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Research article

Rate of convergence to periodic regimes in nonlinear feedback systems with strongly convex backlash characteristics†

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Abstract: This paper considers a class of hysteresis systems consisting of a linear part with an external input and feedback with a backlash nonlinearity. Assuming that the latter is specified by a strongly convex set, we establish estimates for the Lyapunov exponents which quantify the rate of convergence of the system state trajectories to a forced periodic regime when the input is a periodic function of time with a sufficiently large "amplitude". These results employ enhanced dissipation inequalities, arising from differential inclusions with strongly convex sets which were used previously for the Moreau sweeping process.

Keywords: backlash dynamics; strong convexity; dissipation inequality; periodic regime; Lyapunov exponent

1. Introduction

Forced and self-induced periodic motions of dynamical systems often arise and play an important role in electrical and mechanical engineering applications. Existence and uniqueness of periodic regimes and the rate of convergence to them are well-known for stable linear systems subject to periodic inputs. Such systems may result from a stabilising linear feedback applied to an unstable linear plant (with the linear models often obtained by linearising nonlinear dynamics). If the feedback is implemented using mechanical elements (for example, gears, levers, spring-damper units, or inerters [27]), it can be susceptible to backlashes. These effects make the resulting closed-loop system nonlinear and augment its state.

An idealised mathematical model of the backlash is considered in the theory of ordinary differential equations with hysteresis nonlinearities [20] (see also [4]). This model describes the backlash as a closed convex set whose spatial location is specified by a time-varying input, while the output is contained by this set and moves only on its contact with the boundary in a normal direction (in an inertialess sliding fashion without friction). Such dynamics form a special yet practically important case of the Moreau sweeping process [22, 23] generated by a convex-set-valued function of time. Although the resulting ODE has a discontinuous right-hand side, the initial value problem is well-posed due to the convexity of the backlash set and the properties of normal cones and metric projections onto such sets. Well-posedness and other aspects (such as stability and periodic regimes) of systems with additional dynamics as a perturbation of the Moreau process with closed convex constraint sets are discussed, for example, in [16] and references therein, including [2, 3, 21].

The presence of strong convexity [5, 6, 13, 14, 20, 24, 25], when the supporting hyperplanes and the corresponding half-spaces can be replaced with balls of finite radius, leads to the enhancement of properties which employ convexity. In particular, this yields improved (quadratic instead of linear) convergence rates for the numerical solution of differential games [11, 12] with geometric constraints specified by strongly convex sets. The condition of strong convexity for the set-valued map in the Moreau process mentioned above leads to an exponentially fast convergence [29] to periodic solutions, provided the map is periodic and has no equilibria. The corresponding Lyapunov exponent involves the arc length of the solution over the period, so that the more the periodic solution "moves", the "faster" it attracts the other trajectories. This property is an example of dissipativity which is of differential geometric nature rather than coming from energy dissipation (due to resistance or friction) in the context of electromechanical systems. Note that, beyond the strong convexity, periodic solutions and convergence to them in the sweeping processes under different conditions (such as polyhedral constraints) were studied, for example, in [7, 8, 15].

The present paper (its brief conference version was published without proofs in [28]) extends the results of [29] to a class of closed-loop systems which consist of a linear part and a feedback involving a backlash nonlinearity. The linear subsystem is a plant governed by a linear ODE with constant coefficients, which is driven by an external input and the backlash output as forcing terms. In turn, the plant output is an input to the nonlinear feedback with a strongly convex backlash. Self-induced oscillations in an autonomous version of such systems (with no external input and without the strong convexity) were studied, for example, in [30]. Assuming that in the absence of the backlash (when the backlash set is reduced to a singleton) the resulting linear system is stable, the trajectories of this linearised system (more precisely, their tubular neighbourhoods) provide a localization for those of the nonlinear system.

If the system is subject to a periodic input with a sufficiently large "amplitude", the tubular localization leads to a strictly positive lower bound for the arc length of the backlash output path over the period. Under the strong convexity condition, this bound gives rise to dissipation inequalities which involve an interplay between the (energy-related) dissipation in the linear plant, the geometric dissipativity of the strongly convex backlash, and the plant-backlash coupling. In combination with the Gronwall-Bellman lemma, these differential inequalities lead to estimates for the Lyapunov exponents quantifying the rate of convergence to periodic regimes in the nonlinear system. By an appropriate scaling, the assumption of large amplitudes for periodic external inputs can be replaced here with that of smallness of the strong convexity constant for the backlash set.

The paper is organised as follows. Section 2 specifies the class of closed-loop systems with a backlash in the feedback being considered. Section 3 discusses those initial conditions for the backlash output which remain stationary over a bounded time interval. Section 4 provides a tubular localization for trajectories of the nonlinear system about those of its linearization. Section 5 establishes asymptotic bounds for the path length and time derivative of the backlash output for periodic inputs of large amplitudes. Section 6 considers dissipation inequalities for the deviation of the system state trajectories from each other in the case of a strongly convex backlash set. Section 7 employs these dissipation relations in order to obtain bounds for the rates of convergence to periodic trajectories. Section 8 summarizes the results of the paper and outlines further directions of research. In addition to the proofs of lemmas and theorems in the main body of the paper, Appendices A and B provide auxiliary lemmas on inward normal cones for convex and strongly convex sets and a spectral bound for a class of real symmetric matrices.

2. Nonlinear systems being considered

We consider a nonlinear continuous time invariant system consisting of a linear part and a nonlinear feedback with a backlash, as shown in Figure 1 and specified below.

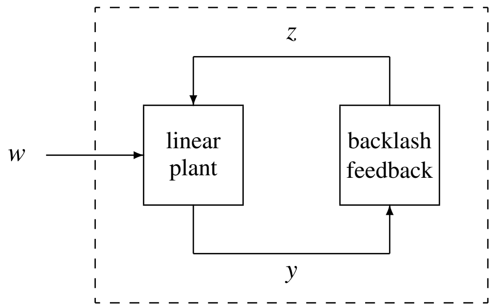

Figure 1. The closed-loop system (2.1)–(2.3) consisting of a linear part with an external input w and output y, and a nonlinear feedback with a backlash output z.

The linear part of the closed-loop system has an $\mathbb{R}^n$-valued internal state x and an $\mathbb{R}^p$-valued output y, which are driven by an $\mathbb{R}^m$-valued external input w and an $\mathbb{R}^p$-valued feedback output z according to the equations

$$\dot{x} = Ax + Bw + Ez, \qquad y = Cx, \tag{2.1}$$

where $\dot{(\ )} := \mathrm{d}/\mathrm{d}t$ is the time derivative (the time arguments will often be omitted for brevity), and $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, $E \in \mathbb{R}^{n\times p}$ are given constant matrices. At any time, the output z of the nonlinear feedback satisfies

$$z \in y + \Theta, \tag{2.2}$$

where Θ is a given convex compact in $\mathbb{R}^p$. Here, $K+L := \{u+v :\ u \in K, v \in L\}$ is the Minkowski sum of sets $K, L \subset \mathbb{R}^p$, which reduces to the usual sum of vectors if the sets are singletons. The inclusion (2.2) by itself does not specify a particular trajectory for z and can be satisfied by many functions of time. For example, any $z = y + s$, obtained by the translation of y by a constant vector $s \in \Theta$, trivially satisfies (2.2) at every moment of time. However, the effect of mechanical backlash consists in that z remains at rest while it is in the interior $\mathrm{int}(y + \Theta) = y + \mathrm{int}\Theta$ of the set on the right-hand side of (2.2) and can only be moved on its contact with the boundary $\partial(y + \Theta) = y + \partial\Theta$. Such behaviour, in a simplified (inertialess and frictionless) fashion, is reflected by the multidimensional play as one of the hysteron models [20, Chapter 4] governed by [20, Eq (16.19)]

$$\dot{z} = P_{N_{y+\Theta}(z)}(\dot{y}) = P_{N_\Theta(q)}(\dot{y}), \tag{2.3}$$

where

$$q := z - y \tag{2.4}$$

(which corresponds to the related multidimensional stop in [20, Eqs (16.23) and (16.25)]) is a function of time with values in Θ in accordance with (2.2). Here, for a closed convex set $S \subset \mathbb{R}^r$,

$$N_S(u) := \{s \in \mathbb{R}^r : \ s^{\mathrm{T}}(v - u) \geqslant 0 \text{ for all } v \in S\} \tag{2.5}$$

is the closed convex cone of inward normals to S at a point $u \in \mathbb{R}^r$ (it is always nonempty since $0 \in N_S(u)$), and

$$P_S(u) := \operatorname*{argmin}_{v \in S} |u - v| \tag{2.6}$$

is the metric projection of u onto S, with $|u| := \sqrt{u^{\mathrm{T}}u}$ the standard Euclidean norm generated by the inner product $u^{\mathrm{T}}v$ of vectors $u, v \in \mathbb{R}^r$, where $(\cdot)^{\mathrm{T}}$ is the transpose (vectors are assumed to be organised as columns unless indicated otherwise). The second equality in (2.3) follows from (2.4) and the invariance $N_S(u) = N_{S+d}(u + d)$ of the cone (2.5) with respect to the translation of S and u by a common vector $d \in \mathbb{R}^r$. The projection map (2.6) satisfies $P_{S+d}(u+d) = d+P_S(u)$ and is nonexpanding in the sense that

$$|P_S(u) - P_S(v)| \leqslant |u - v|, \qquad u, v \in \mathbb{R}^r. \tag{2.7}$$

In view of (2.3) and (2.4), $\dot{q} = P_{N_\Theta(q)}(\dot{y}) - \dot{y}$ takes values in the tangent cone $\{v \in \mathbb{R}^p : s^{\mathrm{T}}v \geqslant 0 \text{ for all } s \in N_\Theta(q)\}$ to the set Θ at the current point $q \in \Theta$, thus securing the fulfillment of (2.2) at any time. Also, (2.3) implies the differential inclusion $\dot{z} \in N_\Theta(q)$ which, in combination with (2.1), corresponds to a perturbation of the Moreau process with a constraint set Θ (see, for example, [16] and references therein on well-posedness of such systems with closed convex constraint sets). Since an augmented state of the system (2.1)–(2.3) is provided by the pair (x, z), or alternatively, by (x, q), which are related to each other by an invertible linear transformation

$$\begin{bmatrix} x \\ q \end{bmatrix} = U \begin{bmatrix} x \\ z \end{bmatrix}, \qquad U := \begin{bmatrix} I_n & 0 \\ -C & I_p \end{bmatrix} \tag{2.8}$$

(with I_r the identity matrix of order r), the initial conditions are specified by $x(0) \in \mathbb{R}^n$ and $z(0) \in Cx(0) + \Theta$ (or, equivalently, $q(0) \in \Theta$). The input w is assumed to be locally integrable:

$$\int_0^T |w(t)|\mathrm{d}t < +\infty \tag{2.9}$$

for any time horizon $T > 0$. Accordingly, x, y, z are absolutely continuous functions of time, which can be verified by using the property

$$|\dot{z}| = |P_{N_\Theta(q)}(\dot{y}) - \underbrace{P_{N_\Theta(q)}(0)}_{0}| \leqslant |\dot{y}| \tag{2.10}$$

(following from (2.3) and (2.7) applied to the cone $S := N_\Theta(q) \ni 0$) and an appropriate Gronwall-Bellman lemma estimate.

With the backlash dynamics (2.3) being a particular case of the Moreau sweeping process [22, 23] (see also [20, pp. 158–160] and [29]), the properties of the strong solutions of the latter make the backlash output z well-defined for any absolutely continuous input y (see [20] for details). This allows the solution for (2.1)–(2.3) to be obtained as the limit of the Picard iterations or, alternatively, by using the discretizations $\widehat{(\cdot)}_k$ to the moments of time t_k, forming a grid $0 = t_0 < t_1 < \ldots < t_N = t$ of the interval $[0, t]$, and consecutive projections (2.6) as in the catching-up algorithm [23, pp. 365-366] for the Moreau process (see also [17, Section 6.2]):

$$\begin{aligned}\widehat{x_k} &= \mathrm{e}^{\tau_k A}\widehat{x_{k-1}} + \int_{t_{k-1}}^{t_k} \mathrm{e}^{(t_k - s)A}(Bw(s) + E\widehat{z_{k-1}})\mathrm{d}s \\ &= \mathrm{e}^{\tau_k A}\widehat{x_{k-1}} + \int_{t_{k-1}}^{t_k} \mathrm{e}^{(t_k - s)A}Bw(s)\mathrm{d}s + \tau_k \Psi(\tau_k A)E\widehat{z_{k-1}}, \qquad \widehat{y_k} = C\widehat{x_k}, \qquad (2.11)\\ \widehat{z_k} &= P_{\widehat{y_k}+\Theta}(\widehat{z_{k-1}}) = \widehat{y_k} + P_\Theta(\widehat{q_{k-1}} + \widehat{y_{k-1}} - \widehat{y_k}), \qquad k = 1, 2, \ldots, N, \qquad (2.12)\end{aligned}$$

where $\tau_k := t_k - t_{k-1}$, and $N \to +\infty$, with $\max_{1\leqslant k\leqslant N} \tau_k \to 0$. In (2.11) (which, in practice, can be replaced with the Euler approximation $\widehat{x_k} = \widehat{x_{k-1}} + \tau_k(A\widehat{x_{k-1}} + Bw(t_{k-1}) + E\widehat{z_{k-1}})$), use is also made of an entire function

$$\Psi(u) := \int_0^1 \mathrm{e}^{uv}\mathrm{d}v = \begin{cases} 1 & \text{if } u = 0 \\ \frac{\mathrm{e}^u - 1}{u} & \text{if } u \neq 0 \end{cases} = \sum_{k\geqslant 0} \frac{u^k}{(k+1)!}, \qquad u \in \mathbb{C}. \qquad (2.13)$$

Its extension from the complex plane to square matrices [9] describes the solution $\xi(t) := t\Psi(tA)\omega$ of the ODE $\dot{\xi} = A\xi + \omega$ with a constant forcing term $\omega \in \mathbb{R}^n$ and zero initial condition $\xi(0) = 0$. If $\det A \neq 0$, this solution reduces to $\xi(t) = (\mathrm{e}^{tA} - I_n)A^{-1}\omega$. Also, (2.12) admits an equivalent form:

$$\widehat{q_k} = P_\Theta(\widehat{q_{k-1}} + \widehat{y_{k-1}} - \widehat{y_k}).$$

The resulting solution of (2.1)–(2.3) depends continuously on the initial conditions $x(0)$, $z(0)$ (or $q(0)$).

3. Stationary initial conditions for the backlash output

As mentioned above, the backlash output z can be displaced only on its contact with the boundary $y + \partial\Theta$ of the set on the right-hand side of (2.2). Indeed, $\dot{z} = 0$ as long as $z \in y + \mathrm{int}\Theta$, which follows from (2.3)–(2.6) and the fact that the cone (2.5) reduces to the singleton $N_S(u) = \{0\}$ for any $u \in \mathrm{int}S$. Until such a contact of z with $y + \partial\Theta$ takes place, the closed-loop system (2.1) behaves like a linear one with a constant z. Moreover, for any time horizon $T > 0$, all those initial conditions $z(0)$ of the backlash, for which its output z in (2.1)–(2.4) remains constant over the interval $[0, T]$, form a set

$$\begin{aligned}\Re(T) &:= \{z(0) \in \mathbb{R}^p : z(t) = z(0) \text{ for all } t \in [0, T]\} \\ &= \Big\{z(0) \in \mathbb{R}^p : z(0) \in \bigcap_{0\leqslant t\leqslant T} (y(t) + \Theta)\Big\}. \qquad (3.1)\end{aligned}$$

Here, the intersection inherits from Θ the property of being a convex compact in $\mathbb{R}^p$ and (in addition to its dependence on T) depends on the initial conditions $x(0)$, $z(0)$ and the past history $w|_{[0,T]}$ of the external input through the function y from (2.1). Accordingly, the set $\Re(T)$ in (3.1) depends on $x(0)$ and $w|_{[0,T]}$. Any point $z(0) \in \Re(T)$ gives rise to a constant forcing term $Ez(0)$ for the ODE in (2.1) over the time interval $[0,T]$ and will be referred to as a *T-stationary initial condition* for the backlash output. A representation for such initial conditions is provided by the following theorem.

Theorem 1. *For a given time horizon $T > 0$, suppose the matrix*

$$\Phi(t) := I_p - tC\Psi(tA)E, \tag{3.2}$$

associated with the matrices A, C, E in (2.1) and the matrix extension of the function Ψ from (2.13), satisfies

$$\det \Phi(t) \neq 0, \qquad t \in [0,T]. \tag{3.3}$$

Then the set (3.1) of T-stationary initial conditions for the backlash output z can be represented as

$$\Re(T) = \bigcap_{0 \leqslant t \leqslant T} (\Phi(t)^{-1}(Cx_*(t) + \Theta)), \tag{3.4}$$

where

$$x_*(t) := \mathrm{e}^{tA}x(0) + \int_0^t \mathrm{e}^{(t-s)A}Bw(s)\mathrm{d}s \tag{3.5}$$

is the solution of the ODE in (2.1) without the term Ez on its right-hand side (that is, in the absence of the feedback).

Proof. In view of (3.1), the inclusion $z(0) \in \Re(T)$ is equivalent to

$$z(0) \in y(t) + \Theta, \qquad 0 \leqslant t \leqslant T, \tag{3.6}$$

with the linear subsystem (2.1) being driven by the external input w and the constant backlash output $z(0)$ over the time interval $[0,T]$. Therefore, similarly to (2.11),

$$\begin{aligned} x(t) &= \mathrm{e}^{tA}x(0) + \int_0^t \mathrm{e}^{(t-s)A}(Bw(s) + Ez(0))\mathrm{d}s \\ &= x_*(t) + t\Psi(tA)Ez(0), \qquad 0 \leqslant t \leqslant T, \end{aligned} \tag{3.7}$$

where the function x_* is given by (3.5). Substitution of (3.7) into the second equality in (2.1) represents (3.6) as

$$\begin{aligned} z(0) &\in C(x_*(t) + t\Psi(tA)Ez(0)) + \Theta \\ &= tC\Psi(tA)Ez(0) + Cx_*(t) + \Theta, \qquad 0 \leqslant t \leqslant T. \end{aligned} \tag{3.8}$$

In view of (3.2), the inclusion in (3.8) holds if and only if $\Phi(t)z(0) \in Cx_*(t) + \Theta$, which is equivalent to

$$z(0) \in \Phi(t)^{-1}(Cx_*(t) + \Theta), \qquad 0 \leqslant t \leqslant T, \tag{3.9}$$

under the condition (3.3). In turn, (3.9) is equivalent to $z(0) \in \bigcap_{0 \leqslant t \leqslant T}(\Phi(t)^{-1}(Cx_*(t) + \Theta))$, which establishes (3.4). □

The definition of the set $\mathfrak{R}(T)$ in (3.1) (or its representation (3.4)) shows that the set-valued function $\mathfrak{R}$ is nonincreasing: $\mathfrak{R}(t) \subset \mathfrak{R}(s)$ for all $t \geqslant s \geqslant 0$. Therefore, if $\mathfrak{R}(T) = \emptyset$ for some $T > 0$, then $\mathfrak{R}(t) = \emptyset$ for all $t \geqslant T$. In this case, the backlash has no T-stationary initial conditions and its output is activated by the contact with the boundary over the time interval $[0, T]$, thus leading to

$$\int_0^T |\dot{z}(t)|\mathrm{d}t \geqslant \int_{t_1}^{t_2} |\dot{z}(t)|\mathrm{d}t \geqslant |z(t_1) - z(t_2)| > 0, \tag{3.10}$$

where t_1, t_2 are any moments of time such that $0 \leqslant t_1 < t_2 \leqslant T$ and $z(t_1) \neq z(t_2)$. A specific lower estimate for the path length of the backlash output in (3.10) can be obtained by using a localization of the system trajectories.

4. Tubular localization of system trajectories

If the backlash set Θ were a singleton ($\{0\}$ for simplicity) then (2.2) would lead to $z = y$, and the system would be governed by a linear ODE

$$\dot{\xi} = A\xi + Bw + E\eta = F\xi + Bw, \qquad \eta := C\xi, \tag{4.1}$$

with the dynamics matrix

$$F := A + EC. \tag{4.2}$$

Here, the term EC pertains to the feedback, which is absent, for example, if at least one of the matrices C or E vanishes. Although the original system is nonlinear in the case of a nontrivial set Θ, the state x of its linear subsystem (2.1) admits a localization about the trajectory of the "linearised" closed-loop system specified by (4.1), (4.2) with the same initial condition $\xi(0) := x(0)$. To this end, we will use the bijective correspondence between convex compacts $S \subset \mathbb{R}^r$ and their support functions [26]

$$\sigma_S(u) := \max_{v \in S}(u^{\mathrm{T}}v), \qquad u \in \mathbb{R}^r. \tag{4.3}$$

These functions lend themselves to closed-form computation, for example, in the case of ellipsoids. Indeed, for an ellipsoid $S := \{v \in \mathbb{R}^r : \|v - c\|_{\Sigma^{-1}} \leqslant 1\}$ with centre $c \in \mathbb{R}^r$ and "matrix radius" $\sqrt{\Sigma}$ (where $\Sigma = \Sigma^{\mathrm{T}} \in \mathbb{R}^{r\times r}$ is a positive definite matrix), the support function (4.3) takes the form $\sigma_S(u) = \max_{v\in\mathbb{R}^r:\ |v|\leqslant 1}(u^{\mathrm{T}}(c + \sqrt{\Sigma}v)) = u^{\mathrm{T}}c + \|u\|_\Sigma$, where $\|u\|_\Sigma := |\sqrt{\Sigma}u| = \sqrt{u^{\mathrm{T}}\Sigma u}$ is a "weighted" Euclidean norm. The following lemma employs a replacement of a linear ODE with a differential inclusion (with a convex compact-valued right-hand side) from [30, Theorem 1] (extensions of similar ideas to the case of nonlinear ODEs can be found in [18] and references therein).

Lemma 1. *Suppose the input w of the system (2.1)–(2.3) is locally integrable. Then, at any time $t \geqslant 0$, the system state satisfies*

$$x(t) \in \xi(t) + \Xi(t). \tag{4.4}$$

Here,

$$\xi(t) := \mathrm{e}^{tF}x(0) + \int_0^t \mathrm{e}^{(t-s)F}Bw(s)\mathrm{d}s \tag{4.5}$$

is the solution of the ODE in (4.1) with the initial condition $\xi(0) := x(0)$. Also, $\Xi(t)$ is a time-varying convex compact in $\mathbb{R}^n$ whose support function (4.3) is linearly related to that of the convex compact backlash set Θ:

$$\sigma_{\Xi(t)}(u) = \int_0^t \sigma_\Theta(E^{\mathrm{T}} \mathrm{e}^{sF^{\mathrm{T}}} u)\mathrm{d}s, \qquad u \in \mathbb{R}^n. \tag{4.6}$$

Proof. In view of (4.2), the ODE in (2.1) can be represented as

$$\dot{x} = Ax + Bw + E(y + q) = Fx + Bw + Eq, \tag{4.7}$$

where q is the Θ-valued function of time given by (2.4). At any time $t \geqslant 0$, the solution of (4.7) satisfies

$$x(t) - \xi(t) = \int_0^t \mathrm{e}^{sF} Eq(t-s)\mathrm{d}s \tag{4.8}$$

in view of (4.5) and the linear superposition property for solutions of linear ODEs. Now, similarly to [30, Proof of Theorem 1], for any $u \in \mathbb{R}^n$,

$$\sup_q \Big(u^{\mathrm{T}} \int_0^t \mathrm{e}^{sF} Eq(t-s)\mathrm{d}s\Big) \leqslant \int_0^t \max_{v\in\Theta}(u^{\mathrm{T}} \mathrm{e}^{sF} Ev)\mathrm{d}s = \int_0^t \sigma_\Theta(E^{\mathrm{T}} \mathrm{e}^{sF^{\mathrm{T}}} u)\mathrm{d}s, \tag{4.9}$$

where the supremum is over measurable functions $q : [0,t] \to \Theta$. Moreover, for any given $t > 0$, the inequality in (4.9) holds as an equality, and its right-hand side is the support function of a convex compact $\Xi(t) \subset \mathbb{R}^n$ which is the closure of the bounded convex set of values of $J(v) := \int_0^t M(s)v(s)\mathrm{d}s$ at measurable functions $v : [0,t] \to \Theta$, where $M \in C([0,t], \mathbb{R}^{n\times p})$ is an auxiliary continuous matrix-valued function given by $M(s) := \mathrm{e}^{sF}E$. The same set $\Xi(t)$ results from using a narrower class of piece-wise constant functions v given by $v(s) = v_k \in \Theta$ for any $s \in [s_{k-1}, s_k)$ and $k = 1, \ldots, \nu$, with $0 = s_0 < s_1 < \ldots < s_\nu = t$. In this case, $J(v) = \sum_{k=1}^{\nu} M_k v_k$ can take any value from the Minkowski sum $J_\nu := \oplus_{k=1}^{\nu}(M_k\Theta)$ of linearly transformed copies of the set Θ, where $M_k := \int_{s_{k-1}}^{s_k} M(s)\mathrm{d}s$, so that $\sigma_{J_\nu}(u) = \sum_{k=1}^{\nu} \sigma_\Theta(M_k^{\mathrm{T}} u) \to \int_0^t \sigma_\Theta(M(s)^{\mathrm{T}} u)\mathrm{d}s$, as $\nu \to +\infty$ and $\max_{1\leqslant k\leqslant \nu}(s_k - s_{k-1}) \to 0$, with the convergence to the integral being uniform with respect to u over any compact in $\mathbb{R}^n$ due to the uniform continuity of M on $[0,t]$. Therefore, the vector on the right-hand side of (4.8) belongs to $\Xi(t)$ with the support function (4.6), thus establishing (4.4). □

The above proof shows that Lemma 1 employs only the inclusion (2.2) and is valid regardless of the specific backlash dynamics (2.3). For simplicity, it is assumed in what follows that

$$0 \in \Theta, \tag{4.10}$$

and hence, $\sigma_\Theta(u) \geqslant 0$ for all $u \in \mathbb{R}^p$. In this case, the right-hand side of (4.6) is nondecreasing in time t, and so also is the set-valued map Ξ (that is, $\Xi(t) \supset \Xi(s)$ for all $t \geqslant s \geqslant 0$). This leads to a bounded limit

$$\Xi_\infty := \operatorname{clos} \bigcup_{t\geqslant 0} \Xi(t) \tag{4.11}$$

(with $\operatorname{clos}(\cdot)$ the closure of a set), provided the matrix F in (4.2) is Hurwitz. In view of (4.6), the support function of the limit set (4.11) is

$$\sigma_{\Xi_\infty}(u) = \int_0^{+\infty} \sigma_\Theta(E^{\mathrm{T}} \mathrm{e}^{sF^{\mathrm{T}}} u)\mathrm{d}s, \qquad u \in \mathbb{R}^n. \tag{4.12}$$

Note that the convergence in (4.11) is exponentially fast. More precisely, (4.6) and (4.12) imply that

$$\begin{aligned} D(\Xi_\infty, \Xi(t)) &= \max_{u\in\mathcal{S}^{n-1}} |\sigma_{\Xi_\infty}(u) - \sigma_{\Xi(t)}(u)| \\ &= \max_{u\in\mathcal{S}^{n-1}} \int_t^{+\infty} \sigma_\Theta(E^{\mathrm{T}}\mathrm{e}^{sF^{\mathrm{T}}}u)\mathrm{d}s \\ &\leqslant \int_t^{+\infty} \max_{u\in\mathcal{S}^{n-1}} \sigma_\Theta(E^{\mathrm{T}}\mathrm{e}^{sF^{\mathrm{T}}}u)\mathrm{d}s \leqslant D(\Theta,\{0\})\|E\| \int_t^{+\infty} \|\mathrm{e}^{sF}\|\mathrm{d}s, \end{aligned} \tag{4.13}$$

where $\mathcal{S}^{n-1} := \{u \in \mathbb{R}^n :\ |u| = 1\}$ is the unit sphere in $\mathbb{R}^n$. Here, use is made of the Hausdorff deviation

$$D(L, M) := \sup_{u\in L} \rho(u, M) \tag{4.14}$$

of a set $L \subset \mathbb{R}^r$ from another set $M \subset \mathbb{R}^r$, with

$$\rho(u, M) := \inf_{v\in M} |u - v| \tag{4.15}$$

denoting the distance from a point $u \in \mathbb{R}^r$ to M. In particular, in view of (4.3), (4.14) and (4.15),

$$D(\Theta, \{0\}) = \max_{v\in\Theta} |v| = \max_{u\in\mathcal{S}^{n-1}} \sigma_\Theta(u) \tag{4.16}$$

quantifies the deviation of the backlash set Θ from the origin. Also, $\|\cdot\|$ in (4.13) is the operator norm of matrices (induced by the Euclidean vector norm $|\cdot|$), and its submultiplicativity is used. From (4.13), it follows that $\limsup_{t\to+\infty}(\frac{1}{t}\ln D(\Xi_\infty, \Xi(t))) \leqslant \mu$, where

$$\mu := \max_{1\leqslant k\leqslant n} \mathrm{Re}\lambda_k = \ln \mathbf{r}(\mathrm{e}^F) < 0 \tag{4.17}$$

is the largest real part of the eigenvalues $\lambda_1, \dots, \lambda_n$ of the Hurwitz matrix F in (4.2), with $\mathbf{r}(\cdot)$ the spectral radius of a square matrix. Accordingly, $\frac{1}{|\mu|}$ quantifies the largest decay time for transient processes in the linearised system (4.1). Therefore, on time scales $t \gg \frac{1}{|\mu|}$, the inclusion

$$x(t) \in \xi(t) + \Xi_\infty \tag{4.18}$$

(obtained by replacing the set $\Xi(t)$ in (4.4) with its limit Ξ_∞ from (4.11)) is only slightly more conservative than (4.4). The right-hand side of (4.18) can be viewed as a "tube" about the trajectory ξ of the linearised system in (4.5), with its "cross section" being specified by the set Ξ_∞. Note that Ξ_∞ does not depend on the input w which enters the right-hand side of (4.18) only through ξ.

5. Backlash output activation for periodic inputs

Consider the system dynamics when the external input w is a T-periodic bounded function of time, with $T > 0$. More precisely, suppose w satisfies $w(t + T) = w(t)$ for all $t \geqslant 0$, and $w|_{[0,T]}$ belongs to the Banach space $L^\infty([0, T], \mathbb{R}^m)$ with the norm $\|w\|_\infty := \operatorname{ess\,sup}_{0\leqslant t\leqslant T} |w(t)|$, whereby the local integrability condition (2.9) is also satisfied. With the matrix F in (4.2) being assumed to be Hurwitz, the ODE in (4.1) has a unique T-periodic solution ξ_T with the initial condition

$$\xi_T(0) = (I_n - \mathrm{e}^{TF})^{-1} \int_0^T \mathrm{e}^{(T-t)F} Bw(s)\mathrm{d}s. \tag{5.1}$$

Substitution of (5.1) for $x(0)$ into (4.5) represents this solution in the form

$$\begin{aligned}\xi_T(t) &= \mathrm{e}^{tF}(I_n - \mathrm{e}^{TF})^{-1}\int_0^T \mathrm{e}^{(T-s)F}Bw(s)\mathrm{d}s + \int_0^t \mathrm{e}^{(t-s)F}Bw(s)\mathrm{d}s\\ &= \mathrm{e}^{TF}(I_n - \mathrm{e}^{TF})^{-1}\int_0^T \mathrm{e}^{(t-s)F}Bw(s)\mathrm{d}s + \int_0^t \mathrm{e}^{(t-s)F}Bw(s)\mathrm{d}s\\ &= ((I_n - \mathrm{e}^{TF})^{-1} - I_n)\int_0^T \mathrm{e}^{(t-s)F}Bw(s)\mathrm{d}s + \int_0^t \mathrm{e}^{(t-s)F}Bw(s)\mathrm{d}s\\ &= \int_0^T ((I_n - \mathrm{e}^{TF})^{-1} - \chi_{[t,T]}(s)I_n)\mathrm{e}^{(t-s)F}Bw(s)\mathrm{d}s \end{aligned} \tag{5.2}$$

for all $t \in [0, T]$, where use is made of the identity $\mathrm{e}^{tF}(I_n - \mathrm{e}^{TF})^{-1}\mathrm{e}^{TF} = ((I_n - \mathrm{e}^{TF})^{-1} - I_n)\mathrm{e}^{tF}$, and $\chi_S(\cdot)$ is the indicator function of a set S. Since the linearised system (4.1) is stable and the external input w is T-periodic, then ξ_T in (5.2) is a T-periodic global attractor for the system state ξ in the sense that

$$\lim_{t\to+\infty} |\xi(t) - \xi_T(t)| = 0 \tag{5.3}$$

for any initial condition $\xi(0)$. A similar property holds for the corresponding T-periodic output

$$\eta_T := C\xi_T \tag{5.4}$$

with respect to the output η of the linear system (4.1). In view of (4.18), all those initial conditions $x(0)$, $z(0)$, which give rise to T-periodic trajectories of the nonlinear system (2.1)–(2.3), belong to the convex compact

$$x(0) \in \xi_T(0) + \Xi_\infty, \qquad z(0) \in Cx(0) + \Theta \subset \eta_T(0) + C\Xi_\infty + \Theta. \tag{5.5}$$

The Poincare map, associated with the nonlinear system, is continuous and, by the Brouwer fixed-point theorem [1], has at least one fixed point in the set (5.5). We will be concerned with conditions which secure uniqueness for the forced T-periodic regime in the nonlinear system and exponentially fast convergence to it. To this end, the following theorem obtains asymptotic bounds for the backlash output path length (see also [29] for a similar bound) and its time derivative by using the tubular localization from Section 4. For its formulation, we denote by

$$\mathrm{diam}(S) := \sup_{u,v\in S} |u - v| \tag{5.6}$$

the diameter of a bounded set $S \subset \mathbb{R}^r$. Accordingly, the oscillation of a vector-valued function f on a set K is the diameter

$$\Omega_K(f) := \sup_{s,t\in K} |f(s) - f(t)| = \mathrm{diam}(f(K)) \tag{5.7}$$

of the image $f(K) := \{f(t) : t \in K\}$ of K under the map f.

Theorem 2. *Suppose the matrix F of the linearised system in (4.1), (4.2) is Hurwitz, and the backlash set Θ satisfies (4.10). Also, let the nonlinear system (2.1)–(2.3) be driven by a T-periodic bounded input w. Then, for any initial condition of the system, the path length of the backlash output z satisfies*

$$\liminf_{\tau\to+\infty} \int_\tau^{\tau+T} |\dot z(t)|\mathrm{d}t \geqslant (\mho - d)_+, \tag{5.8}$$

where the positive cutoff $(\cdot)_+ := \max(\cdot, 0)$ *is used since the left-hand side of the inequality is nonnegative. Here,*

$$\mho := \Omega_{[0,T]}(\eta_T) \tag{5.9}$$

is the oscillation (5.7) of the T*-periodic output* η_T *of the linearised system given by (5.2), (5.4), and*

$$d := \mathrm{diam}(C\Xi_\infty + \Theta) \tag{5.10}$$

is the diameter (5.6) of the set $C\Xi_\infty + \Theta$ *associated with* Ξ_∞ *in (4.11) and (4.12). Furthermore, the backlash output satisfies*

$$\limsup_{\tau\to+\infty} \int_\tau^{\tau+T} |\dot{z}(t)|^2 \mathrm{d}t \leqslant (\|\dot{\eta}_T\|_2 + \sqrt{T}\Delta)^2, \tag{5.11}$$

$$\limsup_{t\to+\infty} |\dot{z}(t)| \leqslant \|\dot{\eta}_T\|_\infty + \Delta, \tag{5.12}$$

where $\|\zeta\|_2 := \sqrt{\int_0^T |\zeta(t)|^2 \mathrm{d}t}$ *is the norm in the Hilbert space* $L^2([0,T], \mathbb{R}^p)$ *of square integrable functions, and*

$$\Delta := D(C(F\Xi_\infty + E\Theta), \{0\}) \tag{5.13}$$

is defined similarly to (4.16).

Proof. From (2.4) and the second equalities in (2.1) and (4.1), it follows that

$$z - \eta = C(x - \xi) + q \in C\Xi_\infty + \Theta, \tag{5.14}$$

where use is made of (4.18) under the conditions (4.10) and the matrix F in (4.2) being Hurwitz. A combination of the triangle inequality with (5.14) and (5.10) leads to

$$\begin{aligned} |\eta(s) - \eta(t)| &= |z(s) - z(t) + z(t) - \eta(t) - (z(s) - \eta(s))| \\ &\leqslant |z(s) - z(t)| + |z(t) - \eta(t) - (z(s) - \eta(s))| \\ &\leqslant |z(s) - z(t)| + d \end{aligned} \tag{5.15}$$

for all $s, t \geqslant 0$. By taking the supremum on both sides of (5.15) with respect to $s, t \in K$ over a time interval K, it follows that the corresponding oscillations (5.7) of the functions η, z satisfy

$$\Omega_K(\eta) \leqslant \Omega_K(z) + d \leqslant \int_K |\dot{z}(t)| \mathrm{d}t + d, \tag{5.16}$$

so that

$$\int_K |\dot{z}(t)| \mathrm{d}t \geqslant (\Omega_K(\eta) - d)_+. \tag{5.17}$$

Since η_T in (5.4) is a T-periodic attractor for the output η of the stable linear system (4.1) with the T-periodic external input w (in the sense that $\lim_{t\to+\infty} |\eta(t) - \eta_T(t)| = 0$), then

$$\lim_{\tau\to+\infty} \Omega_{[\tau,\tau+T]}(\eta) = \Omega_{[0,T]}(\eta_T). \tag{5.18}$$

The inequality (5.8) is now obtained by combining (5.17) with (5.18) and using (5.9). In order to prove (5.12), we note that

$$\begin{aligned}\dot{y} &= C\dot{x} = C(Fx + Bw + Eq)\\ &= C(F\xi_T + Bw + F(\xi - \xi_T) + F(x - \xi) + Eq)\\ &= C\dot{\xi}_T + C(F(\xi - \xi_T) + F(x - \xi) + Eq)\\ &\in \dot{\eta}_T + CF(\xi - \xi_T) + C(F\Xi_\infty + E\Theta),\end{aligned} \tag{5.19}$$

where use is made of (4.1), (4.7) and (4.18). Since $\dot{\eta}_T$ is a T-periodic bounded function of time, a combination of (5.19) with (5.3) leads to

$$\limsup_{t\to+\infty} |\dot{y}(t)| \leqslant \|\dot{\eta}_T\|_\infty + \Delta, \tag{5.20}$$

where the rightmost term is given by (5.13). The relation (5.12) follows from (5.20) in view of (2.10). By a similar reasoning, (2.10), (5.3), (5.13), and (5.19) yield

$$\begin{aligned}\limsup_{\tau\to+\infty} \int_\tau^{\tau+T} |\dot{z}(t)|^2 \mathrm{d}t &\leqslant \limsup_{\tau\to+\infty} \int_\tau^{\tau+T} |\dot{y}(t)|^2 \mathrm{d}t\\ &\leqslant \int_0^T (|\dot{\eta}_T(t)| + \Delta)^2 \mathrm{d}t \leqslant (\|\dot{\eta}_T(t)\|_2 + \sqrt{T}\Delta)^2,\end{aligned}$$

which establishes (5.11). □

The lower bound (5.8) provided by Theorem 2 pertains to a general problem of finding the minimum arc length of a curve in a tubular neighbourhood of another curve. A similar argument, using (5.15) and (5.16), leads to

$$\liminf_{\tau\to+\infty} \int_\tau^{\tau+T} |\dot{z}(t)| \mathrm{d}t \geqslant \sup \sum_{k=1}^N (|\eta_T(t_k) - \eta_T(t_{k-1})| - d)_+, \tag{5.21}$$

where the supremum is taken over all partitions $0 = t_0 < t_1 < \ldots < t_N = T$ of the interval $[0, T]$ into $N = 1, 2, 3, \ldots$ subintervals. In particular, if the points $\eta_T(t_1), \ldots, \eta_T(t_N)$ are centres of pairwise disjoint open balls of radius $\epsilon > 0$, thus giving rise to an ϵ-packing of the set $\eta_T([0, T]) \subset \mathbb{R}^p$, then the corresponding sum on the right-hand side of (5.21) satisfies $\sum_{k=1}^N (|\eta_T(t_k) - \eta_T(t_{k-1})| - d)_+ \geqslant (2\epsilon - d)_+ N$. Therefore,

$$\liminf_{\tau\to+\infty} \int_\tau^{\tau+T} |\dot{z}(t)| \mathrm{d}t \geqslant \sup_{\epsilon > d/2} ((2\epsilon - d) N_\epsilon), \tag{5.22}$$

where N_ϵ is the largest cardinality of an ϵ-packing of the set $\eta_T([0, T])$ (so that $\log_2 N_\epsilon$ is the ϵ-capacity [19] of this set). Note that the right-hand side of (5.22) is amenable to asymptotic analysis as $d \to 0+$.

Now, the right-hand side of the inequality (5.8) is positive (and thus provides a nontrivial lower bound for its left-hand side) if the external input w has a sufficiently large "amplitude" in the sense that the oscillation $\mho$ in (5.9) exceeds the quantity d in (5.10) which does not depend on w; see Figure 2. In this case, the backlash output z has no stationary initial conditions (see Section 3) and is eventually forced to move (its path length over any sufficiently distant time interval of duration T is separated from zero). This backlash motion gives rise to dissipation inequalities for the system trajectories under additional geometric constraints considered below.

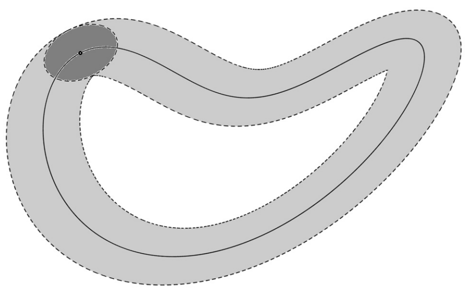

Figure 2. An illustration of the tubular localization (the grey area between the dashed lines) for periodic trajectories z of the original nonlinear system about the periodic trajectory η_T (solid curve) for the linearised system. The ellipse exemplifies the set $C\Xi_\infty + \Theta$ in (5.10) associated with Ξ_∞ from (4.11) and (4.12).

6. Dissipation relations using strong convexity

For what follows, suppose the backlash set Θ in (2.2), (2.3) is a strongly convex subset of $\mathbb{R}^p$ with the *strong convexity constant*

$$R := \inf\Big\{r \geqslant 0 :\ \Theta \subset \bigcap_{u\in\partial\Theta,\ v\in N_\Theta(u)\bigcap \mathcal{S}^{p-1}} \overline{B}_r(u + rv)\Big\} < +\infty, \tag{6.1}$$

where $\overline{B}_r(c) := \operatorname{clos} B_r(c)$ is the closed ball with centre c and radius r. Therefore, R is the smallest r such that for any supporting hyperplane for the set Θ at any given boundary point $u \in \partial\Theta$ with a unit inward normal $v \in N_\Theta(u)\bigcap \mathcal{S}^{p-1}$ (see (2.5)), the closed ball $\overline{B}_r(u + rv)$, supported at u by the same hyperplane, contains Θ. This definition of strong convexity is essentially equivalent to those in [5, 24] and [20, pp. 164-165]. Alternatively, the quantity R in (6.1) is the smallest r for which Θ can be represented as an intersection of closed balls of radius r.

Now, consider two trajectories of the underlying nonlinear system (2.1)–(2.3) driven by the same input w, but with different initial conditions $x_k(0) \in \mathbb{R}^n$ and $z_k(0) \in y_k(0) + \Theta$ (or $q_k(0) \in \Theta$), where $k = 1, 2$. The deviations of the corresponding functions x_k, y_k, z_k, q_k for these trajectories are given by

$$X := x_1 - x_2, \qquad Y := y_1 - y_2 = CX, \tag{6.2}$$

$$Z := z_1 - z_2, \qquad Q := q_1 - q_2 = Z - Y, \tag{6.3}$$

which, as functions of time, satisfy the ODEs

$$\dot X = \dot x_1 - \dot x_2 = AX + EZ = FX + EQ, \tag{6.4}$$

$$\dot Z = \dot z_1 - \dot z_2 = P_{N_\Theta(q_1)}(\dot y_1) - P_{N_\Theta(q_2)}(\dot y_2) \tag{6.5}$$

in accordance with (2.3), (4.2) and (4.7). Also, we associate with the backlash outputs z_1, z_2 a locally integrable nonnegative function of time by

$$\gamma := \frac{1}{2R}(|\dot{z}_1| + |\dot{z}_2|), \tag{6.6}$$

where R is the strong convexity constant from (6.1). The function γ, originating from the strong convexity of the backlash set Θ, plays an important role (similar to [29, Eqs (5)–(7)]) in the following dissipation relations which will be used in order to quantify the rate of convergence of the system trajectories to each other.

Lemma 2. *Under the strong convexity condition (6.1) on the backlash set Θ, the function Q, defined by (6.2) and (6.3), satisfies a differential inequality*

$$\frac{1}{2}(|Q|^2)^{\cdot} \leqslant -X^{\mathrm{T}}F^{\mathrm{T}}C^{\mathrm{T}}Q - (\alpha + \gamma)|Q|^2, \tag{6.7}$$

with $\alpha \in \mathbb{R}$ depending on the coupling between the linear part of the system and the backlash in (2.1) as

$$\alpha := \lambda_{\min}(\mathbf{S}(CE)). \tag{6.8}$$

Here, $\lambda_{\min}(\cdot)$ is the smallest eigenvalue of a real symmetric matrix, $\mathbf{S}(M) := \frac{1}{2}(M + M^{\mathrm{T}})$ is the symmetrizer of square matrices, and γ is given by (6.6).

Proof. Due to the strong convexity of the backlash set Θ with the constant R and the inclusions $\dot{z}_k \in N_\Theta(q_k)$ in the corresponding inward normal cones for $k = 1, 2$, application of (A.4) of Lemma 3 to (6.5) leads to

$$Q^{\mathrm{T}}\dot{Z} \leqslant -\gamma|Q|^2, \tag{6.9}$$

where γ is given by (6.6). By substituting $Z = Q + Y$ from (6.3) into (6.9) and using the identity $Q^{\mathrm{T}}\dot{Q} = \frac{1}{2}(|Q|^2)^{\cdot}$, it follows that

$$\frac{1}{2}(|Q|^2)^{\cdot} + Q^{\mathrm{T}}\dot{Y} \leqslant -\gamma|Q|^2. \tag{6.10}$$

Since a combination of (6.2) with (6.4) yields $\dot{Y} = CFX + CEQ$, then (6.10) can be represented as

$$\frac{1}{2}(|Q|^2)^{\cdot} \leqslant -X^{\mathrm{T}}F^{\mathrm{T}}C^{\mathrm{T}}Q - Q^{\mathrm{T}}(\mathbf{S}(CE) + \gamma I_p)Q. \tag{6.11}$$

Here, use is also made of the identity $v^{\mathrm{T}}Mv = v^{\mathrm{T}}\mathbf{S}(M)v$ for any compatibly dimensioned square matrix M and vector v. It now remains to note that $Q^{\mathrm{T}}\mathbf{S}(CE)Q \geqslant \alpha|Q|^2$ in view of (6.8), and hence, (6.11) implies (6.7). □

Note that the usual (rather than strong) convexity of the set Θ corresponds formally to letting $R \to +\infty$, in which case, (6.6) yields $\gamma = 0$ and the inequality (6.9) reduces to its weaker counterpart $Q^{\mathrm{T}}\dot{Z} \leqslant 0$, whereby (6.7) is replaced with $\frac{1}{2}(|Q|^2)^{\cdot} \leqslant -X^{\mathrm{T}}F^{\mathrm{T}}C^{\mathrm{T}}Q - \alpha|Q|^2$. Moreover, this can be justified rigorously by using the inequality (A.1) from Lemma 3 for the convexity case in the proof of Lemma 2 instead of its enhanced version (A.4) which corresponds to strong convexity.

Returning to the strong convexity case under consideration, we will now use the condition that the matrix F in (4.2) is Hurwitz, thus giving rise to the negative quantity μ in (4.17). Then for any scalar λ such that

$$0 < \lambda < |\mu|, \tag{6.12}$$

the matrix $F_\lambda := F + \lambda I_n$ is also Hurwitz, and hence, there exists a real positive definite symmetric matrix Π of order n satisfying the algebraic Lyapunov inequality $\Pi F + F^{\mathrm{T}}\Pi + 2\lambda\Pi = \Pi F_\lambda + F_\lambda^{\mathrm{T}}\Pi \prec 0$, which is equivalent to

$$\mathbf{S}(\Pi F) = \frac{1}{2}(\Pi F + F^{\mathrm{T}}\Pi) \prec -\lambda\Pi. \tag{6.13}$$

For such a matrix Π, whose choice depends on the matrix F and the quantity λ from (6.12) through the condition (6.13), consider the following Lyapunov function candidate for the system trajectories in (6.2) and (6.3):

$$V := \frac{1}{2}(\|X\|_\Pi^2 + |Q|^2) = \frac{1}{2}\left\|\begin{bmatrix} X \\ Q \end{bmatrix}\right\|_\Gamma^2, \tag{6.14}$$

where

$$\Gamma := \begin{bmatrix} \Pi & 0 \\ 0 & I_p \end{bmatrix} = \Gamma^{\mathrm{T}} \succ 0 \tag{6.15}$$

is an auxiliary matrix of order $n + p$, which inherits its positive definiteness from Π. Another auxiliary $(n \times p)$-matrix to be used in what follows is

$$H := \frac{1}{2}(\Pi E - F^{\mathrm{T}}C^{\mathrm{T}}). \tag{6.16}$$

In order to establish a dissipation inequality for V, we will also use a function $\psi : \mathbb{R}_+ \to \mathbb{R}$ (with $\mathbb{R}_+ := [0, +\infty)$), which is given by

$$\psi(u) := \sqrt{\beta + \frac{1}{4}(\alpha - \lambda + u)^2} - \frac{1}{2}(\alpha + \lambda + u), \qquad u \geqslant 0, \tag{6.17}$$

and specified by the parameters α, λ from (6.8) and (6.12) and a nonnegative constant β defined by

$$\beta := \|\Pi^{-1/2}H\|^2 = \lambda_{\max}(H^{\mathrm{T}}\Pi^{-1}H) \tag{6.18}$$

in terms of the matrices $\Pi \succ 0$ from (6.13) and $H \in \mathbb{R}^{n\times p}$ from (6.16). Here, $\lambda_{\max}(\cdot)$ is the largest eigenvalue of a real symmetric matrix.

Theorem 3. *Suppose the backlash set Θ is strongly convex in the sense of (6.1), and the matrix F in (4.2) is Hurwitz. Then, at any time $t \geqslant 0$, the function V in (6.14), associated with the trajectory deviations X, Q in (6.2), (6.3) and parameterized by λ, Π subject to (6.12) and (6.13), satisfies the inequality*

$$\sqrt{V(t)} \leqslant \mathrm{e}^{\int_0^t \psi(\gamma(s))\mathrm{d}s}\sqrt{V(0)}. \tag{6.19}$$

Here, the integrand is the composition of the function of time γ from (6.6) with the function ψ in (6.17).

Proof. By differentiating (6.14) with respect to time and using (6.4) along with (6.7) of Lemma 2, it follows that

$$\begin{aligned}
\dot{V} &= X^{\mathrm{T}}\Pi\dot{X} + \frac{1}{2}(|Q|^2)^{\cdot} \\
&= X^{\mathrm{T}}\mathbf{S}(\Pi F)X + X^{\mathrm{T}}\Pi EQ + \frac{1}{2}(|Q|^2)^{\cdot} \\
&\leqslant X^{\mathrm{T}}\mathbf{S}(\Pi F)X + X^{\mathrm{T}}\Pi EQ - X^{\mathrm{T}}F^{\mathrm{T}}C^{\mathrm{T}}Q - (\alpha+\gamma)|Q|^2 \\
&= X^{\mathrm{T}}\mathbf{S}(\Pi F)X + 2X^{\mathrm{T}}HQ - (\alpha+\gamma)|Q|^2 \\
&= \begin{bmatrix} X^{\mathrm{T}} & Q^{\mathrm{T}} \end{bmatrix} G \begin{bmatrix} X \\ Q \end{bmatrix},
\end{aligned} \tag{6.20}$$

where

$$G := \begin{bmatrix} \mathbf{S}(\Pi F) & H \\ H^{\mathrm{T}} & -(\alpha+\gamma)I_p \end{bmatrix}, \tag{6.21}$$

and H is the matrix from (6.16). Since the matrix square root and its inverse for the matrix Γ in (6.15) are given by

$$\sqrt{\Gamma} = \begin{bmatrix} \sqrt{\Pi} & 0 \\ 0 & I_p \end{bmatrix}, \qquad \Gamma^{-1/2} = \begin{bmatrix} \Pi^{-1/2} & 0 \\ 0 & I_p \end{bmatrix},$$

then (6.21) implies that

$$\begin{aligned}
\Lambda &:= \Gamma^{-1/2}G\Gamma^{-1/2} \\
&= \begin{bmatrix} \Pi^{-1/2}\mathbf{S}(\Pi F)\Pi^{-1/2} & \Pi^{-1/2}H \\ H^{\mathrm{T}}\Pi^{-1/2} & -(\alpha+\gamma)I_p \end{bmatrix} \\
&\preccurlyeq \begin{bmatrix} -\lambda I_n & \Pi^{-1/2}H \\ H^{\mathrm{T}}\Pi^{-1/2} & -(\alpha+\gamma)I_p \end{bmatrix},
\end{aligned} \tag{6.22}$$

where the inequality is due to (6.13). From the dependence of the matrix Λ on Γ, G in (6.22), it follows that

$$G = \sqrt{\Gamma}\Lambda\sqrt{\Gamma} \preccurlyeq \phi\Gamma, \qquad \phi := \lambda_{\max}(\Lambda). \tag{6.23}$$

Note that ϕ is the largest generalised eigenvalue of the matrix pencil associated with the pair (G,Γ) and depends on time through the function γ in (6.22). By applying (B.2) of Lemma 4, it follows from (6.22) and (6.23) that

$$\begin{aligned}
\phi &\leqslant \lambda_{\max}\left(\begin{bmatrix} -\lambda I_n & \Pi^{-1/2}H \\ H^{\mathrm{T}}\Pi^{-1/2} & -(\alpha+\gamma)I_p \end{bmatrix}\right) \\
&= \sqrt{\beta + \frac{1}{4}(\alpha-\lambda+\gamma)^2} - \frac{1}{2}(\alpha+\lambda+\gamma) = \psi(\gamma),
\end{aligned} \tag{6.24}$$

where use is made of the function ψ from (6.17) along with one of its parameters β given by (6.18). A combination of (6.23), (6.24) with (6.14) allows the right-hand side of (6.20) to be bounded from above as

$$\dot{V} \leqslant \phi \begin{bmatrix} X^{\mathrm{T}} & Q^{\mathrm{T}} \end{bmatrix} \Gamma \begin{bmatrix} X \\ Q \end{bmatrix} = 2\phi V \leqslant 2\psi(\gamma)V. \tag{6.25}$$

It now remains to note that (6.19) is obtained from (6.25) by applying the Gronwall-Bellman lemma. □

From (6.24) and the remark at the end of Appendix B, it follows that ψ, defined by (6.17), is a nonincreasing convex function which satisfies

$$\psi(0) = \sqrt{\beta + \frac{1}{4}(\alpha - \lambda)^2} - \frac{\alpha + \lambda}{2} \geqslant \lim_{u \to +\infty} \psi(u) = -\lambda < 0 \tag{6.26}$$

in view of the condition (6.12). Moreover, if

$$\beta > 0 \tag{6.27}$$

in (6.18), which is a generic case equivalent to $H \neq 0$ in (6.16), then the function ψ is strictly decreasing, strictly convex and infinitely differentiable. In the context of (6.19) of Theorem 3, the quantity $\psi(0)$ in (6.26) corresponds to the limiting case $\gamma = 0$ of a convex (but not necessarily strongly convex) compact backlash set Θ as discussed immediately after the proof of Lemma 2.

Also, while the Lyapunov function V in (6.14) is defined using X, Q, the inequality (6.19) can also be reformulated in terms of X, Z. Indeed, in view of (6.2) and (6.3), these pairs are related by the same transformation as in (2.8):

$$\begin{bmatrix} X \\ Q \end{bmatrix} = U \begin{bmatrix} X \\ Z \end{bmatrix}.$$

Hence, in accordance with the equivalence of norms in $\mathbb{R}^{n+p}$ (as any other Euclidean space), V admits bilateral bounds

$$\frac{1}{\|U^{-1}\Gamma^{-1/2}\|} \left\| \begin{bmatrix} X \\ Z \end{bmatrix} \right\| \leqslant \sqrt{2V} = \left\| \begin{bmatrix} X \\ Z \end{bmatrix} \right\|_{U^{\mathrm{T}}\Gamma U} \leqslant \| \sqrt{\Gamma} U \| \left\| \begin{bmatrix} X \\ Z \end{bmatrix} \right\| \tag{6.28}$$

using the standard Euclidean distance $\sqrt{|X|^2 + |Z|^2}$ between the system state trajectories and the nonsingularity of the matrices U, Γ in (2.8) and (6.15).

7. Rates of convergence to periodic regimes

If the quantity $\psi(0)$, given by the first equality in (6.26), satisfies $\psi(0) < 0$, which is equivalent to

$$\alpha > \frac{\beta}{\lambda} \tag{7.1}$$

for the parameters (6.8), (6.12) and (6.18), then (6.19) implies that

$$\sqrt{V(t)} \leqslant \mathrm{e}^{\psi(0)t} \sqrt{V(0)}, \qquad t \geqslant 0, \tag{7.2}$$

thus securing an exponentially fast decay for the Lyapunov function (6.14), as $t \to +\infty$, regardless of the strong convexity of the backlash set Θ. Therefore, the condition (7.1) is sufficient for the nonlinear system (2.1)–(2.3) subjected to a T-periodic input w to have a unique forced T-periodic regime to which all the system trajectories converge at an exponential rate. In this case, by a combination of (7.2) with (6.28), the corresponding Lyapunov exponent admits the following upper bound:

$$\limsup_{t \to +\infty} \Big(\frac{1}{t} \ln \left\| \begin{bmatrix} X(t) \\ Z(t) \end{bmatrix} \right\| \Big) \leqslant \psi(0) < 0.$$

However, in the opposite case, when $\psi(0) \geqslant 0$, the upper bound (6.19) can lead to an exponential decay for V only due to the strong convexity of the backlash set and effective movement of the backlash output captured in the function γ in (6.6). This is discussed in the following theorem which employs the quantities

$$\gamma_1 := \frac{1}{TR}(\mho - d)_+, \tag{7.3}$$

$$\gamma_\infty := \frac{1}{R}(\|\dot{\eta}_T\|_\infty + \Delta) \tag{7.4}$$

associated with the bounds (5.8) and (5.12) from Theorem 2. Together with (6.17), they give rise to

$$\theta := \psi(0) - \frac{\gamma_1}{\gamma_\infty}(\psi(0) - \psi(\gamma_\infty)) \tag{7.5}$$

(the trivial case $\gamma_\infty = 0$ is not considered), which satisfies $\theta \leqslant \psi(0)$ because, as mentioned above, the function ψ is nonincreasing, and hence,

$$\psi(0) \geqslant \psi(\gamma_\infty). \tag{7.6}$$

Theorem 4. *Suppose the matrix F in (4.2) is Hurwitz, the backlash set Θ satisfies (4.10) along with the strong convexity condition (6.1), and the system (2.1)–(2.3) is driven by a T-periodic bounded input w. Also, let the quantities (7.3) and (7.4) satisfy*

$$\frac{\gamma_1}{\gamma_\infty} > \frac{\psi(0)}{\psi(0) - \psi(\gamma_\infty)}, \tag{7.7}$$

with the function ψ from (6.17). Then the system has a unique forced T-periodic regime, and the Lyapunov exponent for the convergence of the state trajectories in (6.2), (6.3) admits an upper bound in terms of (7.5):

$$\limsup_{t\to+\infty}\Big(\frac{1}{t}\ln\left\|\begin{bmatrix} X(t)\\ Z(t)\end{bmatrix}\right\|\Big) \leqslant \theta < 0. \tag{7.8}$$

Proof. Since the conditions of Theorems 2 and 3 are fulfilled, then it follows from (5.8) and (5.12) that the function γ in (6.6) satisfies

$$\frac{1}{T}\liminf_{\tau\to+\infty}\int_\tau^{\tau+T}\gamma(t)\mathrm{d}t \geqslant \gamma_1, \tag{7.9}$$

$$\limsup_{t\to+\infty}\gamma(t) \leqslant \gamma_\infty \tag{7.10}$$

in view of (7.3) and (7.4), so that

$$0 \leqslant \gamma_1 \leqslant \gamma_\infty. \tag{7.11}$$

The relation (7.10) implies that for any $\delta > 0$, there exists $\tau \geqslant 0$ such that $\sup_{t\geqslant\tau}\gamma(t) \leqslant \gamma_\infty + \delta$, and hence,

$$\psi(\gamma(t)) \leqslant \psi(0) - \frac{\gamma(t)}{\gamma_\infty + \delta}(\psi(0) - \psi(\gamma_\infty + \delta)), \qquad t \geqslant \tau, \tag{7.12}$$

by the convexity of the function ψ in (6.17). Since the Cesaro type partial limits do not depend on the behaviour of the integrand over a fixed bounded interval, then it follows from (7.12) that

$$\begin{aligned} \nu &:= \limsup_{t\to+\infty}\Big(\frac{1}{t}\int_0^t \psi(\gamma(s))\mathrm{d}s\Big) \\ &\leqslant \psi(0) - \frac{\ell}{\gamma_\infty+\delta}(\psi(0)-\psi(\gamma_\infty+\delta)), \end{aligned} \tag{7.13}$$

where

$$\ell := \liminf_{t\to+\infty}\Big(\frac{1}{t}\int_0^t \gamma(s)\mathrm{d}s\Big), \tag{7.14}$$

and use is also made of the fact that the function ψ is nonincreasing on $\mathbb{R}_+$. The inequality (7.13) holds for any $\delta > 0$, and its left-hand side does not depend on δ. Hence, by using the continuity of ψ and letting $\delta \to 0+$, it follows that

$$\nu \leqslant \psi(0) - \frac{\ell}{\gamma_\infty}(\psi(0)-\psi(\gamma_\infty)). \tag{7.15}$$

This corresponds to an affine upper bound for the nonincreasing convex function ψ over the interval $[0,\gamma_\infty]$:

$$\psi(u) \leqslant \psi(0) - \frac{u}{\gamma_\infty}(\psi(0)-\psi(\gamma_\infty)), \qquad 0 \leqslant u \leqslant \gamma_\infty, \tag{7.16}$$

which is represented by the chord in Figure 3.

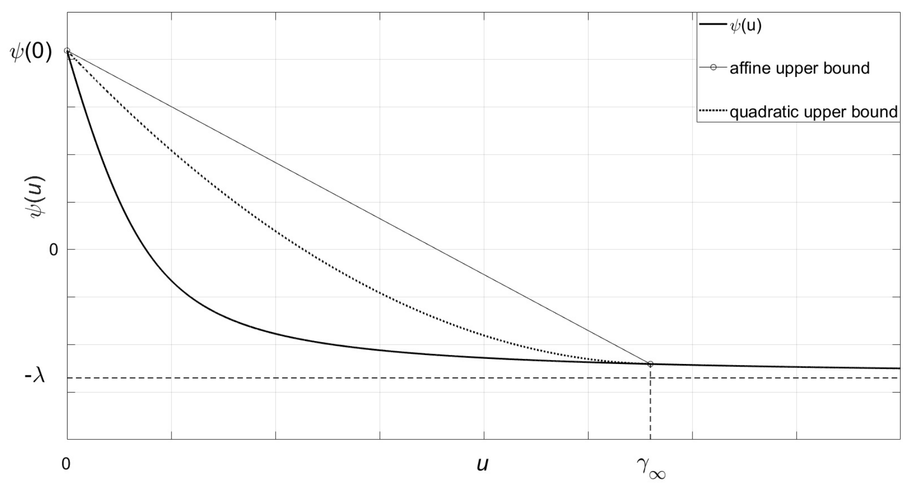

Figure 3. A typical graph of the function ψ in (6.17) (in the generic case (6.27)) along with its extreme values (6.26). In this illustration, $\psi(0) > 0$. The chord represents the affine upper bound (7.16) for the restriction $\psi|_{[0,\gamma_\infty]}$ which results from (7.12) in the limit, as $\delta \to 0+$. The dotted curve depicts a stronger quadratic bound provided by (7.21).

Now, (7.14) admits a lower bound in terms of (7.9):

$$\ell = \frac{1}{T}\liminf_{t\to+\infty}\Big(\frac{1}{\lfloor t/T\rfloor}\sum_{k=1}^{\lfloor t/T\rfloor}\int_{t-kT}^{t-(k-1)T}\gamma(s)\mathrm{d}s\Big) \geqslant \gamma_1, \tag{7.17}$$

where $\lfloor\cdot\rfloor$ is the floor function. The inequalities (7.15) and (7.17) (along with (7.6) used above) allow ν in (7.13) to be bounded as

$$\nu \leqslant \theta, \tag{7.18}$$

where θ, given by (7.5), is a convex combination of $\psi(0)$, $\psi(\gamma_\infty)$ in view of (7.11). By combining (7.18) with (6.19) and (6.28), it follows that the Lyapunov exponent for the deviations (6.2) and (6.3) of the state trajectories satisfies

$$\limsup_{t\to+\infty}\Big(\frac{1}{t}\ln\left\|\begin{bmatrix}X(t)\\ Z(t)\end{bmatrix}\right\|\Big) = \frac{1}{2}\limsup_{t\to+\infty}\Big(\frac{1}{t}\ln V(t)\Big) \leqslant \nu \leqslant \theta, \tag{7.19}$$

which establishes the first inequality in (7.8). The second inequality in (7.8) for θ in (7.5) is secured by the condition (7.7) and implies the uniqueness of a forced T-periodic regime in the nonlinear system. □

Some remarks are in order concerning the applicability of Theorem 4 when the function ψ, defined by (6.17), satisfies $\psi(0) > 0$. In this case, the condition (7.7) does not contradict (7.11) only if

$$\psi(\gamma_\infty) < 0, \tag{7.20}$$

which makes the right-hand side of (7.7) strictly less than 1. In view of the negative limit in (6.26), the inequality (7.20) holds for all sufficiently large γ_∞. Moreover, $\psi(\gamma_\infty)$ can be made arbitrarily close to the limit value $-\lambda$ by a suitable choice of the T-periodic external input w. In particular, this can be achieved by increasing the quantity $\|\dot{\eta}_T\|_\infty$ in (7.4), which is associated with the T-periodic output (5.4) of the linearised system (4.1) through (5.2) and also quantifies the "amplitude" of w. The condition (7.7) can then be satisfied by appropriately shaping the input w.

Also note that Theorem 4 and its proof have employed the local affine upper bound (7.16) for the function ψ, as illustrated by Figure 3 for the case (6.27). However, in the latter case, ψ is strongly convex over $[0,\gamma_\infty]$ (or any other bounded interval in $\mathbb{R}_+$), as mentioned above, and admits a quadratic upper bound

$$\begin{aligned}\psi(u) &\leqslant \psi(0) - \frac{u}{\gamma_\infty}(\psi(0)-\psi(\gamma_\infty)) - \frac{1}{2}\varpi(\gamma_\infty)u(\gamma_\infty - u)\\ &= \psi(0) - \Big(\frac{\psi(0)-\psi(\gamma_\infty)}{\gamma_\infty} + \frac{1}{2}\varpi(\gamma_\infty)\gamma_\infty\Big)u + \frac{1}{2}\varpi(\gamma_\infty)u^2, \qquad 0 \leqslant u \leqslant \gamma_\infty.\end{aligned} \tag{7.21}$$

This bound for ψ is stronger than (7.16) and uses a continuous function $\varpi : (0,+\infty) \to (0,+\infty)$ associated with (6.17) by

$$\begin{aligned}\varpi(v) &:= \frac{2}{v^2}\inf_{0<s<1}\frac{(1-s)\psi(0)+s\psi(v)-\psi(sv)}{s(1-s)}\\ &\geqslant \varpi_*(v) := \min_{0\leqslant u\leqslant v}\psi''(u)\\ &= \frac{1}{4}\beta\min_{0\leqslant u\leqslant v}\frac{1}{(\beta+\frac{1}{4}(\alpha-\lambda+u)^2)^{3/2}}\\ &= \frac{1}{4}\beta\Big(\beta + \frac{1}{4}\max(|\alpha-\lambda|,|\alpha-\lambda+v|)^2\Big)^{-3/2} > 0, \qquad v > 0,\end{aligned} \tag{7.22}$$

and satisfying $\varpi(0+) = \psi''(0)$. Its geometric meaning is that $\frac{1}{2}\varpi(v)$ is the leading coefficient for the lowest quadratic parabola contained by the epigraph of the function ψ on the interval $[0, v]$ and passing through the same endpoints $(0, \psi(0))$ and $(v, \psi(v))$, as illustrated by the dotted curve in Figure 3 for the case of $v = \gamma_\infty$. Accordingly, $\varpi_*(v)$ in (7.22) is a guaranteed (yet conservative since the function ψ is not quadratic) lower bound for $\varpi(v)$, which makes the function $\psi(u) - \frac{1}{2}\varpi_*(v)u^2$ convex with respect to $u \in [0, v]$ (moreover, $\varpi_*(v)$ is the largest constant of strong convexity for ψ on $[0, v]$). The quadratic upper bound (7.21) can be combined with an additional quantity

$$\gamma_2 := \frac{1}{R}\Big(\frac{1}{\sqrt{T}}\|\dot{\eta}_T\|_2 + \Delta\Big), \tag{7.23}$$

coming from the estimate (5.11) for the second-order moment of the backlash output and giving rise to

$$\kappa := \psi(0) - \Big(\frac{\psi(0) - \psi(\gamma_\infty)}{\gamma_\infty} + \frac{1}{2}\varpi(\gamma_\infty)\gamma_\infty\Big)\gamma_1 + \frac{1}{2}\varpi(\gamma_\infty)\gamma_2^2 \tag{7.24}$$

instead of (7.5) as discussed below.

Theorem 5. *As in Theorems 2–4, suppose the matrix F in (4.2) is Hurwitz, the backlash set Θ satisfies (4.10) along with the strong convexity condition (6.1), and the system (2.1)–(2.3) is driven by a T-periodic bounded input w. Also, suppose the condition (6.27) is fulfilled, and the quantities (7.3), (7.4), and (7.23) satisfy*

$$\frac{\gamma_1}{\gamma_\infty} > \frac{\psi(0) - \frac{1}{2}\varpi(\gamma_\infty)\gamma_2(\gamma_\infty - \gamma_2)}{\psi(0) - \psi(\gamma_\infty)}, \tag{7.25}$$

with the functions ψ, ϖ from (6.17) and (7.22). Then the system has a unique forced T-periodic regime, and the Lyapunov exponent for the convergence of the state trajectories in (6.2), (6.3) admits an upper bound in terms of (7.24):

$$\limsup_{t\to+\infty}\Big(\frac{1}{t}\ln\left\|\begin{bmatrix} X(t)\\ Z(t)\end{bmatrix}\right\|\Big) \leqslant \kappa < 0. \tag{7.26}$$

Proof. By Jensen's inequality applied to the convex function $\mathbb{R} \ni \zeta \to \zeta^2$, it follows from (6.6) that

$$\gamma^2 \leqslant \frac{1}{2R^2}(|\dot{z}_1|^2 + |\dot{z}_2|^2)$$

whose combination with (5.11) yields

$$\frac{1}{T}\limsup_{\tau\to+\infty}\int_\tau^{\tau+T}\gamma(t)^2\mathrm{d}t \leqslant \frac{1}{TR^2}(\|\dot{\eta}_T\|_2 + \sqrt{T}\Delta)^2 = \gamma_2^2 \tag{7.27}$$

in view of (7.23). Since $\|\zeta\|_2 \leqslant \sqrt{T}\|\zeta\|_\infty$ for any function $\zeta \in L^\infty([0, T], \mathbb{R}^p)$, a comparison of (7.23) with (7.4) leads to

$$\gamma_2 \leqslant \gamma_\infty. \tag{7.28}$$

Furthermore, from (7.9), the Cauchy-Bunyakovsky-Schwarz inequality and (7.27), it follows that

$$\gamma_1 \leqslant \frac{1}{T}\limsup_{\tau\to+\infty}\int_\tau^{\tau+T}\gamma(t)\mathrm{d}t \leqslant \frac{1}{\sqrt{T}}\limsup_{\tau\to+\infty}\sqrt{\int_\tau^{\tau+T}\gamma(t)^2\mathrm{d}t} \leqslant \gamma_2,$$

which, together with (7.28), allows (7.11) to be complemented as

$$0 \leqslant \gamma_1 \leqslant \gamma_2 \leqslant \gamma_\infty. \tag{7.29}$$

The rest of the proof is similar to that of Theorem 4. The relations (7.27) are used similarly to (7.17) in order to obtain

$$\limsup_{t\to+\infty}\Big(\frac{1}{t}\int_0^t \gamma(s)^2 \mathrm{d}s\Big) = \frac{1}{T}\limsup_{t\to+\infty}\Big(\frac{1}{\lfloor t/T\rfloor}\sum_{k=1}^{\lfloor t/T\rfloor}\int_{t-kT}^{t-(k-1)T}\gamma(s)^2\mathrm{d}s\Big) \leqslant \gamma_2^2. \tag{7.30}$$

Then a combination of (7.9), (7.10), and (7.30) with the quadratic upper bound (7.21) applied to a larger interval $[0, \gamma_\infty + \delta]$ and followed by taking the limit, as $\delta \to 0+$, leads to a refined upper bound for the quantity ν from (7.13):

$$\begin{aligned}\nu &\leqslant \psi(0) + \lim_{\delta\to 0+}\Big(-\Big(\frac{\psi(0)-\psi(\gamma_\infty+\delta)}{\gamma_\infty+\delta} + \frac{1}{2}\varpi(\gamma_\infty+\delta)(\gamma_\infty+\delta)\Big)\gamma_1 + \frac{1}{2}\varpi(\gamma_\infty+\delta)\gamma_2^2\Big)\\ &= \kappa,\end{aligned} \tag{7.31}$$

where use is also made of the continuity of the functions ψ, ϖ in (6.17) and (7.22), and κ is given by (7.24). In view of (7.31), the first inequality in (7.26) can now be obtained by replacing θ in (7.19) with κ. The condition (7.25) is equivalent to κ in (7.24) being negative, which completes the proof of (7.26) and implies the uniqueness of a forced T-periodic regime in the nonlinear system. □

Note that the condition (7.25) is less restrictive than (7.7) since the right-hand side of (7.25) has a smaller numerator

$$\psi(0) - \frac{1}{2}\varpi(\gamma_\infty)\gamma_2(\gamma_\infty - \gamma_2) \leqslant \psi(0)$$

in view of (7.29) and since the function ϖ in (7.22) is positive. For the same reason, the upper bound provided by Theorem 5 for the Lyapunov exponent in (7.26) is stronger than that in (7.8) of Theorem 4: $\kappa \leqslant \theta$.

8. Conclusions

We have considered a class of nonlinear systems, where a linear plant forms a feedback loop with a backlash specified by a strongly convex compact. We have discussed a tubular localization for the trajectories of the system about those of its linearised counterpart without the backlash. This property has been combined with dissipation relations in order to study the establishment of periodic regimes in the system subject to periodic inputs of relatively large amplitudes. This approach, which is based on enhanced differential inequalities and was used previously for the Moreau sweeping process in [29], is applicable to similar differential inclusions with strongly convex sets. Other directions of research may include extension of these results to system interconnections with several strongly convex backlashes and also backlash models which take into account the friction and inertial effects.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

References

1. E. Allgower, K. Georg, Simplicial and continuation methods for approximating fixed points and solutions to systems of equations, *SIAM Rev.*, **22** (1980), 28–85. https://doi.org/10.1137/1022003
2. B. Brogliato, Absolute stability and the Lagrange-Dirichlet theorem with monotone multivalued mappings, *Syst. Control Lett.*, **51** (2004), 343–353. https://doi.org/10.1016/j.sysconle.2003.09.007
3. B. Brogliato, W. P. M. H. Heemels, Observer design for Lur'e systems with multivalued mappings: a passivity approach, *IEEE Trans. Automat. Contr.*, **54** (2009), 1996–2001. https://doi.org/10.1109/TAC.2009.2023968
4. M. Brokate, A. Pokrovskii, D. Rachinskii, O. Rasskazov, Differential equations with hysteresis via a canonical example, In: G. Bertotti, I. D. Mayergoyz, *The science of hysteresis*, Vol. I, Academic Press, 2006, 125–291. https://doi.org/10.1016/B978-012480874-4/50005-1
5. H. Frankowska, C. Olech, *R*-convexity of the integral of set-valued functions, In: *Contributions to Analysis and Geometry*, Baltimore MD: Johns Hopkins University Press, 1981, 117–129.
6. V. V. Goncharov, G. E. Ivanov, Strong and weak convexity of closed sets in a Hilbert space, In: N. Daras, T. Rassias, *Operations research, engineering, and cyber security*, Springer, **113** (2017), 259–297. https://doi.org/10.1007/978-3-319-51500-7_12
7. I. Gudoshnikov, O. Makarenkov, D. Rachinskii, Formation of a nontrivial finite-time stable attractor in a class of polyhedral sweeping processes with periodic input, *ESAIM: COCV*, **29** (2023), 84. https://doi.org/10.1051/cocv/2023074
8. I. Gudoshnikov, O. Makarenkov, Structurally stable families of periodic solutions in sweeping processes of networks of elastoplastic springs, *Phys. D*, **406** (2020), 132443. https://doi.org/10.1016/j.physd.2020.132443
9. N. J. Higham, *Functions of matrices: theory and computation*, Philadelphia: SIAM, 2008.
10. R. A. Horn, C. R. Johnson, *Matrix analysis*, New York: Cambridge University Press, 2007.
11. G. E. Ivanov, *Quadratic convergence of algorithms for solving linear differential games*, Ph.D. Thesis, Moscow Institute of Physics and Technology, 1994.
12. G. E. Ivanov, E. S. Polovinkin, On strongly convex differential games, *Differ. Uravn.*, **31** (1995), 1641–1648.
13. G. E. Ivanov, *Weakly convex sets and functions: theory and applications*, Moscow: Fizmatlit, 2006.

14. G. E. Ivanov, Nonlinear images of sets. I: strong and weak convexity, *J. Convex Anal.*, **27** (2020), 361–380.

15. M. Kamenskii, O. Makarenkov, On the response of autonomous sweeping processes to periodic perturbations, *Set-Valued Var. Anal.*, **24** (2016), 551–563. https://doi.org/10.1007/s11228-015-0348-1

16. M. Kamenskii, O. Makarenkov, L. N. Wadippuli, P. R. de Fitte, Global stability of almost periodic solutions to monotone sweeping processes and their response to non-monotone perturbations, *Nonlinear Anal.: Hybrid Syst.*, **30** (2018), 213–224. https://doi.org/10.1016/j.nahs.2018.05.007

17. M. Kamenskii, O. Makarenkov, L. N. Wadippuli, A continuation principle for periodic BV-continuous state-dependent sweeping processes, *SIAM J. Math. Anal.*, **52** (2020), 5598–5626. https://doi.org/10.1137/19M1248613

18. P. E. Kloeden, V. S. Kozyakin, The inflation of attractors and their discretization: the autonomous case, *Nonlinear Anal.*, **40** (2000), 333–343. https://doi.org/10.1016/S0362-546X(00)85020-8

19. A. N. Kolmogorov, On certain asymptotic characteristics of completely bounded metric spaces, *Dokl. Akad. Nauk SSSR*, **108** (1956), 385–388.

20. M. A. Krasnosel'skiĭ, A. V. Pokrovskiĭ, *Systems with hysteresis*, Springer-Verlag, 1989. https://doi.org/10.1007/978-3-642-61302-9

21. R. I. Leine, N. van de Wouw, Uniform convergence of monotone measure differential inclusions: with application to the control of mechanical systems with unilateral constraints, *Int. J. Bifur. Chaos*, **18** (2008), 1435–1457. https://doi.org/10.1142/S0218127408021099

22. J. J. Moreau, Rafle par un convexe variable, I, *Sém. Anal. Convexe Montpellier*, 1971.

23. J. J. Moreau, Evolution problem associated with a moving convex set in a Hilbert space, *J. Differ. Equations*, **26** (1977), 347–374.

24. A. Plis, Accessible sets in control theory, *Int. Conf. Differ. Equ.*, 1975, 646–650. https://doi.org/10.1016/B978-0-12-059650-8.50052-1

25. E. S. Polovinkin, M. V. Balashov, *Elements of convex and strongly convex analysis*, Moscow: Fizmatlit, 2004.

26. R. T. Rockafellar, *Convex analysis*, Princeton University Press, 1970. https://doi.org/10.1515/9781400873173

27. M. C. Smith, Synthesis of mechanical networks: the inerter, *IEEE Trans. Automat. Contr.*, **47** (2002), 1648–1662. https://doi.org/10.1109/TAC.2002.803532

28. I. G. Vladimirov, I. R. Petersen, Convergence to periodic regimes in nonlinear feedback systems with a strongly convex backlash, *2020 European Control Conference (ECC)*, 2020, 71–76. https://doi.org/10.23919/ECC51009.2020.9143648

29. I. G. Vladimirov, On strong solutions of the Moreau process, *Doklady Math.*, **55** (1997), 167–169.

30. I. G. Vladimirov, V. V. Chernorutski, On self-induced oscillations in systems with play, *Autom. Remote Control*, **56** (1995), 1538–1544.

Appendix

Appendix A. Inequalities for inward normals

The following lemma compares two inequalities for the inward normals to convex and strongly convex sets and is used in dissipation relations of Section 6, including (6.9) in the proof of Lemma 2.

Lemma 3. *Suppose S is a closed convex subset of $\mathbb{R}^p$. Then for any given points $x, y \in S$, the inequality*

$$(u - v)^{\mathrm{T}}(x - y) \leqslant -2r(|u| + |v|) \tag{A.1}$$

holds for any $u \in N_S(x)$ and $v \in N_S(y)$ in the corresponding inward normal cones (2.5), where

$$r := \rho(c, \partial S) \tag{A.2}$$

is the distance (4.15) from the midpoint

$$c := \frac{1}{2}(x + y) \tag{A.3}$$

(of the line segment with the endpoints x, y) to the boundary ∂S. Moreover, if S is strongly convex with the constant R (as in (6.1)), then

$$(u - v)^{\mathrm{T}}(x - y) \leqslant -\frac{1}{2R}(|u| + |v|)|x - y|^2. \tag{A.4}$$

Proof. Under the assumption that $S \subset \mathbb{R}^p$ is closed and convex, the closed ball $\overline{B}_r(c)$ of radius (A.2) centred at the midpoint (A.3) of the line segment with the endpoints $x, y \in S$, is contained by S. In combination with the definition (2.5) of the inward normal cone, the inclusion $\overline{B}_r(c) \subset S$ implies that for any $u \in N_S(x)$,

$$\begin{aligned} 0 &\leqslant \inf_{z \in S}(u^{\mathrm{T}}(z - x)) \leqslant \min_{z \in \overline{B}_r(c)} (u^{\mathrm{T}}(z - x)) \\ &= u^{\mathrm{T}}(c - x) + \min_{w \in \overline{B}_r(0)} (u^{\mathrm{T}} w) = \frac{1}{2} u^{\mathrm{T}}(y - x) - r|u|, \end{aligned}$$

and hence,

$$u^{\mathrm{T}}(x - y) \leqslant -2r|u|. \tag{A.5}$$

By a similar reasoning, for any $v \in N_S(y)$,

$$v^{\mathrm{T}}(y - x) \leqslant -2r|v|. \tag{A.6}$$

By taking the sum of (A.5) and (A.6) and recalling (A.2) and (A.3), we arrive at (A.1). Now, suppose the set S is strongly convex. In this case, (A.4) cannot be obtained directly from (A.1) by using the lower bound [5]

$$r \geqslant R - \sqrt{R^2 - \frac{1}{4}|x - y|^2} \geqslant \frac{1}{8R}|x - y|^2 \tag{A.7}$$

for the radius (A.2) in terms of the strong convexity constant R of the set S, because (A.7) leads to a more conservative factor $\frac{1}{4}$ than $\frac{1}{2}$ in (A.4). In order to prove (A.4), suppose $x \in \partial S$ and $u \in N_S(x) \setminus \{0\}$. Then, by using the unit inward normal

$$\vartheta := \frac{1}{|u|} u, \tag{A.8}$$

it follows from (6.1), applied to the set S, that $S \subset \overline{B}_R(x + R\vartheta)$. This inclusion is equivalent to $|x + R\vartheta - y| \leqslant R$ for all $y \in S$, and hence,

$$0 \geqslant |x + R\vartheta - y|^2 - R^2 = |x - y|^2 + 2R\vartheta^{\mathrm{T}}(x - y), \tag{A.9}$$

where use is made of the property $|\vartheta| = 1$ which follows from (A.8). Therefore, multiplication of (A.9) by $|u|$ leads to

$$2Ru^{\mathrm{T}}(x - y) \leqslant -|u||x - y|^2. \tag{A.10}$$

Since the cone $N_S(x)$ reduces to the singleton $\{0\}$ for any interior point $x \in \mathrm{int}S$, the inequality (A.10) remains valid for any $x, y \in S$ and any $u \in N_S(x)$. Similarly, if $v \in N_S(y)$, then

$$2Rv^{\mathrm{T}}(y - x) \leqslant -|v||y - x|^2. \tag{A.11}$$

The sum of (A.10) and (A.11) yields (A.4), thus completing the proof. □

Appendix B. A spectral bound for a class of symmetric matrices

The dissipation relations of Section 6, including (6.24) in the proof of Theorem 3, employ a spectral bound below.

Lemma 4. *For any $a, b \in \mathbb{R}$ and $g \in \mathbb{R}^{n\times p}$, the largest eigenvalue of the following real symmetric matrix*

$$M := \begin{bmatrix} aI_n & g \\ g^{\mathrm{T}} & bI_p \end{bmatrix} \tag{B.1}$$

is computed as

$$\lambda_{\max}(M) = \frac{a+b}{2} + \sqrt{\|g\|^2 + \Big(\frac{a-b}{2}\Big)^2}. \tag{B.2}$$

Proof. Since the matrix (B.1) can be obtained as a "scalar shift"

$$M = cI_{n+p} + N, \qquad c := \frac{a+b}{2} \tag{B.3}$$

of a real symmetric matrix N of order $n + p$ given by

$$N := \begin{bmatrix} dI_n & g \\ g^{\mathrm{T}} & -dI_p \end{bmatrix}, \qquad d := \frac{a-b}{2}, \tag{B.4}$$

then

$$\lambda_{\max}(M) = c + \lambda_{\max}(N). \tag{B.5}$$

By the variational representation of the eigenvalues of real symmetric (or complex Hermitian) matrices [10] applied to N in (B.4),

$$\begin{aligned}\lambda_{\max}(N) &= \max_{z\in\mathbb{R}^{n+p}:\ |z|=1} (z^{\mathrm{T}} N z)\\ &= \max_{u\in\mathbb{R}^{n},\ v\in\mathbb{R}^{p}:\ |u|^2+|v|^2=1} (d(|u|^2 - |v|^2) + 2u^{\mathrm{T}} g v)\\ &= \max_{u\in\mathbb{R}^{n},\ v\in\mathbb{R}^{p}:\ |u|^2+|v|^2=1} (d(|u|^2 - |v|^2) + 2\|g\||u||v|)\\ &= \max_{\varphi\in[0,\pi/2]} (d\cos(2\varphi) + \|g\|\sin(2\varphi)) = \sqrt{d^2 + \|g\|^2},\end{aligned} \tag{B.6}$$

where repeated use is made of the Cauchy-Bunyakovsky-Schwarz inequality, including the relations

$$\max_{u\in\mathbb{R}^{n},\ v\in\mathbb{R}^{p}:\ |u|=\alpha,\ |v|=\beta} (u^{\mathrm{T}} g v) = \alpha \max_{v\in\mathbb{R}^{p}:\ |v|=\beta} |gv| = \alpha\beta\|g\|, \qquad \alpha,\beta \geqslant 0,$$

combined with the change of variables $|u| = \cos\varphi$ and $|v| = \sin\varphi$ and the trigonometric identities $(\cos\varphi)^2 - (\sin\varphi)^2 = \cos(2\varphi)$ and $2\cos\varphi\sin\varphi = \sin(2\varphi)$. Substitution of (B.6) into (B.5) yields (B.2) in view of (B.3). □

Note that, due to the affinity of the map $(a, b) \mapsto M$ in (B.1), the right-hand side of (B.2) depends on a, b in a convex (and nondecreasing) fashion, inheriting this property from the function $\lambda_{\max}$ for real symmetric (or complex Hermitian) matrices.