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Research article

Composed perturbations to a coupled ODEs system with hysteresis†

Shengda Zeng[1] **and Sergey A. Timoshin**[2,*]

[1] National Center for Applied Mathematics in Chongqing, and School of Mathematical Sciences, Chongqing Normal University, Chongqing 401331, China
[2] School of Mathematics and Physics, Xi'an Jiaotong-Liverpool University, Suzhou, Jiangsu 215123, China

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* **Correspondence:** Email: sergey.timoshin@gmail.com.

Abstract: In this paper, we address a system of two ODEs including a hysteresis nonlinearity of generalized play type. Our system is subject to a composed perturbation, giving under particular choices of composants various types of common multivalued perturbations. We prove the existence of a solution to this system. The theoretical analysis is complemented by a discussion of a mechanical model illustrating potential applicability of our results and the physical meaning of the underlying assumptions.

Keywords: evolution system; hysteresis; composed perturbation; nonconvex-valued set-mapping; convex-valued set-mapping; mechanical system

‡*Dedicated to Pavel Krejčí on the occasion of his 70th birthday.*

1. Introduction

Let $T > 0$ be a fixed final time. Consider the nonlinear system described by two ordinary differential equations of the following form

$$\dot{y}(t) + \partial I_{D(x(t))}(y(t)) + U(t, x(t), y(t)) + V(t, x(t), y(t)) \ni h_1(x(t), y(t)), \tag{1.1}$$

$$a_1(x(t), y(t))\dot{x}(t) + a_2(x(t), y(t))\dot{y}(t) = h_2(x(t), y(t)), \tag{1.2}$$

defined on the interval $[0, T]$ and supplied with the initial conditions

$$x(0) = x_0, \qquad y(0) = y_0. \tag{1.3}$$

Here, $D(x) = [f_1(x), f_2(x)]$ is an interval in $\mathbb{R}$, $\partial I_{D(x)}$ is the subdifferential in the sense of convex analysis of the indicator function $I_{D(x)}$ of this interval, $f_i(\cdot) : \mathbb{R} \to \mathbb{R}$, $a_i(\cdot,\cdot)$, $h_i(\cdot,\cdot) : \mathbb{R}^2 \to \mathbb{R}$, $i = 1, 2$, are given single-valued functions, $U, V : [0, T] \times \mathbb{R}^2 \rightrightarrows \mathbb{R}$ are two multivalued mappings. The precise properties of these functions and mappings, under which we study our system, are postulated in the next section. Furthermore, $(x_0, y_0) \in \mathbb{R}^2$ are given initial conditions such that $y_0 \in D(x_0)$.

When $U = V \equiv 0$ and $h_2 \equiv 0$ in (1.1), the resulting inclusion leads to the differential representation of hysteresis operator of the generalized play type (cf. [1,2]), with the graphs of $f_1(x)$ and $f_2(x)$ forming the corresponding hysteresis loop. In this case, the unknowns $x(\cdot)$ and $y(\cdot)$ play the roles of the input and output functions, respectively, and the maximal monotone graph $\partial I_{D(v)}$ entails a nonlinear constrained dynamics of the output $y(\cdot)$ in the sense that $y(\cdot)$ must satisfy the following input-dependent constraints

$$f_1(x(t)) \leq y(t) \leq f_2(x(t)), \quad t \in [0, T]. \tag{1.4}$$

We note that the differential inclusion of this type is not just a particular instance of hysteresis relations. In fact (see [3]), *every scalar return point memory hysteresis operator* can be represented by first order differential inclusions with a one-parameter family of indicator functions.

In the past three decades, hysteresis operators have been widely applied to model irreversible processes across different disciplines of natural sciences and engineering. Among these, we mention population dynamics (see, e.g., [4–7] and references therein), filtration problems [8–11], phase transitions [12–16], concrete carbonation [17–19], and thermostat models [20–23]. In the same vein, differential system with hysteresis of the type (1.1)–(1.3) are used, for example, in economics for modeling macroeconomic processes, or in mechanics for modeling porous media flow, see [9,24]. Such systems are also effective to describe mechanical systems where a body's movement is constrained, for instance by stops, and is subject to various forces, as we discuss in Section 5.

In the analysis of dynamical systems, external forces or control actions are often cast in the form of perturbations. Frequently, such forces are described by multivalued laws, for example, in the case of dry friction or relay controls. Thus, along with traditionally considered single-valued perturbations more readily amenable to mathematical treatment, multivalued perturbations emerged in the theory of evolutionary equations. Initially, such perturbations were assumed to have convex values, again for analytical convenience. However, in many real-world applications, nonconvexity of the perturbation's values is a natural assumption. In this regard, a control system similar to (1.1)–(1.3) with $V \equiv 0$ and different assumptions on a nonconvex-valued perturbation U was studied in [25,26] from the existence of solutions and their relaxation properties point of view.

The present work extends these previous studies by considering a *composed perturbation*, represented by the sum $U + V$. Such a structure of perturbation is motivated by applications where a system can be simultaneously affected by different types of external influences, for example, a well-defined (but possibly nonconvex) control action represented by U and a less predictable external disturbance represented by V. While existence theorems for differential inclusions with a single multivalued term are well-established, the analysis of systems with a sum of multivalued maps with different types of values (convex only, nonconvex only, or mixed) and the corresponding different continuity properties (upper semicontinuous, lower semicontinuous, or combined) remains largely understudied. In our approach, we explore the distinct properties of U and V treating the Lipschitz-continuous mapping U using a contraction argument and the more general mapping V using a convexification technique and Ky Fan's fixed-point theorem. The main contribution of this paper is to demonstrate how these two different analytical techniques can be blended to establish the existence of a solution for the composedly perturbed system, a novel topic in the context of hysteresis.

Recently, composed perturbations of evolution equations, represented by the sum of multivalued mappings with different structural properties, were addressed, mainly in the context of sweeping processes, see [27, 28]. Along these lines, in this article we prove the existence of a solution to systems (1.1)–(1.3) under different types of assumptions on the multivalued perturbations U and V.

The paper is organized as follows. Section 2 introduces notation and the main hypotheses. In Section 3, we establish some properties of an auxiliary system crucial for the main proof. Section 4 contains the proof of our main existence theorem. Finally, in Section 5 we provide a concrete example from mechanics to illustrate the applicability of our abstract results and discuss the physical meaning of the assumptions.

2. Notation, assumptions, and preliminary results

In this section, we present the notation used throughout the paper and posit the hypotheses under which we consider our systems (1.1)–(1.3). We also provide some auxiliary facts that we require for the proof of our existence theorem in Section 4.

The set of positive real numbers we denote by $\mathbb{R}^+$. Let X, Z be two Banach spaces. We call a multivalued mapping $F : [0, T] \rightrightarrows Z$ measurable if the set $\{t \in [0, T] : F(t) \cap B \neq \emptyset\}$ belongs to the σ-algebra $\mathcal{L}$ of Lebesgue measurable sets from $[0, T]$ for any closed set $B \subset Z$. A multivalued mapping $F : [0, T] \times X \rightrightarrows Z$ is called jointly measurable if $\{(t, x) \in [0, T] \times X : F(t, x) \cap B \neq \emptyset\}$ belongs to the σ-algebra $\mathcal{L} \otimes \mathcal{B}_X$ for any closed set $B \subset Z$, where $\mathcal{B}_X$ is the σ-algebra of Borel sets from X.

We say that a multivalued mapping $F : X \rightrightarrows Z$ has a closed graph at a point $x_0 \in X$ if the convergence of a sequence (x_n, z_n), $z_n \in F(x_n)$, $n \geq 1$, to the point (x_0, z_0) implies that $z_0 \in F(x_0)$.

A multivalued mapping $F : X \rightrightarrows Z$ with closed values is called upper semicontinuous at a point $x_0 \in X$, if for any open set $B \subset Z$ with $F(x_0) \subset B$, there exists a neighborhood A of x_0 such that $F(x) \subset B$ for all $x \in A$. A multivalued mapping $F : X \rightrightarrows Z$ with closed values is called lower semicontinuous at a point $x_0 \in X$, if for any open set $B \subset Z$ with $F(x_0) \cap B \neq \emptyset$, there exists a neighborhood A of x_0 such that $F(x) \cap B \neq \emptyset$ for all $x \in A$. A multivalued mapping $F : X \rightrightarrows Z$ is called upper (lower) semicontinuous on some $Y \subset X$, if it is upper (lower) semicontinuous at every point of Y.

If for some neighborhood A of a point x_0 there exists a compact set $B \subset Z$, such that $F(x) \subset B$ for all $x \in A$, then for the multivalued mapping $F : X \rightrightarrows Z$ with closed values, the closedness of its graph at the point x_0 implies the upper semicontinuity of F at this point.

For a set $C \subset X$ and a multivalued mapping $F : [0, T] \times X \rightrightarrows Z$, we denote

$$\|C\| = \sup\{\|x\| : \ x \in C\}$$

and

$$F(t, C) = \left\{ \bigcup F(t, x) : x \in C \right\},$$

where $\| \cdot \|$ is the norm in X.

Proposition 2.1. *([29, Theorem 2.1]) Let X, Z be two finite dimensional spaces and a multivalued mapping $F : [0, T] \times X \rightrightarrows Z$ with compact values satisfy the properties:*

(1) *The mapping $(t, x) \to F(t, x)$ is jointly measurable;*

(2) *For a.e.* $t \in [0,T]$ *and any* $x \in X$*, either the mapping* $F(t,\cdot)$ *is upper semicontinuous at* x *and the set* $F(t,x)$ *is convex, or the restriction of* $F(t,\cdot)$ *to some neighborhood of* x *is lower semicontinuous;*

(3) *There exists* $l(\cdot) \in L^p(0,T;\mathbb{R}^+)$*,* $1 \le p < \infty$*, such that*

$$F(t,x) \bigcap l(t)(1+\|x\|)\,\overline{B} \neq \emptyset, \quad x \in X,$$

where $\|\cdot\|$ *is the norm in* X *and* $\overline{B}$ *is the closed unit ball in* Z*.*

Then, for any $\varepsilon > 0$ *and any compact set* $\mathcal{K} \subset C([0,T];X)$*, there exists an upper semicontinuous in the weak topology of the space* $L^p(0,T;Z)$ *multivalued mapping* $\mathcal{M} : \mathcal{K} \rightrightarrows L^p(0,T;Z)$ *with convex weakly compact values such that for any* $x(\cdot) \in \mathcal{K}$ *and any* $f(\cdot) \in \mathcal{M}(x)$ *we have*

$$f(t) \in F(t,x(t))$$

and

$$\|f(t)\| \le l(t)(1+\|x(t)\|) + \varepsilon$$

for a.e. $t \in [0,T]$*.*

We note that if we consider the multivalued Nemytskii operator $\mathcal{F} : C([0,T];X) \rightrightarrows L^p(0,T;Z)$ generated by the mapping $F : [0,T] \times X \rightrightarrows Z$:

$$\mathcal{F}(x) := \{v(\cdot) \in L^p(0,T;Z) : v(t) \in F(t,x(t)) \ \text{ for a.e. } t \in [0,T]\},$$

then the assumptions of Proposition 2.1 imply that $\mathcal{F}$ has nonempty closed, not necessarily convex, values and the conclusion of Proposition 2.1 says that the restriction on the nonconvex-valued mapping $\mathcal{F}$ to $\mathcal{K}$ has a convex-valued selector $\mathcal{M}$.

Given $x \in X$ and $A \subset X$, denote by $d_X(x,A)$ the distance from x to A and define the Hausdorff distance on the space of closed bounded subsets of X as follows:

$$\text{haus}_X(A,B) = \max\{\sup_{x \in B} d_X(x,A), \sup_{y \in A} d_X(y,B)\}, \quad A, B \subset X.$$

For a convex, lower semicontinuous function $\varphi : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ with $\text{dom}\,\varphi = \{x \in \mathbb{R} : \varphi(x) < +\infty\} \neq \emptyset$, denote by $\partial\varphi(x)$ its subdifferential at a point $x \in \mathbb{R}$. This is, in general, a set-valued function defined as

$$\partial\varphi(x) = \{h \in \mathbb{R} : h(y-x) \le \varphi(y) - \varphi(x) \quad \text{for all } y \in \text{dom}\,\varphi\}. \tag{2.1}$$

In the case when $\varphi(\cdot) = I_{D(x)}(\cdot)$, the indicator function of the closed interval $D(x) := [f_1(x), f_2(x)]$, $x \in \mathbb{R}$:

$$I_{D(x)}(y) := \begin{cases} 0 & \text{if } \ y \in D(x), \\ +\infty & \text{otherwise,} \end{cases}$$

this function has the form:

$$\partial I_{D(x)}(y) = \begin{cases} \emptyset & \text{if } \ y \notin D(x), \\ [0,+\infty) & \text{if } \ y = f_2(x) > f_1(x), \\ \{0\} & \text{if } \ f_1(x) < y < f_2(x), \\ (-\infty,0] & \text{if } \ y = f_1(x) < f_2(x), \\ (-\infty,+\infty) & \text{if } \ y = f_1(x) = f_2(x). \end{cases} \tag{2.2}$$

We make the following assumptions on the data describing systems (1.1)–(1.3):

H(f_i): The functions f_i, f_i', $i = 1, 2$, are Lipschitz continuous, $f_1 \le f_2$, $f_1', f_2' \ge 0$ on $\mathbb{R}$. In addition, there exists a number $k_0 > 0$ such that $f_1(x) = f_2(x)$ for $x \in (-\infty, -k_0] \cup [k_0, +\infty)$;

H(a_i, h_i): The functions a_i, h_i are Lipschitz continuous on $\mathbb{R}^2$ and there exists a constant $c_0 > 0$ such that $a_i \ge c_0$, $i = 1, 2$;

H(U): The multivalued mapping $U : [0, T] \times \mathbb{R}^2 \rightrightarrows \mathbb{R}$ has closed, not necessarily convex, values and

1) the mapping $t \mapsto U(t, x, y)$ is measurable;
2) there exists $l_U(\cdot) \in L^2(0, T; \mathbb{R}^+)$ such that

$$\|U(t, x, y)\| := \sup\{|u| : u \in U(t, x, y)\} \le l_U(t)(1 + |x| + |y|)$$

   for a.e. $t \in [0, T]$, $x, y \in \mathbb{R}$;
3) there exists $k(\cdot) \in L^2(0, T; \mathbb{R}^+)$ such that

$$\text{haus}_{\mathbb{R}}(U(t, x_1, y_1), U(t, x_2, y_2)) \le k(t)(|x_1 - x_2| + |y_1 - y_2|)$$

   for a.e. $t \in [0, T]$, $x_i, y_i \in \mathbb{R}$, $i = 1, 2$;

H(V): The multivalued mapping $V : [0, T] \times \mathbb{R}^2 \rightrightarrows \mathbb{R}$ has closed values and

1) the mapping $(t, x, y) \mapsto V(t, x, y)$ is jointly measurable;
2) there exists $l_V(\cdot) \in L^2(0, T; \mathbb{R}^+)$ such that

$$d(0, V(t, x, y)) < l_V(t)(1 + |x| + |y|)$$

   for a.e. $t \in [0, T]$, $x, y \in \mathbb{R}$;
3) for a.e. $t \in [0, T]$ and any $(x, y) \in \mathbb{R}^2$, either the mapping $V(t, \cdot, \cdot)$ has a closed graph at (x, y) and the set $V(t, x, y)$ is convex, or the restriction of $V(t, \cdot, \cdot)$ to some neighborhood of (x, y) is lower semicontinuous.

3. Existence and properties of solutions of an auxiliary system

In this section, we prove some auxiliary facts necessary to establish the existence of a solution to our composedly perturbed evolution systems (1.1)–(1.3). To this end, first we define a notion of solution to this system. Then, we consider an auxiliary system stemming from our definition of solution and prove the existence of a solution to this latter system. Finally, we explore some continuity properties of the corresponding solution operator.

Definition 3.1 (solution)**.** *By a solution of systems* (1.1)–(1.3)*, we mean a pair* $(x(\cdot), y(\cdot))$ *of functions from* $W^{1,2}([0, T]; \mathbb{R})$ *with* $x(0) = x_0$*,* $y(0) = y_0$*,* $y_0 \in D(x_0)$ *such that there exist* $u(\cdot), v(\cdot) \in L^2(0, T; \mathbb{R})$ *and we have*

$$\dot{y}(t) + \partial I_{D(x(t))}(y(t)) + u(t) + v(t) \ni h_1(x(t), y(t)), \tag{3.1}$$

$$a_1(x(t), y(t))\dot{x}(t) + a_2(x(t), y(t))\dot{y}(t) = h_2(x(t), y(t)), \tag{3.2}$$

$$u(t) \in U(t, x(t), y(t)), \tag{3.3}$$

$$v(t) \in V(t, x(t), y(t)) \tag{3.4}$$

for a.e. $t \in [0, T]$.

Now, given $u(\cdot), v(\cdot) \in L^2(0,T;\mathbb{R})$, not necessarily satisfying (3.3) and (3.4), we consider systems (3.1) and (3.2). The notion of a solution to this system naturally extends from Definition 3.1.

Lemma 3.1. *For any $u(\cdot), v(\cdot) \in L^2(0,T;\mathbb{R})$, $\|u\|_{L^2(0,T;\mathbb{R})} + \|v\|_{L^2(0,T;\mathbb{R})} \leq M$, $M > 0$, there exists a unique solution $(x(u,v), y(u,v))$ to systems (3.1) and (3.2), such that*

$$|x(u,v)| + |y(u,v)| \leq N, \tag{3.5}$$

$$\|\dot{x}(u,v)\|_{L^2(0,T;\mathbb{R})} + \|\dot{y}(u,v)\|_{L^2(0,T;\mathbb{R})} \leq N, \tag{3.6}$$

$t \in [0,T]$, for some $N > 0$ depending on M, (x_0, y_0) and the fixed quantities in Hypotheses $H(f_i)$, $H(a_i, h_i)$ only. Moreover, for any two solutions $(x_i, y_i) := (x(u_i, v_i), y(u_i, v_i))$, $u_i, v_i \in L^2(0,T;\mathbb{R})$, $\|u_i\|_{L^2(0,T;\mathbb{R})} + \|v_i\|_{L^2(0,T;\mathbb{R})} \leq M$, $i = 1,2$, we have

$$|x_1(t) - x_2(t)| + |y_1(t) - y_2(t)| \leq L \int_0^t (|u_1(s) - u_2(s)| + |v_1(s) - v_2(s)|)\, ds, \tag{3.7}$$

$t \in [0,T]$, for some $L > 0$ depending only on the quantities as specified above.

Proof. Under our hypotheses on f_i, a_i, h_i, $i = 1,2$, the proof of the a priori bounds (3.5), (3.6), and the inequality (3.7) is similar to the proof of their counterparts in [25]. For the reader's convenience, we outline below the essential parts of this proof indicating the changes due.

The first estimate in (3.5) follows from the nonnegativity of coefficients a_i, $i = 1,2$, in (3.1) and the nonnegativity and boundedness of the derivatives f_i', $i = 1,2$, of the functions describing the hysteresis region of our system. The second estimate is a consequence of (2.2). By virtue of estimate (3.5), we can assume that the Lipschitz continuous functions h_i and a_i, $i = 1,2$, are bounded on $\mathbb{R}^2$. In particular, the boundedness of h_i implies the estimates (3.6).

For simplicity of notations we denote by L_0, chosen so that $L_0 > 1$, a common Lipschitz constant of the functions a_i, h_i, f_i, $i = 1,2$, and by $R > 0$ a common bound of the functions a_i, $i = 1,2$.

Denote by $H : \mathbb{R} \to \{0,1\}$ the Heaviside function, that is, $H(z) = 1$ if $z > 0$, $H(z) = 0$ if $z \leq 0$ and define the function $\operatorname{sign}(z) := H(z) - H(-z)$. To derive the inequality (3.7), we first test the difference of (3.2) for (x_1, y_1) and (x_2, y_2) by $\operatorname{sign}(x_1 - x_2)$ to obtain

$$\begin{aligned} &\frac{d}{dt}\{a_1(x_1,y_1)|x_1 - x_2|\} + a_2(x_1,y_1)(\dot{y}_1 - \dot{y}_2)\operatorname{sign}(x_1 - x_2) \\ &\quad \leq L_0(1 + |\dot{x}_1| + |\dot{x}_2| + |\dot{y}_1| + |\dot{y}_2|)(|x_1 - x_2| + |y_1 - y_2|) \end{aligned} \tag{3.8}$$

a.e. on $[0,T]$.

Next, we claim that

$$(\dot{y}_1 - h_1(x_1,y_1) + u_1 + v_1)H(y_1 - y_2) \leq (\dot{y}_1 - h_1(x_1,y_1) + u_1 + v_1)H(x_1 - x_2), \tag{3.9}$$

$$-(\dot{y}_1 - h_1(x_1,y_1) + u_1 + v_1)H(y_2 - y_1) \leq -(\dot{y}_1 - h_1(x_1,y_1) + u_1 + v_1)H(x_2 - x_1). \tag{3.10}$$

In fact, the statement is obvious if $\dot{y}_1 - h_1(x_1,y_1) + u_1 + v_1 = 0$ or $\dot{y}_1 - h_1(x_1,y_1) + u_1 + v_1 > 0$ and $y_1 \leq y_2$. Assume now that $\dot{y}_1 - h_1(x_1,y_1) + u_1 + v_1 > 0$ and $y_1 > y_2$. If $x_1 \leq x_2$, then from (2.2), (3.1), and $H(f_i)$ it follows that $y_1 = f_1(x_1) \leq f_1(x_2) \leq y_2$, which is a contradiction with (1.4). Hence, necessarily we have $x_1 > x_2$, for which the claim obviously holds. Similarly, in the unobvious case

$\dot{y}_1 - h_1(x_1, y_1) + u_1 + v_1 < 0$ and $y_1 < y_2$, $x_1 \geq x_2$, we obtain $y_1 = f_2(x_1) \geq f_2(x_2) \geq y_2$, a contradiction again. Therefore, (3.9) and (3.10) always hold.

Now, we add (3.9) and (3.10) to have

$$(\dot{y}_1 - h_1(x_1, y_1) + u_1 + v_1)\,\mathrm{sign}(y_1 - y_2) \leq (\dot{y}_1 - h_1(x_1, y_1) + u_1 + v_1)\,\mathrm{sign}(x_1 - x_2). \tag{3.11}$$

Interchanging the indices 1 and 2 of the functions x_i, y_i, u_i, v_i in (3.11) and adding the resulting inequality to (3.11), we see that

$$\begin{aligned}(\dot{y}_1 - \dot{y}_2)\,\mathrm{sign}(y_1 - y_2) &\leq (\dot{y}_1 - \dot{y}_2)\,\mathrm{sign}(x_1 - x_2)\\ &\quad + 2L_0(|x_1 - x_2| + |y_1 - y_2|) + 2(|u_1 - u_2| + |v_1 - v_2|).\end{aligned} \tag{3.12}$$

Multiplying this inequality by $a_2(x_1, y_1) \geq c_0 > 0$, in view of the fact that $\dot{z}\,\mathrm{sign}(z) = \frac{d}{dt}|z|$, yields

$$\begin{aligned}a_2(x_1, y_1)(\dot{y}_1 - \dot{y}_2)\,\mathrm{sign}(x_1 - x_2) \geq{}& \frac{d}{dt}\{a_2(x_1, y_1)|y_1 - y_2|\}\\ &- (2a_2(x_1, y_1) + 1)L_0(1 + |\dot{x}_1| + |\dot{y}_1|)(|x_1 - x_2| + |y_1 - y_2|)\\ &- 2a_2(x_1, y_1)(|u_1 - u_2| + |v_1 - v_2|).\end{aligned}$$

Combining this inequality with (3.8) and taking account of (3.6) and the bound for a_i, $i = 1, 2$, we have

$$\begin{aligned}&\frac{d}{dt}\{a_1(x_1, y_1)|x_1 - x_2| + a_2(x_1, y_1)|y_1 - y_2|\}\\ &\qquad\leq 2L_0(1 + 4N)(R + 1)(|x_1 - x_2| + |y_1 - y_2|)\\ &\qquad\quad + 2R(|u_1 - u_2| + |v_1 - v_2|).\end{aligned}$$

Integrating this inequality from 0 to $t \in [0, T]$, using $H(a_i, h_i)$, and applying Gronwall's inequality finally leads to (3.7).

We note that (3.7) entails the uniqueness of a solution to (3.1) and (3.2). The existence of this solution follows closely the lines of the proof of [25, Theorem 4.1], using double approximation of systems (3.1) and (3.2). In the first approximation, given $\varepsilon > 0$, one changes the function $y(\cdot)$ under derivative in (3.2) for the ε−mollification J_ε of its projection J onto the set $D(x) = [f_1(x), f_2(x)]$, $J(x, y) := y - (y - f_2(x))^+ + (f_1(x) - y)^+$, where the sign plus stands to denote the positive part. The second approximation stems from the standard replacement of the subdifferential $\partial I_{D(x)}(y)$ with its Moreau-Yosida regularisation

$$\partial I^\lambda_{D(x)}(y) := \frac{1}{\lambda}(y - f_2(x))^+ + \frac{1}{\lambda}(f_1(x) - y)^+, \quad \lambda > 0.$$

Under our assumptions on the system's coefficients and right-hand sides, the existence of a unique solution to the resulting doubly approximate system is a classical fact of ODE theory. To derive a priori estimates independent of ε, first, and of λ, second, one uses suitable testings of system's equations and employs specific properties of projection and Moreau-Yosida regularization. These a priori bounds for solutions of approximate problems and their derivatives imply by the Arzelá-Ascoli theorem the convergence of approximate solutions in the space $C([0, T]; \mathbb{R})$ of continuous functions equipped with the uniform norm. This convergence is enough to pass to the limit in the nonlinear coefficients and right-hand sides describing Eqs (3.1) and (3.2). To show that the limits satisfy the subdifferential inclusion (3.1) as well, one uses particular structure of the subdifferential of indicator function coupled with the properties of its Moreau-Yosida regularization and, thus, finally establishes the existence of a solution to systems (3.1) and (3.2). □

Next, we introduce the solution operator $\mathcal{T} : L^2(0,T;\mathbb{R}^2) \to C([0,T];\mathbb{R}^2)$ that to given $u, v \in L^2(0,T;\mathbb{R})$ associates the corresponding unique solution $(x(u,v), y(u,v))$ to systems (3.1) and (3.2):

$$(x(u,v), y(u,v)) = \mathcal{T}(u,v). \tag{3.13}$$

The existence and uniqueness of such a solution is proved in Lemma 3.1. The following lemma explores a continuity-type property of the operator $\mathcal{T}$ which along with the results of Proposition 2.1 are instrumental in proving the main result of the paper on the existence of a solution to systems (1.1)–(1.3) in the next section.

Lemma 3.2. *The operator* $\mathcal{T} : L^2(0,T;\mathbb{R}^2) \to C([0,T];\mathbb{R}^2)$ *is weak-strong continuous.*

Proof. Take an arbitrary sequence $(u_n(\cdot), v_n(\cdot)) \in L^2(0,T;\mathbb{R}^2)$, $n \geq 1$, weakly convergent in $L^2(0,T;\mathbb{R}^2)$ to some $(u(\cdot), v(\cdot)) \in L^2(0,T;\mathbb{R}^2)$ and let $(x_n, y_n) := (x(u_n, v_n), y(u_n, v_n))$, $n \geq 1$, be the sequence of solutions to (3.1) and (3.2) corresponding to (u_n, v_n). We show below that (x_n, y_n) converges strongly in $C([0,T];\mathbb{R}^2)$ to $(x(u,v), y(u,v))$, the solution of (3.1) and (3.2) corresponding to (u,v), as $n \to \infty$.

The weak convergence of u_n, v_n, $n \geq 1$, in $L^2(0,T;\mathbb{R})$ implies that $\|u_n\|_{L^2(0,T;\mathbb{R})} + \|v_n\|_{L^2(0,T;\mathbb{R})} \leq M$ for some $M > 0$. Then, Lemma 3.1 says that the estimates (3.5) and (3.6) are valid for the functions (x_n, y_n), $n \geq 1$, and by virtue of the Arzelá-Ascoli theorem we deduce the existence of subsequences $x(u_{n_k}, v_{n_k}) =: x_k$ and $y(u_{n_k}, v_{n_k}) =: y_k$, $k \geq 1$, of x_n and y_n, $n \geq 1$, respectively, such that

$$(x_k, y_k) \to (x, y) \quad \text{strongly in } C([0,T];\mathbb{R}^2), \tag{3.14}$$

$$(\dot{x}_k, \dot{y}_k) \to (\dot{x}, \dot{y}) \text{ weakly in } L^2(0,T;\mathbb{R}^2)$$

for some $(x, y) \in W^{1,2}[0,T];\mathbb{R}^2)$. From the properties of functions a_i, $i = 1, 2$, h_2 we infer that the pair (x, y) satisfies Eq (3.2).

Denote

$$\psi_k(t) := h_1(x_k(t), y_k(t)) - u_k(t) - v_k(t), \; k \geq 1, t \in [0,T].$$

Then, by the definition of (x_k, y_k), $k \geq 1$, in view of Eq (3.1), we have

$$-\dot{y}_k(t) + \psi_k(t) \in \partial I_{D(x_k(t))}(y_k(t)) \tag{3.15}$$

for a.e. $t \in [0,T]$ and from our assumption, $H(a_i, h_i)$ and (3.14) it also follows that

$$\psi_k \to \psi \; \text{ weakly in } L^2(0,T;\mathbb{R}), \tag{3.16}$$

where

$$\psi(t) = h_1(x(t), y(t)) - u(t) - v(t), \; t \in [0,T]. \tag{3.17}$$

Define on the space $L^2(0,T;\mathbb{R})$ the function

$$I_{\mathcal{D}(x)}(y) = \int_0^T I_{D(x(\tau))}(y(\tau))\, d\tau,$$

which is the indicator function of the set

$$\mathcal{D}(x) = \{y \in L^2(0, T; \mathbb{R}) : y(t) \in D(x(t)) \text{ for a.e. } t \in [0, T]\}.$$

From (3.15), by virtue of [30, Proposition 0.3.3], we see that

$$-\dot{y}_k + \psi_k \in \partial I_{\mathcal{D}(x_k)}(y_k), \quad k \geq 1. \tag{3.18}$$

Take an arbitrary function $z \in L^2(0, T; \mathbb{R})$ with $z \in D(x) = [f_1(x), f_2(x)]$ a.e. on $[0, T]$ and for $\sigma, x \in \mathbb{R}$ define

$$P_x(\sigma) := \max\{\min\{\sigma, f_2(x)\}, f_1(x)\}.$$

Let $z_k := P_{x_k}(z)$, $k \geq 1$. Then, $z_k \in D(x_k) = [f_1(x_k), f_2(x_k)]$ a.e. on $[0, T]$, $k \geq 1$, and $z_k \to z$ in $L^2(0, T; \mathbb{R})$ as $k \to \infty$. The (2.1) of sub-differential together with (3.18) imply that

$$(-\dot{y}_k + \psi_k, z_k - y_k)_{L^2(0,T;\mathbb{R})} \leq I_{\mathcal{D}(x_k)}(z_k) - I_{\mathcal{D}(x_k)}(y_k) = 0, \quad k \geq 1.$$

Passing to the limit as $k \to \infty$ in this inequality, on account of (3.14) and (3.16), we obtain

$$(-y + \psi, z - y)_{L^2(0,T;\mathbb{R})} \leq 0 = I_{\mathcal{D}(x)}(z) - I_{\mathcal{D}(x)}(y).$$

Since $z \in \operatorname{dom} I_{\mathcal{D}(x)}$ is arbitrary, from this inequality, by the definition of subdifferential, it follows that

$$-y + \psi \in \partial I_{\mathcal{D}(x)}(y).$$

By [30, Proposition 0.3.3] again, this inclusion yields

$$-\dot{y}(t) + \psi(t) \in \partial I_{D(x(t))}(y(t))$$

for a.e. $t \in [0, T]$ and consequently, recalling (3.17) we conclude that the pair (x, y) satisfies Eq (3.1) as well. This finally proves that $(x, y) = (x(u, v), y(u, v))$. □

4. Main result

In this section, equipped with the continuity of solution operator $\mathcal{T}$ from Section 3 and the properties of Nemytskii operator from Section 2, we establish our existence result through a multivalued fixed point argument.

We are now in a position to state and prove the main result of the paper.

Theorem 4.1. *Systems* (1.1)–(1.3) *have a solution.*

Proof. We fix $\varepsilon > 0$ and consider the differential equation

$$\dot{r}(t) = L\left((l_U(t) + l_V(t))(1 + r(t)) + \varepsilon\right), \quad \dot{r}(0) = r_0, \tag{4.1}$$

where L is the constant from (3.7), $l_U(\cdot)$ and $l_V(\cdot)$ are the functions from Hypotheses $H(U)2$) and $H(V)2$), respectively, and $r_0 := \sup\{|x(0, 0)(t)| + |y(0, 0)(t)| : t \in [0, T]\}$ for the solution $(x(0, 0), y(0, 0))$

of (3.1) and (3.2) corresponding to $u = v \equiv 0$. Equation (4.1) has a unique solution $r(t) \geq 0$ defined on the interval $[0, T]$. From (3.7) it follows that

$$|x(u, v)(t)| + |y(u, v)(t)| \leq r_0 + L \int_0^t (|u(s)| + |v(s)|)\, ds, \tag{4.2}$$

$t \in [0, T]$, $u(\cdot), v(\cdot) \in L^2(0, T, \mathbb{R})$. Let

$$S_U = \{u(\cdot) \in L^2(0, T, \mathbb{R}) : |u(t)| \leq l_U(t)(1 + r(t)) \text{ for a.e. } t \in [0, T]\}, \tag{4.3}$$

$$S_V = \{v(\cdot) \in L^2(0, T, \mathbb{R}) : |v(t)| \leq l_V(t)(1 + r(t)) + \varepsilon \text{ for a.e. } t \in [0, T]\}. \tag{4.4}$$

Lemma 3.1 implies that for any $u(\cdot) \in S_U$, $v(\cdot) \in S_V$, the systems (3.1) and (3.2) have a unique solution $(x(u, v), y(u, v)) = \mathcal{T}(u, v)$ (cf. (3.13)). From (4.2), we see that

$$|x(u, v)(t)| + |y(u, v)(t)| \leq r_0 + L \int_0^t ((l_U(s) + l_V(s))(1 + r(s)) + \varepsilon)\, ds = r(t)$$

for any $u(\cdot) \in S_U$, $v(\cdot) \in S_V$, $t \in [0, T]$, so that for the set

$$\mathcal{T}(S_U, S_V)(t) = \{(x(u, v)(t), y(u, v)(t)) : u(\cdot) \in S_U,\ v(\cdot) \in S_V\}, \quad t \in [0, T],$$

we have

$$\|\mathcal{T}(S_U, S_V)(t)\| \leq r(t), \quad t \in [0, T]. \tag{4.5}$$

Then, the set $V(t, \mathcal{T}(S_U, S_V)(t)) \bigcap (l_V(t)(1 + \|\mathcal{T}(S_U, S_V)(t)\|) + \varepsilon)\overline{B}$ is relatively compact for a.e. $t \in [0, T]$. Consequently, the multivalued mapping

$$t \mapsto \mathcal{V}(t) := \overline{\text{co}}\Big(V(t, \mathcal{T}(S_U, S_V)(t)) \bigcap (l_V(t)(1 + \|\mathcal{T}(S_U, S_V)(t)\|) + \varepsilon)\,\overline{B}\Big), \tag{4.6}$$

where $\overline{\text{co}}$ stands for the closed convex hull, has convex compact values. In addition, Lemma 3.1 in [29] guarantees that for any $u(\cdot) \in S_U$, $v(\cdot) \in S_V$ the multivalued mapping

$$t \mapsto V(t, x(u, v)(t), y(u, v)(t)) \bigcap \left(l_V(t)\left(1 + \left(|x(u, v)(t)|^2 + |y(u, v)(t)|^2\right)^{1/2}\right) + \varepsilon\right)\overline{B}$$

is measurable. Therefore, the set

$$S_{\mathcal{V}} = \{v(\cdot) \in L^2(0, T; \mathbb{R}) : v(t) \in \mathcal{V}(t) \text{ for a.e. } t \in [0, T]\} \tag{4.7}$$

is a nonempty convex compact subset of the space $L^2(0, T, \mathbb{R})$ endowed with the weak topology.

Now, define the set

$$\mathcal{U}(u, v) := \{f(\cdot) \in L^2(0, T, \mathbb{R}) : f(t) \in U(t, \mathcal{T}(u, v)(t)) \text{ for a.e. } t \in [0, T]\}$$

and introduce on the space $L^2(0, T; \mathbb{R})$ the following norm

$$P(u) = \left(\int_0^T \exp\left(-4L^2 \int_0^t k^2(\tau)\, d\tau\right) u^2(t) dt\right)^{1/2}, \quad u(\cdot) \in L^2(0, T; \mathbb{R}),$$

where L is from (3.7) and $k(\cdot)$ is from $H(U)3)$. This norm is equivalent to the standard norm on $L^2(0,T;\mathbb{R})$. Denoting by haus_P the Hausdorff metric generated by the norm P on the space of closed bounded subsets of $L^2(0,T;\mathbb{R})$ and using $H(U)3)$ and (3.7), we obtain

$$\begin{aligned}
&\text{haus}_P(\mathcal{U}(u_1,v),\mathcal{U}(u_2,v)) \\
&\le \left(\int_0^T \exp\left(-4L^2\int_0^t k^2(\tau)\,d\tau\right)\text{haus}^2_{\mathbb{R}}\left(U(t,\mathcal{T}(u_1,v)(t)),U(t,\mathcal{T}(u_2,v)(t))\right)dt\right)^{1/2} \\
&\le \left(\int_0^T \exp\left(-4L^2\int_0^t k^2(\tau)\,d\tau\right)L^2k^2(t)\left(\int_0^t |u_1(\tau)-u_2(\tau)|^2 d\tau\right)dt\right)^{1/2}
\end{aligned}$$

for $u_1(\cdot),u_2(\cdot),v(\cdot)\in L^2(0,T;\mathbb{R})$. The integration by parts of this inequality yields

$$\text{haus}_P(\mathcal{U}(u_1,v),\mathcal{U}(u_2,v)) \le \frac{1}{2}P(u_1-u_2). \tag{4.8}$$

In view of the properties $H(U)1)-3)$ and (3.7), it is a standard matter to show that for any $u(\cdot),v(\cdot)\in L^2(0,T,\mathbb{R})$ the set $\mathcal{U}(u,v)$ is a nonempty bounded and closed subset of the space $L^2(0,T;\mathbb{R})$. Moreover, it is decomposable in $L^2(0,T;\mathbb{R})$. Recall that a subset of $L^2(0,T;\mathbb{R})$ is said to be decomposable if together with any $g_1,g_2\in L^2(0,T;\mathbb{R})$ it contains the function $g_1\chi_E+g_2\chi_{[0,T]\setminus E}$ for any measurable $E\subset[0,T]$, where χ_F is the characteristic function of a set $F\subset[0,T]$.

Invoking now the contraction mapping principle for multivalued mappings with decomposable values (see, e.g., [31, Chapter 13]) we derive the existence of a fixed point for the mapping $\mathcal{U}$, i.e., of $u(\cdot)\in L^2(0,T;\mathbb{R})$ such that

$$u(t)\in U(t,\mathcal{T}(u,v)(t)) \tag{4.9}$$

for a.e. $t\in[0,T]$, $v(\cdot)\in L^2(0,T;\mathbb{R})$. Furthermore, considering $v(\cdot)\in L^2(0,T;\mathbb{R})$ as a parameter in the reasoning above we observe from $H(U)3)$ and Lemma 3.2 that the mapping $v\mapsto\mathcal{U}(u,v)$ is lower semicontinuous from $S_{\mathcal{V}}$ (cf. (4.7)) endowed with the weak topology of the space $L^2(0,T;\mathbb{R})$ to $L^2(0,T;\mathbb{R})$. Then, in view of (4.8), from [32, Theorem 3.1] we conclude that there exists a weak-strong continuous function $u: S_{\mathcal{V}}\to L^2(0,T;\mathbb{R})$ such that

$$u(v)(t)\in U(t,\mathcal{T}(u(v),v)(t)) \tag{4.10}$$

for a.e. $t\in[0,T]$. From Lemma 3.2, it follows that the superposition mapping $v\mapsto\mathcal{T}(u(v),v)$ is weak-strong continuous from $S_{\mathcal{V}}$ to the space $C([0,T];\mathbb{R}^2)$. The fact that $S_{\mathcal{V}}$ is compact in the weak topology of the space $L^2(0,T;\mathbb{R})$ implies then that the set

$$\mathcal{K}:=\{\mathcal{T}(u(v),v): v(\cdot)\in S_{\mathcal{V}}\}$$

is compact in the space $C([0,T];\mathbb{R}^2)$. Under our Hypotheses $H(V)$, we apply Proposition 2.1 to the multivalued mapping $V:[0,T]\times\mathbb{R}^2\rightrightarrows\mathbb{R}$ and deduce the existence of an upper semicontinuous in the weak topology of the space $L^p(0,T;\mathbb{R})$ multivalued mapping $\mathcal{M}:\mathcal{K}\rightrightarrows L^2(0,T;\mathbb{R})$ with convex weakly compact values such that for any $v(\cdot)\in S_{\mathcal{V}}$ and any $f(\cdot)\in\mathcal{M}(\mathcal{T}(u(v),v))$ we have

$$f(t)\in V(t,\mathcal{T}(u(v),v)(t)) \tag{4.11}$$

and

$$|f(t)| \le l_V(t)(1 + \|\mathcal{T}(u(v), v)(t)\|) + \varepsilon \tag{4.12}$$

for a.e. $t \in [0, T]$, where ε is the fixed number from the beginning of proof. Consider now the multivalued mapping $\mathcal{M}(\mathcal{T}(u(\cdot), \cdot)) : S_{\mathcal{V}} \rightrightarrows L^2(0, T; \mathbb{R})$. From the continuity properties of $\mathcal{M}$, $\mathcal{T}$, and $u(\cdot)$ it follows that this mapping is weak-weak upper semicontinuous with convex compact values. In addition, from (4.6), (4.7), (4.11), and (4.12), we see that $\mathcal{M}(\mathcal{T}(u(v), v)) \subset S_{\mathcal{V}}$ for any $v(\cdot) \in S_{\mathcal{V}}$ provided that

$$v(\cdot) \in S_V \quad \text{and} \quad u(v)(\cdot) \in S_U. \tag{4.13}$$

Now, the first inclusion in (4.13) follows from (4.4)–(4.7). To prove the second inclusion, for an arbitrary $v(\cdot) \in S_{\mathcal{V}}$ we use (4.2), (4.10), and $H(U)2$) to obtain

$$\begin{aligned} |x(u(v), v)(t)| + |y(u(v), v)(t)| &\le r_0 + L \int_0^t (|u(v)(s)| + |v(s)|)\, ds \\ &\le r_0 + L \int_0^t l_U(s)(1 + |x(u(v), v)(s)| + |y(u(v), v)(s)|)\, ds \\ &\quad + L \int_0^t (l_V(s)(1 + r(s)) + \varepsilon)\, ds. \end{aligned} \tag{4.14}$$

Consider the differential equation

$$\dot\rho(t) = L(l_U(t)(1 + \rho(t)) + l_V(t)(1 + r(t)) + \varepsilon), \quad \dot\rho(0) = r_0, \tag{4.15}$$

where r_0 is from Eq (4.1). Then, (4.14) and (4.15) imply that

$$|x(u(v), v)(t)| + |y(u(v), v)(t)| \le \rho(t), \quad t \in [0, T]. \tag{4.16}$$

On the other hand, taking the difference of (4.15) and (4.1) we see that

$$\dot r(t) - \dot\rho(t) \le l_U(t)(r(t) - \rho(t)), \quad t \in [0, T].$$

Consequently, $r(t) = \rho(t)$ and from (4.16) we have

$$|x(u(v), v)(t)| + |y(u(v), v)(t)| \le r(t), \quad t \in [0, T].$$

This inequality combined with (4.10) and Hypothesis $H(U)2$) finally yield the second inclusion in (4.13).

We have thus proved that $\mathcal{M}(\mathcal{T}(u(\cdot), \cdot))$ is an upper semicontinuous in the weak topologies self-mapping on $S_{\mathcal{V}}$ with convex compact values. Ky Fan's theorem [33] then implies that there exists a fixed point $v_*(\cdot)$ of this mapping:

$$v_* \in \mathcal{M}(\mathcal{T}(u(v_*), v_*)).$$

Setting $u_*(\cdot) = u(v_*)(\cdot)$, $(x_*, y_*) = \mathcal{T}(u_*, v_*)$ and taking into account (4.9) and (4.11), we obtain

$$u_*(t) \in U(t, x_*, y_*)(t),$$

$$v_*(t) \in V(t, x_*, y_*)(t)$$

for a.e. $t \in [0, T]$. Therefore, (x_*, y_*) is a solution to systems (1.1)–(1.3). □

5. Example and discussion

To illustrate the applicability of our theoretical results, we consider a simplified mechanical system whose dynamics can be described by systems (1.1)–(1.3).

Consider a mass moving in a highly viscous medium (e.g., a piston in a cylinder filled with oil), such that its motion is overdamped. Let $y(t)$ be the position of the mass. The movement of the mass is constrained by two stops, whose positions depend on the ambient temperature $x(t)$. The allowed region for the position $y(t)$ is therefore $y(t) \in [f_1(x(t)), f_2(x(t))]$. When the mass reaches a stop, a reaction force prevents it from moving further. This constraint corresponds to the term $\partial I_{D(x(t))}(y(t))$ in (1.1).

The mass is subject to a composed external force. The first component is a *feedback control force* $u(t)$ designed to regulate the system, which can switch between two different control laws, $g_1(t, x, y)$ and $g_2(t, x, y)$. The set of available control forces at any instant is then the nonconvex value of control constraints mapping $\{g_1(t, x, y), g_2(t, x, y)\}$. The second component is an *unpredictable external disturbance* $v(t)$, which is known only to lie within a certain range that may depend on the state, i.e., $v(t) \in [v_{min}(t, x, y), v_{max}(t, x, y)]$. This is a convex-valued mapping.

The temperature $x(t)$ of the system evolves based on two effects: an external heat source/sink $q(t, x, y)$ and a reversible thermomechanical coupling. This can occur if the piston and cylinder act as a thermocouple junction; moving the piston ($\dot{y} \neq 0$) then causes heat to be either absorbed or released at a rate proportional to its velocity. This thermomechanical coupling is described by the second equation of our system.

We can now link this physical model to our abstract systems (1.1)–(1.3) in the following way. The equation of motion for the mass (in an overdamped regime) is: $\beta\dot{y}(t) + \partial I_{[f_1(x(t)), f_2(x(t))]}(y(t)) \ni$ *combined external force*, where $\beta > 0$ is the viscous damping coefficient. We set $h_1(x, y) \equiv 0$, redefine the perturbations to be $-(u + v)/\beta$ and place it to the left-hand side. Assuming, for simplicity, $\beta = 1$ we then have

$$\dot{y}(t) + \partial I_{D(x(t))}(y(t)) + u(t) + v(t) \ni 0$$

with

$$u(t) \in U(t, x(t), y(t)) = \{-g_1(t, x(t), y(t)), -g_2(t, x(t), y(t))\}$$

and

$$v(t) \in V(t, x(t), y(t)) = [-v_{max}(t, x(t), y(t)), -v_{min}(t, x(t), y(t))].$$

The heat balance equation is

$$c_m\dot{x}(t) = q(t, x, y) - \gamma\dot{y}(t),$$

where c_m is the thermal capacity and $\gamma > 0$ represents the Peltier coefficient for the system. This equation describes how motion actively cools ($-\gamma\dot{y} < 0$ for $\dot{y} > 0$) or heats the system. To match the general form of our system, we write this as

$$c_m\dot{x}(t) + \gamma\dot{y}(t) = q(t, x, y),$$

which corresponds to (1.2) with $a_1(x, y) = c_m$, $a_2(x, y) = \gamma$, and $h_2(x, y) = q(t, x, y)$.

In this model, our abstract assumptions can be physically interpreted as follows:

H(f_i): The Lipschitz continuous functions f_1, f_2 define the stop positions changing smoothly with temperature. The condition $f_i' \geq 0$ means that the stops move in the same direction (e.g., expand) as temperature increases, while $f_1(x) = f_2(x)$ for large $|x|$ means that at extreme temperatures the gap between the stops closes.

H(a_i, h_i): The functions a_1, a_2 are material parameters (thermal capacity, thermomechanical coupling coefficient) and can be assumed constant, hence Lipschitz. The Lipschitz continuity of external heat source $h_2 = q$ in (x, y) is a standard regularity assumption. The condition $a_i \geq c_0 > 0$ has a clear physical meaning in our model: The thermal capacity $a_1 = c_m$ is naturally positive and the Peltier coefficient $a_2 = \gamma$ can also be assumed to be a positive material constant.

H(U): The nonconvex set U represents a choice between two distinct control actions g_1, g_2 (a two-point set), which is typical for switched systems. The Hausdorff-Lipschitz condition means that these control laws must depend smoothly on the state variables (x, y).

H(V): The set V models an uncertain disturbance. If v_{min} and v_{max} are continuous functions of the state, then the corresponding mapping V is upper semicontinuous with convex values. Our setting also allows for more complex (lower semicontinuous) perturbations caused, e.g., by abruptly appearing forces. The linear growth condition is a standard assumption to prevent solutions from exploding in finite time.

To compare our method with alternative approaches we note that it allows to handle the *sum* of two structurally different multivalued perturbations. When only the perturbation U is present ($V \equiv 0$), the differential inclusion of our system takes the form

$$\dot{y}(t) + \partial I_{D(x(t))}(y(t)) + U(t, x(t), y(t)) \ni h_1(x(t), y(t)).$$

Since U is assumed to be Hausdorff-Lipschitz, one could prove the existence of solutions by defining the appropriate solution operator and showing that it is a contraction in a suitable function space relying on the Lipschitz property of U, as shown, e.g., in [25,26]. If only the perturbation V is present ($U \equiv 0$), this inclusion is given by

$$\dot{y}(t) + \partial I_{D(x(t))}(y(t)) + V(t, x(t), y(t)) \ni h_1(x(t), y(t)).$$

Since V is not necessarily Lipschitz, a contraction argument fails. Instead, one could use an existence approach for differential inclusions with upper semicontinuous right-hand sides based on Ky Fan's type fixed-point argument after a potential convexification step, as in [29] for example.

In our case of a composed perturbation $U + V$ neither of the above methods alone works. The sum of a Lipschitz mapping and an upper semicontinuous mapping is, in general, neither Lipschitz nor upper semicontinuous. Our strategy separates the treatment of the two perturbations. We first use the Lipschitz property of U to construct a solution map for any given square-integrable perturbation v. Then, we treat this map as part of a larger fixed-point problem for the selection $v \in V(t, x, y)$. This two-stage fixed-point argument combining a contraction principle for U with a Ky Fan's argument for V allows us to establish the existence of a solution for the composedly perturbed system, a result that is not attainable by applying standard existence theorems to the sum $U + V$ directly.

6. Conclusions

In this paper, we established the existence of a solution for a system of two ordinary differential equations with a generalized play-type hysteresis operator and a composed multivalued perturbation.

This perturbation allows for the representation of typical choices of multivalued perturbations with different structural properties. Our theoretical results are applied to a mechanical model incorporating the overdamped motion of a mass between temperature-dependent stops, subject to a composed external force that includes both a nonconvex feedback control and a convex-valued external disturbance. This application illustrates the physical meaning of the underlying mathematical assumptions and demonstrates the value of our analytical method for systems where standard existence theorems for multivalued perturbations are not directly applicable.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

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