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Research article

A further remark on the density estimate for degenerate Allen-Cahn equations: Δ_p-type equations for $1 < p < \frac{n}{n-1}$ with rough coefficients

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Abstract: In this short remark on a previous paper [1], we continue the study of Allen-Cahn equations associated with Ginzburg-Landau energies

$$J(v,\Omega) = \int_{\Omega} \Big\{ F(\nabla v, v, x) + W(v, x) \Big\} dx,$$

involving a Dirichlet energy $F(\vec{\xi}, \tau, x) \sim |\vec{\xi}|^p$ and a degenerate double-well potential $W(\tau, x) \sim (1-\tau^2)^m$. In contrast to [1], we remove all regularity assumptions on the Ginzburg-Landau energy. Then, with further assumptions that $1 < p < \frac{n}{n-1}$ and that $W(\tau, x)$ is monotone in τ on both sides of 0, we establish a density estimate for the level sets of nontrivial minimizers $|u| \leq 1$.

Keywords: Ginzburg-Landau theory; phase transitions; degenerate Allen-Cahn equations; density estimate; weak Harnack principle

1. Introduction

The Ginzburg-Landau energy was developed from the theory of Van der Waals (see [2]) by Landau, Ginzburg and Pitaevskiĭ in [3–5] to describe phase transitions in thermodynamics (see also [6, 7]). In this paper, we study a global minimizer $u : \mathbb{R}^n \to [-1, 1]$ of the Ginzburg-Landau energy

$$J(v,\Omega) = \int_{\Omega} \Big\{ F(\nabla v, v, x) + W(v, x) \Big\} dx, \tag{1.1}$$

in which v represents the mean field of the spin of the particles. A minimizer of (1.1) is defined as follows:

Definition 1.1. *Let $\Omega \subseteq \mathbb{R}^n$. We say that $u \in W^{1,p}(\Omega, [-1, 1])$ is a minimizer of* (1.1)*, if for every bounded open set $\Omega' \subset\subset \Omega$, and any $v \in W^{1,p}_{loc}(\Omega, [-1, 1])$ such that $u = v$ in $\Omega \setminus \Omega'$, we have*

$$J(u, \Omega') \leq J(v, \Omega').$$

Remark 1.1. *When $F(\vec{\xi}, \tau, x)$ and $W(\tau, x)$ have good regularity, in particular, when*

$$F(\vec{\xi}, \tau, x) \equiv |\vec{\xi}|^p, \quad W(\tau, x) \equiv (1 - v^2)^m,$$

then the Euler-Lagrange equation of a minimizer, namely the Allen-Cahn equation, is the following:

$$p\Delta_p u = p \cdot \operatorname{div}(|\nabla u|^{p-2}\nabla u) = W'(u) = -2m(1 - u^2)^{m-1}u. \tag{1.2}$$

However, in this paper, we avoid the use of the Euler-Lagrange equation because we do not impose any regularity assumptions. Instead, we assume that there exists some universal constant $\lambda > 1$ such that:

(A) For every vector $\vec{\xi}$, $\tau \in [-1, 1]$ and $x \in \mathbb{R}^n$, we have

$$\lambda^{-1}|\vec{\xi}|^p \leq F(\vec{\xi}, \tau, x) \leq \lambda|\vec{\xi}|^p,$$

where the exponent p is universal and it satisfies $1 < p < \frac{n}{n-1}$;

(B) For every $\tau \in [-1, 1]$, we have that

$$\lambda^{-1}(1 - \tau^2)^m \leq W(\tau, x) \leq \lambda(1 - \tau^2)^m,$$

where the exponent m is universal and it satisfies $m > p$;

(C) For each fixed x, $W(\tau, x)$ is increasing when $-1 \leq \tau \leq 0$, and it is decreasing when $0 \leq \tau \leq 1$.

Apart from the trivial minimizers $u \equiv \pm 1$, the more complicated and interesting question is to study minimizers or critical points representing phase transitions, i.e., a solution that can be sufficiently close to both 1 and -1 (two steady states), but with a phase field region $|u| \leq 1 - \varepsilon$ in between.

It is well known that phase transitions modeled by minimizers u defined in a large ball B_R are closely related to minimal surfaces. More precisely, the rescaling of the transition region $\frac{1}{R}\{|u| \leq 1 - \varepsilon\}$ of u from B_R to the unit ball B_1 is well approximated by a minimal surface in B_1. In the classical case $p = m = 2$ the approximation is made rigorous in three main steps:

(1) The Γ-convergence result established by Modica and Mortola in [8, 9], see also [10, 11].
(2) The density estimate obtained by Caffarelli-Córdoba in [12], see also [13–17].
(3) The convergence of $\{u_R = 0\}$ to ∂E in the stronger $C^{2,\alpha}_{loc}(B_1)$ sense, i.e., the improvement of flatness technique, see [18–20].

The heteroclinical solution is the monotone one-dimensional solution that connects the stable phases -1 and $+1$ as x ranges from $-\infty$ to ∞. The rate of decay of this solution to the limits ± 1 depends on the values of m and p. Precisely, if $m < p$ then the limits are achieved outside a finite interval, producing a free boundary of Alt-Phillips type for the region $u \neq \pm 1$. If $m = p$ the rate of decay is exponential. On the other hand, the case $m > p$ produces less stable minimal points (still at ± 1) for an infinitesimal potential energy $W(v) \sim (1-v^2)^m$, and the rate of decay is polynomial. To see this, one can multiply $u'(t)$ on both sides of the one-dimensional version of (1.2) and integrate. It follows that the heteroclinical solution satisfies the first-order ODE

$$u'(t) = \Big(\frac{(1 - u^2)^m}{p - 1}\Big)^{1/p}.$$

The decay rate is then obtained by integrating the equation $\frac{u'(t)}{(1-u^2)^{m/p}} \sim 1$. Recently, the decay rate estimate for the heteroclinical solution has been extended to the nonlocal Allen-Cahn equation in De Pas et al. [21]. It is then natural to investigate whether the results of Γ-convergence, density estimate, and improvement of flatness mentioned above extend to these types of degenerate Ginzburg-Landau energies.

In [13], Dipierro et al. considered Q-minimizers (a relaxation of the terminology minimizer) of such degenerate energies and obtained the density estimates for a certain range of m's depending on the dimension n. Precisely, the authors considered general Ginzburg-Landau type energies

$$J(v) = \int_\Omega E(\nabla v, v, x)dx, \quad E(\vec{\xi}, \tau, x) \sim (|\vec{\xi}|^p + |\tau+1|^m). \tag{1.3}$$

If $\frac{pm}{m-p} > n$ and $\left|\{u \geq 0\} \cap B_1\right| > c$ for some positive c, then the authors showed that

$$\left|\{u \geq 0\} \cap B_R\right| \geq \delta R^n, \quad \text{for some } \delta > 0 \text{ depending on } E(\cdot,\cdot,\cdot) \text{ and } c.$$

Notice that the energy (1.1) with assumptions (A)–(C) on $F(\vec{\xi}, \tau, x)$ and $W(\tau, x)$ is a special case of the energy (1.3). We also remark that the density estimates in the non-degenerate case $0 < m \leq p$ were obtained in the earlier work Farina-Valdinoci [14].

In Savin-Zhang [1], by further assuming that there exists an initial ball B_ρ of a fixed large radius in which the density estimate holds (see (1.4) below), the authors removed the assumption $\frac{pm}{m-p} > n$ and obtained a new version of the density estimate in [1, Theorem 1.1]. We state its simplified version below as a lemma:

Lemma 1.1. *Let u be a minimizer of the energy* (1.3) *in $\mathbb{R}^n$. Given any $\varepsilon > 0$, there exist $r_0 = r_0(\varepsilon)$ large and $\delta = \delta(\varepsilon)$, so that if*

$$\left|\{u \geq 0\} \cap B_r\right| \geq \varepsilon r^n \tag{1.4}$$

for some $r \geq r_0$, then $\left|\{u \geq 0\} \cap B_R\right| \geq \delta R^n$, for all $R \geq r$.

As an application, the authors proved the density estimate for a class of degenerate Allen-Cahn equations with further regularity assumptions on F and W. Their strategy is to translate the origin to a specific point x^* where $u(x^*)$ is sufficiently close to 1, and to verify the condition (1.4) there. Recently, in [22, 23], Dipierro et al. have obtained a similar result in the nonlocal setting.

In this paper, the main result is the following density estimate for a class of degenerate Allen-Cahn equations, whose Ginzburg-Landau energy has little regularity.

Theorem 1.1. *Assume that $u : \mathbb{R}^n \to [-1, 1]$ is a minimizer of the energy* (1.1) *in $\mathbb{R}^n$, such that $F(\vec{\xi}, \tau, x)$ and $W(\tau, x)$ satisfy assumptions (A)–(C). If $u(0) = 0$, then there exist some universal constants $\delta, R_0 > 0$, such that for every $R \geq R_0$, we have*

$$\left|B_R \cap \{u \geq 0\}\right| \geq \delta R^n \text{ and } \left|B_R \cap \{u \leq 0\}\right| \geq \delta R^n.$$

Similar to [1], the strategy in proving Theorem 1.1 is to translate the origin to some point x^* and to verify the assumption (1.4) in Lemma 1.1. In the first step, we prove that $\max_{B_R} u = u(x^*)$ is sufficiently close to 1 for a uniform radius R. In the second step, we prove that the density of the positive set is large in a ball centered at x^*, thus verifying the condition (1.4). The two key steps mentioned above are derived via variants of the weak Harnack principles.

2. Proof of Theorem 1.1

As was mentioned in the Introduction, it suffices to verify that the assumption (1.4) is satisfied in a fixed ball close to the origin. As a preliminary lemma, we prove the energy estimate of a minimizer:

Lemma 2.1. *Let u be a minimizer to* (1.1) *satisfying the assumptions (A)–(C). Then there exists some universal constant C, such that $J(u, B_R) \leq CR^{n-1}$ for all $R \geq 1$.*

Remark 2.1. *In fact, such an energy estimate was already proven in [13], and it holds true both not only when the Ginzburg-Landau energy* (1.1) *is degenerate ($m > p$), but also when it is non-degenerate ($m \leq p$).*

Proof of Lemma 2.1. Let $v(x) = v(|x|)$, such that ("med" stands for the median of the three quantities):

$$v(r) = \mathrm{med}(-1, 1, r - R - 1).$$

Let $\Omega = \{u \geq v\}$, then we have $B_R \subseteq \Omega$ and $\overline{\Omega} \subseteq B_{R+2}$. By the minimality of u, we have

$$J(u, B_R) \leq J(u, \Omega) \leq J(v, \Omega) \leq J(v, B_{R+2}) \leq CR^{n-1}.$$

Here, in the last step of the inequality above, we have used the fact that the infinitesimal energy

$$F(\nabla v, v, x) + W(v, x) \leq C$$

for some uniform constant C, and that it is supported only in the annulus $B_{R+2} \setminus B_R$. □

Now, let us prove Lemmas 2.2 and 2.3, which are the key steps to proving Theorem 1.1. During the proof, C denotes some universal constant, which might change from line to line.

In the first key step, we show that u is close to 1 at some $x^* \in B_\rho$ for some sufficiently large ρ.

Lemma 2.2. *Let u with $u(0) = 0$ be a minimizer to* (1.1) *satisfying the assumptions (A)–(C). Given any $h < 1$, then there exists some $\rho = \rho(h)$ such that $\max_{B_\rho} u \geq 1 - h$.*

Proof. Let us assume that $\max_{B_R} u \leq 1 - h$ for some radius $R = 2^L$ (without loss of generality, assume that $L \geq 2$ is an integer), then it suffices to find an upper bound of L depending on h. For the given h, we can choose a fixed $t_\infty = t_\infty(h)$, such that:

$$-1 < t_\infty < 0, \quad W(t_\infty) \leq W(1 - h).$$

For all $k \geq 0$, we make the following notations:

$$t_k = (1 - 2^{-k-1})t_\infty - 2^{-k-1}, \quad r_k = \frac{1 + 2^{-k}}{2}R, \quad A_k = B_{r_k} \cap \{u \geq t_k\}.$$

We consider a sequence of competitors ϕ_k. When $k \geq L - 1$, we let

$$\phi_k(x) = \mathrm{med}\Big(t_k, 1, 1 + \frac{2^{k+2}}{R}(1 - t_k)(|x| - r_k)\Big).$$

It follows that $\phi_k \equiv 1$ outside B_{r_k} and $\phi_k \equiv t_k$ inside $B_{r_{k+1}}$. Besides, as $|\nabla\phi_k| \leq C\frac{2^k}{R}$ in the annulus $B_{r_k} \setminus B_{r_{k+1}}$, we use the assumptions (A) and (B) on $F(\cdot,\cdot,\cdot)$ and $W(\cdot,\cdot)$, and have that

$$F(\nabla\phi_k, \phi_k, x) + W(\phi_k, x) \leq \lambda\Big(C\frac{2^k}{R}\Big)^p + \lambda \leq C\frac{2^{kp}}{R^p} \quad \text{in the annulus } B_{r_k} \setminus B_{r_{k+1}}, \tag{2.1}$$

where we have used $\frac{2^k}{R} = \frac{2^k}{2^L} \geq \frac{1}{2}$ in the last step of the inequality above.

When $0 \leq k \leq L-2$, we choose some $N_k \in (r_{k+1}, r_k] \cap \mathbb{Z}$ (the choice of N_k will be specified later), or equivalently:

$$N_k \in \{2^{L-1} + 2^{L-k-2} + 1, 2^{L-1} + 2^{L-k-2} + 2, \cdots, 2^{L-1} + 2^{L-k-1-2} - 1, 2^{L-1} + 2^{L-k-1-2}\}.$$

With such a choice of N_k, we set

$$\phi_k(x) = \operatorname{med}\Big(t_k, 1, 1 + (1-t_k)(|x| - N_k)\Big).$$

We then have $\phi_k \equiv 1$ outside B_{N_k}, $\phi_k \equiv t_k$ inside B_{N_k-1}. Besides, as $|\nabla\phi_k| \leq 2$ in the annulus $B_{r_k} \setminus B_{r_{k+1}}$, we use the assumptions (A) and (B) on $F(\cdot,\cdot,\cdot)$ and $W(\cdot,\cdot)$, and have that

$$F(\nabla\phi_k, \phi_k, x) + W(\phi_k, x) \leq \lambda \cdot 2^p + \lambda \leq C \quad \text{in the annulus } B_{N_k} \setminus B_{N_k-1}.$$

Denote $\Omega_k = \{u > \phi_k\}$ for $k \geq 0$, then since $u \leq 1-h$ in B_R, we see $\overline{\Omega_k} \subseteq B_{r_k}$ for $k \geq L-1$ and $\overline{\Omega_k} \subseteq B_{N_k} \subseteq B_{r_k}$ for $0 \leq k \leq L-2$. It then follows from the minimality of u that:

$$\begin{aligned} \lambda^{-1}\int_{\Omega_k} |\nabla u|^p dx \leq & J(u, \Omega_k) - \int_{\Omega_k} W(u,x)dx \leq J(\phi_k, \Omega) - \int_{\Omega_k} W(u,x)dx \\ \leq & \lambda\int_{\Omega_k} |\nabla\phi_k|^p dx + \int_{\Omega_k} \Big\{W(\phi_k, x) - W(u,x)\Big\}dx. \end{aligned}$$

Since $2^{1-p}|\vec{\xi} - \vec{\eta}|^p \leq |\vec{\xi}|^p + |\vec{\eta}|^p$ for any two vectors, we choose $\vec{\xi} = \nabla u$, $\vec{\eta} = \nabla\phi_k$, and conclude that

$$\int_{\Omega_k} |\nabla(u - \phi_k)|^p dx \leq C\int_{\Omega_k} |\nabla\phi_k|^p dx + C\int_{\Omega_k} \Big\{W(\phi_k, x) - W(u,x)\Big\}dx. \tag{2.2}$$

Since $W(t_\infty) \leq W(1-h)$ and since $u \leq 1-h$ in B_R, by the assumption (C) of $W(\tau, x)$, we see that $W(\phi_k, x) \leq W(u,x)$ when $x \in \Omega_k$ and $\phi_k = t_k \leq t_\infty$. Besides, we have $\Omega_k \subseteq A_k$. Then, we have:

- Case 1: If $0 \leq k \leq L-2$, then:

$$\begin{aligned} \int_{\Omega_k} |\nabla(u-\phi_k)|^p dx \leq & C\int_{(B_{N_k} \setminus B_{N_k-1}) \cap A_k} |\nabla\phi_k|^p dx + C\int_{(B_{N_k} \setminus B_{N_k-1}) \cap A_k} W(\phi_k, x)dx \\ \leq & C|(B_{N_k} \setminus B_{N_k-1}) \cap A_k|. \end{aligned}$$

Moreover, if we choose N_k wisely, then we even have the following estimate:

$$\int_{\Omega_k} |\nabla(u-\phi_k)|^p dx \leq C\frac{2^k}{R}|A_k|. \tag{2.3}$$

In fact, since the width of the annulus $B_{r_k} \setminus B_{r_{k+1}}$ equals $2^{L-k-2} = \frac{R}{2^{k+2}}$, and that

$$\sum_{N=r_{k+1}+1}^{r_k} |(B_N \setminus B_{N-1}) \cap A_k| = |(B_{r_k} \setminus B_{r_{k+1}}) \cap A_k| \leq |A_k|,$$

we can then choose some $N_k \in (r_{k+1}, r_k] \cap \mathbb{Z}$, such that

$$|(B_{N_k} \setminus B_{N_k-1}) \cap A_k| \leq \frac{2^{k+2}}{R}|A_k|.$$

- Case 2: If $k \geq L-1$, using the estimate (2.1), we obtain from (2.2) that:

$$\int_{\Omega_k} |\nabla(u-\phi_k)|^p dx \leq C \int_{\Omega_k} \Big\{|\nabla\phi_k|^p + W(\phi_k, x)\Big\} dx \leq C\frac{2^{kp}}{R^p}|A_k|. \tag{2.4}$$

Now, we first apply the Sobolev inequality, and then apply the Hölder inequality, and obtain that:

$$\int_{\Omega_k} |\nabla(u-\phi_k)|^p dx \geq C\Big\{\int_{\Omega_k} |u-\phi_k|^{\frac{np}{n-p}} dx\Big\}^{\frac{n-p}{n}} \geq C|\Omega_k|^{-\frac{p}{n}} \int_{\Omega_k} |u-\phi_k|^p dx.$$

Recall that $\Omega_k \subseteq A_k$ and that $\phi_k \equiv t_k$ in $B_{r_{k+1}}$, then we conclude that

$$\int_{\Omega_k} |\nabla(u-\phi_k)|^p dx \geq C|A_k|^{-\frac{p}{n}} \int_{\Omega_k \cap B_{r_{k+1}} \cap \{u \geq t_{k+1}\}} |u-t_k|^p dx \geq C|A_k|^{-\frac{p}{n}} \frac{|A_{k+1}|}{2^{kp}}. \tag{2.5}$$

We combine (2.3)–(2.5) and get

$$\begin{aligned} |A_{k+1}| \leq& C\frac{2^{k(1+p)}}{R}|A_k|^{1+\frac{p}{n}}, \quad \text{if } 0 \leq k \leq L-2,\\ |A_{k+1}| \leq& C\frac{4^{kp}}{R^p}|A_k|^{1+\frac{p}{n}}, \quad \text{if } k \geq L-1. \end{aligned}$$

Consequently, by using the observation that $\frac{4^{kp}}{R^p} \geq 2^{1-p} \cdot \frac{2^{k(1+p)}}{R}$ for $k \geq L-1$, we have the following inductive inequality for $|A_k|$'s:

$$|A_{k+1}| \leq C\frac{2^{k(1+p)}}{R}|A_k|^{1+\frac{p}{n}}, \quad \text{for all } k \geq 0. \tag{2.6}$$

Now we divide $R^{\frac{n}{p}}$ on both sides of (2.6), and denote $\beta_k = R^{-\frac{n}{p}}|A_k|$, then we have

$$\beta_{k+1} \leq C2^{k(1+p)}\beta_k^{1+\frac{p}{n}}. \tag{2.7}$$

Notice that for all $x \in A_0 \subseteq B_R$, $u(x) \geq t_0 = \frac{t_\infty - 1}{2}$ and $u(x) \leq 1-h$, then by Lemma 2.1, we have

$$\beta_0 = R^{-\frac{n}{p}}|A_0| \leq R^{-\frac{n}{p}} \frac{1}{\min\{\lambda|\frac{t_\infty+1}{2}|^m, \lambda h^m\}} \int_{B_R} W(u,x) dx \leq R^{-\frac{n}{p}} \cdot \frac{CR^{n-1}}{c(h)}.$$

When $1 < p < \frac{n}{n-1}$, we have $n-1-\frac{n}{p} < 0$. By choosing $\rho = \rho(h)$ sufficiently large, we see that $\max\limits_{B_\rho} u \geq 1-h$. In fact, suppose on the contrary that $\max\limits_{B_R} u \leq 1-h$ for some large R. Then, as

$$\lim_{R\to+\infty} \frac{CR^{n-1-\frac{n}{p}}}{c(h)} = 0,$$

we see that β_0 is sufficiently small. In other words, $\ln\beta_0$ is a negative number with a sufficiently large absolute value. We take the logarithm on both sides of the inductive inequality (2.7), and have that

$$\ln\beta_{k+1} \leq (1+\frac{p}{n})\ln\beta_k + k\cdot\ln(2^{1+p}) + \ln C.$$

As the initial data $\ln\beta_0$ is a sufficiently large negative number, we can inductively show that

$$\ln\beta_k \leq (1+\frac{p}{2n})^k \cdot \ln\beta_0.$$

As a result, $\ln\beta_k \to -\infty$ as $k\to\infty$, or equivalently, $\beta_k \to 0$. In other words, we have that $u \leq t_\infty$ almost everywhere in $B_{R/2}$, which contradicts the assumption $u(0) = 0$. □

In the second key step, we show that the positive set of u is large near some point $x^* \in B_\rho$.

Lemma 2.3. *There exists some universal constant $\sigma > 0$ and a function $h = h(R) > 0$ for all $R \geq 1$, such that the following holds: Assume that u is a minimizer to* (1.1) *satisfying the assumptions (A)–(C) and that $u(0) \geq 1-h$, then $\left|B_R \cap \{u \geq 0\}\right| \geq \sigma R^n$.*

Proof. As $W(\tau,x) \leq \lambda(1-\tau^2)^m$ with $m > p$, we choose $h = h(R) = \min\left\{(2^m\lambda R^p)^{-\frac{1}{m-p}}, \frac{1}{2}\right\}$, then

$$W(\tau,x) \leq h^p R^{-p}, \quad \text{for all } 1-2h \leq \tau \leq 1.$$

For each $h \leq a \leq 2h$, we consider a competitor

$$\phi_a(x) = \min\left\{(1-a) + \frac{4h^2|x|^2}{aR^2}, 1\right\}.$$

We can easily verify that $\{\phi_a(x) < 1\} \subseteq B_R$ and $|\nabla\phi_a| \leq \frac{8h}{R}$ everywhere. Moreover, $W(\phi_a,x) \leq h^pR^{-p}$ everywhere since $\phi_a \geq 1-2h$. Now let us denote

$$\Omega_a = \{u > \phi_a\}, \quad V_a = \int_{\Omega_a}(u-\phi_a)dx.$$

We clearly have $\overline{\Omega_a} \subseteq B_R$. Then, we deduce from the minimality of u that

$$\lambda^{-1}\int_{\Omega_a}|\nabla u|^p dx \leq J(u,\Omega_a) \leq J(\phi_a,\Omega_a) \leq \lambda\int_{\Omega_a}|\nabla\phi_a|^p dx + \int_{\Omega_a}W(\phi_a,x)dx.$$

Since $2^{1-p}|\vec{\xi}-\vec{\eta}|^p \leq |\vec{\xi}|^p + |\vec{\eta}|^p$ for any two vectors, we choose $\vec{\xi} = \nabla u$, $\vec{\eta} = \nabla\phi_a$. By applying the Hölder inequality to the function $|\nabla(u-\phi_a)|$, we conclude that:

$$\left([u-\phi_a]_{W^{1,1}(\Omega_a)}\right)^p|\Omega_a|^{1-p} \leq \int_{\Omega_a}|\nabla(u-\phi_a)|^p dx \leq C\int_{\Omega_a}\left\{|\nabla\phi_a|^p dx + W(\phi_a,x)\right\}dx \leq Ch^pR^{-p}|\Omega_a|.$$

By the Hölder inequality and the Sobolev inequality, we have

$$V_a \leq \|u-\phi_a\|_{L^{\frac{n}{n-1}}(\Omega_a)}|\Omega_a|^{\frac{1}{n}} \leq [u-\phi_a]_{W^{1,1}(\Omega_a)}|\Omega_a|^{\frac{1}{n}} \leq \frac{Ch}{R}|\Omega_a|^{1+\frac{1}{n}}. \tag{2.8}$$

On the other hand, note that $\frac{d}{d\kappa}\Big|_{\kappa=a}\phi_\kappa(x) \le -1$ for any $x \in \Omega_a$, we then have

$$\frac{d}{d\kappa}\Big|_{\kappa=a} V_\kappa = -\int_{\Omega_a} \frac{d}{d\kappa}\Big|_{\kappa=a}\phi_\kappa(x)dx \ge |\Omega_a| \ge c(\frac{R}{h}\cdot V_a)^{\frac{n}{n+1}}, \tag{2.9}$$

where we have used (2.8) in the last step. Recall that the assumption $u(0) \ge 1-h$ implies that $V_a > 0$ for all $a > h$, we can divide $V_a^{\frac{n}{n+1}}$ on both sides of (2.9) when $a > h$, and obtain the following:

$$\frac{d}{d\kappa}\Big|_{\kappa=a} V_\kappa^{\frac{1}{n+1}} \ge c(\frac{R}{h})^{\frac{n}{n+1}}, \quad \text{for all } h < a \le 2h,$$

which implies that $V_{2h} \ge cR^n h$. Notice that $u - \phi_{2h} \le 2h$ and $u \ge 0$ in Ω_{2h}, we then have

$$\Big|B_R \cap \{u \ge 0\}\Big| \ge |\Omega_{2h}| \ge \frac{V_{2h}}{2h} \ge cR^n.$$

This proves the existence of the desired uniform constant $\sigma > 0$. □

With Lemmas 2.2 and 2.3, we now prove the main result.

Proof of Theorem 1.1. Let us choose the density ε in Lemma 1.1 as the universal constant σ in Lemma 2.3. Using the two functions $r_0(\cdot)$ and $\delta(\cdot)$ obtained in Lemma 1.1, we set $\widetilde{r} = \max\{r_0(\sigma), 1\}$ and $\widetilde{\delta} = \delta(\sigma)$. Using the function $h(\cdot)$ obtained in Lemma 2.3, we set $\widetilde{h} = h(\widetilde{r})$. Using the function $\rho(\cdot)$ obtained in Lemma 2.2, we set $\widetilde{\rho} = \rho(\widetilde{h})$. Note that as the constant σ from Lemma 2.3 is universal, all other constants $\widetilde{r}, \widetilde{\delta}, \widetilde{h}, \widetilde{\rho}$ above are also universal.

By Lemma 2.2, there exists some $x^* \in B_{\widetilde{\rho}}$ such that $u(x^*) = 1 - \widetilde{h}$. Then, we apply Lemma 2.3 to the translated function $u(x - x^*)$, and conclude that

$$\Big|B_{\widetilde{r}}(x^*) \cap \{u \ge 0\}\Big| \ge \sigma\widetilde{r}^n.$$

As $\widetilde{r} \ge r_0(\sigma)$, we apply Lemma 1.1 to $u(x - x^*)$, and obtain the following:

$$\Big|B_r(x^*) \cap \{u \ge 0\}\Big| \ge \widetilde{\delta} r^n \quad \text{for all } r \ge \widetilde{r}.$$

Finally, we choose the universal constants δ and R_0 in Theorem 1.1. In fact, we set

$$R_0 = 2(\widetilde{\rho} + \widetilde{r}), \quad \delta = \frac{\sigma}{2^n}.$$

It then follows that for all $R \ge R_0$, we have $B_{R/2}(x^*) \subseteq B_R$ and $R/2 \ge \widetilde{r}$. Then,

$$\Big|B_R \cap \{u \ge 0\}\Big| \ge \Big|B_{R/2}(x^*) \cap \{u \ge 0\}\Big| \ge \sigma(\frac{R}{2})^n = \delta R^n.$$

The other inequality $\Big|B_R \cap \{u \le 0\}\Big| \ge \delta R^n$ can be argued similarly. Therefore, we have finished the proof of Theorem 1.1. □

3. Conclusions

In this paper, we have obtained the density estimate for a class of degenerate Δ_p-type Allen-Cahn equations for $1 < p < \frac{n}{n-1}$ with very little regularity assumption on the coefficients. The idea is to apply a previous result in [1] by verifying its assumptions for equations with rough coefficients. To achieve this, we derive two weak Harnack principles, and then prove the existence of a point near the origin such that the density of the positive set of u near that point is sufficiently large.

Use of Generative-AI tools declaration

The author declares that he has not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares no conflict of interest.

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