_j^p \,dx \right). \end{aligned} \tag{4.18}$$

We then estimate each term in the right-hand side. It is straightforward to see that

$$⨍_{B_j} w_j^p \,dx \leq \frac{1}{|B_j|}\int_{B_j\cap\{u<\ell_j\}} \ell_j^p \,dx \leq \ell_j^p A_j. \tag{4.19}$$

Next, we apply (3.1) to $w_j$ and $B_j$, which gives

$$\begin{aligned} & ⨍_{B_{j+1}} \int_{B_{j+1}} \frac{|w_j(x) - w_j(y)|^p}{|x-y|^{n+tp}} \,dx\,dy \\ &\leq ⨍_{B_{j+1}} \int_{B_{j+1}} \left( a(x,y)\frac{|w_j(x) - w_j(y)|^p}{|x-y|^{n+sp}} + b(x,y)\frac{|w_j(x) - w_j(y)|^p}{|x-y|^{n+tp}} \right) dx\,dy \\ &\leq c\,2^{jp} r^{-p} ⨍_{B_j} \int_{B_j} \left( a(x,y)\frac{(w_j(x))^p}{|x-y|^{n+(s-1)p}} + b(x,y)\frac{(w_j(x))^p}{|x-y|^{n+(t-1)p}} \right) dx\,dy \\ &\quad + c2^{j(n+tp)} T(w_j; r_j, r_j) ⨍_{B_j} w_j \,dx. \end{aligned} \tag{4.20}$$

Similarly to (4.14) and (4.16), we can estimate each term in the right-hand side as

$$\begin{aligned} & 2^{jp} r^{-p} ⨍_{B_j} \int_{B_j} \left( a(x,y)\frac{(w_j(x))^p}{|x-y|^{n+(s-1)p}} + b(x,y)\frac{(w_j(x))^p}{|x-y|^{n+(t-1)p}} \right) dx\,dy \\ &\leq c2^{jp} \frac{|B_j \cap \{u < \ell_j\}|}{|B_j|} \left( r^{-sp}\ell_j^p + r^{-tp}\ell_j^p \right) \\ &\leq c2^{jp} r^{-tp} \ell_j^p A_j, \end{aligned} \tag{4.21}$$

and

$$T(w_j; r_j, r_j) \leq c\left(r^{-sp} + r^{-tp}\right)\ell_j^{p-1} + cg(d) \leq cr^{-tp}\ell_j^{p-1}. \tag{4.22}$$

Combining all the estimates (4.17)–(4.22), we obtain

$$A_{j+1} \leq c_1 2^{j(n+tp+2p)} A_j^{1+\beta}$$

for a constant $c_1 \equiv c_1(\mathtt{data}) > 0$. Therefore, as in **Case 1**, we can choose $\delta$ sufficiently small in order to have $A_j \to 0$, thereby concluding with (4.5). □

## 5. Proof of Theorems 1.3 and 1.4

In this section, we keep on assuming (1.9) and (1.10) on $b(\cdot,\cdot)$.

### *5.1. Hölder regularity*

Here we prove Hölder regularity for (1.1).

*Proof of Theorem 1.3.* Let $B_r \equiv B_r(x_0) \subset \Omega$ be a fixed ball, and set

$$k_0 := 2\|u\|_{L^\infty(B_{r/2})} + 2\left[\frac{1}{G_{B_r}(1)}T(u;r/2,r)\right]^{1/(p-1)} = 2\|u\|_{L^\infty(B_{r/2})} + 2g_{B_r}^{-1}(T(u;r/2,r)).$$

We claim that there exist small constants $\gamma \in (0,1)$ and $\tau \in (0,1)$, both depending only on `data`, such that

$$\operatorname*{ess\,osc}_{B_{\tau^j r/2}} u \le \tau^{\gamma j} k_0 \tag{5.1}$$

holds for every $j \in \mathbb{N}$. We show this by using strong induction on $j$. First, the definition of $k_0$ directly implies that (5.1) holds for $j = 0$. Now, with $i \in \mathbb{N} \cup \{0\}$ being fixed, we assume that (5.1) holds for all $j \in \{0, 1, \ldots, i\}$, and then show that it holds for $j = i + 1$.

For each $j \in \mathbb{N} \cup \{0\}$, we set

$$r_j := \tau^j \frac{r}{2}, \qquad B_j := B_{r_j}(x_0), \qquad k_j := \left(\frac{r_j}{r_0}\right)^\gamma k_0 = \tau^{\gamma j} k_0,$$

where $\gamma \in (0,1)$ and $\tau \in (0,1)$ are free parameters whose values will be chosen later, and then

$$M_j := \operatorname*{ess\,sup}_{B_j} u, \qquad m_j := \operatorname*{ess\,inf}_{B_j} u.$$

Observe that either

$$\left|2B_{i+1} \cap \left\{u - m_i \ge \frac{k_i}{2}\right\}\right| \ge \frac{1}{2}|2B_{i+1}| \quad \text{or} \quad \left|2B_{i+1} \cap \left\{u - m_i < \frac{k_i}{2}\right\}\right| \ge \frac{1}{2}|2B_{i+1}| \tag{5.2}$$

must hold; we accordingly set

$$u_i := \begin{cases} u - m_i & \text{if } (5.2)_1 \text{ holds,} \\ k_i - (u - m_i) & \text{if } (5.2)_2 \text{ holds.} \end{cases}$$

In any case, $u_i$ is a weak solution to (1.1), $u_i \ge 0$ in $B_i$ and

$$\left|2B_{i+1} \cap \left\{u_i \ge \frac{k_i}{2}\right\}\right| \ge \frac{1}{2}|2B_{i+1}|. \tag{5.3}$$

It moreover satisfies

$$|u_i| \le M_j - m_j + k_i \le k_j + k_i \le 2k_j \ \text{ a.e. in } \ B_j \quad \text{for any} \ \ j \in \{0, 1, \ldots, i\},$$

and

$$|u_i| \le |u| + 2k_0 \quad \text{a.e. in } \ \mathbb{R}^n \setminus B_0.$$

From these observations, we estimate

$$\begin{aligned}
T(u_i; r_i, 2r_{i+1}) &\le \sup_{x\in 2B_{i+1}} \sum_{j=1}^{i} \int_{B_{j-1}\setminus B_j} \left( a(x,y)\frac{|u_i(y)|^{p-1}}{|y-x_0|^{n+sp}} + b(x,y)\frac{|u_i(y)|^{p-1}}{|y-x_0|^{n+tp}} \right) dy \\
&\quad + \sup_{x\in 2B_{i+1}} \int_{\mathbb{R}^n\setminus B_0} \left( a(x,y)\frac{|u_i(y)|^{p-1}}{|y-x_0|^{n+sp}} + b(x,y)\frac{|u_i(y)|^{p-1}}{|y-x_0|^{n+tp}} \right) dy \\
&\le \sup_{x\in 2B_{i+1}} \sum_{j=1}^{i} \int_{B_{j-1}\setminus B_j} \left( a(x,y)\frac{(2k_{j-1})^{p-1}}{|y-x_0|^{n+sp}} + b(x,y)\frac{(2k_{j-1})^{p-1}}{|y-x_0|^{n+tp}} \right) dy \\
&\quad + \sup_{x\in 2B_{i+1}} \int_{\mathbb{R}^n\setminus B_0} \left( a(x,y)\frac{(|u(y)|+2k_0)^{p-1}}{|y-x_0|^{n+sp}} + b(x,y)\frac{(|u(y)|+2k_0)^{p-1}}{|y-x_0|^{n+tp}} \right) dy.
\end{aligned}$$

To proceed, we again divide the cases.

**Case 1: $b^+_{2B_{i+1}} \le \nu/4$.**

In this case, we use (1.8), (4.15) and the fact that

$$2r_{j+1} \le 2r_0,\ b^+_{2B_0} \le b^+_{2B_{i+1}} + [b]_\alpha (2r_0)^\alpha \le b^+_{2B_{i+1}} + [b]_\alpha (2r_0)^{(t-s)p} \quad\Longrightarrow\quad G_{2B_0}(1) \lesssim G_{2B_{i+1}}(1)$$

in order to get

$$\begin{aligned}
T(u_i; r_i, 2r_{i+1}) &\le c\sum_{j=1}^{i} (2k_{j-1})^{p-1}\left(r_j^{-sp} + b^+_{2B_{i+1}} r_j^{-tp}\right) + ck_0^{p-1} G_{2B_0}(1) + ck_0^{p-1}\left(r_0^{-sp} + b^+_{2B_{i+1}} r_0^{-tp}\right) \\
&\le c\sum_{j=1}^{i} k_{j-1}^{p-1}\left(r_j^{-sp} + b^+_{2B_{i+1}} r_j^{-tp}\right).
\end{aligned}$$

In turn, we have

$$\begin{aligned}
\frac{1}{G_{2B_{i+1}}(1)} T(u_i; r_i, 2r_{i+1}) &\le c\sum_{j=1}^{i} \frac{r_j^{-sp} + b^+_{2B_{i+1}} r_j^{-tp}}{r_{i+1}^{-sp} + b^+_{2B_{i+1}} r_{i+1}^{-tp}} k_{j-1}^{p-1} \\
&\le c\sum_{j=1}^{i} \left(\frac{r_{i+1}}{r_j}\right)^{sp} k_{j-1}^{p-1} = ck_i^{p-1} \sum_{j=1}^{i} \tau^{(i+1-j)[sp-\gamma(p-1)]}.
\end{aligned}$$

**Case 2: $b^+_{2B_{i+1}} > \nu/4$.**

In this case, with (1.8), we directly estimate

$$\begin{aligned}
T(u_i; r_i, 2r_{i+1}) &\le c\sum_{j=1}^{i} (2k_{j-1})^{p-1}\left(r_j^{-sp} + r_j^{-tp}\right) + ck_0^{p-1} G_{2B_0}(1) + ck_0^{p-1}\left(r_0^{-sp} + r_0^{-tp}\right) \\
&\le c\sum_{j=1}^{i} k_{j-1}^{p-1}\left(r_j^{-sp} + r_j^{-tp}\right) \le c\sum_{j=1}^{i} k_{j-1}^{p-1} r_j^{-tp}
\end{aligned}$$

and so

$$\frac{1}{G_{2B_{i+1}}(1)} T(u_i; r_i, 2r_{i+1}) \le c\sum_{j=1}^{i} \left(\frac{r_{i+1}}{r_j}\right)^{tp} k_{j-1}^{p-1} = ck_i^{p-1} \sum_{j=1}^{i} \tau^{(i+1-j)[tp-\gamma(p-1)]}.$$

Therefore, if $\gamma \in (0,1)$ is so small that

$$\gamma \leq \frac{sp}{2(p-1)} \leq \frac{tp}{2(p-1)}, \tag{5.4}$$

then in any case we have

$$\frac{1}{G_{2B_{i+1}}(1)} T((u_i)_-; r_i, 2r_{i+1}) \leq \frac{1}{G_{2B_{i+1}}(1)} T(u_i; r_i, 2r_{i+1}) \leq c k_i^{p-1} \sum_{j=1}^{i} \tau^{jsp/2} \leq c \frac{\tau^{sp/2}}{1-\tau^{sp/2}} k_i^{p-1}$$

for a constant $c \equiv c(\mathtt{data}) > 0$. We now choose $\tau \equiv \tau(\mathtt{data}) \in (0, 1/8)$ so small that

$$\left[\frac{1}{G_{2B_{i+1}}(1)} T((u_i)_-; r_i, 2r_{i+1})\right]^{1/(p-1)} \leq \left(c \frac{\tau^{sp/2}}{1-\tau^{sp/2}}\right)^{1/(p-1)} k_i \leq \frac{\delta}{2} k_i,$$

where $\delta \equiv \delta(\mathtt{data}) \in (0, 1/4)$ is the constant determined in Lemma 4.2 with the choice $\sigma \equiv 1/2$. Consequently, recalling (5.3), we employ Lemma 4.2 with the choices $k \equiv k_i/2$, $\sigma \equiv 1/2$, $B_R \equiv B_i$ and $B_r \equiv B_{i+1}$, which gives

$$\operatorname*{ess\,inf}_{B_{i+1}} u_{i+1} \geq \frac{\delta k_i}{2}.$$

In turn, we have the following:

(i) If $(5.2)_1$ holds, then $m_{i+1} - m_i \geq \delta k_i/2$ and so

$$M_{i+1} - m_{i+1} \leq M_i - m_i - (m_{i+1} - m_i) = \operatorname*{ess\,osc}_{B_i} u - (m_{i+1} - m_i) \leq \left(1 - \frac{\delta}{2}\right) k_i.$$

(ii) If $(5.2)_2$ holds, then $k_i - M_{i+1} + m_i \geq \delta k_i/2$ and so

$$M_{i+1} - m_{i+1} \leq M_{i+1} - m_i \leq \left(1 - \frac{\delta}{2}\right) k_i.$$

Namely, in any case we obtain

$$\operatorname*{ess\,osc}_{B_{i+1}} u \leq \left(1 - \frac{\delta}{2}\right) \tau^{-\gamma} k_{i+1}.$$

Then we finally fix $\gamma \in (0,1)$ as a small constant, depending only on $\mathtt{data}$, in a way that (5.4) and $1 - \delta/2 \leq \tau^{\gamma}$ are satisfied; this leads to (5.1) for $j = i+1$. Hence, we conclude with

$$\begin{aligned} \operatorname*{ess\,osc}_{B_j} u &\leq c\tau^{\gamma j} \left[\|u\|_{L^\infty(B_{r/2})} + g_{B_r}^{-1}(T(u; r/2, r))\right] \\ &\overset{(1.20)}{\leq} c\tau^{\gamma j} \left[\left(\fint_{B_r} |u|^p \, dx\right)^{1/p} + g_{B_r}^{-1}(T(u; r/2, r))\right] \end{aligned}$$

for any $j \in \mathbb{N}$, and then estimate (1.21) follows by elementary manipulations. □

### *5.2. Harnack inequality*

In the following, we further assume that $a(\cdot,\cdot) \equiv 1$. We first recall a Krylov–Safonov type covering lemma, see [32]. The following version, which employs balls instead of cubes, can be found in [31, Lemma 7.2].

**Lemma 5.1.** *Let $E \subset B_R$ be a measurable set, and let $\bar\delta \in (0,1)$. Define*

$$[E]_{\bar\delta} := \bigcup \left\{ B_{3\rho}(x) \cap B_R : x \in B_R, \rho > 0, |E \cap B_{3\rho}(x)| > \bar\delta |B_\rho(x)| \right\}. \tag{5.5}$$

*Then either $[E]_{\bar\delta} = B_R$, or else $|[E]_{\bar\delta}| \geq (2^n\bar\delta)^{-1}|E|$.*

**Remark 5.2.** It is clear that, in the above definition of $[E]_{\bar\delta}$, we may consider only the balls $B_{3\rho}(x)$ with $\rho \leq 2r/3$.

Using Lemmas 4.2 and 5.1, we obtain a weak Harnack type estimate.

**Lemma 5.3.** *Let $u \in \mathcal{A}(\Omega) \cap \mathcal{T}(\mathbb{R}^n)$ be a weak supersolution to (1.1) under assumptions (1.6)–(1.10) with $a(\cdot,\cdot) \equiv 1$, which is nonnegative in a ball $B_{10r} \equiv B_{10r}(x_0) \Subset \Omega$ with $10r \leq R_0$. Then there exist constants $p_0 \in (0,1)$ and $c \geq 1$, both depending only on* `data`*, such that*

$$\left( \fint_{B_{2r}} u^{p_0}\,dx \right)^{1/p_0} \leq c \operatorname*{ess\,inf}_{B_r} u + c g_{B_r}^{-1}(T(u_-; 10r, 10r)).$$

*Proof.* We set $\bar\delta := 2^{-n-1}$ and $T_0 := c_2 g_{B_r}^{-1}(T(u_-; 10r, 10r))$, with $c_2 \equiv c_2(\texttt{data})$ being a constant whose precise value will be determined in (5.6) below. For each $i \in \mathbb{N} \cup \{0\}$ and $\tau > 0$, we define

$$A_\tau^i := \left\{ x \in B_{2r} : u(x) > \tau\delta^i - \frac{T_0}{1-\delta} \right\},$$

where $\delta \equiv \delta(\texttt{data}) \in (0, 1/4)$ is the constant determined in Lemma 4.2 in the case $\sigma = \bar\delta/6^n$. It is obvious that $A_\tau^{i-1} \subset A_\tau^i$.

Let $x \in B_{2r}$ and $\rho \in (0, 2r/3]$ be such that

$$B_{3\rho}(x) \cap B_{2r} \subset [A_\tau^{i-1}]_{\bar\delta}$$

holds (recall the notation (5.5) and Remark 5.2). Then we get

$$|A_\tau^{i-1} \cap B_{3\rho}(x)| \geq \bar\delta |B_\rho(x)|$$

and so

$$\frac{|A_\tau^{i-1} \cap B_{6\rho}(x)|}{|B_{6\rho}(x)|} \geq \frac{1}{6^n} \frac{|A_\tau^{i-1} \cap B_{3\rho}(x)|}{|B_\rho(x)|} \geq \frac{\bar\delta}{6^n}.$$

Moreover, $u \geq 0$ in $B_{12\rho}(x) \subset B_{10r}$. Therefore we can apply Lemma 4.2 to $u$ on the concentric balls $B_{3\rho}(x) \subset B_{6\rho}(x) \subset B_{12\rho}(x)$ with the choices $k \equiv \tau\delta^{i-1} - T_0/(1-\delta)$, $\sigma \equiv \bar\delta/6^n$ and $d \equiv g_{B_{6\rho}(x)}^{-1}(T(u_-; x, 12\rho, 6\rho))$; this leads to

$$\operatorname*{ess\,inf}_{B_{3\rho}(x)} u \geq \delta \left( \tau\delta^{i-1} - \frac{T_0}{1-\delta} \right) - d.$$

We also observe that, since $u \ge 0$ in $B_{10r}$,

$$\begin{aligned} g_{B_{6\rho}(x)}(d) &= \sup_{B_{6\rho}(x)} \int_{\mathbb{R}^n \setminus B_{12\rho}(x)} \left( \frac{(u_-(y))^{p-1}}{|y-x|^{n-sp}} + b(\cdot, y) \frac{(u_-(y))^{p-1}}{|y-x|^{n+tp}} \right) dy \\ &\le c \sup_{B_{10r}} \int_{\mathbb{R}^n \setminus B_{10r}} \left( \frac{(u_-(y))^{p-1}}{|y-x_0|^{n+sp}} + b(\cdot, y) \frac{(u_-(y))^{p-1}}{|y-x_0|^{n+tp}} \right) dy \\ &= cT(u_-; 10r, 10r) \\ &\le c_2^{p-1} g_{B_{6\rho}(x)}(g_{B_r}^{-1}(T(u_-; 10r, 10r))) = g_{B_{6\rho}(x)}(T_0) \end{aligned} \tag{5.6}$$

holds for a constant $c_2 \equiv c_2(\texttt{data}) > 0$, where we have also used the fact that

$$\rho \le 2r/3, \ b^+_{B_r} \lesssim b^+_{B_{6\rho}(x)} + r^{\alpha} \lesssim b^+_{B_{6\rho}(x)} + r^{(t-s)p} \implies g_{B_r}(\cdot) \lesssim g_{B_{6\rho}(x)}(\cdot).$$

Consequently, we have

$$\operatorname*{ess\,inf}_{B_{3\rho}(x)} u \ge \delta \left( \tau \delta^{i-1} - \frac{T_0}{1-\delta} \right) - T_0 = \tau \delta^i - \frac{T_0}{1-\delta}$$

and this implies $B_{3\rho}(x) \cap B_{2r} \subset A_\tau^i$. In turn, since $B_{3\rho}(x)$ was an arbitrary member of the family making the union in (5.5), we have $[A_\tau^{i-1}]_{\bar{\delta}} \subset A_\tau^i$. We now apply Lemma 5.1 with $E = A_\tau^{i-1}$, and the rest of the proof is exactly the same as the case of fractional $p$-Laplacian [18, 21]. We thus finish the proof here. □

The following lemma allows us to capture the precise tail contribution in the Harnack inequality.

**Lemma 5.4.** *Let $u \in \mathcal{A}(\Omega) \cap \mathcal{T}(\mathbb{R}^n)$ be a weak solution to* (1.1) *under assumptions* (1.6)–(1.10) *with $a(\cdot, \cdot) \equiv 1$, which is nonnegative in a ball $B_R \equiv B_R(x_0) \Subset \Omega$ with $R \le R_0$. Then*

$$T(u_+; r, 2r) \le c g_{B_r} \left( \sup_{B_r} u \right) + cT(u_-; R, r)$$

*holds for any $r \in (0, R/2]$, where $c \equiv c(\texttt{data}) > 0$.*

*Proof.* We set $w := u - 2k$ for $k := \sup_{B_r} u$, and take a cut-off function $\phi \in C_0^\infty(B_r)$ such that $\operatorname{supp} \phi \subseteq B_{3r/4}$, $0 \le \phi \le 1$, $\phi \equiv 1$ in $B_{r/2}$ and $|D\phi| \le 8/r$. Recalling the notation (4.2), we test (1.1) with $w\phi^p$ in order to have

$$\begin{aligned} 0 &= \int_{B_r} \int_{B_r} \Phi_p(u(x) - u(y))(w(x)\phi^p(x) - w(y)\phi^p(y)) \left[ K_{sp}(x, y) + b(x, y) K_{tp}(x, y) \right] dx\, dy \\ &\quad + 2 \int_{\mathbb{R}^n \setminus B_r} \int_{B_r} \Phi_p(u(x) - u(y)) w(x) \phi^p(x) \left[ K_{sp}(x, y) + b(x, y) K_{tp}(x, y) \right] dx\, dy \\ &=: I_1 + I_2. \end{aligned} \tag{5.7}$$

We first estimate $I_1$. By the calculations done in the proof of [9, Lemma 4.2], we have

$$\begin{aligned} &\Phi_p(w(x) - w(y))(w(x)\phi^p(x) - w(y)\phi^p(y)) \\ &\ge \frac{1}{4} |w(x) - w(y)|^p (\phi^p(x) + \phi^p(y)) - c|\phi(x) - \phi(y)|^p (w(x) + w(y))^p \\ &\ge -ck^p \frac{|x-y|^p}{r^p} \end{aligned}$$

and therefore

$$\begin{aligned} I_1 &\geq -c \int_{B_r}\int_{B_r} \left( \frac{k^p}{r^p} \frac{1}{|x-y|^{n+(s-1)p}} + b(x,y)\frac{k^p}{r^p}\frac{1}{|x-y|^{n+(t-1)p}} \right) dx\,dy \\ &\geq -c\left( k^p r^{n-sp} + b^+_{B_r} k^p r^{n-tp} \right) \\ &\geq -cr^n G_{B_r}(k). \end{aligned} \tag{5.8}$$

We next split $I_2$ as

$$\begin{aligned} I_2 &\geq 2\int_{\mathbb{R}^n\setminus B_r}\int_{B_r} k(u(y)-k)_+^{p-1}\phi^p(x)\left[ K_{sp}(x,y) + b(x,y)K_{tp}(x,y) \right] dx\,dy \\ &\quad - 2\int_{\mathbb{R}^n\setminus B_r}\int_{B_r} 2k\chi_{\{u(y)<k\}}(u(x)-u(y))_+^{p-1}\phi^p(x)\left[ K_{sp}(x,y) + b(x,y)K_{tp}(x,y) \right] dx\,dy \\ &=: I_{2,1} - I_{2,2}. \end{aligned} \tag{5.9}$$

We note that

$$\frac{1}{4}|y-x_0| \leq |y-x_0| - |x-x_0| \leq |x-y| \leq |x-x_0| + |y-x_0| \leq \frac{7}{4}|y-x_0|$$

for any $x \in B_{3r/4}$ and $y \in \mathbb{R}^n \setminus B_r$, in order to estimate $I_{2,1}$ as

$$\begin{aligned} I_{2,1} &\geq \frac{k}{c}\int_{\mathbb{R}^n\setminus B_r}\int_{B_r} \left( \frac{(u_+(y))^{p-1}}{|x-y|^{n+sp}} + b(x,y)\frac{(u_+(y))^{p-1}}{|x-y|^{n+tp}} \right)\phi^p(x)\,dx\,dy \\ &\quad - c\int_{\mathbb{R}^n\setminus B_r}\int_{B_r} \left( \frac{k^p}{|x-y|^{n+sp}} + b(x,y)\frac{k^p}{|x-y|^{n+tp}} \right)\phi^p(x)\,dx\,dy \\ &\geq \frac{kr^n}{c}\int_{\mathbb{R}^n\setminus B_r} \left( \frac{(u_+(y))^{p-1}}{|y-x_0|^{n+sp}} + \inf_{B_{r/2}} b(\cdot,y)\frac{(u_+(y))^{p-1}}{|y-x_0|^{n+tp}} \right) dy - cr^n G_{B_r}(k). \end{aligned} \tag{5.10}$$

As for $I_{2,2}$, we observe that

(i) If $x \in B_r$ and $y \in B_R$, then $(u(x)-u(y))_+^{p-1} \leq (u(x))^{p-1}$;
(ii) If $x \in B_r$ and $y \in \mathbb{R}^n \setminus B_R$, then $(u(x)-u(y))_+^{p-1} \leq 2^{p-1}\left[ (u(x))^{p-1} + (u_-(y))^{p-1} \right]$.

We thus estimate

$$\begin{aligned} I_{2,2} &\leq 2k\int_{B_R\setminus B_r}\int_{B_r} \left( \frac{k^{p-1}}{|y-x_0|^{n+sp}} + b(x,y)\frac{k^{p-1}}{|y-x_0|^{n+tp}} \right)\phi^p(x)\,dx\,dy \\ &\quad + 2k\int_{\mathbb{R}^n\setminus B_R}\int_{B_r} \left( \frac{(k+u_-(y))^{p-1}}{|y-x_0|^{n+sp}} + b(x,y)\frac{(k+u_-(y))^{p-1}}{|y-x_0|^{n+tp}} \right)\phi^p(x)\,dx\,dy \\ &\leq cr^n G_{B_r}(k) + ckr^n T(u_-;R,r). \end{aligned} \tag{5.11}$$

Connecting (5.8)–(5.11) to (5.7), and then recalling the definition of $k$, we arrive at

$$\int_{\mathbb{R}^n\setminus B_r} \left( \frac{(u_+(y))^{p-1}}{|y-x_0|^{n+sp}} + \inf_{B_{r/2}} b(\cdot,y)\frac{(u_+(y))^{p-1}}{|y-x_0|^{n+tq}} \right) dy \leq cg_{B_r}\left( \sup_{B_r} u \right) + cT(u_-;R,r).$$

Finally, we use (1.9) and (1.10) to estimate

$$\begin{aligned}
T(u_+;r,2r) &= \sup_{x\in B_{2r}}\int_{\mathbb{R}^n\setminus B_r}\left(\frac{(u_+(y))^{p-1}}{|y-x_0|^{n+sp}}+b(x,y)\frac{(u_+(y))^{p-1}}{|y-x_0|^{n+tp}}\right)dy\\
&\le \int_{\mathbb{R}^n\setminus B_r}\left(\frac{(u_+(y))^{p-1}}{|y-x_0|^{n+sp}}+\sup_{B_{2r}}b(\cdot,y)\frac{(u_+(y))^{p-1}}{|y-x_0|^{n+tp}}\right)dy\\
&\le \int_{\mathbb{R}^n\setminus B_r}\left(\frac{(u_+(y))^{p-1}}{|y-x_0|^{n+sp}}+\inf_{B_{2r}}b(\cdot,y)\frac{(u_+(y))^{p-1}}{|y-x_0|^{n+tp}}\right)dy+cr^{\alpha}\int_{\mathbb{R}^n\setminus B_r}\frac{(u_+(y))^{p-1}}{|y-x_0|^{n+sp}r^{tp-sp}}\,dy\\
&\le c\int_{\mathbb{R}^n\setminus B_r}\left(\frac{(u_+(y))^{p-1}}{|y-x_0|^{n+sp}}+\inf_{B_{r/2}}b(\cdot,y)\frac{(u_+(y))^{p-1}}{|y-x_0|^{n+tp}}\right)dy.
\end{aligned}$$

Combining the last two displays, we conclude with the desired estimate. □

We are now in a position to prove Theorem 1.4.

*Proof of Theorem 1.4.* Let $u$ be a weak solution to (1.1) which is nonnegative in a ball $B_{10r}\equiv B_{10r}(x_0)\Subset\Omega$ with $10r\le R_0$. First, arguing in a way completely similar to the proof of (1.20), we have

$$\sup_{B_{\sigma r}}u\le\frac{c_\varepsilon}{(\tau-\sigma)^{n/p}}\left(\fint_{B_{\tau r}}u^p\,dx\right)^{1/p}+\varepsilon g_{B_r}^{-1}\left(T(u_+;r,2r)\right)$$

for any $1\le\sigma<\tau\le 2$ and $\varepsilon\in(0,1]$, where $c_\varepsilon\equiv c_\varepsilon(\mathtt{data},\varepsilon)>0$. Next, using Young's inequality, we get

$$\begin{aligned}
\sup_{B_{\sigma r}}u&\le\frac{c_\varepsilon}{(\tau-\sigma)^{n/p}}\left(\sup_{B_{\tau r}}u\right)^{(p-p_0)/p}\left(\fint_{B_{2r}}u^{p_0}\,dx\right)^{1/p}+\varepsilon g_{B_r}^{-1}(T(u_+;r,2r))\\
&\le\frac{1}{2}\sup_{B_{\tau r}}u+\frac{c_\varepsilon}{(\tau-\sigma)^{n/p_0}}\left(\fint_{B_{2r}}u^{p_0}\,dx\right)^{1/p_0}+\varepsilon g_{B_r}^{-1}(T(u_+;r,2r)),
\end{aligned}$$

whenever $1\le\sigma<\tau\le 2$, where $p_0\equiv p_0(\mathtt{data})\in(0,1)$ is the constant determined in Lemma 5.3. Then Lemma 2.5 implies

$$\sup_{B_r}u\le c_\varepsilon\left(\fint_{B_{2r}}u^{p_0}\,dx\right)^{1/p_0}+\varepsilon g_{B_r}^{-1}(T(u_+;r,2r)).$$

We now apply Lemmas 5.3 and 5.4 to the first and the second terms in the right-hand side, respectively, thereby obtaining

$$\sup_{B_r}u\le c_*\varepsilon\sup_{B_r}u+c_\varepsilon\inf_{B_r}u+c_\varepsilon g_{B_r}^{-1}(T(u_-;10r,10r))$$

for any $\varepsilon\in(0,1]$, where $c_*\equiv c_*(\mathtt{data})$ and $c_\varepsilon\equiv c_\varepsilon(\mathtt{data},\varepsilon)$ are positive constants. In this last display, we choose $\varepsilon=1/(2c_*)$ and then reabsorb the first term in the right-hand side; this finally yields (1.22). □

## 6. Conclusions

We have investigated a class of nonlinear integro-differential equations driven by a superposition of $(s, p)$-Laplacian and $(t, p)$-Laplacian type operators, where $0 < s < t < 1 < p < \infty$, weighted via two possibly degenerate coefficients $a(\cdot, \cdot) \geq 0$ and $b(\cdot, \cdot) \geq 0$, respectively. Under natural assumptions on the coefficients $a(\cdot, \cdot)$, $b(\cdot, \cdot)$ and the powers $s$, $t$, $p$, we established local boundedness and Hölder continuity of weak solutions. In the special case $a(\cdot, \cdot) \equiv 1$, we further derived a Harnack inequality.

## Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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## Conflict of interest

The authors declare that they have no conflicts of interest.

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