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Research article

Surfactants in the two gradient theory of phase transitions

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Abstract: We investigate the influence of surfactants on stabilizing the formation of interfaces in solid-solid phase transitions. The analysis focuses on singularly perturbed van der Waals-Cahn-Hillard-type energies for gradient vector fields, supplemented with a term that accounts for the interaction between the surfactant and the solid. Assuming the potential term to have only two rank-1 connected wells, we prove that the effective energy for the formation of an interface decreases when the surfactant segregate to the interface.

Keywords: phase transitions; surfactants; Γ-convergence

1. Introduction

Surfactants (surface active agents) play a pivotal role in influencing phase transitions. In essence, the primary mechanism driving these effects is the adsorption of surface active molecules onto phase interfaces. This adsorption alters the surface tension, by decreasing the energy penalty associated with the different chemical environments of the different phases. Consequently surfactants exert a profound influence on the stability and morphology of the physical system. The capacity of surfactants to modulate phase transitions has found practical applications in various fields both in fluid-fluid and in solid-solid phase transitions. In the case of solid-solid transitions we refer to [24], where manganese in an iron-manganese alloy is used as surfactant which favours the formation of transition layers between singular martensite crystals resulting in modified mechanical properties of the material. Other examples are provided by crystal growth, metallurgy, and ceramics processing (see [21] and the references therein).

In this paper we introduce a phase transition model in presence of surfactant working within the framework of the gradient theory of phase transition. More specifically we modify the easiest phase-field model for solid-solid transition introduced in [11] in order to account for the interaction between the surfactant and the solid. The model we introduce draws inspiration from the one proposed by

Perkins, Sekerka, Warren and Langer for fluid-fluid transitions and analyzed in [19] (see also [1, 5, 14] for some extension to more general models of fluid-fluid or multiphase-fluid-fluid phase transitions in presence of surfactants). To fix the ideas, in what follows we first present the latter model. Such a model, motivated by the investigation on foam stability, is a modification of the classical van der Waals-Cahn-Hillard energy functional. More specifically, an integral term accounting for the fluid-surfactant interaction is added to the classical Cahn-Hillard functional, as explained in detail below.

In a given open and bounded set $\Omega \subset \mathbb{R}^N$ (the region occupied by the fluid and the surfactant), one considers a scalar function $u : \Omega \to \mathbb{R}$ and a non-negative function $\rho : \Omega \to [0, +\infty)$ representing the order parameter of the fluid and the density of the surfactant, respectively. As $\varepsilon \to 0$ one is interested in the asymptotic behaviour of the singularly perturbed sequence of energy functionals $\mathcal{E}_\varepsilon : W^{1,2}(\Omega) \times \mathcal{M}(\Omega) \to [0, +\infty)$ defined as

$$\mathcal{E}_\varepsilon(u, \mu) := \int_\Omega \frac{1}{\varepsilon} W(u) + \varepsilon|\nabla u|^2 + \varepsilon(\rho - |\nabla u|)^2 \, dx, \tag{1.1}$$

where $\mu = \rho\mathcal{L}^N$ denotes the surfactant measure and $W : \mathbb{R} \to [0, +\infty)$ is a double-well potential with wells $\{u : W(u) = 0\} = \{0, 1\}$. The first two terms in the energy define the usual Cahn-Hillard energy functional, namely

$$CH_\varepsilon(u) = \int_\Omega \frac{1}{\varepsilon} W(u) + \varepsilon|\nabla u|^2 \, dx,$$

which models the energy cost of a phase separation phenomenon in a two-component immiscible fluid. In few words, within this theory, also known as the gradient theory of phase transitions, the phase separation phenomenon corresponds to the transition from the value 0 to the value 1 of the order parameter u which represents the local concentration of one of the components of the fluid. The variational limit in the sense of Γ-convergence (see [8, 15]) of the Cahn-Hillard functional as $\varepsilon \to 0$ has been considered by Modica and Mortola in [22, 23] (see also [6, 20] for analogous results about the asymptotic behaviour of the Cahn-Hillard functional in the case of vector valued order parameters). In [22], it is proved the pre-compactness in $BV(\Omega; \{0, 1\})$ of sequences of phase-fields u_ε with uniformly bounded energy and it is computed the Γ-limit of CH_ε as $\varepsilon \to 0$ with respect to the L^1 convergence. Roughly speaking, the limit u of a converging subsequence of u_ε will take only the values 0 and 1, partitioning Ω in the two sets $\{u = 0\}$ and $\{u = 1\}$ (the two immiscible phases of the fluid) whose common boundary (the phase interface) will correspond to the jump set S_u of the function u. Since u is of bounded variation, the latter set will have finite $\mathcal{H}^{N-1}$-measure. Up to a multiplicative constant depending on the shape of W the effective asymptotic energy of the system, captured by the Γ-limit of CH_ε, will be proportional to such a perimeter measure. Hence, if we fix the measure of the set $\{u_\varepsilon = 0\}$ to be strictly smaller than the measure of Ω, both phases will be non empty and the minimal Cahn-Hillard energy as $\varepsilon \to 0$ will correspond to the partition of Ω in the two sets having the least perimeter of the common boundary. Such an energy will be achieved along a sequence u_ε of phase fields with ∇u_ε concentrating on S_u. In this perspective, one can understand the role of the additional third term in $\mathcal{E}_\varepsilon(u, \mu)$ which is responsible for the interaction between the surfactant and the fluid. The presence of the additional term will modify the minimizers of CH_ε described above enhancing the phase separation phenomenon to happen in the regions where the surfactant is present. In fact, the last integral term in (1.1) is minimized if $\rho_\varepsilon = \nabla u_\varepsilon$, which corresponds to the situation in which both the surfactant measure $\mu_\varepsilon = \rho_\varepsilon \mathcal{L}^N$ and the approximating phase interface $\{\nabla u_\varepsilon \simeq \frac{1}{\varepsilon}\}$ are concentrating on the same $(N-1)$-dimensional set. As explained in [19] the scaling factor ε multiplying the third integral is chosen in order to observe the effect of this concentration in the asymptotic limit energy. In fact in [19] the authors have proven that, carrying out the Γ-limit of $\mathcal{E}_\varepsilon$ with respect to the strong L^1 convergence of the phase fields and the weak∗-convergence of the surfactant measures, one obtains a limit functional finite for $u \in BV(\Omega, \{0,1\})$ and $\mu \in \mathcal{M}(\Omega)$ (the space of positive Radon measures) where it takes the form

$$\mathcal{E}(u,\mu) := \int_{S_u \cap \Omega} \Psi\left(\frac{d\mu}{d\mathcal{H}^{N-1} \llcorner S_u}\right) d\mathcal{H}^{N-1}. \tag{1.2}$$

Here $\Psi : [0,+\infty) \to [0,+\infty)$ is a decreasing function of the relative density of the surfactant measure with respect to the surface measure of the interface. In other words, according to the limit energy functional, the surface tension between the phases $\{u = 0\}$ and $\{u = 1\}$ can be lowered increasing the surfactant density on the interface, a phenomenon that characterizes surface active agents as already recalled at the beginning of this section. It is worth mentioning that in [2, 3] such a limit energy has been obtained via a variational discrete-to-continuum coarse-graining procedure starting from the microscopic Blume-Emery-Griffiths ternary surfactant model.

In this paper we are interested in extending the results above to the framework of solid-solid phase transition models. Unlike the fluid-fluid ones already introduced, these transitions, and the variational energy models leading to the associated phase separation phenomena, are vectorial problems. The energy functionals we are interested in will be obtained by adding a surfactant-solid interaction term to the functionals $H_\varepsilon : W^{2,2}(\Omega; \mathbb{R}^d)$ defined as

$$H_\varepsilon(u) := \int_\Omega \frac{1}{\varepsilon} W(\nabla u) + \varepsilon |\nabla^2 u|^2 \, dx. \tag{1.3}$$

The latter functionals, which are the analogue for gradient vector fields of the Cahn-Hillard functionals CH_ε mentioned before, commonly arise as higher-order regularizations of non-convex stored energy functional in elasticity as those considered in the seminal paper [7]. Their Γ-convergence as $\varepsilon \to 0$ has been carried out in [11] assuming the wells of W to be rank-1 connected; i.e.,

$$\{W = 0\} = \{A, B\}, \text{ with } A - B = a \otimes \nu, \text{ for some } a \in \mathbb{R}^d, \nu \in \mathbb{S}^{N-1}. \tag{1.4}$$

For further generalizations allowing for frame invariant potentials W, see [12, 13, 17] (the reader interested in vector valued singularly perturbed problems with higher order gradients regularizations would also find interesting the results obtained in [9, 10, 16, 18]). Under additional assumptions on W and Ω, satisfied in particular by prototypical quadratic potentials as $W(\xi) = \min\{|\xi - A|^2, |\xi - B|^2\}$ and by regular convex domains Ω, the authors of [11] compute the Γ-limit of H_ε and prove that the latter is given by a functional H finite on those $u \in W^{1,1}(\Omega; \mathbb{R}^d)$ with $\nabla u \in BV(\Omega; \{A, B\})$. On this set of functions $H(u)$ takes the form

$$H(u) = K \cdot \mathcal{H}^{N-1}(S_{\nabla u} \cap \Omega), \tag{1.5}$$

where the constant $K > 0$ is obtained by solving an asymptotic cell-problem formula.

In the present paper we are going to investigate functionals defined on functions $u \in W^{2,2}(\Omega, \mathbb{R}^d)$ and measures $\mu = \rho\mathcal{L}^N$ of the form

$$E_\varepsilon(u,\mu) := \int_\Omega \frac{1}{\varepsilon} W(\nabla u) + \varepsilon|\nabla^2 u|^2 + \varepsilon(\rho - |\nabla^2 u|)^2 \, dx. \tag{1.6}$$

The main result of this paper is stated in Theorem 2.3 in which we compute, under the same assumptions on W (w.l.o.g. we assume that in (1.4) $\nu = e_N$) and Ω as those considered in [11] (see Section 2.2 for details), the Γ-limit of $E_\varepsilon(u,\mu)$ with respect to the strong $W^{1,p}$ convergence of the deformations u and the weak$*$-convergence of the surfactant measures μ. We have that $\Gamma\text{-}\lim_{\varepsilon\to 0^+} E_\varepsilon(u,\mu) = E(u,\mu)$ where $E(u,\mu)$ is a functional finite on those functions $u \in W^{1,p}(\Omega;\mathbb{R}^d)$ such that $\nabla u \in (\Omega; \{A, B\})$ on which it takes the form

$$E(u,\mu) = \int_{S_{\nabla u}\cap\Omega} \Phi\left(\frac{d\mu}{d\mathcal{H}^{\text{N-1}}\llcorner S_{\nabla u}}\right) d\mathcal{H}^{\text{N-1}} . \tag{1.7}$$

The surface tension Φ above is a nonnegative nonincreasing function given by an asymptotic formula (see Definition 3.2). Roughly speaking $\Phi(\gamma)$ can be interpreted as the effective energy per unit $\mathcal{H}^{\text{N-1}}$-measure associated to the phase separation induced by the deformation $u : Q \to \mathbb{R}^d$ with

$$\nabla u(x) = \begin{cases} -a \otimes e_N, & x_N < 0, \\ a \otimes e_N, & x_N > 0 \end{cases}$$

and in presence of the surfactant measure

$$\mu := \gamma \mathcal{H}^{\text{N-1}} \llcorner \{x_N = 0\}.$$

In order to prove our main result, we first need to show that Φ can be obtained restricting the class of admissible functions in the asymptotic formula to those sharing additional regularity and periodicity assumption as in (3.17). To this end we need to combine some of the arguments in [11, 19], the latter modified to fit in the present vectorial case. Such characterization allows us to compute the Γ-limit on functions with fixed boundary conditions that are periodic in direction of the phase separation. The proof of the $\Gamma\text{-}\liminf$-inequality (Proposition 4.1) is then obtained by a blow-up technique near the interfaces of u. The proof of the Γ-limsup inequality (Proposition 4.2) makes use of a density argument which reduces the construction of a recovery sequence for a generic pair (u,μ) to the case of deformations with a single interface and to constant surfactant densities.

2. Notation, statement of the main result and preliminaries

2.1. General notation

Let $\Omega \subset \mathbb{R}^N$ be a bounded, open set with Lipschitz boundary. We denote by $\mathcal{L}^N$ and $\mathcal{H}^{N-1}$ the N-dimensional Lebesgue measure and the $(N-1)$-dimensional Hausdorff measure in $\mathbb{R}^N$, respectively. We use the notation $|U| := \int_U dx := \mathcal{L}^N(U)$. We denote by $\mathcal{M}(\Omega)$ the space of non negative Radon measures finite on Ω. We set $Q_{x_0,\delta} := (x_0 - \delta, x_0 + \delta)^N$ and we use the notation $Q := Q_{0,\frac{1}{2}}$ for the unitary open cube in $\mathbb{R}^N$ centred at the origin. Given $x \in \mathbb{R}^N$ we label the first $(N-1)$-coordinates as x' and the last coordinate as x_N and we write $x = (x', x_N)$. We also set $Q' = (-\frac{1}{2}, \frac{1}{2})^{N-1}$, hence $Q = Q' \times (-\frac{1}{2}, \frac{1}{2})$. Given a function $u : Q \to \mathbb{R}^d$ such that for all $x_N \in (-\frac{1}{2}, \frac{1}{2})$ it holds that $u(\cdot, x_N)$ is Q'-periodic, we say that u is Q'-periodic in x'. Given a function $u \in L^1(\Omega, \mathbb{R}^d)$, we denote by S_u the approximate discontinuity set of u, i.e., the set of those points $x \in \Omega$ for which no $z \in \mathbb{R}^d$ exists such that $\lim_{r\to 0^+} |B_r(x)|^{-1} \int_{B_r(x)} |u(y) - z| dy = 0$ holds. We denote by $BV(\Omega)$ the set of functions of bounded variation in Ω. We say that a measurable set $E \subset \mathbb{R}^N$ is a set of finite perimeter in Ω if $\chi_E \in BV(\Omega)$. Denoting by $P(E, \Omega)$ the De Giorgi's perimeter in Ω of E, if E is a set of finite perimeter we also write that $P(E, \Omega) = \mathcal{H}^{N-1}(\partial^* E \cap \Omega) < +\infty$ where $\partial^* E$ stands for the reduced boundary of E. If $u \in BV(\Omega; \{a, b\})$ is a function of bounded variation in Ω taking only the two values $a, b \in \mathbb{R}^d$, the $(N-1)$-Hausdorff measure of S_u equals the perimeter of the level set $\{u = a\}$ (and $\{u = b\}$) in Ω or in formula $\mathcal{H}^{\text{N-1}}(S_u) = P(\{u = a\}, \Omega)$. For all properties of functions of bounded variations and of sets of finite perimeter needed in this paper we refer the reader to [4]. Finally we set $\mathbb{S}^{N-1} := \{x \in \mathbb{R}^N : |x| = 1\}$ and we denote by c and C generic real positive constants that may vary from line to line and expression to expression within the same formula.

2.2. The main result

In this section we introduce the energy functional we are interested in and state our main theorems. For $\varepsilon > 0$ we consider the functional $E_\varepsilon : W^{1,1}(\Omega; \mathbb{R}^d) \times \mathcal{M}(\Omega) \times \mathcal{A}(\Omega) \to [0, +\infty]$ defined as

$$E_\varepsilon(u, \mu, U) := \begin{cases} \int_U \frac{1}{\varepsilon} W(\nabla u) + \varepsilon |\nabla^2 u|^2 + \varepsilon(\rho - |\nabla^2 u|)^2 \, dx, & \text{if } u \in W^{2,2}(\Omega; \mathbb{R}^d),\ \mu = \rho\, dx, \\ +\infty, & \text{otherwise in } W^{1,1}(\Omega; \mathbb{R}^d) \times \mathcal{M}(\Omega). \end{cases} \tag{2.1}$$

With a little abuse of notation we will also introduce the functional $E_\varepsilon(u, \mu) : W^{1,1}(\Omega; \mathbb{R}^d) \times \mathcal{M}(\Omega) \to [0, +\infty]$ defined as

$$E_\varepsilon(u, \mu) := E_\varepsilon(u, \mu, \Omega).$$

The asymptotic analysis as $\varepsilon \to 0$ of the functional $E_\varepsilon(u, \mu)$ will be carried over in the ambiance space $W^{1,1}(\Omega) \times \mathcal{M}(\Omega)$ endowed with the convergence $\tau_1 \times \tau_2$ where τ_1 denotes the strong convergence in $W^{1,1}(\Omega; \mathbb{R}^d)$, while τ_2 denotes the weak*-convergence in the space of non-negative bounded Radon measures $\mathcal{M}(\Omega)$.

On the potential $W : \mathbb{R}^{d\times N} \to [0, \infty)$ we make the following set of assumptions:

$$W \text{ is continuous, } W(\xi) = 0 \text{ if and only if } \xi \in \{A, B\}, \quad \text{where } A - B = a \otimes \nu, \text{ for some } a \in \mathbb{R}^d,\ \nu \in \mathbb{S}^{N-1}. \tag{H1}$$

$$\frac{1}{C}|\xi|^p - C \le W(\xi) \le C(|\xi|^p + 1) \text{ for some } C > 1, p \ge 2. \tag{H2}$$

$$\begin{aligned} &c|\xi - A|^p \le W(\xi) \le C|\xi - A|^p, \quad |\xi - A| \le \rho, \\ &c|\xi - B|^p \le W(\xi) \le C|\xi - B|^p, \quad |\xi - B| \le \rho, \\ &\text{for some } \rho > 0 \text{ and } p \ge 2. \end{aligned} \tag{H3}$$

$$W(\xi_1, \ldots, \xi_i, \ldots, \xi_N) = W(\xi_1, \ldots, -\xi_i, \ldots, \xi_N), \quad i = 1, \ldots, N. \tag{H4}$$

Remark 2.1. We observe that assumption (H1) together with the control from below in (H2) for $p > 1$ would suffice to obtain the forthcoming compactness and liminf inequality statements.

Remark 2.2. The following control on the potential energy W is proven in [11, Remark 6.1]. From (H2) and (H3), there exist $C_1, C_2 > 0$ such that

$$C_1|\xi'|^p \le W(\xi) \le C_2(W(\eta) + |\xi - \eta|^p)$$

holds for all $\xi, \eta \in \mathbb{R}^{d\times N}$.

The following two theorems are the main result of this article. Note that a corresponding compactness statement is given in Proposition 3.1.

Theorem 2.3. *Let E_ε be as in (1.6), where $W : \mathbb{R}^{N\times d} \to [0,\infty)$ is a continuous double-well potential as in (1.4) that satisfies growth conditions as in (H1)–(H3) and is even as in (H4). If the domain $\Omega \subset \mathbb{R}^N$ is open, bounded, has a Lipschitz boundary, is simply connected and for all $t \in \mathbb{R}$ it holds that the section $\{(x_1,\dots x_N) \in \Omega : x_N = t\}$ is connected, then in the space $W^{1,1}(\Omega,\mathbb{R}^d) \times \mathcal{M}(\Omega)$ it holds that*

$$\Gamma\text{-}\lim_{\varepsilon\to 0^+} E_\varepsilon(u,\mu) = E(u,\mu).$$

Here, we have written

$$E(u,\mu) := \begin{cases} \int_{S_{\nabla u}} \Phi\left(\frac{d\mu}{d\mathcal{H}^{N-1}\llcorner S_{\nabla u}}\right), & \nabla u \in BV(\Omega,\{A,B\}), \\ +\infty, & \textit{otherwise in } W^{1,1}(\Omega;\mathbb{R}^d)\times\mathcal{M}(\Omega), \end{cases} \tag{2.2}$$

with Φ a nonnegative nonincreasing function as in (3.1).

Proof. The proof follows from Propositions 4.1 and 4.2. □

Theorem 2.4. *Let E_ε be as in (1.6), where $W : \mathbb{R}^{N\times d} \to [0,\infty)$ is a continuous double-well potential as in (1.4) that satisfies growth conditions as in (H1)–(H3) and (H5). If the domain $\Omega \subset \mathbb{R}^N$ is open, bounded, has a Lipschitz boundary, is simply connected and for all $t \in \mathbb{R}$ it holds that the section $\{(x_1,\dots x_N) \in \Omega : x_N = t\}$ is connected, then in the space $W^{1,1}(\Omega,\mathbb{R}^d) \times \mathcal{M}(\Omega)$ it holds that*

$$\Gamma\text{-}\lim_{\varepsilon\to 0^+} E_\varepsilon(u,\mu) = E(u,\mu),$$

where $E(u,\mu)$ is given in (2.2)*. The surface tension Φ in Definition 3.2 is obtained further restricting the admissible set of functions in the cell-problem formula to one-dimensional profiles $u_h(x) = u_h(x_N)$.*

Proof. The proof of the statement follows from the proof of Theorem 2.3 taking into account Proposition 3.7. □

Remark 2.5. Note that for potentials that do not satisfy (H5) Proposition 3.7, and hence Theorem 2.4, are false as shown in [11, Section 8].

2.3. Preliminaries

In what follows we will often make use of the following straightforward identity:

$$\min\{\lambda^2 + w^2, 2w^2\} = w^2 + (\max\{\lambda + w, 0\} - w)^2 \text{ for all } \lambda \leq 0, w \geq 0. \tag{2.3}$$

The next theorem is proved in [11, Theorem 3.3].

Theorem 2.6. *Let $u \in W^{1,1}(\Omega, \mathbb{R}^d)$ be such that $\nabla u \in \mathrm{BV}(\Omega, \{A, B\})$ with $A - B = a \otimes \nu$, for some $a \in \mathbb{R}^d$, $\nu \in \mathbb{S}^{N-1}$. Then u has the form*

$$u(x', x_N) = \gamma_0 + a x_N - 2\psi(x)a$$

for some $\gamma_0 \in \mathbb{R}^d$ such that $\gamma_0 \cdot a = 0$ and for some $\psi \in W^{1,\infty}(\Omega, \mathbb{R}^d)$ such that $\nabla\psi(x) = \chi_E(x)e_N$. The set $E \subseteq \Omega$ has $P(E, \Omega) < \infty$ and moreover

$$\partial^* E = \bigcup_{i=1}^{\infty} \Omega_i \times \{t_i\}$$

with $\Omega_i \subset \mathbb{R}^{N-1}$ connected, open and bounded and $t_i \in \mathbb{R}$. If in addition for each $t \in \mathbb{R}$ the set $\{(x', x_N) \in \Omega \mid x_N = t\}$ is connected, then ψ depends only on x_N.

The following lemma is proved in [19, Lemma 3.2].

Lemma 2.7. *Assume that (X, μ) is a measure space with μ a non-atomic positive measure. Let $g : X \to [0, \infty) \in L^1(X, \mu) \cap L^2(X, \mu)$ and $0 < \gamma \leq \int_X g\, d\mu$ be given. Then, for all $v \geq 0$ with $\int_X v\, d\mu = \gamma$ it holds true that*

$$\int_X (v - g)^2\, d\mu \geq \int_X (\max\{\lambda + g, 0\} - g)^2\, d\mu = \int_X \min\{\lambda^2, g^2\}\, d\mu,$$

where $\lambda \in (-\infty, 0]$ satisfies $\int_X \max\{\lambda + g, 0\}\, d\mu = \gamma$.

3. Compactness and characterization of the asymptotic surface tension

In this section we prove the compactness statement for sequences u_h and $\mu_h = \rho_h\, dx$ with equibounded energy $E_{\varepsilon_h}(u_h.\rho_h)$. We moreover introduce the effective asymptotic (as $\varepsilon_h \to 0$) surface tension of the energy functional given by the variational limit of E_{ε_h} and provide some useful characterization of it. For simplicity of notation and without loss of generality from now on we will assume in $(H1)$ that $A = -B = a \otimes e_N$.

3.1. Compactness

In what follows we state the main compactness result for our functionals. It is a direct consequence of [11, Theorem 3.1]. In all our analysis ε_h denotes a sequence of positive numbers vanishing as $h \to +\infty$.

Proposition 3.1 (Compactness)**.** *Let $W : \mathbb{R}^{d\times N} \to [0, \infty)$ satisfy assumptions* (H1) *and* (H2)*, $\varepsilon_h \to 0^+$, (u_h) be a sequence in $W^{2,2}(\Omega, \mathbb{R}^d)$ and $\mu_h = \rho_h\, dx$ be a sequence in $\mathcal{M}(\Omega)$ such that*

$$\sup_h E_{\varepsilon_h}(u_h, \rho_h) < \infty \text{ and } \sup_h \mu_h(\Omega) < \infty.$$

Then there exist subsequences (u_{h_k})*,* (ρ_{h_k}) *and* $u \in W^{1,p}(\Omega,\mathbb{R}^d)$ *with* $\nabla u \in \mathrm{BV}(\Omega,\{A,B\})$ *and* $\mu \in \mathcal{M}(\Omega)$ *such that*

$$u_{h_k} - \frac{1}{|\Omega|}\int_\Omega u_{h_k}\,dx \to u \text{ in } W^{1,p}(\Omega,\mathbb{R}^d) \text{ and } \mu_{h_k} \overset{*}{\rightharpoonup} \mu \text{ as } k \to +\infty.$$

Proof. The convergence of a subsequence of (u_h) follows as in [11, Remark 3.2 (ii)]. In fact we can apply the proof of [11, Theorem 3.1] to find a subsequence $u_{h_k} \to u$ in $W^{1,1}(\Omega,\mathbb{R}^d)$. By assumption (H2) there exists $L > 0$ such that $W(\xi) \geq c|\xi|^p$ for $p \geq 2$ and for all ξ satisfying $|\xi| \geq L$ and therefore

$$c\int_{\{|\nabla u_{h_k}|\geq L\}} |\nabla u_{h_k}|^p\,dx \leq \int_\Omega W(\nabla u_{h_k})\,dx \to 0.$$

If follows

$$\int_{\{|\nabla u_{h_k}|\geq L\}} |\nabla u_{h_k} - \nabla u|^p\,dx \to 0$$

and together with

$$\int_{\{|\nabla u_{h_k}|\leq L\}} |\nabla u_{h_k} - \nabla u|^p\,dx \leq (L + |A| + |B|)^{p-1}\int_\Omega |\nabla u_{h_k} - \nabla u|\,dx \to 0,$$

it implies the convergence of u_{h_k} in $W^{1,p}(\Omega,\mathbb{R}^d)$. The convergence of a subsequence of (μ_h) in the weak*-topology is a consequence of the weak*-compactness of $\mathcal{M}(\Omega)$. □

In what follows we define the lower semicontinuous envelope of our energy functional on a restricted class of admissible functions. In Section 4, we are going to prove that this lower bound is actually our Γ-liminf functional.

We start by introducing a notation. For every open subset $U \subset \Omega$ we define

$$F_\varepsilon(u,\lambda,U) := \int_U \frac{1}{\varepsilon}W(\nabla u) + \varepsilon|\nabla^2 u|^2 + \varepsilon\min\{\lambda^2, |\nabla^2 u|^2\}\,dx.$$

In the case $U = Q$ (the unitary cube in $\mathbb{R}^N$ centred at the origin) the notation above will be shortened and we will use $F_\varepsilon(u,\lambda) := F_\varepsilon(u,\lambda,Q)$.

Definition 3.2. *For* $\gamma \geq 0$*,* $k > 0$ *and* $\omega \subset \mathbb{R}^{N-1}$ *a bounded open set with* $\mathcal{H}^{N-1}(\partial\omega) = 0$ *we define*

$$F(\gamma,\omega\times(-k,k)) := \inf\Big\{\liminf_{h\to\infty} F_{\varepsilon_h}(u_h,\lambda_h,\omega\times(-k,k)) :\ u_h \to |x_N|a, (u_h,\lambda_h) \in \mathcal{A}(\gamma,\omega\times(-k,k))\Big\},$$

where we have used the notation $\mathcal{A}(\gamma,\omega\times(-k,k))$ *for the set of admissible functions defined as*

$$\mathcal{A}(\gamma,\omega\times(-k,k)) := \Big\{(u,\lambda) \in W^{2,2}(Q,\mathbb{R}^d)\times(-\infty,0] \mid \int_{\omega\times(-k,k)} \max\{\lambda + |\nabla^2 u|, 0\}\,dx \leq \gamma\,\mathcal{H}^{N-1}(\omega)\Big\}.$$

In order to shorten the notation we also set

$$\mathcal{A}(\gamma) := \mathcal{A}(\gamma,Q), \quad \tilde{\Phi}(\gamma) := F(\gamma,Q), \quad \text{and} \quad \Phi(\gamma) := \lim_{\delta\to 0^+}\tilde{\Phi}(\gamma+\delta) \tag{3.1}$$

and we observe that since $\tilde{\Phi}$ is non-increasing, the function Φ is well-defined.

The following lemma, whose proof we omit, can be proved as in [11, Lemma 4.3], the only care being that the rescaling argument used to show assertion (iv) now makes use of the admissible sequence $(\alpha u_n(\frac{x}{\alpha}), \frac{\lambda_n}{\alpha})$.

Lemma 3.3. *For $\gamma \geq 0$ fixed, it holds*

(i) $F(\gamma, x' + \omega \times (-k,k)) = F(\gamma, \omega \times (-k,k))$ *for all* $x' \in \mathbb{R}^{N-1}$;

(ii) $F(\gamma, \omega_1 \times (-k,k)) \leq F(\gamma, \omega_2 \times (-k,k))$ *if* $\omega_1 \subset \omega_2$;

(iii) $F(\gamma, \omega_1 \times (-k,k)) + F(\gamma, \omega_2 \times (-k,k)) \leq F(\gamma, \omega_1 \cup \omega_2 \times (-k,k))$ *if* $\omega_1 \cap \omega_2 = \emptyset$;

(iv) $F(\gamma, \alpha\omega \times (-\alpha k, \alpha k)) = \alpha^{N-1} F(\gamma, \omega \times (-k,k))$ *for* $\alpha > 0$,
$F(\gamma, \alpha\omega \times (-h,h)) \geq \alpha^{N-1} F(\gamma, \omega \times (-k,k))$ *for* $1 > \alpha > 0$;

(v) $F(\gamma, \omega \times (-k,k)) = \mathcal{H}^{N-1}(\omega) F(\gamma, Q' \times (-k,k))$;

(vi) $F(\gamma, \omega \times (-k,k)) = F(\gamma, \omega \times (-k',k'))$ *for all* $k' > 0$,

and analogously for $\lim_{\delta\to 0^+} F(\gamma+\delta, \omega \times (-k,k))$.

In particular it holds that $F(\gamma, \omega \times (-k,k)) = \mathcal{H}^{N-1}(\omega)\tilde{\Phi}(\gamma)$.

3.2. Characterization of the surface tension

In this section we further characterize the surface tension Φ and $\tilde{\Phi}$. More specifically, following [11], we prove that the minimum problem in Definition 3.2 can be restricted to a narrower class of competitors.

Proposition 3.4. *Let* $W : \mathbb{R}^{d\times N} \to [0,\infty)$ *satisfy* (H1)–(H3) *and let* $\gamma \geq 0$ *and* $\delta > 0$ *be given. Then there exist sequences* $\varepsilon_h^\delta \to 0^+$ *and* $(u_h^\delta, \lambda_h^\delta) \in \mathcal{A}(\gamma+\delta)$ *satisfying*

$$u_h^\delta \to |x_N| a \text{ in } W^{1,p}(Q, \mathbb{R}^d),$$
$$u_h^\delta = -x_N a \text{ near } x_N = -\frac{1}{2}, \quad u_h^\delta = x_N a + c_h^\delta \text{ near } x_N = \frac{1}{2}, \quad c_h^\delta \to 0 \text{ as } h \to \infty,$$
$$\lim_{h\to\infty} F_{\varepsilon_h}(u_h^\delta, \lambda_h^\delta) = \tilde{\Phi}(\gamma).$$

Proof. Our proof follows the strategy of the proof of [11, Proposition 6.2]. We fix δ and drop it from the notation. Choosing admissible sequences satisfying

$$\lim_{h\to\infty} F_{\varepsilon_h}(u_h, \lambda_h) = \tilde{\Phi}(\gamma) \tag{3.2}$$

and using the compactness result (3.1) we can assume that $u_h \to |x_N|a =: u_0$ in $W^{1,p}(Q, \mathbb{R}^d)$. We can partition $Q' \times (\frac{1}{6}, \frac{1}{3})$ into $\lfloor \frac{1}{\varepsilon_h} \rfloor$-horizontal layers of height $\lfloor \frac{1}{\varepsilon_h} \rfloor^{-1} \frac{1}{6}$ and choose a layer $L_h = Q' \times (\theta_h - \lfloor \frac{1}{\varepsilon_h} \rfloor^{-1} \frac{1}{6}, \theta_h)$ which satisifies

$$\begin{aligned} &\left\lfloor \frac{1}{\varepsilon_h} \right\rfloor \left(F_{\varepsilon_h}(u_h, \lambda_h, L_h) + \int_{L_h} |\nabla u_h - a \otimes e_N|^p + |u_h - u_0(x)|^p \, dx \right) \\ &\quad \leq F_{\varepsilon_h}\left(u_h, \lambda_h, Q' \times \left(\frac{1}{6}, \frac{1}{3}\right)\right) + \int_{Q' \times (\frac{1}{6}, \frac{1}{3})} |\nabla u_h - a \otimes e_N|^p + |u_h - u_0(x)|^p \, dx \\ &\quad =: \alpha_h \to 0, \end{aligned} \tag{3.3}$$

where we have used Lemma 3.3(vi), which asserts that the energy concentrates near $Q' \times \{0\}$. By the continuity of W, (3.2) and the very definition of the energy functional F_ε there exists $z_h \in (\theta_h -$

$\lfloor \frac{1}{\varepsilon_h} \rfloor^{-1} \frac{1}{6}, \theta_h)$ such that

$$\int_{Q'} \frac{1}{\varepsilon_h} W(\nabla u_h(x', z_h)) + \varepsilon_h |\nabla^2 u_h(x', z_h)|^2 + \varepsilon_h \min\{\lambda_h^2, |\nabla^2 u_h(x', z_h)|^2\} \, dx' + \int_{Q'} |\nabla u_h(x', z_h) - a \otimes e_N|^p + |u_h(x', z_h) - u_0(x', z_h)|^p \, dx' \leq 6\alpha_h. \tag{3.4}$$

Choosing a smooth cut-off function $\varphi_h : \left(-\frac{1}{2}, \frac{1}{2}\right) \to \mathbb{R}$ satisfying $\varphi_h(x_N) = 1$ if $x_N \leq \theta_h - \lfloor \frac{1}{\varepsilon_h} \rfloor^{-1} \frac{1}{6}$, $\varphi_h = 0$ if $x_N \geq \theta_h$, $|\varphi_h'| \leq \frac{c}{\varepsilon_h}$, $|\varphi_h''| \leq \frac{c}{\varepsilon_h^2}$, we define

$$\begin{aligned} v_h(x) &:= u_0(x) + u_h(x', z_h) - u_0(x', z_h) + \varphi_h(x_N)(u_h(x) - u_0(x) - (u_h(x', z_h) - u_0(x', z_h))) \\ &= \varphi_h(x_N) u_h(x) + (1 - \varphi_h(x_N))(u_0(x) + u_h(x', z_h) - u_0(x', z_h)). \end{aligned}$$

We claim that the following limits hold true:

$$(a) \int_{L_h} |v_h - u_0|^p \, dx \to 0, \qquad (b) \frac{1}{\varepsilon_h} \int_{L_h} |\nabla v_h - a \otimes e_N|^p \, dx \to 0,$$

$$(c) F_{\varepsilon_h}(v_h, \lambda_h, L_h) \to 0, \qquad (d) \int_{L_h} \max\{\lambda_h + |\nabla^2 v_h|, 0\} \, dx \to 0.$$

The limit in (a) follows directly from (3.3) and (3.4). We now prove the limit in (b). We can apply Poincaré inequality to the function $u_h(x) - u_0(x) - (u_h(x', z_h) - u_0(x', z_h))$ to obtain

$$\begin{aligned} &\frac{1}{\varepsilon_h} \int_{L_h} |\nabla v_h - a \otimes e_N|^p \, dx \\ &\leq \frac{C}{\varepsilon_h} \int_{L_h} |\nabla u_h(x) - a \otimes e_N|^p \\ &+ |\nabla_{x'} u_h(x', z_h)|^p + \frac{c^p}{\varepsilon_h^p} |u_h(x) - u_0(x) - (u_h(x', z_h) - u_0(x', z_h))|^p \, dx \\ &\leq C \int_{Q'} |\nabla_{x'} u_h(x', z_h)|^p \, dx' + \frac{C}{\varepsilon_h} \int_{L_h} |\nabla u_h - a \otimes e_N|^p \, dx \\ &\leq C \int_{Q'} |\nabla u_h(x', z_h) - a \otimes e_N|^p \, dx' + \frac{C}{\varepsilon_h} \int_{L_h} |\nabla u_h - a \otimes e_N|^p \, dx \to 0, \end{aligned} \tag{3.5}$$

where in the last step we have used (3.3) and (3.4). We now prove claim (c). Thanks to (H2) and (H3) we obtain

$$\begin{aligned} \frac{1}{\varepsilon_h} \int_{L_h} W(\nabla v_h) \, dx &\leq \frac{1}{\varepsilon_h} \int_{L_h \cap \{|\nabla v_h - \nabla u_0| < \rho\}} C |\nabla v_h - a \otimes e_N|^p \, dx \\ &+ \frac{1}{\varepsilon_h} \int_{L_h \cap \{|\nabla v_h - \nabla u_0| \geq \rho\}} C(1 + |\nabla v_h|^p) \, dx \\ &\leq \frac{C}{\varepsilon_h} \int_{L_h} |\nabla v_h - a \otimes e_N|^p \, dx \to 0, \end{aligned} \tag{3.6}$$

where the last limit follows by (b). We also have that

$$
\begin{aligned}
\int_{L_h} \varepsilon_h |\nabla^2 v_h|^2 \, dx \le & C \int_{L_h} \varepsilon_h |\nabla_{x'} u_h(x', z_h)|^2 + \varepsilon_h |\nabla^2 u_h|^2 + \frac{1}{\varepsilon_h} |\nabla u_h - a \otimes e_N|^2 \\
& + \frac{1}{\varepsilon_h^3} |u_h - u_0 - (u_h(x', z_h) - u_0(x', z_h))|^2 \, dx \\
\le & C\varepsilon_h^2 \int_{Q'} |\nabla^2_{x'} u_h(x', z_h)|^2 \, dx' + C\varepsilon_h \int_{L_h} |\nabla^2 u_h|^2 \, dx \\
& + \frac{C}{\varepsilon_h} \int_{L_h} |\nabla u_h - a \otimes e_N|^2 \, dx + C \int_{Q'} |\nabla_{x'} u_h(x', z_h)|^2 \, dx' \\
& + \frac{C\varepsilon_h^{(p-2)/p}}{\varepsilon_h^3} \left(\int_{L_h} |u_h(x) - u_0(x) - (u_h(x', z_h) - u_0(x', z_h))|^p \, dx \right)^{2/p} \\
\le & C\varepsilon_h^2 \int_{Q'} |\nabla^2_{x'} u_h(x', z_h)|^2 \, dx' + C\varepsilon_h \int_{L_h} |\nabla^2 u_h|^2 \, dx + \frac{C}{\varepsilon_h^{2/p}} \left(\int_{L_h} |\nabla u_h - a \otimes e_N|^p \, dx \right)^{2/p} \\
& + C \left(\int_{Q'} |\nabla u_h(x', z_h) - a \otimes e_N|^p \, dx' \right)^{2/p} \to 0,
\end{aligned}
$$

where we have used Hölder's inequality, Poincaré's inequality for $u_h(x) - u_0(x) - (u_h(x', z_h) - u_0(x', z_h))$ and, in the last step, (3.3)–(3.5). By the trivial inequality

$$
\int_{L_h} \varepsilon_h \min\{\lambda_h^2, |\nabla^2 v_h|^2\} \, dx \le \int_{L_h} \varepsilon_h |\nabla^2 v_h|^2 \, dx,
$$

the estimate above shows (c). Finally (d) follows from (3.3) and (3.4) thanks to the estimate

$$
\int_{L_h} \max\{\lambda_h + |\nabla^2 v_h|, 0\} \, dx \le \int_{L_h} |\nabla^2 v_h| \, dx \le C\varepsilon_h^{1/2} \left(\int_{L_h} |\nabla^2 v_h|^2 \, dx \right)^{1/2} \to 0.
$$

In the next step, we choose a smooth cut-off function $\psi : \left(-\frac{1}{2}, \frac{1}{2}\right) \to \mathbb{R}$ such that $\psi(x_N) = 1$ for $x_N \le \frac{1}{3}$ and $\psi = 0$ near $x_N = \frac{1}{2}$. We moreover may assume that $\|\psi'\|_\infty \le C$ and $\|\psi''\|_\infty \le C$ and define

$$
w_h(x) := u_0(x) + c_h + \psi(x_N)(u_h(x', z_h) - u_0(x', z_h) - c_h),
$$

where

$$
c_h := \int_{Q'} u_h(x', z_h) - u_0(x', z_h) \, dx' \to 0
$$

by (3.4). We write $\hat{Q}_h := Q' \times (\theta_h, 1/2)$ and claim that

$$
(a') \int_{\hat{Q}_h} |w_h - u_0|^p \, dx \to 0, \quad (b') \frac{1}{\varepsilon_h} \int_{\hat{Q}_h} |\nabla w_h - a \otimes e_N|^p \, dx \to 0, \quad (c') \, F_{\varepsilon_h}(w_h, \lambda_h, \hat{Q}_h) \to 0,
$$

$$
(d') \int_{\hat{Q}_h} \max\{\lambda_h + |\nabla^2 w_h|, 0\} \, dx \le \int_{\hat{Q}_h} \max\{\lambda_h + |\nabla^2 u_h|, 0\} \, dx + d_h, \text{where } d_h \to 0 \text{ as } h \to \infty.
$$

The first claim follows from (3.4). In order to prove claim (b') we use Poincaré's inequality and Remark 2.2. We obtain

$$\frac{1}{\varepsilon_h}\int_{\hat{Q}_h} |\nabla w_h - a\otimes e_N|^p \, dx \le \frac{C}{\varepsilon_h}\int_{\hat{Q}_h} |\nabla_{x'} u_h(x',z_h)|^p + |u_h(x',z_h) - u_0(x',z_h) - c_h|^p \, dx \qquad (3.7)$$
$$\le \frac{C}{\varepsilon_h}\int_{Q'} |\nabla_{x'} u_h(x',z_h)|^p \, dx' \le C\int_{Q'} \frac{1}{\varepsilon_h} W(\nabla u_h(x',z_h))\, dx' \to 0,$$

where the last limit follows by (3.4). Similarly, it holds that

$$\int_{\hat{Q}_h} \varepsilon_h |\nabla^2 w_h|^2 \, dx \le C\varepsilon_h \int_{\hat{Q}_h} |\nabla^2_{x'} u_h(x',z_h)|^2 + |\nabla_{x'} u_h(x',z_h)|^2 \, dx$$
$$\le C\varepsilon_h \int_{Q'} |\nabla^2_{x'} u_h(x',z_h)|^2 \, dx' + C\varepsilon_h \left(\int_{Q'} |\nabla u_h(x',z_h) - a\otimes e_N|^p \, dx'\right)^{2/p} \to 0.$$

Again, as shown for the claim (c), the last two estimates, together with (b') and (3.6), give (c'). To prove the last claim we use the estimate

$$|\nabla^2 w_h(x)| \le C(|u_h(x',z_h) - u_0(x',z_h) - c_h| + |\nabla_{x'} u_h(x',z_h)|) + |\nabla^2_{x'} u_h(x',z_h)|$$

to find that

$$\int_{\hat{Q}_h} \max\{\lambda_h + |\nabla^2 w_h|, 0\}\, dx \le \int_{\hat{Q}_h} \max\{\lambda_h + |\nabla^2_{x'} u_h(x',z_h)|, 0\}\, dx + C\int_{\hat{Q}_h} |\nabla_{x'} u_h(x',z_h)| \, dx$$
$$\le \int_{\hat{Q}_h} \max\{\lambda_h + |\nabla^2 u_h|, 0\}\, dx + C\left(\int_{\hat{Q}_h} |\nabla_{x'} u_h(x',z_h)|^2 \, dx\right)^{\frac{1}{2}}.$$

The second summand tends to zero as in (3.7), hence (d') holds true. Let us define

$$U_h := \begin{cases} u_h, & x_N < \theta_h - \lfloor \frac{1}{\varepsilon_h} \rfloor^{-1} \frac{1}{6}, \\ v_h, & \theta_h - \lfloor \frac{1}{\varepsilon_h} \rfloor^{-1} \frac{1}{6} \le x_N \le \theta_h, \\ w_h, & x_N > \theta_h. \end{cases}$$

Our claims show that $U_h \to u_0$ in $W^{1,p}(Q,\mathbb{R}^d)$ and that we may assume $(U_h,\lambda_h) \in \mathcal{A}(\gamma + \delta/2)$ for h large enough. Since by Lemma 3.3(vi), we have that

$$F_{\varepsilon_h}\left(u_h, \lambda_h, Q' \times \left(\frac{1}{6}, \frac{1}{2}\right)\right) \to 0,$$

from our claims it also follows that $\lim_{h\to\infty} F_{\varepsilon_h}(U_h,\lambda_h) = \tilde{\Phi}(\gamma)$. The proof is completed on observing that the construction above can be repeated on $Q' \times (-\frac{1}{2}, 0)$. □

In the next proposition, which is analogous to [11, Proposition 6.3], we show how to further modify the construction of the sequence of functions in the previous proposition to enforce periodic boundary condition in the x' variable.

Proposition 3.5. *Let* $W : \mathbb{R}^{d\times N} \to [0,\infty)$ *satisfy* (H1)–(H4) *and let* $\gamma \geq 0$ *and* $\delta > 0$ *be given. Then there exist sequences* $\varepsilon_h^\delta \to 0^+$ *and* $(u_h^\delta, \lambda_h^\delta) \in \mathcal{A}(\gamma+\delta)$ *satisfying*

$$u_h^\delta \in W^{2,\infty}(Q,\mathbb{R}^d), \quad u_h^\delta \to |x_N|a \text{ in } L^1(Q,\mathbb{R}^d),$$
$$\nabla u_h^\delta = \pm a \otimes e_N \text{ near } x_N = \pm\frac{1}{2} \text{ and } u_h^\delta \text{ is } Q'\text{-periodic for all } x_N,$$
$$\lim_{h\to\infty} F_{\varepsilon_h}(u_h^\delta, \lambda_h^\delta) = \tilde{\Phi}(\gamma).$$

Proof. Fixing δ and dropping it from the notation, we are going to show how to proceed as in the proof of Proposition 6.3 in [11] and show that there exist sequences $\varepsilon_h \to 0^+$, $v_h \in W^{2,\infty}(\mathbb{R}^N,\mathbb{R}^d)$, $\lambda_h \leq 0$ such that

$$(v_h,\lambda_h) \in \mathcal{A}\Big(\gamma+\delta, 2Q' \times \Big(-\frac{1}{2},\frac{1}{2}\Big)\Big),$$
$$v_h(\cdot,x_N) \text{ is } 2Q'\text{-periodic for all } x_N, \quad \nabla v_h = \pm a\otimes e_N \text{ near } x_N = \pm\frac{1}{2},$$
$$\lim_{h\to\infty} F_{\varepsilon_h}\Big(v_h,\lambda_h, 2Q'\times\Big(-\frac{1}{2},\frac{1}{2}\Big)\Big) = 2^{N-1}\tilde{\Phi}(\gamma), \quad \lim_{h\to\infty}\int_{2Q'\times(-\frac{1}{2},\frac{1}{2})} |v_h - |x_N|a|\,dx = 0. \tag{3.8}$$

With (3.8) at hand we can extend v_h linearly to $2Q$, define $u_h(x) := \frac{1}{2}v_h(2x)$ and complete the proof noting that

$$(u_h, 2\lambda_h) \in \mathcal{A}(\gamma+\delta),$$
$$u_h(\cdot,x_N) \text{ is } Q'\text{-periodic for all } x_N, \quad \nabla u_h = \pm a\otimes e_N \text{ near } x_N = \pm\frac{1}{2},$$
$$\lim_{h\to\infty} F_{\frac{\varepsilon_h}{2}}(u_h,\lambda_h) = \tilde{\Phi}(\gamma),$$
$$\lim_{h\to\infty}\int_Q |u_h - |x_N|a|\,dx = 0.$$

In what follows we prove (3.8). A sketch illustrating the notation of the following step 1 of the proof is given in Figure 1.

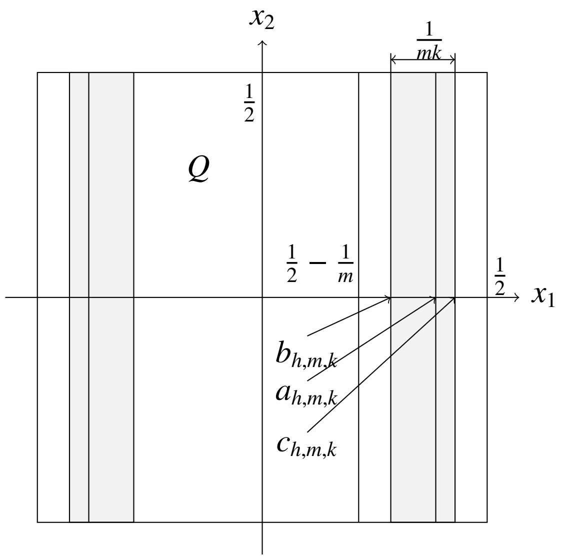

Figure 1. Sketch of the construction in Step 1.

Step 1. We assume that $N = 2$ and set $u_0 := |x_2|a$. By (3.4) we find sequences $\varepsilon_h \to 0^+, (u_h, \lambda_h) \in \mathcal{A}(\gamma + \delta/2)$ such that $u_h \to u_0$ in $L^1(Q, \mathbb{R}^d)$, $\nabla u_h = \pm a \otimes e_2$ near $x_2 = \pm\frac{1}{2}$ and $\lim_{h\to\infty} F_{\varepsilon_h}(u_h, \lambda_h) = \tilde{\Phi}(\gamma)$. Moreover, we may assume $u_h \in C^2(Q, \mathbb{R}^d)$: This follows from (H2) and the fact that

$$\int_\Omega \max\{\lambda + |\rho_\varepsilon * u|, 0\}\, dx \to \int_\Omega \max\{\lambda + |u|, 0\}\, dx$$

as $\varepsilon \to 0$ for $u \in L^1(\mathbb{R}^m, \mathbb{R}^n)$ which can be shown by an application of the Vitali dominated convergence theorem (see [4, Exercise 1.18]). We can therefore conclude as in the proof of [11, Proposition 6.2]. Setting

$$I_m := ((-\frac{1}{2}, -\frac{1}{2} + \frac{1}{m}) \cup (\frac{1}{2} - \frac{1}{m}, \frac{1}{2})) \times (-\frac{1}{2}, \frac{1}{2}),$$

we have

$$\tilde{\Phi}(\gamma) = \lim_{h\to\infty} F_{\varepsilon_h}(u_h, \lambda_h) \geq \liminf_{h\to\infty} F_{\varepsilon_h}(u_h, \lambda_h, Q \backslash I_m) \geq \tilde{\Phi}(\gamma)\Big(1 - \frac{2}{m}\Big),$$

and therefore

$$\limsup_{h\to\infty} F_{\varepsilon_h}(u_h, \lambda_h, I_m) \leq \tilde{\Phi}(\gamma)\frac{2}{m}.$$

By the compactness result stated in Theorem 3.1 we have $u_h \to u_0$ in $W^{1,p}(Q, \mathbb{R}^d)$. Therefore, for h sufficiently large, it holds that

$$F_{\varepsilon_h}(u_h, \lambda_h, I_m) + \int_{I_m} m^p |\nabla u_h - \nabla u_0|^p + |u_h - u_0|\, dx \leq \tilde{\Phi}(\gamma)\frac{3}{m}. \tag{3.9}$$

Let us subdivide $(-\frac{1}{2}, -\frac{1}{2} + \frac{1}{m}) \times (-\frac{1}{2}, \frac{1}{2})$ and $(\frac{1}{2} - \frac{1}{m}, \frac{1}{2}) \times (-\frac{1}{2}, \frac{1}{2})$ into k strips of equal width and order them in pairs. By (3.9) we find a pair of strips

$$R^+_{h,m,k} = (b_{h,m,k}, c_{h,m,k}) \times \left(-\frac{1}{2}, \frac{1}{2}\right),$$

$$R^-_{h,m,k} = (-b_{h,m,k}, -c_{h,m,k}) \times \left(-\frac{1}{2}, \frac{1}{2}\right),$$

such that for h sufficiently large it holds that

$$F_{\varepsilon_h}(u_h, \lambda_h, R^-_{h,m,k} \cup R^+_{h,m,k}) + \int_{R^-_{h,m,k} \cup R^+_{h,m,k}} m^p |\nabla u_h - \nabla u_0|^p + |u_h - u_0|\, dx \leq \tilde{\Phi}(\gamma)\frac{3}{mk}. \tag{3.10}$$

In particular we have that, setting $J_{h,m,k} := (\frac{b_{h,m,k}+c_{h,m,k}}{2}, c_{h,m,k})$,

$$\begin{aligned}
&\int_{J_{h,m,k}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{\varepsilon_h} W(\nabla u_h(x)) + \varepsilon_h |\nabla^2 u_h(x)|^2 + m^p |\nabla u_h(x) - \nabla u_0(x)|^p + |u_h(x) - u_0(x)| \\
&\quad + \frac{1}{\varepsilon_h} W(\nabla u_h(-x)) + \varepsilon_h |\nabla^2 u_h(-x)|^2 + m^p |\nabla u_h(-x) - \nabla u_0(-x)|^p + |u_h(-x) - u_0(-x)|\, dx \\
&\leq \tilde{\Phi}(\gamma)\frac{3}{mk}
\end{aligned}$$

and this shows that there exists $a_{h,m,k} \in \left(b_{h,m,k}, \frac{b_{h,m,k}+c_{h,m,k}}{2}\right)$ satisfying

$$\begin{aligned}\int_{-\frac{1}{2}}^{\frac{1}{2}} & \frac{1}{\varepsilon_h} W(\nabla u_h(a_{h,m,k}, x_2)) + \varepsilon_h |\nabla^2 u_h(a_{h,m,k}, x_2)|^2 \\ & + m^p |\nabla u_h(a_{h,m,k}, x_2) - \nabla u_0(a_{h,m,k}, x_2)|^p + |u_h(a_{h,m,k}, x_2) - u_0(a_{h,m,k}, x_2)| \\ & + \frac{1}{\varepsilon_h} W(\nabla u_h(-a_{h,m,k}, x_2)) + \varepsilon_h |\nabla^2 u_h(-a_{h,m,k}, x_2)|^2 \\ & + m^p |\nabla u_h(-a_{h,m,k}, x_2) - \nabla u_0(-a_{h,m,k}, x_2)|^p + |u_h(-a_{h,m,k}, x_2) - u_0(-a_{h,m,k}, x_2)| \, dx \\ \leq {} & 6\tilde{\Phi}(\gamma). \end{aligned} \tag{3.11}$$

Next we modify u_h in order to obtain a new function that coincides with u_h near $x_1 = -a_{h,m,k} + \frac{1}{2mk}$ and with $u_h(-a_{h,m,k}, \cdot)$ near $x_1 = -a_{h,m,k}$ (note that by construction $(-a_{h,m,k}, -a_{h,m,k} + \frac{1}{2mk}) \subset (-c_{h,m,k}, -b_{h,m,k})$). To this end we choose a smooth cut-off function $\varphi_{h,m,k} : \left(-\frac{1}{2}, \frac{1}{2}\right) \to \mathbb{R}$ such that $\varphi_{h,m,k} = 1$ if $x_1 > -a_{h,m,k} + \frac{1}{2mk}$, $\varphi_{h,m,k} = 0$ if $x_1 < -a_{h,m,k}$, $|\varphi'_{h,m,k}|_\infty \leq cmk$, $|\varphi''_{h,m,k}|_\infty \leq cm^2k^2$ and define

$$w_{h,m,k}(x) := \varphi_{h,m,k}(x_1) u_h(x) + (1 - \varphi_{h,m,k}(x_1)) u_h(-a_{h,m,k}, x_2).$$

We have that $w_{h,m,k} \in W^{2,\infty}(Q, \mathbb{R}^d)$ and $\nabla w_{h,m,k} = \pm a \otimes e_2$ near $x_2 = \pm\frac{1}{2}$. We are going to show that

$$\begin{aligned} & \limsup_{h\to\infty} \limsup_{k\to\infty} \limsup_{m\to\infty} F_{\varepsilon_h}(w_{h,m,k}, \lambda_h) \leq \tilde{\Phi}(\gamma), \\ & \limsup_{h\to\infty} \limsup_{k\to\infty} \limsup_{m\to\infty} \int_Q |w_{h,m,k} - u_0| \, dx = 0 \quad \text{and} \\ & \limsup_{h\to\infty} \limsup_{k\to\infty} \limsup_{m\to\infty} \int_Q \max\{\lambda_h + |\nabla^2 w_{h,m,k}|, 0\} \, dx \leq \gamma + \frac{\delta}{2}. \end{aligned} \tag{3.12}$$

With (3.12) at hand, we can repeat the same modification procedure close to $x_1 = a_{h,m,k}$ and obtain, using a diagonal argument, a sequence $(w_h, \lambda_h) \in \mathcal{A}(\gamma + \delta)$, $w_h \in W^{2,\infty}(Q, \mathbb{R}^d)$ where $w_h = w_h(\pm a_h, \cdot)$ near $x_1 = \pm\frac{1}{2}$, $\nabla w_h = \pm a \otimes e_2$ near $x_2 = \pm\frac{1}{2}$, $w_h \to u_0$ in $L^1(Q, \mathbb{R}^d)$ and $F_{\varepsilon_h}(w_h, \lambda_h) \to \tilde{\Phi}(\gamma)$. We can now reflect w_h with respect to the axis $x_1 = \frac{1}{2}$, translate it such that it is defined on $(-1, 1) \times (-\frac{1}{2}, \frac{1}{2})$ and denote it by v_h. Using property (H4) of W, we obtain that we have found a sequence of $2Q'$-periodic functions as desired. For later use, we also note that $v_h \in W^{2,\infty}_{\mathrm{loc}}(\mathbb{R} \times (-\frac{1}{2}, \frac{1}{2}), \mathbb{R}^d)$ if extended periodically.

In order to show (3.12), we first claim that

$$\begin{aligned} & \limsup_{h\to\infty} \limsup_{k\to\infty} \limsup_{m\to\infty} \int_{R^-_{h,m,k}} \frac{1}{\varepsilon_h} W(\nabla w_{h,m,k}) \, dx = 0, \\ & \limsup_{h\to\infty} \limsup_{k\to\infty} \limsup_{m\to\infty} \int_{R^-_{h,m,k}} \varepsilon_h |\nabla^2 w_{h,m,k}|^2 \, dx = 0 \quad \text{and} \\ & \limsup_{h\to\infty} \limsup_{k\to\infty} \limsup_{m\to\infty} \int_{R^-_{h,m,k}} |w_{h,m,k} - u_h| \, dx = 0. \end{aligned} \tag{3.13}$$

We are going to exploit that, from the very definition of $w_{h,m,k}$, we have that

$$\nabla w_{h,m,k}(x) = \varphi_{h,m,k}(x_1)\nabla u_h(x) + (1-\varphi_{h,m,k}(x_1))\Big(0 \mid \frac{\partial u_h}{\partial x_2}(-a_{h,m,k}, x_2)\Big) + (u_h(x_1,x_2) - u_h(a_{h,m,k}, x_2)) \otimes \varphi'_{h,m,k}(x_1)e_1,$$

and, by assumption (H2), it holds true that $W(\xi) \le C(1+|\xi|^p) \le C(1 + 2^{p-1}|\nabla u_0|^p + 2^{p-1}|\xi - \nabla u_0|^p)$. As a consequence of that, using (3.9)–(3.11), we get

$$\begin{aligned}
\int_{R^-_{h,m,k}} \frac{1}{\varepsilon_h} W(\nabla w_{h,m,k})\,dx &\le \frac{C}{\varepsilon_h}\int_{R^-_{h,m,k}} 1 + |\nabla u_0|^p + |\nabla w_{h,m,k} - \nabla u_0|^p\,dx \\
&\le \frac{C}{\varepsilon_h mk}(1 + |a\otimes e_2|^p) + \frac{C}{\varepsilon_h}\int_{R^-_{h,m,k}} |\nabla u_h - \nabla u_0|^p + \left|\nabla u_0 - \left(0 \mid \frac{\partial u_h}{\partial x_2}(-a_{h,m,k}, x_2)\right)\right|^p\,dx \\
&\quad + \frac{C}{\varepsilon_h}\int_{R^-_{h,m,k}} m^p k^p |u_h(x) - u_h(-a_{h,m,k}, x_2)|^p\,dx \\
&\le \frac{C}{\varepsilon_h mk} + \frac{C}{\varepsilon_h}\int_{R^-_{h,m,k}} |\nabla u_h - \nabla u_0|^p\,dx + \frac{C}{\varepsilon_h mk}\int_{-\frac{1}{2}}^{\frac{1}{2}} |\nabla(u_0 - u_h)(-a_{h,m,k}, x_2)|^p\,dx_2 \\
&\quad + \frac{Cm^pk^p}{\varepsilon_h}\int_{R^-_{h,m,k}} |u_h(x) - u_h(-a_{h,m,k}, x_2)|^p\,dx \\
&\le \frac{C}{\varepsilon_h mk} + \frac{3C\tilde\Phi(\gamma)}{\varepsilon_h m^{p+1}k} + \frac{6C\tilde\Phi(\gamma)}{\varepsilon_h m^{p+1}k} + \frac{Cm^pk^p}{\varepsilon_h}\int_{R^-_{h,m,k}} |u_h(x) - u_h(-a_{h,m,k}, x_2)|^p\,dx.
\end{aligned}$$

By Hölder's inequality we infer that

$$\begin{aligned}
|u_h(x) - u_h(-a_{h,m,k}, x_2)|^p &\le \left(\int_{-b_{h,m,k}}^{-c_{h,m,k}} \left|\frac{\partial u_h}{\partial x_1}(s,x_2)\right| ds\right)^p \\
&\le \frac{C}{(mk)^{p-1}}\int_{-b_{h,m,k}}^{-c_{h,m,k}} \left|\frac{\partial u_h}{\partial x_1}(s,x_2)\right|^p ds \le \frac{C}{(mk)^{p-1}}\int_{-b_{h,m,k}}^{-c_{h,m,k}} |\nabla(u_h - u_0)(s,x_2)|^p ds.
\end{aligned}$$

The latter estimate together with (3.10) gives

$$\begin{aligned}
\int_{R^-_{h,m,k}} |u_h(x) - u_h(-a_{h,m,k}, x_2)|^p\,dx &\le \frac{C}{(mk)^{p-1}}\int_{R^-_{h,m,k}}\int_{-b_{h,m,k}}^{-c_{h,m,k}} |\nabla(u_h - u_0)(s,x_2)|^p ds\,dx \\
&\le \frac{C}{(mk)^p}\int_{R^-_{h,m,k}} |\nabla u_h - \nabla u_0|^p\,dx \le \frac{C}{m^{2p+1}k^{p+1}},
\end{aligned} \tag{3.14}$$

which shows the first equation in (3.13). To show the second equation, we remark that, by the very definition of $\varphi_{h,m,k}$, in $R^-_{h,m,k}$, we have that

$$\begin{aligned}
|\nabla^2 w_{h,m,k}(x)|^2 \le{}& C\left|\nabla^2 u_h\right|^2 + C\left|\frac{\partial^2 u_h}{\partial x_2^2}(-a_{h,m,k}, x_2)\right|^2 + Cm^4k^4|u_h(x) - u_h(-a_{h,m,k}, x_2)|^2 \\
&+ Cm^2k^2\left|\nabla u_h(x) - \left(0 \mid \frac{\partial u_h}{\partial x_2}(-a_{h,m,k}, x_2)\right)\right|^2.
\end{aligned}$$

Hence, by (3.10) and (3.11) it follows that

$$\begin{aligned}\int_{R^-_{h,m,k}} \varepsilon_h |\nabla^2 w_{h,m,k}|^2\, dx &\leq C \int_{R^-_{h,m,k}} \varepsilon_h |\nabla^2 u_h|^2\, dx + \frac{C\varepsilon_h}{mk} \int_{-\frac{1}{2}}^{\frac{1}{2}} |\nabla^2 u_h(-a_{h,m,k}, x_2)|^2\, dx_2 \\ &\quad + C\varepsilon_h m^4 k^4 \int_{R^-_{h,m,k}} |u_h(x) - u_h(-a_{h,m,k}, x_2)|^2\, dx \\ &\quad + C\varepsilon_h m^2 k^2 \int_{R^-_{h,m,k}} \left|\nabla u_h(x) - \left(0 \mid \frac{\partial u_h}{\partial x_2}(-a_{h,m,k}, x_2)\right)\right|^2\, dx \\ &\leq \frac{C}{mk} + C\varepsilon_h m^4 k^4 \int_{R^-_{h,m,k}} |u_h(x) - u_h(-a_{h,m,k}, x_2)|^2\, dx \\ &\quad + C\varepsilon_h m^2 k^2 \int_{R^-_{h,m,k}} \left|\nabla u_h(x) - \left(0 \mid \frac{\partial u_h}{\partial x_2}(-a_{h,m,k}, x_2)\right)\right|^2\, dx.\end{aligned}$$

We are left with an estimate of the two integral terms in the previous expression. The first can be estimated noting that Hölder's inequality and (3.14) yield

$$\begin{aligned}\int_{R^-_{h,m,k}} |u_h(x) - u_h(-a_{h,m,k}, x_2)|^2\, dx &\leq |R^-_{h,m,k}|^{(p-2)/p} \left(\int_{R^-_{h,m,k}} |u_h(x) - u_h(-a_{h,m,k}, x_2)|^p\, dx\right)^{2/p} \\ &\leq \frac{1}{(mk)^{(p-2)/p}} \left(\frac{C}{m^{2p+1}k^{p+1}}\right)^{2/p} = \frac{C}{m^5 k^3}.\end{aligned} \tag{3.15}$$

In order to estimate the second term we first observe that the following inequality holds true

$$\begin{aligned}\left|\nabla u_h(x) - \left(0 \mid \frac{\partial u_h}{\partial x_2}(-a_{h,m,k}, x_2)\right)\right|^2 &= \left|\frac{\partial u_h}{\partial x_1}(x)\right|^2 + \left|\frac{\partial u_h}{\partial x_2}(x) - \frac{\partial u_h}{\partial x_2}(-a_{h,m,k}, x_2)\right|^2 \\ &\leq |\nabla u_h(x) - \nabla u_0(x)|^2 + \left(\int_{-c_{h,m,k}}^{-b_{h,m,k}} \left|\frac{\partial^2 u_h}{\partial x_1 x_2}(s, x_2)\right| ds\right)^2 \\ &\leq |\nabla u_h(x) - \nabla u_0(x)|^2 + \frac{1}{mk} \int_{-c_{h,m,k}}^{-b_{h,m,k}} |\nabla^2 u_h(s, x_2)|^2 ds.\end{aligned}$$

Combining it with (3.10), we find that

$$\begin{aligned}&\int_{R^-_{h,m,k}} \left|\nabla u_h(x) - \left(0 \mid \frac{\partial u_h}{\partial x_2}(-a_{h,m,k}, x_2)\right)\right|^2\, dx \\ &\leq \int_{R^-_{h,m,k}} |\nabla u_h - \nabla u_0|^2\, dx + \frac{1}{mk} \int_{R^-_{h,m,k}} \int_{-c_{h,m,k}}^{-b_{h,m,k}} |\nabla^2 u_h(s, x_2)|^2 ds\, dx \\ &\leq \frac{1}{(mk)^{(p-2)/p}} \left(\int_{R^-_{h,m,k}} |\nabla u_h - \nabla u_0|^p\, dx\right)^{2/p} + \frac{1}{m^2 k^2} \int_{R^-_{h,m,k}} |\nabla^2 u_h|^2\, dx \\ &\leq \frac{C}{m^3 k} + \frac{C}{\varepsilon_h m^3 k^3}.\end{aligned}$$

This completes the proof of the second equation in (3.13). The third equation follows noting that from (3.15) it holds that

$$\int_{R^-_{h,m,k}} |w_{h,m,k} - u_h|\,dx \le \frac{1}{mk}\left(\int_{R^-_{h,m,k}} |u_h(x) - u_h(-a_{h,m,k}, x_2)|^2\,dx\right)^{1/2} \le \frac{C}{m^{7/2}k^{5/2}}.$$

We now prove (3.12). We first observe that from Remark 2.2 it follows that

$$\begin{aligned} W\Big(0, \frac{\partial u_h}{\partial x_2}(-a_{h,m,k}, x_2)\Big) &\le C\,W(\nabla u_h(-a_{h,m,k}, x_2)) + C\left|\frac{\partial u_h}{\partial x_1}(-a_{h,m,k}, x_2)\right|^p \\ &\le C\,W(\nabla u_h(-a_{h,m,k}, x_2)). \end{aligned}$$

Setting $J := (-\frac{1}{2}, -b_{h,m,k}) \times (-\frac{1}{2}, \frac{1}{2})$, the previous estimate together with (3.11) gives

$$\begin{aligned} F_{\varepsilon_h}(w_{h,m,k}, \lambda_h, J\backslash R_{h,m,k}) &\le \int_{J\backslash R_{h,m,k}} \frac{1}{\varepsilon_h} W(\nabla w_{h,m,k}) + 2\varepsilon_h |\nabla^2 w_{h,m,k}|^2\,dx \\ &\le \frac{2}{m}\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{\varepsilon_h} W\left(0, \frac{\partial u_h}{\partial x_2}(-a_{h,m,k}, x_2)\right) + \varepsilon_h \left|\frac{\partial^2 u_h}{\partial x_2^2}(-a_{h,m,k}, x_2)\right|^2\,dx \\ &\le \frac{C}{m}\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{\varepsilon_h} W(\nabla u_h(-a_{h,m,k}, x_2)) + \varepsilon_h \left|\frac{\partial^2 u_h}{\partial x_2^2}(-a_{h,m,k}, x_2)\right|^2\,dx \le \frac{C}{m}. \end{aligned} \tag{3.16}$$

This shows that

$$F_{\varepsilon_h}(w_{h,m,k}, \lambda_h) \le F_{\varepsilon_h}(u_h, \lambda_h, Q\backslash J) + F_{\varepsilon_h}(w_{h,m,k}, \lambda_h, R^-_{h,m,k}) + \frac{C}{m}$$

which, combined with (3.13), implies the first equation in (3.12). Again by (3.11) it holds that

$$\begin{aligned} \int_Q |w_{h,m,k} - u_0|\,dx &\le \int_{Q\backslash J} |u_h - u_0|\,dx + \int_{R^-_{h,m,k}} |w_{h,m,k} - u_0|\,dx \\ &\quad + \frac{1}{m}\int_{-\frac{1}{2}}^{\frac{1}{2}} |u_h(-a_{h,m,k}, x_2) - u_0(x)|\,dx \\ &\le \int_{Q\backslash J} |u_h - u_0|\,dx + \int_{R^-_{h,m,k}} |w_{h,m,k} - u_0|\,dx + \frac{C}{m}, \end{aligned}$$

which, together with (3.13), implies the second equation in (3.12). To show the third equation, we first note that by the previous inequalities leading to the proof of (3.13), we have that

$$\int_{R^-_{h,m.k}} |\nabla^2 w_{h,m,k}|^2\,dx \le \frac{C}{\varepsilon_h mk} + \frac{Ck}{m}.$$

From that and (3.16) it we get

$$\begin{aligned} \int_J \max\{\lambda_h + |\nabla^2 w_{h,m,k}|, 0\}\,dx &\le \int_J |\nabla^2 w_{h,m,k}|\,dx \\ &\le \frac{1}{m}\int_{-\frac{1}{2}}^{\frac{1}{2}} \left|\frac{\partial^2 u_h}{\partial x_2^2}(-a_{h,m,k}, x_2)\right| + |R^-_{h,m,k}|^{1/2}\left(\int_{R^-_{h,m.k}} |\nabla^2 w_{h,m,k}|^2\,dx\right)^{1/2} \end{aligned}$$

$$\leq \frac{C}{\varepsilon_h m} + \left(\frac{C}{\varepsilon_h m^2 k^2} + \frac{C}{m^2}\right)^{1/2}.$$

This completes the first part of the proof. □

Step 2. For $N \geq 3$ one can repeat the argument in Step 1, leading to the definition of the sequence of functions (w_h) (from which the sequence (u_h) is obtained), in each coordinate direction. We give here only the main idea (see Step 2 in the proof of Proposition 6.3 [11] for additional details). Starting from a sequence $(u_h) \subset W^{2,2}(Q, \mathbb{R}^d)$, we modify u_h as in Step 1 and obtain, by reflection with respect to the hyperplane $\{x_1 = \frac{1}{2}\}$, the functions $w_h \in W^{2,\infty}((-1,1) \times (-\frac{1}{2}, \frac{1}{2})^{N-1}, \mathbb{R}^d)$ that are $(-1,1)$-periodic in x_1. The desired sequence (w_h) is then obtained repeating the same construction with respect to the variables $x_2, \ldots, x_{N-1}$.

We define for $\gamma \geq 0$,

$$\tilde{\Phi}_p(\gamma) := \inf\{F_{1/L}(u, \lambda) : L > 0, u \in W^{2,\infty}(Q, \mathbb{R}^d), \nabla u = \pm a \otimes e_N \text{ near } x_N = \pm\frac{1}{2}, \tag{3.17}$$

$$u \text{ periodic of period one in } x', (u, \lambda) \in \mathcal{A}(\gamma)\} \tag{3.18}$$

and

$$\Phi_p(\gamma) := \lim_{\delta \to 0^+} \tilde{\Phi}_p(\gamma + \delta). \tag{3.19}$$

Proposition 3.6. *Let $W : \mathbb{R}^{d\times N} \to [0, \infty)$ satisfy (H1)–(H4). Then it holds that $\Phi_p(\gamma) = \Phi(\gamma)$.*

Proof. From the propositions above it follows that $\Phi_p \leq \Phi$. The other inequality can be shown by an application of the same rescaling argument we are going to use in the proof of Proposition 4.2, which allows us to define from an admissible pair (u, λ) for Φ_p an admissible sequence for Φ. □

In analogy with [11, Proposition 5.3] if we replace the assumption on the potential (H4) with the following

$$W(\xi) \geq W(0, \xi_N), \quad \xi = (\xi', \xi_N) \in \mathbb{R}^{d\times N}, \tag{H5}$$

we can show that the sequence of functions from Proposition 3.4 can be chosen to depend only on the x_N-variable. More precisely, it holds

Proposition 3.7. *Let $W : \mathbb{R}^{d\times N} \to [0, \infty)$ satisfy* (H1)–(H3) *and* (H5) *and let $\gamma \geq 0$ and $\delta > 0$ be given. Then there exist sequences $\varepsilon_h^\delta \to 0^+$ and $(u_h^\delta, \lambda_h^\delta) \in \mathcal{A}(\gamma + \delta)$ satisfying*

$$\begin{aligned}
&u_h^\delta \in W^{2,\infty}(Q, \mathbb{R}^d),\\
&u_h^\delta \to |x_N| a \text{ in } L^1(Q, \mathbb{R}^d),\\
&\nabla u_h^\delta = \pm a \otimes e_N \text{ near } x_N = \pm\frac{1}{2},\\
&u_h \text{ depends only on } x_N,\\
&\lim_{h\to\infty} F_{\varepsilon_h}(u_h^\delta, \lambda_h^\delta) = \tilde{\Phi}(\gamma).
\end{aligned}$$

Proof. As in the proof of Proposition 3.5 we obtain the existence of sequences $\varepsilon_h \to 0^+, (u_h, \lambda_h) \in \mathcal{A}(\gamma + \delta)$ and $u_h \in C^2(Q, \mathbb{R}^d)$ such that $u_h \to u_0$ in $L^1(Q, \mathbb{R}^d)$, $\nabla u_h = \pm a \otimes e_N$ near $x_2 = \pm\frac{1}{2}$ and

$\lim_{h\to\infty} F_{\varepsilon_h}(u_h, \lambda_h) = \tilde{\Phi}(\gamma)$. Because we have that $(x_N \mapsto u_h(0, x_N), \lambda) \in \mathcal{A}(\gamma + \delta)$, it is enough to observe that

$$\begin{aligned} F_{\varepsilon_h}(u_h, \lambda_h) &= \int_Q \frac{1}{\varepsilon_h} W(\nabla u_h) + \varepsilon_h |\nabla^2 u_h|^2 + \varepsilon_h \min\{\lambda_h^2, |\nabla^2 u_h|^2\}\, dx \\ &\geq \int_Q \frac{1}{\varepsilon_h} W(0, \partial_{x_N} u_h) + \varepsilon_h |\nabla^2 u_h(0, x_N)|^2 + \varepsilon_h \min\{\lambda_h^2, |\nabla^2 u_h(0, x_N)|^2\}\, dx \\ &= F_{\varepsilon_h}(u_h(0, \cdot), \lambda_h). \end{aligned}$$

□

4. Γ-convergence

In this section we state and prove the two propositions 4.1 and 4.2, which together imply our Γ-convergence result.

Proposition 4.1 (Γ-liminf inequality)**.** *Let $W : \mathbb{R}^{d\times N} \to [0, \infty)$ satisfies (H1) and (H2), $\varepsilon_h \to 0^+$, and let (u_h) be a sequence in $W^{2,2}(\Omega, \mathbb{R}^d)$ and $\mu_h = \rho_h\, dx$ be a sequence in $\mathcal{M}(\Omega)$ such that*

$$\varepsilon_h \to 0^+, \quad u_h \to u \text{ in } W^{1,1}(\Omega, \mathbb{R}^d), \quad \mu_h \overset{*}{\rightharpoonup} \mu \in \mathcal{M}(\Omega).$$

Then it holds that

$$E(u, \mu) \leq \liminf_{h\to\infty} E_{\varepsilon_h}(u_h, \rho_h).$$

Proof. Thanks to Proposition 3.1, up to choosing a subsequence, we may assume that

$$\begin{aligned} &\liminf_{h\to\infty} E_{\varepsilon_h}(u_h, \rho_h) = \lim_{h\to\infty} E_{\varepsilon_h}(u_h, \rho_h), \\ &\nabla u_h \to \nabla u \text{ a.e. and } \nabla u \in \mathrm{BV}(\Omega, \{A, B\}) \quad \text{and} \\ &\Big(\frac{1}{\varepsilon_h} W(\nabla u_h) + \varepsilon_h (|\nabla^2 u_h|^2 + (\rho_h - |\nabla^2 u_h|)^2)\Big)\mathcal{L}^N \overset{*}{\rightharpoonup} \sigma. \end{aligned}$$

To conclude it is enough to show the following claim:

$$\frac{d\sigma}{d\mathcal{H}^{N-1} \llcorner S_{\nabla u}} \geq \Phi\Big(\frac{d\mu}{d\mathcal{H}^{N-1} \llcorner S_{\nabla u}}\Big) \quad \mathcal{H}^{N-1}\text{-a.e. in } S_{\nabla u}.$$

We can use Theorem 2.6 together with the Besikovitch differentiation theorem [4, Theorem 2.22] and [4, Proposition 1.62] to see that for $\mathcal{H}^{N-1}$-a.e. $x_0 \in S_{\nabla u}$ and for all but at most countably many $\delta \ll 1$ it holds that

$$\begin{aligned} &\mu(\partial Q_{x_0,\delta}) = 0 \Rightarrow \lim_{h\to\infty} \int_{Q_{x_0,\delta}} \rho_h\, dx = \mu(Q_{x_0,\delta}), \\ &\sigma(\partial Q_{x_0,\delta}) = 0 \Rightarrow \lim_{h\to\infty} E_{\varepsilon_h}(u_h, \rho_h, Q_{x_0,\delta}) = \sigma(Q_{x_0,\delta}), \\ &\mathcal{H}^{N-1} \llcorner S_{\nabla u}(Q_{x_0,\delta}) = \delta^{N-1}, \\ &\lim_{\delta\to 0^+} \frac{\mu(Q_{x_0,\delta})}{\mathcal{H}^{N-1} \llcorner S_{\nabla u}(Q_{x_0,\delta})} = \frac{d\mu}{d\mathcal{H}^{N-1} \llcorner S_{\nabla u}}(x_0) =: \rho, \end{aligned}$$

$$\lim_{\delta\to 0^+}\frac{\sigma(Q_{x_0,\delta})}{\mathcal{H}^{N-1}\llcorner S_{\nabla u}(Q_{x_0,\delta})}=\frac{d\sigma}{d\mathcal{H}^{N-1}\llcorner S_{\nabla u}}(x_0).$$

In particular, given $r>0$, we have

$$\int_{Q_{x_0,\delta}}\rho_h\,dx\le(1+2r)\delta^{N-1}\rho,\tag{4.1}$$

$$(1+2r)\frac{d\sigma}{d\mathcal{H}^{N-1}\llcorner S_{\nabla u}}(x_0)\ge E_{\varepsilon_h}(u_h,\rho_h,Q_{x_0,\delta})\delta^{1-N},\tag{4.2}$$

for all but at most countably many $\delta\ll 1$ and h sufficiently large. We define λ_h such that

$$\int_{Q_{x_0,\delta}}\max\{\lambda_h+|\nabla^2 u_h|,0\}\,dx=\int_{Q_{x_0,\delta}}\rho_h\,dx\tag{4.3}$$

and consider the following subsequences (if they exist) of (λ_h) that we do not relabel:

Case 1 (λ_h satisfies $\lambda_h\ge 0$). We have by (4.1) and (4.3),

$$\int_{Q_{x_0,\delta}}|\nabla^2 u_h|\,dx\le\int_{Q_{x_0,\delta}}\rho_h\,dx\le(1+2r)\delta^{N-1}\rho.$$

This means that $(u_h,0)\in\mathcal{A}((1+2r)\rho,Q_{x_0,\delta})$ and we obtain

$$\lim_{h\to\infty}E_{\varepsilon_h}(u_h,\rho_h,Q_{x_0,\delta})\ge\liminf_{h\to\infty}F_{\varepsilon_h}(u_h,0,Q_{x_0,\delta})\ge\tilde\Phi((1+2r)\rho)\delta^{N-1},$$

where in the last inequality we have used Lemma 3.3. By (4.2) we obtain that

$$(1+2r)\frac{d\sigma}{d\mathcal{H}^{N-1}\llcorner S_{\nabla u}}(x_0)\ge\tilde\Phi((1+2r)\rho)$$

and, letting $r\to 0^+$,

$$\frac{d\sigma}{d\mathcal{H}^{N-1}\llcorner S_{\nabla u}}(x_0)\ge\Phi(\rho).$$

Case 2 (λ_h satisfies $\lambda_h\le 0$). By (4.3) we have $(u_h,\lambda_h)\in\mathcal{A}((1+2r)\rho,Q_{x_0,\delta})$ and, since $\lambda_h\le 0$,

$$\int_{Q_{x_0,\delta}}\rho_h\,dx\le\int_{Q_{x_0,\delta}}|\nabla^2 u_h|\,dx$$

and we can apply Lemma 2.7 to get

$$\begin{aligned}E_{\varepsilon_h}(u_h,\rho_h,Q_{x_0,\delta})&=\int_{Q_{x_0,\delta}}\frac{1}{\varepsilon_h}W(\nabla u_h)+\varepsilon_h|\nabla^2 u_h|^2+\varepsilon_h(\rho_h-|\nabla^2 u_h|)^2\,dx\\&\ge\int_{Q_{x_0,\delta}}\frac{1}{\varepsilon_h}W(\nabla u_h)+\varepsilon_h|\nabla^2 u_h|^2+\varepsilon_h\min\{\lambda_h^2,|\nabla^2 u_h|^2\}\,dx\\&=F_{\varepsilon_h}(u_h,\lambda_h,Q_{x_0,\delta}).\end{aligned}$$

This implies

$$\lim_{h\to\infty}E_{\varepsilon_h}(u_h,\rho_h,Q_{x_0,\delta})\ge\tilde\Phi((1+2r)\rho)\delta^{N-1}.$$

We conclude as in the previous case. □

From now on, we assume that Ω is open, bounded, with Lipschitz boundary, and, in addition, we assume it to be simply connected. We write $\Omega_t := \{(x', x_N) \in \Omega \,|x_N = t\}$ and set

$$\alpha := \inf\{x_N : \Omega_{x_N} \neq \emptyset\}, \quad \beta := \sup\{x_N : \Omega_{x_N} \neq \emptyset\}.$$

We assume moreover that the sets Ω_{x_N} are connected for any $x_N \in (\alpha, \beta)$.

In what follows, given a set $A \subset \mathbb{R}^n$ we define $A_\delta := \{x \in \mathbb{R}^n \,|d(x, A) < \delta\}$. For the sake of simplicity we also use the notation $\Omega_{x_N,\delta} := (\Omega_{x_N})_\delta$.

Proposition 4.2 (Γ-limsup inequality)**.** *Let $W : \mathbb{R}^{d\times N} \to [0,\infty)$ satisfy (H1)–(H4). Given a measure $\mu \in \mathcal{M}(\Omega)$ and $u \in W^{1,1}(\Omega, \mathbb{R}^d)$ with $\nabla u \in \mathrm{BV}(\Omega, \{A, B\})$, and given any sequence $\varepsilon_h \to 0^+$, there exist sequences $(\mu_h) \subset \mathcal{M}(\Omega)$ and $(u_h) \subset W^{2,2}(\Omega, \mathbb{R}^d)$ satisfying*

$$\mu_h \overset{*}{\rightharpoonup} \mu, \quad u_h \to u \text{ in } W^{1,1}(\Omega, \mathbb{R}^d), \quad \limsup_{h\to\infty} E_{\varepsilon_h}(u_h, \mu_h) \leq E(u, \mu).$$

Proof. We will prove the statement in several steps. At each step we assume u and μ to be of increasing generality and provide for them a recovery sequence.

Step A.

Step A.1. We assume that u and μ are given by

$$u(x) = |x_N| a \quad \text{and} \quad \mu = \gamma_1 \chi_K \mathcal{H}^{N-1} \llcorner S_{\nabla u} + \beta \delta_{x_0}, \tag{4.4}$$

where $\gamma_1, \beta \geq 0$, $x_0 \in \Omega \backslash S_{\nabla u}$ and $K \subset \Omega$ is a compact set. Note that in this case there is only one connected interface, namely $S_{\nabla u} = \Omega_0$. We choose $h > 0$ such that $[-4h, 4h] \subset [\alpha, \beta]$ and assume moreover without loss of generality that $K = K' \times [-\frac{h}{2}, \frac{h}{2}]$, with $K' \subset \Omega_0$ compact. Given $\varepsilon, \delta, \tilde{\delta} > 0$, thanks to Proposition 3.5, which guarantees in particular that $\Phi_p \leq \Phi$, and the very definition of $\tilde{\Phi}_p$ and $\tilde{\Phi}$, we can choose $L_1 > 0$ and

$$(v^1, \lambda^1) \in \mathcal{A}(\gamma_1 + \tilde{\delta}) \cap W^{2,\infty}(Q, \mathbb{R}^d)$$
$$\nabla v^1 = \pm a \otimes e_N \text{ near } x_N = \pm\frac{1}{2}, \; v^1 \text{ periodic of period one in } x'$$

that we assume to be extended periodically in x' such that

$$F_{1/L^1}(v^1, \lambda^1) \leq \tilde{\Phi}_p(\gamma_1 + \tilde{\delta}) + \tilde{\delta} \leq \Phi(\gamma_1) + \tilde{\delta}.$$

We set

$$z^1_\varepsilon(x) := \begin{cases} \varepsilon L^1 v^1(\frac{x'}{\varepsilon L^1}, -\frac{1}{2}) - a(x_N + \frac{\varepsilon L^1}{2}), & x_N < -\frac{\varepsilon L^1}{2}, \\ \varepsilon L^1 v^1(\frac{x}{\varepsilon L^1}), & |x_N| \leq \frac{\varepsilon L^1}{2}, \\ \varepsilon L^1 v^1(\frac{x'}{\varepsilon L^1}, \frac{1}{2}) + a(x_N - \frac{\varepsilon L^1}{2}), & x_N > \frac{\varepsilon L^1}{2}, \end{cases}$$

and

$$\lambda^1_\varepsilon := \frac{\lambda^1}{\varepsilon L^1}.$$

It holds that

$$\nabla z^1_\varepsilon(x) = \begin{cases} -a \otimes e_N, & x_N < -\frac{\varepsilon L^1}{2}, \\ \nabla v^1(\frac{x}{\varepsilon L^1}), & |x_N| < \frac{\varepsilon L^1}{2}, \\ a \otimes e_N, & x_N > \frac{\varepsilon L^1}{2}. \end{cases}$$

Moreover, by the periodicity of ∇v^1 in the x'-variable, we also have that

$$\int_Q \max\{|\nabla^2 z^1_\varepsilon| + \lambda^1_\varepsilon, 0\}\, dx = \int_{Q'} \int_{-1/2}^{1/2} \max\left\{\left|\nabla^2 v^1\Big(\frac{x'}{\varepsilon L^1}, x_N\Big)\right| + \lambda^1, 0\right\} dx_N\, dx'$$
$$\to_{\varepsilon\to 0^+} \int_Q \max\{|\nabla^2 v^1| + \lambda^1, 0\}\, dx \le \gamma_1 + \tilde{\delta},$$

where the last inequality follows by the assumption on (v^1, λ^1). We repeat the same construction, up to replacing the index 1 by 2, for $\gamma_2 := 0$ and obtain $(z^2_\varepsilon, \lambda^2_\varepsilon)$. We choose a smooth cut-off function $\varphi_\delta : \Omega_{0,2\delta} \times (-\frac{h}{2}, \frac{h}{2}) \to \mathbb{R}$ satisfying

$$\varphi_\delta|_K = 1, \quad \varphi_\delta|_{\Omega_{0,2\delta}\times(-\frac{h}{2},\frac{h}{2})\setminus K_\delta} = 0, \quad |\nabla\varphi_\delta| \le \frac{C}{\delta}, \quad |\nabla^2\varphi_\delta| \le \frac{C}{\delta^2}$$

and set $z_{\varepsilon,\delta} := \varphi_\delta z^1_\varepsilon + (1 - \varphi_\delta) z^2_\varepsilon$. We also choose a smooth cut-off function $\psi_\delta : \Omega \to \mathbb{R}$ satisfying

$$\psi_\delta|_{\Omega_{0,\delta}\times(-h/3,h/3)} = 1, \quad \psi_\delta|_{\Omega\setminus\Omega_{0,2\delta}\times(-h/2,h/2)} = 0, \quad |\nabla\psi_\delta| \le \frac{C}{\delta}, \quad |\nabla^2\psi_\delta| \le \frac{C}{\delta^2}$$

and set

$$u_{\varepsilon,\delta} := \psi_\delta z_{\varepsilon,\delta} + (1 - \psi_\delta) u.$$

An example of this situation is shown in Figure 2.

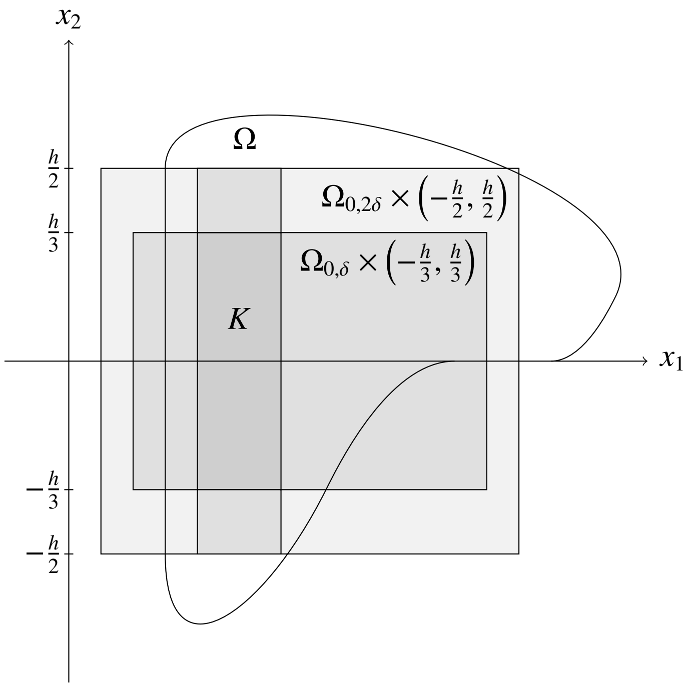

Figure 2. Sketch of the construction in Step A.1.

We claim that, uniformly with respect to $\tilde{\delta}$,

$$\lim_{\delta\to 0^+} \lim_{\varepsilon\to 0^+} \|u_{\varepsilon,\delta} - u\|_{W^{1,p}(\Omega,\mathbb{R}^d)} = 0.$$

To prove the claim we first observe that

$$|\nabla u_{\varepsilon,\delta} - \nabla u| \leq |\nabla\psi_\delta||z_{\varepsilon,\delta} - u| + |\nabla z_{\varepsilon,\delta} - \nabla u|.$$

Since for $|x_N| \geq \max\{\frac{\varepsilon L^1}{2}, \frac{\varepsilon L^2}{2}\}$ it holds that $\nabla z^1_\varepsilon = \nabla z^2_\varepsilon = \nabla u$, for such x we have that $\nabla z_{\varepsilon,\delta} - \nabla u = \nabla\varphi_\delta(z^1_\varepsilon - z^2_\varepsilon)$ and, recalling that v^1 and v^2 are bounded in L^∞, it follows that

$$|\nabla u_{\varepsilon,\delta} - \nabla u| \leq |\nabla\psi_\delta||z_{\varepsilon,\delta} - u| + |\nabla z_{\varepsilon,\delta} - \nabla u| = |\nabla\psi_\delta||z_{\varepsilon,\delta} - u| + |\nabla\varphi_\delta(z^1_\varepsilon - z^2_\varepsilon)| \leq \frac{C}{\delta}\varepsilon.$$

In the set $\{\psi_\delta = 1\}$ it holds

$$|\nabla u_{\varepsilon,\delta} - \nabla u| \leq |\nabla\varphi_\delta|(|z^1_\varepsilon| + |z^2_\varepsilon|) + |\nabla z^1_\varepsilon| + |\nabla z^2_\varepsilon| \leq \frac{C}{\delta}\varepsilon + C$$

and on $\{\psi_\delta = 0\}$ it holds

$$|\nabla u_{\varepsilon,\delta} - \nabla u| = 0.$$

Without loss of generality we may assume that $\max\{\frac{\varepsilon L^1}{2}, \frac{\varepsilon L^2}{2}\} \leq \frac{h}{3}$, which implies that φ_δ is constant on $\{|x_N| \leq \max\{\frac{\varepsilon L^1}{2}, \frac{\varepsilon L^2}{2}\}\} \cap \{0 < \psi_\delta < 1\}$ and we have

$$|\nabla u_{\varepsilon,\delta} - \nabla u| \leq |\nabla\psi_\delta|(|z^1_\varepsilon| + |z^2_\varepsilon|) + |\nabla z^1_\varepsilon| + |\nabla z^2_\varepsilon| \leq \frac{C}{\delta}\varepsilon + C.$$

It follows

$$\int_\Omega |\nabla u_{\varepsilon,\delta} - \nabla u|^p\, dx \leq \frac{C}{\delta^p}\varepsilon^p + \left|\frac{C}{\delta}\varepsilon + C\right|^p \max\left\{\frac{\varepsilon L^1}{2}, \frac{\varepsilon L^2}{2}\right\}\mathcal{H}^{N-1}(\Omega_{0,2\delta}).$$

Since $v^{1,2}_{\varepsilon,\delta} \in L^\infty$, it holds that $\|z^{1,2}_\varepsilon - u\|_\infty \leq C\varepsilon$ and the claim follows.

We define a sequence of measures converging to μ. We set $\mu^1_\varepsilon := \rho^1_\varepsilon \mathcal{L}^N$, where

$$\rho^1_\varepsilon(x) := \begin{cases} \max\{|\nabla^2 z^1_\varepsilon| + \lambda^1_\varepsilon, 0\}, & |x_N| < \frac{\varepsilon L^1}{2}, \\ \frac{1}{2\sqrt{\varepsilon}}(\gamma_1 + \tilde{\delta} - \int_Q \max\{|\nabla^2 v^1| + \lambda^1, 0\}), & \frac{\varepsilon L^1}{2} < |x_N| < \frac{\varepsilon L^1}{2} + \sqrt{\varepsilon}, \\ \frac{\beta}{\sqrt{\varepsilon}|B_1|}, & x \in B_{\varepsilon^{\frac{1}{2N}}}(x_0), \\ 0, & \text{otherwise.} \end{cases}$$

We have that $\mu^1_\varepsilon \stackrel{*}{\rightharpoonup} (\gamma_1 + \tilde{\delta})\mathcal{H}^{N-1} \llcorner S_{\nabla u} + \beta\delta_{x_0}$: For any open cylinder $\tilde{\Omega} \subset\subset \Omega$ of the type $\tilde{\Omega} = \tilde{\Omega}' \times (b, c)$ with $b < 0 < c$ we have, by the periodicity of ∇v^1, that

$$\begin{aligned} \int_\Omega \chi_{\tilde{\Omega}} d\mu^1_\varepsilon &= \int_{-\varepsilon L^1/2}^{\varepsilon L^1/2} \int_{\tilde{\Omega}'} \max\{|\nabla^2 z^1_\varepsilon| + \lambda^1_\varepsilon, 0\}\, dx \\ &\quad + \frac{1}{\sqrt{\varepsilon}} \int_{\varepsilon L^1/2}^{(\varepsilon L^1/2) + \sqrt{\varepsilon}} \int_{\tilde{\Omega}'} (\gamma_1 + \tilde{\delta} - \int_Q \max\{|\nabla^2 v^1| + \lambda^1, 0\})\, dx + \beta\chi_{\tilde{\Omega}}(x_0) \\ &= \int_{-1/2}^{1/2} \int_{\tilde{\Omega}'} \max\left\{\left|\nabla^2 v^1\left(\frac{x'}{\varepsilon L^1}, x_N\right)\right| + \lambda^1, 0\right\}\, dx \\ &\quad + \mathcal{H}^{N-1}(\tilde{\Omega}')(\gamma_1 + \tilde{\delta} - \int_Q \max\{|\nabla^2 v^1| + \lambda^1, 0\}) + \beta\chi_{\tilde{\Omega}}(x_0) \end{aligned}$$

$$\to_{\varepsilon\to 0^+} \mathcal{H}^{N-1}(\tilde{\Omega}')(\gamma_1+\tilde{\delta}) + \beta\chi_{\tilde{\Omega}}(x_0) = \mathcal{H}^{N-1}(S_{\nabla u}\cap\tilde{\Omega})(\gamma_1+\tilde{\delta}) + \beta\chi_{\tilde{\Omega}}(x_0).$$

If instead $0 \notin (b, c)$ we have that

$$\int_\Omega \chi_{\tilde{\Omega}} d\mu^1_\varepsilon \to \beta\chi_{\tilde{\Omega}}(x_0).$$

To obtain the claimed weak* convergence it is enough to recall that compactly supported continuous functions can be approximated uniformly by piecewise constant functions on cylinders of the type of $\tilde{\Omega}$. Finally, one can analogously define ρ^2_ε for $\gamma_2 := 0$. Eventually, setting $\rho_{\varepsilon,\delta} := \varphi_\delta\rho^1_\varepsilon + (1-\varphi_\delta)\rho^2_\varepsilon$ and $\mu_{\varepsilon,\delta} := \rho_{\varepsilon,\delta}\mathcal{L}^N$ we get

$$\mu_{\varepsilon,\delta} \overset{*}{\rightharpoonup}_{\varepsilon\to 0^+} \varphi_\delta(\gamma_1+\tilde{\delta})\mathcal{H}^{N-1}\llcorner S_{\nabla u} + \beta\delta_{x_0},$$

which implies that $\lim_{\tilde{\delta}\to 0^+}\lim_{\delta\to 0^+}\lim_{\varepsilon\to 0^+}\mu_{\varepsilon,\delta}\mathcal{L}^N = \mu$ with respect to the weak* convergence. Let us observe that in particular $\mu_{\varepsilon,\delta}(\Omega)$ is uniformly bounded in ε, δ and that the bound decreases for decreasing $\tilde{\delta}$. Next we claim that

$$\limsup_{\tilde{\delta}\to 0^+}\limsup_{\delta\to 0^+}\limsup_{\varepsilon\to 0^+} E_\varepsilon(u_{\varepsilon,\delta},\rho_{\varepsilon,\delta}) \le E(u,\mu).$$

We have that

$$|\nabla^2 u_{\varepsilon,\delta}| \le |\nabla^2\psi_\delta||z_{\varepsilon,\delta}-u| + 2|\nabla\psi_\delta||\nabla z_{\varepsilon,\delta}-\nabla u| + |\nabla^2 z_{\varepsilon,\delta}|.$$

Since for $|x_N| \ge \max\{\frac{\varepsilon L^1}{2}\frac{\varepsilon L^2}{2}\}$ it holds that $\nabla z^1_\varepsilon = \nabla z^2_\varepsilon = \nabla u$, for such x and for δ sufficiently small we obtain

$$|\nabla^2 u_{\varepsilon,\delta}| \le \frac{C}{\delta^2}\varepsilon.$$

In the set $\{\psi_\delta = 1\}\cap\{0<\varphi_\delta<1\}$, recalling that $v^{1,2} \in W^{2,\infty}(\Omega,\mathbb{R}^d)$, for δ sufficiently small, it holds

$$\begin{aligned}|\nabla^2 u_{\varepsilon,\delta}| &= |\nabla^2 z_{\varepsilon,\delta}| \le |\nabla^2\varphi_\delta||z^1_\varepsilon - z^2_\varepsilon| + 2|\nabla\varphi_\delta||\nabla z^1_\varepsilon - \nabla z^2_\varepsilon| + |\nabla^2 z^1_\varepsilon| + |\nabla^2 z^2_\varepsilon| \\ &\le \frac{C}{\delta^2}\varepsilon + \frac{C}{\delta} + \frac{C}{\varepsilon} \le \frac{C}{\delta^2} + \frac{C}{\varepsilon}.\end{aligned}$$

On $\{\psi_\delta = 1\}\cap\{\varphi_\delta = 1\}$ it holds

$$|\nabla u_{\varepsilon,\delta} - \nabla u| = |\nabla z^1_\varepsilon| \le C, \quad \nabla^2 u_{\varepsilon,\delta} = \nabla^2 z^1_\varepsilon$$

and similarly on $\{\psi_\delta = 1\}\cap\{\varphi_\delta = 0\}$. On $\{\psi_\delta = 0\}$ it holds $\nabla u_{\varepsilon,\delta} = \nabla u$, hence

$$|\nabla^2 u_{\varepsilon,\delta}| = 0.$$

In the next estimates we may assume that $\max\{\frac{\varepsilon L^1}{2}, \frac{\varepsilon L^2}{2}\} \le \frac{h}{3}$. As a result, φ_δ is constant on $\{|x_N| \le \max\{\frac{\varepsilon L^1}{2}, \frac{\varepsilon L^2}{2}\}\}\cap\{0<\psi_\delta<1\}$, hence

$$\begin{aligned}|\nabla^2 u_{\varepsilon,\delta}| &\le |\nabla^2\psi_\delta||z_{\varepsilon,\delta}-u| + 2|\nabla\psi_\delta||\nabla z_{\varepsilon,\delta}-\nabla u| + |\nabla^2 z_{\varepsilon,\delta}| \\ &\le \frac{C}{\delta^2}\varepsilon + \frac{C}{\delta} + \frac{C}{\varepsilon} \le \frac{C}{\delta^2} + \frac{C}{\varepsilon}\end{aligned}$$

for δ sufficiently small. As a consequence of the previous estimates, setting

$$J_\varepsilon := \{|x_N| > \max\{\frac{\varepsilon L^1}{2}, \frac{\varepsilon L^2}{2}\}\} \text{ and } K_\varepsilon := \{\frac{\varepsilon L^1}{2} < |x_N| < \frac{\varepsilon L^1}{2} + \sqrt{\varepsilon}\},$$

we get

$$\begin{aligned}
&\int_{\Omega\cap J_\varepsilon} \frac{1}{\varepsilon}|\nabla u_{\varepsilon,\delta} - \nabla u|^p + \varepsilon|\nabla^2 u_{\varepsilon,\delta}|^2 + \varepsilon(\rho_{\varepsilon,\delta} - |\nabla^2 u_{\varepsilon,\delta}|)^2\,dx \\
&\quad\le \frac{C}{\delta^p}\varepsilon^{p-1} + 3\frac{C}{\delta^4}\varepsilon^3 + 2\varepsilon\int_{\Omega\cap K_\varepsilon}\left(\frac{1}{2\sqrt{\varepsilon}}\left(\gamma_1 + \tilde{\delta} - \int_Q \max\{|\nabla^2 v^1| + \lambda^1, 0\}\right)\right)^2 dx \\
&\quad\quad + 2\varepsilon\beta^2 + 2\varepsilon\int_{\Omega\cap K_\varepsilon}\left(\frac{1}{2\sqrt{\varepsilon}}\left(\gamma_2 + \tilde{\delta} - \int_Q \max\{|\nabla^2 v^2| + \lambda^2, 0\}\right)\right)^2 dx \\
&\quad\le \frac{C}{\delta^p}\varepsilon^{p-1} + \frac{C}{\delta^4}\varepsilon^3 + 2\varepsilon\beta^2 + C\sqrt{\varepsilon},
\end{aligned}$$

$$\begin{aligned}
&\int_{\{\psi_\delta=1\}\cap\{0<\varphi_\delta<1\}\cap J_\varepsilon} \frac{1}{\varepsilon}|\nabla u_{\varepsilon,\delta} - \nabla u|^p + \varepsilon|\nabla^2 u_{\varepsilon,\delta}|^2 + \varepsilon(\rho_{\varepsilon,\delta} - |\nabla^2 u_{\varepsilon,\delta}|)^2\,dx \\
&\quad\le \left|K_\delta\backslash K \cap \left\{|x_N| < \max\left\{\frac{\varepsilon L^1}{2}, \frac{\varepsilon L^2}{2}\right\}\right\}\right|\left(\frac{1}{\varepsilon}\left(\frac{C}{\delta}\varepsilon - C\right)^p + 3\varepsilon\left(\frac{C}{\delta^2} + \frac{C}{\varepsilon}\right)^2\right) \\
&\quad\quad + 2\varepsilon\left(\mu_{\varepsilon,\delta}\left(K_\delta\backslash K \cap \left\{|x_N| < \max\left\{\frac{\varepsilon L^1}{2}, \frac{\varepsilon L^2}{2}\right\}\right\}\right)\right)^2 \\
&\quad\le \max\{\varepsilon L^1, \varepsilon L^2\}\delta^{N-1}\left(\frac{1}{\varepsilon}\left(\frac{C}{\delta}\varepsilon - C\right)^p + 3\varepsilon\left(\frac{C}{\delta^2} + \frac{C}{\varepsilon}\right)^2\right) + \varepsilon C
\end{aligned}$$

and

$$\int_{\{\psi_\delta=0\}} \frac{1}{\varepsilon}|\nabla u_{\varepsilon,\delta} - \nabla u|^p + \varepsilon|\nabla^2 u_{\varepsilon,\delta}|^2 + \varepsilon(\rho_{\varepsilon,\delta} - |\nabla^2 u_{\varepsilon,\delta}|)^2\,dx \le \varepsilon C.$$

Moreover, recalling that we have assumed that $\max\{\frac{\varepsilon L^1}{2}, \frac{\varepsilon L^2}{2}\} \le \frac{h}{3}$, we also get

$$\begin{aligned}
&\int_{\{0<\psi_\delta<1\}\cap J_\varepsilon} \frac{1}{\varepsilon}|\nabla u_{\varepsilon,\delta} - \nabla u|^p + \varepsilon|\nabla^2 u_{\varepsilon,\delta}|^2 + \varepsilon(\rho_{\varepsilon,\delta} - |\nabla^2 u_{\varepsilon,\delta}|)^2\,dx \\
&\quad\le \left|(\Omega_{0,2\delta}\backslash\Omega_{0,\delta}) \times \left(-\max\left\{\frac{\varepsilon L^1}{2}, \frac{\varepsilon L^2}{2}\right\}, \max\left\{\frac{\varepsilon L^1}{2}, \frac{\varepsilon L^2}{2}\right\}\right)\right|\left(\frac{1}{\varepsilon}\left(\frac{C}{\delta}\varepsilon + C\right)^p + 3\varepsilon\left(\frac{C}{\delta^2} + \frac{C}{\varepsilon}\right)^2\right) \\
&\quad\quad + 2\varepsilon\left(\mu_{\varepsilon,\delta}\left(\Omega_{0,2\delta}\backslash\Omega_{0,\delta} \times \left(-\max\left\{\frac{\varepsilon L^1}{2}, \frac{\varepsilon L^2}{2}\right\}, \max\left\{\frac{\varepsilon L^1}{2}, \frac{\varepsilon L^2}{2}\right\}\right)\right)\right)^2 \\
&\quad\le \mathcal{H}^{N-1}(\Omega_{0,2\delta}\backslash\Omega_{0,\delta})\max\{\varepsilon L^1, \varepsilon L^2\}\left(\frac{1}{\varepsilon}\left(\frac{C}{\delta}\varepsilon + C\right)^p + 3\varepsilon\left(\frac{C}{\delta^2} + \frac{C}{\varepsilon}\right)^2\right) + \varepsilon C.
\end{aligned}$$

Choosing $\eta = \pm a \otimes e_N$ in Remark 2.2 we observe that

$$W(\xi) \le C|\min\{\xi - a\otimes e_N, \xi + a\otimes e_N\}|^p,$$

which, together with the above estimates, implies that

$$\limsup_{\delta\to 0^+}\limsup_{\varepsilon\to 0^+} E_\varepsilon(u_{\varepsilon,\delta}, \rho_{\varepsilon,\delta}, \Omega\backslash\{\psi_\delta = 1\} \cap (\{\varphi_\delta = 0\} \cup \{\varphi_\delta = 1\})) = 0. \tag{4.5}$$

We are left to estimate the energy on the set $\{\psi_\delta = 1\} \cap (\{\varphi_\delta = 0\} \cup \{\varphi_\delta = 1\})$. To this end, without loss of generality we may suppose that $x_0 \notin \{|x_N| \le \frac{\varepsilon L^1}{2}\}$ and $\tilde{\delta} \le C$. We have that

$$\begin{aligned}
&E_\varepsilon(u_{\varepsilon,\delta}, \rho_{\varepsilon,\delta}, \{\psi_\delta = 1\} \cap \{\varphi_\delta = 1\}) \\
&\le \int_{K_\delta} \frac{1}{\varepsilon} W(\nabla u_{\varepsilon,\delta}) + \varepsilon|\nabla^2 u_{\varepsilon,\delta}|^2 + \varepsilon(\rho_{\varepsilon,\delta} - |\nabla^2 u_{\varepsilon,\delta}|^2)\, dx \\
&= \int_{K_\delta} \frac{1}{\varepsilon} W(\nabla z^1_\varepsilon) + \varepsilon|\nabla^2 z^1_\varepsilon|^2 + \varepsilon(\rho_{\varepsilon,\delta} - |\nabla^2 z^1_\varepsilon|)^2\, dx \\
&= \int_{K_\delta \cap \{|x_N| < \frac{\varepsilon L^1}{2}\}} \frac{1}{\varepsilon} W(\nabla z^1_\varepsilon) + \varepsilon|\nabla^2 z^1_\varepsilon|^2 + \varepsilon(\max\{|\nabla^2 z^1_\varepsilon| + \lambda^1_\varepsilon, 0\} - |\nabla^2 z^1_\varepsilon|)^2\, dx \\
&\quad + \int_{K_\delta \cap \{\frac{\varepsilon L^1}{2} < |x_N| < \frac{\varepsilon L^1}{2} + \sqrt{\varepsilon}\}} \varepsilon \left(\frac{1}{2\sqrt{\varepsilon}} \left(\gamma_1 + \tilde{\delta} - \int_Q \max\{|\nabla^2 v^1| + \lambda^1, 0\} \right) \right)^2 dx + \sqrt{\varepsilon}\beta\chi_K(x_0) \\
&\le \int_{K_\delta \cap \{|x_N| < \frac{\varepsilon L^1}{2}\}} \frac{1}{\varepsilon} W(\nabla z^1_\varepsilon) + \varepsilon(\min\{|\nabla^2 z^1_\varepsilon|^2 + (\lambda^1_\varepsilon)^2, 2|\nabla^2 z^1_\varepsilon|^2\}\, dx + C\sqrt{\varepsilon} \\
&= \int_{K'_\delta \times (-\frac{1}{2}, \frac{1}{2})} L^1 W\left(\nabla v^1 \left(\frac{x'}{\varepsilon L^1}, x_N \right) \right) + \frac{1}{L^1} \min \left\{ \left| \nabla^2 v^1 \left(\frac{x'}{\varepsilon L^1}, x_N \right) \right|^2 + (\lambda^1)^2, 2 \left| \nabla^2 v^1 \left(\frac{x'}{\varepsilon L^1}, x_N \right) \right|^2 \right\} dx + C\sqrt{\varepsilon}.
\end{aligned}$$

Using the periodicity of ∇v^1 in the estimate above we get

$$\lim_{\varepsilon \to 0^+} E_\varepsilon(u_{\varepsilon,\delta}, \rho_{\varepsilon,\delta}, \{\psi_\delta = 1\} \cap \{\varphi_\delta = 1\}) \le \mathcal{H}^{N-1}(K'_\delta) F_{1/L^1}(v^1, \lambda^1) \le \mathcal{H}^{N-1}(K'_\delta)(\Phi(\gamma_1) + \tilde{\delta}).$$

An analogous argument leads to the estimate of the energy on the set $\{\psi_\delta = 1\} \cap \{\varphi_\delta = 0\}$, namely

$$\lim_{\varepsilon \to 0^+} E_\varepsilon(u_{\varepsilon,\delta}, \rho_{\varepsilon,\delta}, \{\psi_\delta = 1\} \cap \{\varphi_\delta = 0\}) \le \mathcal{H}^{N-1}(\Omega_{0,2\delta} \backslash K')(\Phi(\gamma_1) + \tilde{\delta}).$$

The last two inequalities together with (4.5) imply that

$$\lim_{\tilde{\delta} \to 0^+} \lim_{\delta \to 0^+} \lim_{\varepsilon \to 0^+} E_\varepsilon(u_{\varepsilon,\delta}, \rho_{\varepsilon,\delta}) \le E(u, \mu).$$

Using a diagonal argument, we have shown the assertion of the theorem if u and μ are as in (4.4).

Step A.2. We consider the case of finitely many interfaces, namely $S_{\nabla u} = \cup_{h=1}^n \Omega_{s_h}$, and

$$\mu = \sum_{i=0}^n \gamma_i \chi_{K_i} + \sum_{i=0}^m \beta_i \delta_{x_i}, \quad \gamma_i, \beta_i \ge 0, K_i \subset S_{\nabla u} \text{ compact and pairwise disjoint, } x_i \in \Omega \backslash S_{\nabla u}.$$

By Theorem 2.6, we can suppose that near an interface Ω_{s_h}, u takes the form

$$u(x) = \pm|x_n - s_h| + c_h.$$

Therefore, we can (up to adding constants) apply Step A.1 to obtain a recovery sequence near the interface Ω_{s_h} for any summand in the definition of μ. Since this construction is local and the sets Ω_{s_h} as well as the sets K_i and $\{x_i\}$ have positive distance from each other, these local constructions can be glued to give a recovery sequence for μ near Ω_{s_h} and then also for u, μ in Ω. We leave the details to the reader.

Step A.3. We now consider the case of infinitely many interfaces, namely $S_{\nabla u} = \cup_{h=1}^{\infty}\Omega_{s_h}$ and we let μ be a finite sum of terms as in Step A.2. As in the proof of [11, Theorem 5.5, Step 2], only α and β can be accumulation points of the sequence (s_h) (otherwise one could find a cylinder $Z \subset \Omega$ with axis in direction of e_N that intersects infinitely many of the Ω_{s_h}, and this would contradict the fact that $\sum_h \mathcal{H}^{N-1}(\Omega_{s_h}) < \infty$). We can choose a decreasing sequence $\delta_k \to 0^+$ such that $\{\alpha+\delta_k, \beta-\delta_k\}_k \cap \{s_h\}_h = \emptyset$ and such that $\text{supp}(\mu) \subset\subset \{\alpha + \delta_k < x_N < \beta - \delta_k\}$ for all k. Then it follows that u has only finitely many interfaces in

$$U_k := \Omega \cap \{\alpha + \delta_k < x_N < \beta - \delta_k\}.$$

Therefore, we can apply Step A.2 and find for any given sequence $\varepsilon_h \to 0^+$ sequences $(u_h^k)_h$ and $(\mu_h)_h$ such that

$$\lim_{h\to\infty} u_h^k = u \text{ in } W^{1,1}(\Omega, U_k), \quad \mu_h \overset{*}{\rightharpoonup} \mu \text{ in } \Omega, \quad \lim_{h\to\infty} E_{\varepsilon_h}(u_h^k, \mu_h, U_k) \le E(u, \mu, U_k), \tag{4.6}$$

where we used that $\text{supp}(\mu) \subset\subset U_k$ and that therefore the sequences μ_h which approximate μ in Step A.2 can be chosen in such a way that they not depend on k. By construction it holds

$$E_{\varepsilon_h}(u_h^k, \mu_h, U_k) = E_{\varepsilon_h}(u_h^k, \mu_h, \Omega)$$

which implies by (4.6) that

$$\lim_{k\to\infty}\lim_{h\to\infty} E_{\varepsilon_h}(u_h^k, \mu_h, \Omega) \le \lim_{k\to\infty} E(u, \mu, U_k) \le E(u, \mu, \Omega). \tag{4.7}$$

The existence of a subsequence $u_h^{k(h)}$ with the desired properties follows by a diagonal argument.

Step B. Now we assume that μ is a finite positive Radon measure. It holds

$$0 \le g := \frac{d\mu}{d\mathcal{H}^{\text{N-1}} \llcorner S_{\nabla u}} \in L^1(S_{\nabla u}, \mathcal{H}^{\text{N-1}}).$$

We observe that there exist functions

$$g_k = \sum_{i=0}^{n_k} \gamma_{k,i}\chi_{K'_{k,i}}, \quad K'_{k,i} \subset S_{\nabla u} \text{ compact, pairwise disjoint}$$

which satisfy

$$\lim_{k\to\infty} g_k = g \text{ in } L^1(S_{\nabla u}, \mathcal{H}^{N-1}), \lim_{k\to\infty} \mathcal{H}^{N-1}(\{g_k < g\}) = 0.$$

By the boundedness of μ, we find measures $\sum_{i=0}^{m_k} \beta_{k,i}\delta_{x_{k,i}}$ supported outside $S_{\nabla u}$ such that

$$\sum_{i=0}^{m_k} \beta_{k,i}\delta_{x_{k,i}} \overset{*}{\rightharpoonup} \mu - g\mathcal{H}^{N-1} \llcorner S_{\nabla u}.$$

It follows that

$$\mu_k := g_k \mathcal{H}^{N-1} \llcorner S_{\nabla u} + \sum_{i=0}^{m_k} \beta_{k,i}\delta_{x_{k,i}} \overset{*}{\rightharpoonup} \mu.$$

Since the Γ- lim sup is lower semicontinuous and Φ is non-increasing and bounded, we can use the result of Step A to obtain

$$\begin{aligned}\Gamma\text{-}\limsup_{h\to\infty} E_{\varepsilon_h}(u,\mu) &\leq \liminf_{k\to\infty} \Gamma\text{-}\limsup_{h\to\infty} E_{\varepsilon_h}(u,\mu_k)\\ &\leq \liminf_{k\to\infty} E(u,\mu_k) = \liminf_{k\to\infty} \int_\Omega \Phi(g_k)d\mathcal{H}^{N-1}\\ &\leq \liminf_{k\to\infty} \int_\Omega \Phi(g)d\mathcal{H}^{N-1} + C\lim_{k\to\infty} \mathcal{H}^{N-1}(\{g_k < g\}) = E(u,\mu)\end{aligned}$$

for any sequence $\varepsilon_h \to 0^+$. This concludes the proof of the theorem. □

5. Conclusions

We investigated a family of Cahn-Hilliard type energies for gradient vector fields with a double-well potential W having two rank-one connected wells A and B, supplemented by an additional term modeling the interaction with a surfactant density.

We proved the Γ-convergence of these functionals to a limit energy $E(u,\mu)$, finite for deformations u with $\nabla u \in BV(\Omega,\{A,B\})$ and for surfactant measures μ. The limit energy is of perimeter type and can be expressed as

$$E(u,\mu) = \int_{S_{\nabla u}} \Phi\left(\frac{d\mu}{d\mathcal{H}^{\text{N-1}}\llcorner S_{\nabla u}}\right) d\mathcal{H}^{\text{N-1}},$$

where Φ is a nonincreasing surface tension density determined through an asymptotic cell problem.

The lim inf inequality follows by the blow-up method, while the lim sup construction exploits the fact that the asymptotic problem defining Φ can be restricted to classes of functions with additional regularity and periodicity.

From a modeling perspective, our Γ-convergence result shows that phase transitions are promoted in regions where surfactant accumulates. This mechanism is fully consistent with the scalar fluid-fluid case introduced by Perkins, Sekerka, Warren and Langer and considered in [19], and provides a rigorous variational analysis of surfactant-driven solid-solid transitions. In this sense, our result extends the classical gradient theory of phase transitions in presence of surfactant to a vectorial setting, contributing to the mathematical understanding of phenomena relevant, for instance, in crystal growth, metallurgy, and ceramics processing.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

Prof. Marco Cicalese is an editorial board member for Mathematics in Engineering and was not involved in the editorial review and the decision to publish this article. Both authors declare no conflicts of interest in this paper.

References

1. E. Acerbi, G. Bouchitté, A general class of phase transition models with weighted interface energy, *Ann. Inst. H. Poincaré C Anal. Non Linéaire*, **25** (2008), 1111–1143. https://doi.org/10.1016/j.anihpc.2007.09.004
2. R. Alicandro, A. Braides, M. Cicalese, M. Solci, *Discrete variational problems with interfaces*, Cambridge Monographs on Applied and Computational Mathematics, Cambridge University Press, 2024. https://doi.org/10.1017/9781009298766
3. R. Alicandro, M. Cicalese, L. Sigalotti, Phase transitions in presence of surfactants: from discrete to continuum, *Interfaces Free Bound.*, **14** (2012), 65–103. https://doi.org/10.4171/ifb/274
4. L. Ambrosio, N. Fusco, D. Pallara, *Functions of bounded variation and free discontinuity problems*, Oxford Academic, 2000. https://doi.org/10.1093/oso/9780198502456.001.0001
5. M. Baía, A. C. Barroso, J. Matias, A model for phase transitions with competing terms, *Q. J. Math.*, **68** (2017), 957–1000. https://doi.org/10.1093/qmath/hax009
6. S. Baldo, Minimal interface criterion for phase transitions in mixtures of Cahn-Hilliard fluidss, *Ann. Inst. H. Poincaré C Anal. Non Linéaire*, **7** (1990), 67–90. https://doi.org/10.1016/s0294-1449(16)30304-3
7. J. M. Ball, R. D. James, Fine phase mixtures as minimizers of energy, *Arch. Rational Mech. Anal.*, **100** (1987), 13–52. https://doi.org/10.1007/BF00281246
8. A. Braides, *Gamma-convergence for beginners*, Oxford Lecture Series in Mathematics and its Applications, Oxford University Press, 2002. https://doi.org/10.1093/acprof:oso/9780198507840.001.0001
9. M. Chermisi, G. Dal Maso, I. Fonseca, G. Leoni, Singular perturbation models in phase transitions for second-order materials, *Indiana Univ. Math. J.*, **60** (2011), 367–409. https://doi.org/10.1512/iumj.2011.60.4346
10. M. Cicalese, E. N. Spadaro, C. I. Zeppieri, Asymptotic analysis of a second-order singular perturbation model for phase transitions, *Calc. Var. Partial Differential Equations*, **41** (2011), 127–150. https://doi.org/10.1007/s00526-010-0356-9
11. S. Conti, I. Fonseca, G. Leoni, A Γ-convergence result for the two-gradient theory of phase transitions, *Commun. Pure Appl. Math.*, **55** (2002), 857–936. https://doi.org/10.1002/cpa.10035
12. S. Conti, B. Schweizer, A sharp-interface limit for a two-well problem in geometrically linear elasticity, *Arch. Ration. Mech. Anal.*, **179** (2006), 413–452. https://doi.org/10.1007/s00205-005-0397-y
13. S. Conti, B. Schweizer, Rigidity and gamma convergence for solid-solid phase transitions with SO(2) invariance, *Commun. Pure Appl. Math.*, **59** (2006), 830–868. https://doi.org/10.1002/cpa.20115
14. R. Cristoferi, I. Fonseca, A. Hagerty, C. Popovici, A homogenization result in the gradient theory of phase transitions, *Interfaces Free Bound.*, **21** (2019), 367–408. https://doi.org/10.4171/ifb/426
15. G. Dal Maso, *An introduction to* Γ*-convergence*, Progress in Nonlinear Differential Equations and their Application, Vol. 8, Birkhäuser Boston, 1993. https://doi.org/10.1007/978-1-4612-0327-8

16. G. Dal Maso, I. Fonseca, G. Leoni, Asymptotic analysis of second order nonlocal Cahn-Hilliard-type functionals, *Trans. Amer. Math. Soc.*, **370** (2018), 2785–2823. https://doi.org/10.1090/tran/7151
17. E. Davoli, M. Friedrich, Two-well rigidity and multidimensional sharp-interface limits for solid-solid phase transitions, *Calc. Var. Partial Differential Equations*, **59** (2020), 44. https://doi.org/10.1007/s00526-020-1699-5
18. I. Fonseca, C. Mantegazza, Second order singular perturbation models for phase transitions, *SIAM J. Math. Anal.*, **31** (2000), 1121–1143. https://doi.org/10.1137/S0036141099356830
19. I. Fonseca, M. Morini, V. Slastikov, Surfactants in foam stability: a phase-field model, *Arch. Rational Mech. Anal.*, **183** (2007), 411–456. https://doi.org/10.1007/s00205-006-0012-x
20. I. Fonseca, L. Tartar, The gradient theory of phase transitions for systems with two potential wells, *Proc. Roy. Soc. Edinb.: Sec. A Math.*, **111** (1989), 89–102. https://doi.org/10.1017/S030821050002504X
21. D. Myers, *Surfactant science and technology*, John Wiley & Sons, Inc., 2020. https://doi.org/10.1002/9781119465829
22. L. Modica, The gradient theory of phase transitions and the minimal interface criterion, *Arch. Rational Mech. Anal.*, **98** (1987) 123–142. https://doi.org/10.1007/BF00251230
23. L. Modica, S. Mortola, Un esempio di Γ^--convergenza, *Boll. Un. Mat. Ital. B*, **14** (1977), 285–299.
24. D. Raabe, S. Sandlöbes, J. Millán, D. Ponge, H. Assadi, M. Herbig, et al., Segregation engineering enables nanoscale martensite to austenite phase transformation at grain boundaries: a pathway to ductile martensite, *Acta Mater.*, **61** (2013), 6132–6152. https://doi.org/10.1016/j.actamat.2013.06.055