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Research article

A volume constraint problem for the nonlocal doubly nonlinear parabolic equation†

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Abstract: We consider a volume constraint problem for the nonlocal doubly nonlinear parabolic equation, called the nonlocal p-Sobolev flow, and introduce a nonlinear intrinsic scaling, converting a prototype nonlocal doubly nonlinear parabolic equation into the nonlocal p-Sobolev flow. This paper is dedicated to Giuseppe Mingione on the occasion of his 50th birthday, who is a maestro in the regularity theory of PDEs.

Keywords: nonlocal p-Sobolev flow; nonlinear intrinsic scaling; doubly nonlinear equations

1. Introduction

We are concerned with the following volume constraint problem of a nonlocal doubly nonlinear parabolic equation of the type

$$\begin{cases} \partial_t\Big(|u|^{q-1}u\Big) + (-\Delta)_p^s u = \lambda(t)|u|^{q-1}u & \text{in} \quad \Omega_T := \Omega \times (0,T), \\ \displaystyle\int_{\mathbb{R}^n} |u(t)|^{q+1}\,\mathrm{d}x = 1 & \text{for any } t \geq 0, \\ u = 0 & \text{on } (\mathbb{R}^n \setminus \Omega) \times (0,T), \\ u = u_0 & \text{in} \quad \Omega \times \{0\}, \end{cases} \tag{1.1}$$

with $p > 1$, $s \in (0,1)$ satisfying $sp < n$ and $q := p_s^* - 1 := \frac{np}{n-sp} - 1$, where Ω is a bounded open subset of $\mathbb{R}^n$ for $n \geq 2$ and $T > 0$, whereas the initial datum u_0 belongs to $W_0^{s,p}(\Omega) \cap L^{q+1}(\Omega)$ satisfying $\|u_0\|_{L^{q+1}(\Omega)} = 1$. Here the nonlocal term $(-\Delta)_p^s u$ is the fractional p-Laplacian defined as

$$
\begin{aligned}
(-\Delta)_p^s u(x,t) &:= 2\,\mathrm{PV.}\int_{\mathbb{R}^n} \frac{\Phi_p(u(x,t)-u(y,t))}{|x-y|^{n+sp}}\,\mathrm{d}y \\
&:= 2\lim_{\varepsilon\downarrow 0}\int_{\mathbb{R}^n\setminus B_\varepsilon(x)} \frac{\Phi_p(u(x,t)-u(y,t))}{|x-y|^{n+sp}}\,\mathrm{d}y,
\end{aligned}
\tag{1.2}
$$

where the function $\Phi_p : \mathbb{R} \to \mathbb{R}$ is defined by $\Phi_p(w) := |w|^{p-2}w$ for short. Moreover, the fractional p-Laplacian $(-\Delta)_p^s$ stems from the an energy structure. The energy functional on $W_0^{s,p}(\Omega)$

$$
W_0^{s,p}(\Omega) \ni w \quad \mapsto \quad \mathcal{E}(w) := \frac{1}{p}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|w(x)-w(y)|^p}{|x-y|^{n+sp}}\,\mathrm{d}x\mathrm{d}y,
$$

has Gâteaux derivative given by

$$
\left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} \mathcal{E}(w+\varepsilon\varphi) = \langle (-\Delta)_p^s w, \varphi\rangle,
$$

where the symbol $\langle\cdot,\cdot\rangle$ denotes the pairing on $W_0^{s,p}(\Omega)\times W^{-s,p'}(\Omega)$ with $\frac{1}{p}+\frac{1}{p'} = 1$. Here $W^{-s,p'}(\Omega)$ means the dual space of $W_0^{s,p}(\Omega)$. The fractional p-Laplace operator defined as (1.2) is naturally characterized in the distribution sense, that is, a bounded linear functional on Sobolev-Slobodeckiĭ space $\mathcal{D}_0^{s,p}(\Omega)$ defined as the completion of $C_0^\infty(\Omega)$ in the seminorm $(\mathcal{E}(\cdot))^{1/p}$ of $W^{s,p}(\mathbb{R}^n)$. In addition, if the boundary of domain Ω is smooth, then $W_0^{s,p}(\Omega)$ is identical to $\mathcal{D}_0^{s,p}(\mathbb{R}^n)$, which is verified by the density of $C_0^\infty(\Omega)$ in $W_0^{s,p}(\Omega)$ (refer to [17] and [23, Theorem 1.4.2.2]). In this way, $(1.1)_1$ can be interpreted as the nonlinear generalization of the usual gradient vector flow $\partial_t w = -\nabla\mathcal{E}(w)$. Furthermore, the volume constraint in $(1.1)_2$ yields that the constant $\lambda(t)$ appearing in $(1.1)_1$ is the Lagrange multiplier. Indeed, multiplying (1.1) by the solution u itself and integration by parts, one can check that

$$
\lambda(t) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x,t)-u(y,t)|^p}{|x-y|^{n+sp}}\,\mathrm{d}x\mathrm{d}y.
$$

We call the system (1.1) as *nonlocal p-Sobolev flow*. Prior to stating the main results, we briefly comment on the previous results with regard to the local or nonlocal doubly nonlinear evolutionary problems. The doubly nonlinear parabolic equations appear in a model of some physical phenomena like plasma physics or turbulent filtration of liquids; for instance, see [39] and references therein for a detailed explanation. They also describe the gradient flow associated with the p-Sobolev type inequality. Accordingly, we shall consider the prototype doubly nonlinear parabolic equation:

$$
\partial_t\Big(|u|^{q-1}u\Big) - \Delta_p u = 0, \quad \text{with } \ p > 1 \text{ and } \ q > 0.
\tag{1.3}
$$

Here, $\Delta_p u := \operatorname{div}\Big(|Du|^{p-2}Du\Big)$ is the p-Laplacian, where by $Du = (u_{x_i})_{1\leq i\leq n}$ we denote the spatial gradient of u with respect to x. The existence of solutions to the Cauchy-Dirichlet problem of (1.3) is firstly shown by Alt and Luckhaus [3] in the case $q \geq 1$. Employing a variational method, the existence of nonnegative solutions is obtained by Bögelein et al. [5] and, they extended the result to the more general form

$$\partial_t b(u) - \mathrm{div} D_\xi f(x,u,Du) = -D_u f(x,u,Du) \quad \text{in } \Omega_T, \tag{1.4}$$

where $f : \Omega \times \mathbb{R} \times \mathbb{R}^n \to (-\infty,\infty]$ is a Carathéodory function fulfilling some convexity and coercivity assumptions. The prototype of the integrand f is of the form $f \equiv f(x,Du) = \alpha(x)|Du|^p + \beta(x)|Du|^q$ with $1 < p < q$ and $\alpha(\cdot)$, $\beta(\cdot)$ being nonnegative functions such that $\alpha(x)+\beta(x) \geq \nu > 0$. The historical notes and the quick overview of regularity theory for the elliptic equation associated with (1.4) with the non-standard growth condition similarly as above are summarized in transparent paper by Mingione and Rădulescu [32]. The first and second authors proved the existence of (possibly singed) weak solutions to (1.3) for all $p > 1$ and $q > 0$ via energy estimates for approximate equations of Rothe type and the integral strong convergence of spatial gradient of approximate solutions [34]. The Hölder regularity issues of singed weak solutions to (1.3) are partially solved by three results: Bögelein et al. [9] ($q = p-1$, $p > 1$), Bögelein et al. [10] ($p > 2$, $0 < q < p-1$) and Liao & Schätzler [30] ($1 < p < 2$, $0 < p-1 < q$). To the best of our knowledge, in the other combinations of (p,q), the regularity issues are still open problems. The doubly nonlinear equation having a power nonlinearity of solution itself such as (1.3) is not translation invariant with respect to unknown function and that is why we directly consider a signed solution for Hölder regularity estimates of solution itself. The regularity method is expected to be developed for doubly nonlinear parabolic equations in all cases of $p > 1$ and $q > 0$. The method will be adopted to doubly nonlinear parabolic fractional equations as $(1.1)_1$, with the nonlocal tail effect at infinity. We shall pursue the regularity problems for doubly nonlinear parabolic fractional equations in our future work.

We briefly explain our motivation for studying the nonlocal p-Sobolev flow (1.1). Focusing on the Sobolev critical case where $q+1 = \frac{np}{n-p}$ with $n \geq 3$ and $2 \leq p < n$ and taking $s \to 1$ formally, Eq (1.1) under such a choice is deeply related to Yamabe flow in a compact Riemannian manifold. Kuusi & first and second authors [27, 28] proved the global existence of a positive weak solution with the spatial gradient regularity. This result covers that of Yamabe flow in the Euclidean setting of which the scalar curvature is zero. In those results, we employed the so-called nonlinear intrinsic scaling method, transforming the doubly nonlinear parabolic equation (1.3) to the p-Sobolev flow. In the present work we employ the same program in a nonlocal context. In the fractional framework, Eq (1.1) is also related to a geometric flow in Differential Geometry as well. As a matter of fact, through the stereographic projection on the Euclidean sphere $\mathbb{S}^n = \left\{x \in \mathbb{R}^{n+1} : |x| = 1\right\}$, the fractional Yamabe flow on $\mathbb{S}^n$ is represented as

$$\partial_t \left(u^{\frac{n+2s}{n-2s}}(t)\right) + (-\Delta)^s u(t) = r_s^g u^{\frac{n+2s}{n-2s}}(t) \quad \text{in } \mathbb{R}^n, \tag{1.5}$$

where $(-\Delta)^s u$ denotes the usual fractional Laplacian defined by (1.2) with $p = 2$. The quantity $r_s^g := \fint_M R_s^g \, \mathrm{dvol}_g$ appearing in (1.5) is the integral average of the fractional order curvature R_s^g.

At least from analytical point of view, the nonlocal p-Sobolev flow (1.1) can be interpreted as a generalization of the fractional Yamabe flow (1.5) in the L^p-setting.

In this paper, we also focus on the Cauchy-Dirichlet problem for the prototype nonlocal fractional doubly nonlinear parabolic equation because this is an auxiliary equation vital to establishing our main results for (1.1):

$$\begin{cases} \partial_\tau\Big(|v|^{q-1}v\Big) + (-\Delta)_p^s v = 0 & \text{in} \quad \Omega_S, \\ v = 0 & \text{on} \quad (\mathbb{R}^n \setminus \Omega) \times (0, S), \\ v = v_0 & \text{in} \quad \Omega \times \{\tau = 0\}, \end{cases} \tag{1.6}$$

where the initial datum v_0 belongs to the class $W_0^{s,p}(\Omega) \cap L^{q+1}(\Omega)$. The definition of weak solution and its properties are summarized in Section 3.

For the nonlocal p-Laplace equation, that is, (1.6) with $q = 1$, there are many remarkable literatures [1,16,26,29,31,38,42] and references therein. In particular, Strömqvist [40,41] established the local boundedness and a Harnack inequality with nonlocal tail for weak solutions to $(1.6)_1$ with $q = 1$ and $p > 2$, who first extended the breakthrough techniques by Di Castro et al. [18, 19] to the parabolic setting. Brasco et al. [16] showed the Hölder regularity of $(1.6)_1$ with $q = 1$ and $p > 2$ by beginning a slightly weaker notion of solutions; we remark that the proof relies on the iterated discrete differentiation method and a certain Morrey type embedding, which is completely different to previous approaches. In the homogenous doubly nonlinear case of $(1.6)_1$, i.e., $q = p - 1$ with $p > 2$, Banerjee et al. [4] first proved the local boundedness with the nonlocal tail of positive solutions. Up to now, as far as we know, and as stated as before, the regularity problems in the doubly nonlinear case ($q \neq 1$) for (1.6) are still left to investigate.

Recently, mixed local and nonlocal problems become a subject of engaging investigation in terms of not only purely mathematics but also biological viewpoint; see for instance, the linear case is addressed in [12–14,21]. Very recently, in the nonlinear case, the gradient regularity is established by De Filippis and Mingione [20] and the further topics are addressed in references therein. It is of its own interest how the local effect of p-Laplacian controls the nonlocal behavior with the tail at infinity of solutions in mixed local and nonlocal equations.

This paper introduces the *nonlinear intrinsic scaling* technique adopted to the nonlocal p-Sobolev flow. The idea of this nonlinear intrinsic scaling is to mimic the aforementioned p-Sobolev flow case. Indeed, a nonlinear intrinsic scaling is introduced to convert the nonlocal doubly nonlinear parabolic equation (1.6) to the nonlocal p-Sobolev flow (1.1) (see Theorem 1.1). Our nonlinear intrinsic scaling generates p-Sobolev flow and nonlocal one, the gradient flow of a constrained energy-minimization problem as stated before, from the corresponding prototype equation. It is worth remaking that the energy identity for the prototype equation plays a crucial role in both local and nonlocal setting, see [25, 34]. On the other hand, the difference of equations between the local case and nonlocal one is just reduced to the corresponding prototype equations. The technical detail of our nonlinear intrinsic scaling is addressed in a common way to both the local and nonlocal setting (see [28]). The outline of Theorem 1.1 will be given in Section 4.

Our result reads as follows:

Theorem 1.1 (Nonlinear Intrinsic Scaling)**.** *Fix $p > 1$, $s \in (0, 1)$ satisfying $sp < n$ and $q + 1 = p_s^*$. Let $v = v(x, \tau)$ be a nonnegative weak solution to* (1.6) *with $v_0 = u_0$ being the initial datum as in* (1.1)*. Let $S^* < +\infty$ be a finite extinction time of v depending only on n, s and p. Then, there exists a unique pair of solutions*

$$(\Lambda, g) \in C^1[0, \infty) \times C^1[0, \infty)$$

fulfilling

$$\begin{cases} \Lambda'(\tau) = (S^*)^{-1}\left[\displaystyle\int_{\mathbb{R}^n} v^{q+1}\left(x, S^*\left(1 - e^{-\Lambda(\tau)}\right)\right) \mathrm{d}x\right]^{\frac{sp}{n}} \\ \Lambda(0) = 0 \end{cases}$$

and

$$\begin{cases} g'(t) = e^{\Lambda(g(t))} \\ g(0) = 0 \end{cases}$$

such that the following statement holds true: Let

$$\tau(t) = S^*\left(1 - e^{-\Lambda(g(t))}\right)$$

and consider the following composite function:

$$u(x,t) := \frac{v\,(x, \tau(t))}{\left(\displaystyle\int_{\mathbb{R}^n} v^{q+1}\,(x, \tau(t))\;\mathrm{d}x\right)^{1/(q+1)}}. \tag{1.7}$$

Then, u is a weak solution of the nonlocal p-Sobolev flow (1.1) *in the sense of Definition 1, where*

$$\lambda(t) := \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x,t) - u(y,t)|^p}{|x-y|^{n+sp}}\,\mathrm{d}x\mathrm{d}y.$$

Theorem 1.1 actually treats a very special case of the choice of exponents (p, q) but the intrinsic scaling transformation will be well-worked for more general case that $p > 1$ and $q > 0$.

The following global existence result then follows directly from Theorem 1.1:

Theorem 1.2 (Global existence)**.** *Let $p > 1$, $q > 0$, $s \in (0, 1)$ be such that $sp < n$ and $q + 1 = p_s^*$. Suppose that the initial datum u_0 belongs to $W_0^{s,p}(\Omega) \cap L^{q+1}(\Omega)$ and satisfies $\|u_0\|_{L^{q+1}(\Omega)} = 1$. Then there exists a global in time nonnegative weak solution to* (1.1) *in the sense of Definition 1 possessing the regularity*

$$\partial_t\Big(|u|^{\frac{q-1}{2}} u\Big) \in L^2(\Omega_T) \tag{1.8}$$

for every positive $T < \infty$.

2. Basic setup

For convenience, we first fix some notation that will be used throughout the paper, then we define fractional Sobolev spaces. After that, we introduce some elementary inequalities and list the property of exponential mollification. Finally, we state the definition of weak solutions to (1.1).

2.1. Notation

In what follows, $\Omega \subset \mathbb{R}^n$ with $n \geq 2$ denotes a bounded domain with Lipschitz boundary. For $T \in (0, \infty]$, $\Omega_T := \Omega \times (0, T)$ describes a space-time cylinder. If $E \subset \mathbb{R}^k$ with $k \geq 1$ is a measurable subset satisfying $0 < |E| < \infty$, then for any $g \in L^1(E)$, we will denote

$$(g)_E := \fint_E g(x)\,\mathrm{d}x := \frac{1}{|E|}\int_E g(x)\,\mathrm{d}x.$$

Finally, we record the general notation. By C we denote a general positive constant, which varies from line to line and only depends on the parameters indicated in the statement to be revealed. By putting the corresponding parameters in parentheses, we will emphasize relevant dependencies on parameters; for instance, $C \equiv C(n, s, p, q)$ means that C depends on n, s, p and q. Furthermore, the symbol $(\ )_i$ denotes the i-th line of the display $(\ )$.

2.2. Fractional materials

In this subsection, we summarize fractional spaces and some useful tools. We start with recalling the fractional space. For $1 \leq p < +\infty$ and $s \in (0, 1)$, the fractional Sobolev space $W^{s,p}(\mathbb{R}^n)$ is defined via

$$W^{s,p}(\mathbb{R}^n) := \Big\{ w \in L^p(\mathbb{R}^n) : [w]_{W^{s,p}(\mathbb{R}^n)} < +\infty \Big\},$$

where

$$[w]_{W^{s,p}(\mathbb{R}^n)} := \left(\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|w(x) - w(y)|^p}{|x-y|^{n+sp}} \,\mathrm{d}x\mathrm{d}y \right)^{1/p}$$

is the Gagliardo-Slobodeckiĭ seminorm. $W^{s,p}(\mathbb{R}^n)$ is a Banach space endowed with the norm

$$\|w\|_{W^{s,p}(\mathbb{R}^n)} := \|w\|_{L^p(\mathbb{R}^n)} + [w]_{W^{s,p}(\mathbb{R}^n)}.$$

In a similar fashion, the fractional Sobolev spaces $W^{s,p}(\Omega)$ in a domain $\Omega \subset \mathbb{R}^n$ can be defined. The fractional Sobolev space with zero boundary values is defined as

$$W^{s,p}_0(\Omega) := \Big\{ w \in W^{s,p}(\Omega) : w = 0 \text{ on } \mathbb{R}^n \setminus \Omega \Big\}.$$

See [37] for the fundamental topics and tools related to these spaces and the references therein. Further a precise description of the completeness of these spaces are addressed in [15].

The next inequality is retrieved from [37, Theorem 6.5].

Lemma 2.1 (Fractional Sobolev inequality)**.** *Assume that $p > 1$ and $s \in (0, 1)$ satisfy $sp < n$. Then there exists a constant $C_{\mathrm{Sob}} \equiv C_{\mathrm{Sob}}(n, s, p)$ such that*

$$C_{\mathrm{Sob}} \left(\int_{\mathbb{R}^n} |w(x)|^{p_s*} \,\mathrm{d}x \right)^{\frac{p}{p_s^*}} \leq \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|w(x) - w(y)|^p}{|x-y|^{n+sp}} \,\mathrm{d}x\mathrm{d}y$$

holds whenever $w \in W^{s,p}_0(\Omega)$ with denoting $p_s^* := \frac{np}{n-sp}$.

2.3. Some elementary inequalities

First, we collect the property of the *boundary integral term*, devised by Bögelein et al. [5]: For $v, k \in \mathbb{R}$

$$\mathbf{B}[v, k] := \frac{1}{q+1} \left(|k|^{q+1} - |v|^{q+1} \right) - |v|^{q-1} v(k - v). \tag{2.1}$$

This boundary term is a crucial quantity for our argument. We state the estimates for the boundary term $\mathbf{B}$, whose precise proof is presented in [5, Lemma 2.5] and also in [8, Lemma 3.4].

Lemma 2.2. *Fix $q > 0$ and let $\mathbf{B}[v, k]$ be the algebraic quantity given by (2.1) for $v, k \in \mathbb{R}$. Then, there exists a positive constant $C \equiv C(q)$ such that the following estimates hold:*

$$\mathbf{B}[v, k] \leq \left(|k|^{q-1}k - |v|^{q-1}v\right)(k - v) \tag{2.2}$$

and

$$C^{-1}\left||v|^{\frac{q-1}{2}}v - |k|^{\frac{q-1}{2}}k\right|^2 \leq \mathbf{B}[v, k] \leq C\left||v|^{\frac{q-1}{2}}v - |k|^{\frac{q-1}{2}}k\right|^2. \tag{2.3}$$

In particular,

$$\mathbf{B}[v, k] \geq 0 \tag{2.4}$$

whenever $v, k \in \mathbb{R}$.

We recall the some algebraic inequalities. The next is addressed in [2, Lemma 2.2] in the case $0 < \beta < 1$ and in [22, inequality (2.4)] in the case $\beta > 1$.

Lemma 2.3. *For every $\beta > 0$ there exists a constant $C \equiv C(\beta)$ such that*

$$C^{-1}\left||\xi|^{\beta-1}\xi - |\eta|^{\beta-1}\eta\right| \leq (|\xi| + |\eta|)^{\beta-1}|\xi - \eta| \leq C\left||\xi|^{\beta-1}\xi - |\eta|^{\beta-1}\eta\right|$$

holds whenever $\xi, \eta \in \mathbb{R}$.

The above algebraic inequality is recasted in the following:

Lemma 2.4. *For all $\alpha \in (1, \infty)$*

$$(|\xi|^{\alpha-2}\xi - |\eta|^{\alpha-2}\eta)(\xi - \eta) \geq 0$$

holds whenever $\xi, \eta \in \mathbb{R}$.

2.4. Exponential mollification in time

We employ throughout the paper that the technique of the *exponential mollification in time*, devised in [24]. This mollification is a key ingredient that can overcome the difficulty of weak differentiable in time of weak solutions to Eq (1.1). Indeed, this technique is available for various doubly nonlinear parabolic equations; for instance, see [7–9, 11, 35, 36] and also the references therein. Let $E \subset \mathbb{R}^k$ be an open set with $k \geq 1$ and set $E_T := E \times (0, T)$. For a function $v \in L^1(E_T)$ and a number $h \in (0, T)$, we define

$$[v]_h(x, t) := \frac{1}{h}\int_0^t e^{\frac{\vartheta - t}{h}} v(x, \vartheta)\,\mathrm{d}\vartheta, \quad (x, t) \in E \times [0, T]. \tag{2.5}$$

Analogously, we define the reversed version of $[v]_h$ by

$$[v]_{\bar{h}}(x, t) := \frac{1}{h}\int_t^T e^{\frac{t - \vartheta}{h}} v(x, \vartheta)\,\mathrm{d}\vartheta, \quad (x, t) \in E \times [0, T].$$

In this setting, we summarize the properties of $[v]_h$ and $[v]_{\bar{h}}$ displayed below, whose detailed proof of (i)–(iii) can be seen in the literatures [24, Lemma 2.2] and [6, Appendix B], We are going to apply Lemma 2.5 with $E = \Omega$ or $E = \Omega \times \Omega$.

Lemma 2.5. *Let $E \subset \mathbb{R}^k$ be an open, bounded set. Assume that $v \in L^1(E_T)$ and $p \in [1, \infty)$. Then the mollifications $[v]_h$ and $[v]_{\bar{h}}$ have the following properties:*

(i) *If $v \in L^p(E_T)$, then $[v]_h \in L^p(E_T)$ and the inequality holds true:*

$$\|[v]_h\|_{L^p(E_T)} \le \|v\|_{L^p(E_T)}.$$

Furthermore, $[v]_h \to v$ strongly in $L^p(E_T)$ as $h \downarrow 0$. A same statement for $[v]_{\bar{h}}$ holds true.

(ii) *If $v \in L^p(E_T)$, then $[v]_h$ and $[v]_{\bar{h}}$ have weak time derivatives being in $L^p(E_T)$ and follow the ODE:*

$$\partial_t [v]_h = -\frac{[v]_h - v}{h}, \qquad \partial_t [v]_{\bar{h}} = \frac{[v]_{\bar{h}} - v}{h}.$$

(iii) *If $v \in L^p(0,T\,; L^p(E))$, then $[v]_h$ and $[v]_{\bar{h}}$ belong to $C([0,T]\,; L^p(E))$.*

A simple manipulation validates the following result.

Lemma 2.6. *Let $w \in L^p(0,T\,; W^{s,p}(\mathbb{R}^n))$. Then $[w]_h$ belongs to $L^p(0,T\,; W^{s,p}(\mathbb{R}^n))$; in particular,*

$$\int_0^T [[w(x,t)]_h]^p_{W^{s,p}(\mathbb{R}^n)}\,\mathrm{d}t \le \int_0^T [w(x,t)]^p_{W^{s,p}(\mathbb{R}^n)}\,\mathrm{d}t.$$

We conclude this section by defining of weak solutions of (1.1) used in the course of the paper.

Definition 1. Let $p > 1$, $q > 0$ and $s \in (0,1)$ be such that $sp < n$ and $q+1 = p_s^*$. We identify a function

$$u \in L^\infty(0,T\,; W^{s,p}(\mathbb{R}^n))$$

as a *weak solution* of (1.1) if and only if the volume constraint $(1.1)_2$ is in force and there exists $\lambda(\cdot) \in L^1(0,T)$ such that the identity

$$\begin{aligned} &-\iint_{\Omega_T} |u|^{q-1} u \varphi_t \,\mathrm{d}x\mathrm{d}t + \int_0^T \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{\Phi_p(u(x,t) - u(y,t))}{|x-y|^{n+sp}} (\varphi(x,t) - \varphi(y,t))\,\mathrm{d}x\mathrm{d}y\mathrm{d}t \\ &\quad = \iint_{\Omega_T} \lambda(t) |u|^{q-1} u \varphi \,\mathrm{d}x\mathrm{d}t \end{aligned}$$

holds whenever $\varphi \in \mathscr{T}_T$, where the class of testing functions $\mathscr{T}_T$ are defined by

$$\mathscr{T}_T := \left\{ \varphi \in L^p\left(0,T\,; W_0^{s,p}(\Omega)\right) \cap W^{1,q+1}\left(0,T\,; L^{q+1}(\Omega)\right) \;\middle|\; \begin{array}{c} \varphi(x,0) = \varphi(x,T) = 0 \\ \text{for a.e. } x \in \Omega \end{array} \right\}.$$

Moreover, u attains the prescribed initial condition $u(0) = u_0$ in the $W^{s,p}$ -sense, that is,

$$\lim_{t \downarrow 0} \|u(t) - u_0\|_{W^{s,p}(\mathbb{R}^n)} = 0$$

and u satisfies the boundary condition in the following sense:

$$u(t) \in W_0^{s,p}(\Omega) \quad \text{for a.e. } t \in (0,T).$$

3. Prototype nonlocal doubly nonlinear parabolic equation

In this section we shall collect the results on the prototype nonlocal doubly nonlinear parabolic equation (1.6). To begin, we prepare the notion of weak solutions to (1.6), whose definition is retrieved from [25]:

Definition 2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and fix $S \in (0, \infty]$. Suppose that the initial datum v_0 is in the class $W_0^{s,p}(\Omega) \cap L^{q+1}(\Omega)$. Let $\mathscr{T}_S$ be the class of test functions as in Definition 1, replaced T with S. We say that a measurable function $v = v(x, \tau)$ defined on a whole space-time region $\mathbb{R}^n \times (0, S)$ is a* weak solution *to* (1.6) *provided that the following conditions are satisfied:*

- $v \in L^\infty(0, S\,; W^{s,p}(\mathbb{R}^n)) \cap L^\infty\left(0, S\,; L^{q+1}(\mathbb{R}^n)\right)$.
- *There holds*

$$-\iint_{\Omega_S} |v|^{q-1} v\, \varphi_\tau \,\mathrm{d}x\mathrm{d}\tau + \int_0^S \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{\Phi_p\left(v(x,\tau) - v(y,\tau)\right)}{|x-y|^{n+sp}} (\varphi(x,\tau) - \varphi(y,\tau)) \,\mathrm{d}x\mathrm{d}y\mathrm{d}\tau = 0 \tag{3.1}$$

  for every $\varphi \in \mathscr{T}_S$.
- *v attains the initial datum v_0 continuously in the fractional Sobolev space:*

$$\lim_{\tau \downarrow 0} \|v(\tau) - v_0\|_{W^{s,p}(\mathbb{R}^n)} = 0.$$

  and satisfies the boundary condition in the sense that

$$v(\tau) \in W_0^{s,p}(\Omega) \quad \textit{for almost every } \tau \in (0, S).$$

3.1. Existence of a weak solution

In this section we recall the result of the existence for the problem (1.6).

Theorem 3.1 ([25, Theorem 1.1]). *Let $p > 1$, $q > 0$ and $s \in (0, 1)$ be given and assume that the initial datum v_0 belongs to $W_0^{s,p}(\Omega) \cap L^{q+1}(\Omega)$. Then there exists a global in time weak solution to* (1.6) *in the sense of Definition 2 fulfilling the following energy structures and regularity:*

$$\sup_{0<\tau<\infty} \int_\Omega |v(\tau)|^{q+1} \,\mathrm{d}x + \frac{q+1}{q} \int_0^\infty \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|v(x,\tau) - v(y,\tau)|^p}{|x-y|^{n+sp}} \,\mathrm{d}x\mathrm{d}y\mathrm{d}\tau \le \int_\Omega |v_0|^{q+1} \,\mathrm{d}x, \tag{3.2}$$

$$\sup_{0<\tau<\infty} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|v(x,\tau) - v(y,\tau)|^p}{|x-y|^{n+sp}} \,\mathrm{d}x\mathrm{d}y \le C \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|v_0(x) - v_0(y)|^p}{|x-y|^{n+sp}} \,\mathrm{d}x\mathrm{d}y, \tag{3.3}$$

and

$$\iint_{\Omega_\infty} \left| \partial_\tau \left(|v|^{\frac{q-1}{2}} v \right) \right|^2 \,\mathrm{d}x\mathrm{d}\tau \le C \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|v_0(x) - v_0(y)|^p}{|x-y|^{n+sp}} \,\mathrm{d}x\mathrm{d}y \tag{3.4}$$

with a positive constant $C \equiv C(p, q)$.

Proof. Here, we only give the sketch of the proof. As discussed precisely in [34] or [25], we first introduce the approximating equation of Rothe type for (1.6), which is constructed by a direct method in the calculus of variations. Subsequently, we prove that these approximate solutions fulfill certain energy estimates of the form (3.2)–(3.4) through some algebraic estimates associated to the difference quotient. Next, we exploit a truncation of the approximate solutions and make some integral estimates of spatial gradients and time-derivative of them. Since the integral bounds are available for the truncated approximating solutions above, combining the Fatou and Vitali convergence lemmas, we can pass to the limit of them. The most crucial step in this limiting procedure is how to deal with the time derivative of power-nonlinearity of solution itself. We notice that a simple transformation of unknown function such as $w = |v|^{q-1}v$ does not work well, because the positivity of solutions is not a priorily guaranteed. So we are forced to make use of the approximation of Rothe type. Indeed, the difference quotient in time effectively works to obtain the energy inequalities (3.2)–(3.4). □

Remark 3.2. Some apriori esimates for (1.6) can be verified by virtue of the exponential mollification in time. The time continuity in $L^{q+1}(\Omega)$ and the energy inequalities apriorily hold for weak solutions of (1.6). See Theorem 3.5 and Proposition 3.6. However, we are not sure to a priori get the energy estimate (3.3) containing space-time integral of time-derivative. Other approximations can be probably considered for (1.6); see for instance, [33].

3.2. Extinction in the finite time

The phenomenon of extinction of (nonnegative) solutions to (1.6) in the finite time naturally occurs, which is in the subsequent proposition.

Proposition 3.3 (Extinction in the finite time)**.** *Let $p > 1$, $s \in (0,1)$ be satisfy $sp < n$ and $q + 1 = p_s^*$. Let v be a nonnegative weak solution to the problem* (1.6) *in the sense of Definition 2. Suppose that the initial datum v_0 belongs to $W_0^{s,p}(\Omega) \cap L^{p_s^*}(\Omega)$. Then, there exists a finite time*

$$S^* \leq \frac{(p_s^*-1)(p_s^*-p)}{(p_s^*)^2 C_{\mathrm{Sob}}} \|v_0\|_{L^{p_s^*}(\Omega)}^{\frac{sp}{n}p_s^*}$$

such that, there holds that

$$v(\cdot,\tau) \equiv 0 \quad \textit{whenever} \quad \tau \geq S^*.$$

Proof. The proof is quite similar to [1,33], although we provide the details for the sake of completeness. Let v be a weak nonnegative solution to the problem (1.6). We multiply $(1.6)_1$ by the solution v and integrating over $\mathbb{R}^n$ render that

$$\frac{q}{q+1}\frac{\mathrm{d}}{\mathrm{d}\tau}\int_{\mathbb{R}^n} v(\tau)^{q+1}\,\mathrm{d}x + \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|v(x,\tau)-v(y,\tau)|^p}{|x-y|^{n+sp}}\,\mathrm{d}x\mathrm{d}y = 0, \tag{3.5}$$

where this procedure can be justified rigorously by using the independent Proposition 3.6 below. This combined with the fractional Sobolev inequality (Lemma 2.1) yields

$$\frac{p_s^*-1}{p_s^*}\frac{\mathrm{d}}{\mathrm{d}\tau}\int_{\mathbb{R}^n} v(\tau)^{p_s^*}\,\mathrm{d}x + C_{\mathrm{Sob}}\left(\int_{\mathbb{R}^n} v(\tau)^{p_s^*}\,\mathrm{d}x\right)^{\frac{p}{p_s^*}} \leq 0$$

with $C_{\mathrm{Sob}} \equiv C_{\mathrm{Sob}}(n,s,p)$ being the fractional Sobolev best constant, where we used the condition $q+1=p_s^*$. We shall shorten $W(\tau) := \int_{\mathbb{R}^n} v(\tau)^{p_s^*}\,\mathrm{d}x$ and therefore, the above display is rewritten as

$$W(\tau)^{-\frac{p}{p_s^*}} W'(\tau) \le -\frac{p_s^*}{p_s^*-1} C_{\mathrm{Sob}}.$$

Integrating this over $[0,\tau]$ leads to

$$\begin{aligned} 0 \le W(\tau)^{1-\frac{p}{p_s^*}} &\le W(0)^{1-\frac{p}{p_s^*}} - \frac{p_s^*}{p_s^*-1}\frac{p_s^*}{p_s^*-p} C_{\mathrm{Sob}}\tau \\ &= \left(\int_\Omega v_0^{p_s^*}\,\mathrm{d}x\right)^{\frac{sp}{n}} - \frac{p_s^*}{p_s^*-1}\frac{p_s^*}{p_s^*-p} C_{\mathrm{Sob}}\tau. \end{aligned}$$

This plainly assures that if

$$\tau \ge S^* := \frac{(p_s^*-1)(p_s^*-p)}{(p_s^*)^2 C_{\mathrm{Sob}}} \|v_0\|_{L^{p_s^*}(\Omega)}^{\frac{sp}{n}p_s^*}$$

then $v(x,\tau) \equiv 0$ holds and therefore the desired result follows. □

3.3. *Continuity in $L^{q+1}(\Omega)$*

In this subsection we prove the time continuity of solution via the technique of the exponential mollification. The following energy identity is an essential tool in this approach. Notice that the assertion holds for every exponent $q>0$. The argument below follows from [11, 35, 39].

Lemma 3.4 (Energy identity)**.** *Given $p>1$, $q>0$ and $s \in (0,1)$, let $\zeta = \zeta(\tau)$ be a nonnegative piecewise smooth function, defined on $[0,S]$, such that $\zeta(0)=\zeta(S)=0$. Set*

$$\mathcal{X} := \left\{ f \in L^p(0,S\,;W_0^{s,p}(\Omega)) \;:\; f,\, \partial_\tau f \in L^{q+1}(\Omega_S) \right\}.$$

Then there holds the quantitative identity

$$\begin{aligned} &-\iint_{\Omega_S} \zeta\left(|v|^{q-1}v - |w|^{q-1}w\right)\partial_\tau w\,\mathrm{d}x\mathrm{d}\tau + \iint_{\Omega_S} \zeta' \mathbf{B}[v,w]\,\mathrm{d}x\mathrm{d}\tau \\ &+ \int_0^S \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{\Phi_p\left(v(x,\tau)-v(y,\tau)\right)}{|x-y|^{n+sp}}\Big[\left(w(x,\tau)-v(x,\tau)\right) - \left(w(y,\tau)-v(y,\tau)\right)\Big]\,\mathrm{d}x\mathrm{d}y\mathrm{d}\tau = 0 \end{aligned}$$

for every weak solution v to (1.6) *and $w \in \mathcal{X}$, where $\mathbf{B}[v,w]$ is as in* (2.1).

Proof. The proof is similar to [35, Lemma 3.3], although we will give the full proof for the reader's convenience. First, the function space $\mathcal{X}$ allows us to choose $w = [v]_{\bar{h}}$ later. Let $w \in \mathcal{X}$. Testing the identity (3.1) with $\varphi \equiv \varphi_h := \zeta\,(w - [v]_h)$, which is admissible by Lemma 2.6, it is

$$\begin{aligned} &-\iint_{\Omega_S} |v|^{q-1}v \cdot \partial_\tau \varphi_h\,\mathrm{d}x\mathrm{d}\tau \\ &+ \int_0^S \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{\Phi_p\left(v(x,\tau)-v(y,\tau)\right)}{|x-y|^{n+sp}}\Big[\varphi_h(x,\tau)-\varphi_h(y,\tau)\Big]\,\mathrm{d}x\mathrm{d}y\mathrm{d}\tau = 0. \end{aligned} \tag{3.6}$$

For the evolutionary term we split it into four terms as follows:

$$
\begin{aligned}
&-\iint_{\Omega_S} |v|^{q-1} v \partial_\tau \varphi_h \,\mathrm{d}x\mathrm{d}\tau \\
&= -\iint_{\Omega_S} \Big[\zeta' |v|^{q-1} v\,(w-[v]_h) + \zeta |v|^{q-1} v\,(\partial_\tau w - \partial_\tau [v]_h)\Big]\,\mathrm{d}x\mathrm{d}\tau \\
&= -\iint_{\Omega_S} \zeta' |v|^{q-1} v\,(w-[v]_h)\,\mathrm{d}x\mathrm{d}\tau - \iint_{\Omega_S} \zeta |v|^{q-1} v\,\partial_\tau w \,\mathrm{d}x\mathrm{d}\tau \\
&\quad + \iint_{\Omega_S} \zeta \left(|v|^{q-1} v - |[v]_h|^{q-1}[v]_h\right)\partial_\tau [v]_h \,\mathrm{d}x\mathrm{d}\tau + \iint_{\Omega_S} \zeta |[v]_h|^{q-1}[v]_h \partial_\tau [v]_h \,\mathrm{d}x\mathrm{d}\tau \\
&=: (\mathrm{O}) + (\mathrm{I}) + (\mathrm{II}) + (\mathrm{III}).
\end{aligned}
$$

By Lemma 2.5-(v) and Lemma 2.4 we obtain

$$
(\mathrm{II}) = \iint_{\Omega_S} \zeta \left(|v|^{q-1} v - |[v]_h|^{q-1}[v]_h\right) \frac{v-[v]_h}{h}\,\mathrm{d}x\mathrm{d}\tau \geq 0
$$

and, in view of integration by parts, there holds

$$
(\mathrm{III}) = \iint_{\Omega_S} \zeta \partial_\tau \left(\tfrac{1}{q+1}\big|[v]_h\big|^{q+1}\right)\mathrm{d}x\mathrm{d}\tau = -\iint_{\Omega_S} \zeta'\, \tfrac{1}{q+1}\big|[v]_h\big|^{q+1}\,\mathrm{d}x\mathrm{d}\tau,
$$

where we used the fact that $\zeta(0) = \zeta(S) = 0$. Since $[v]_h \to v$ in $L^{q+1}(\Omega_S)$ by Lemma 2.5-(i), the combination of the preceding estimates above validates that

$$
\begin{aligned}
&\liminf_{h\downarrow 0}\left(-\iint_{\Omega_S} |v|^{q-1} v \partial_\tau \varphi_h \,\mathrm{d}x\mathrm{d}\tau\right) \\
&\geq \liminf_{h\downarrow 0}\Big[(\mathrm{O}) + (\mathrm{I}) + (\mathrm{III})\Big] \\
&= \liminf_{h\downarrow 0}\left(-\iint_{\Omega_S} \Big[\zeta' |v|^{q-1} v\,(w-[v]_h) + \zeta |v|^{q-1} v \partial_\tau w + \zeta'\, \tfrac{1}{q+1}\big|[v]_h\big|^{q+1}\Big]\,\mathrm{d}x\mathrm{d}\tau\right) \\
&= -\iint_{\Omega_S} \Big[\zeta' |v|^{q-1} v\,(w-v) + \zeta |v|^{q-1} v \partial_\tau w + \zeta'\, \tfrac{1}{q+1}|v|^{q+1}\Big]\,\mathrm{d}x\mathrm{d}\tau \\
&= -\iint_{\Omega_S} \Big[\zeta \left(|v|^{q-1} v - |w|^{q-1} w\right)\partial_\tau w + \zeta \tfrac{1}{q+1}\partial_\tau |w|^{q+1} \\
&\qquad\qquad + \zeta' \left(|v|^{q-1} v(w-v) + \tfrac{1}{q+1}|v|^{q+1}\right)\Big]\,\mathrm{d}x\mathrm{d}\tau \\
&= -\iint_{\Omega_S} \Big[\zeta \left(|v|^{q-1} v - |w|^{q-1} w\right)\partial_\tau w + \zeta' \left(\tfrac{1}{q+1}\left(|v|^{q+1} - |w|^{q+1}\right) - |v|^{q-1} v(v-w)\right)\Big]\,\mathrm{d}x\mathrm{d}\tau
\end{aligned}
$$

$$= -\iint_{\Omega_S} \zeta\left(|v|^{q-1}v - |w|^{q-1}w\right)\partial_\tau w \,\mathrm{d}x\mathrm{d}\tau + \iint_{\Omega_S} \zeta' \mathbf{B}[v,w]\,\mathrm{d}x\mathrm{d}\tau. \tag{3.7}$$

Here, in the fifth line, we used integration by parts combined with $\zeta(0) = \zeta(S) = 0$ again. A similar procedure to [35, (3.5)] assures that

$$\begin{aligned} &\int_0^S \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{\Phi_p\left(v(x,\tau)-v(y,\tau)\right)}{|x-y|^{n+sp}}\Big[\varphi_h(x,\tau)-\varphi_h(y,\tau)\Big]\,\mathrm{d}x\mathrm{d}y\mathrm{d}\tau \\ &\to \int_0^S \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{\Phi_p\left(v(x,\tau)-v(y,\tau)\right)}{|x-y|^{n+sp}}\zeta(\tau)\Big[\big(w(x,\tau)-v(x,\tau)\big)-\big(w(y,\tau)-v(y,\tau)\big)\Big]\,\mathrm{d}x\mathrm{d}y\mathrm{d}\tau \end{aligned} \tag{3.8}$$

as $h \downarrow 0$. Merging the last inequality (3.7) with (3.8) and then, sending $h \downarrow 0$ in (3.6) leads to

$$\begin{aligned} &-\iint_{\Omega_S} \zeta\left(|v|^{q-1}v - |w|^{q-1}w\right)\partial_\tau w\,\mathrm{d}x\mathrm{d}\tau + \iint_{\Omega_S} \zeta'\mathbf{B}[u,w]\,\mathrm{d}x\mathrm{d}\tau \\ &+\int_0^S \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{\Phi_p\left(v(x,\tau)-v(y,\tau)\right)}{|x-y|^{n+sp}}\Big[\big(w(x,\tau)-v(x,\tau)\big)-\big(w(y,\tau)-v(y,\tau)\big)\Big]\,\mathrm{d}x\mathrm{d}y\mathrm{d}\tau \le 0. \end{aligned}$$

The reversed sign "$\geq$" is confirmed by taking $\phi = \zeta(w - [v]_{\bar{h}})$ in (3.1) and the proof is complete. □

As a consequence of Lemma 3.4, we can show the time continuity in the Lebesgue space L^{q+1}:

Theorem 3.5 (**Time continuity in L^{q+1}**)**.** *Given $p > 1$, $q > 0$ and $s \in (0,1)$ let v be a weak solution to* (1.6) *in the sense of Definition 2. Then $v \in C([0,S]; L^{q+1}(\Omega))$.*

Proof. The argument follows that of [11, Lemma 3.11] or [35, Proposition 3.4], but we give full details here for the sake of completeness.

For every $S \in (0,\infty)$, let $\psi = \psi(\tau) : \mathbb{R} \to \mathbb{R}$ be a piecewise smooth function satisfying

$$\psi(\tau) := \begin{cases} 1, & \tau \le S/2, \\ 0, & \tau > 3S/4 \end{cases} \quad ; \quad 0 \le \psi \le 1 \quad ; \quad |\psi'| \le \frac{C}{S}. \tag{3.9}$$

Given $\bar{\tau} \in (0, S/2)$, set

$$\chi_\varepsilon(\tau) := \begin{cases} 0, & \tau < \bar{\tau}, \\ \dfrac{1}{\varepsilon}(\tau-\bar{\tau}), & \bar{\tau} \le \tau \le \bar{\tau}+\varepsilon, \\ 1, & \tau > \bar{\tau}+\varepsilon, \end{cases}$$

for $\varepsilon > 0$ small enough so that $\bar{\tau} + \varepsilon < S/2$. We take $\zeta \equiv \chi_\varepsilon(\tau)\psi(\tau)$ and $w \equiv [v]_{\bar{h}} \in \mathcal{X}$ in Lemma 3.4 to observe that

$$\begin{aligned} &\fint_{\bar{\tau}}^{\bar{\tau}+\varepsilon}\int_\Omega \mathbf{B}[v,[v]_{\bar{h}}]\,\mathrm{d}x\mathrm{d}\tau \\ &= -\iint_{\Omega_S} \mathbf{B}[v,[v]_{\bar{h}}]\chi_\varepsilon\psi'\,\mathrm{d}x\mathrm{d}\tau + \iint_{\Omega_S} \chi_\varepsilon\psi\left(|v|^{q-1}v - |[v]_{\bar{h}}|^{q-1}[v]_{\bar{h}}\right)\partial_\tau[v]_{\bar{h}}\,\mathrm{d}x\mathrm{d}\tau \end{aligned}$$

$$-\int_0^S\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{\Phi_p}{|x-y|^{n+sp}}\chi_\varepsilon(\tau)\psi(\tau)\Big[([v]_{\bar h}(x,\tau)-v(x,\tau))-([v]_{\bar h}(y,\tau)-v(y,\tau))\Big]\,\mathrm{d}x\mathrm{d}y\mathrm{d}\tau,$$

where we shortened $\Phi_p\equiv\Phi_p\,(v(x,\tau)-v(y,\tau))$. We use Lemma 2.5-(v) and Lemma 2.4 to obtain

$$\begin{aligned}&\iint_{\Omega_S}\chi_\varepsilon\psi\left(|v|^{q-1}v-|[v]_{\bar h}|^{q-1}[v]_{\bar h}\right)\partial_\tau[v]_{\bar h}\,\mathrm{d}x\mathrm{d}\tau\\&=\iint_{\Omega_S}\chi_\varepsilon\psi\left(|v|^{q-1}v-|[v]_{\bar h}|^{q-1}[v]_{\bar h}\right)\frac{[v]_{\bar h}-v}{h}\,\mathrm{d}x\mathrm{d}\tau\le 0,\end{aligned}$$

therefore combining this with the previous display and the definition (3.9), we obtain

$$\begin{aligned}&\fint_{\bar\tau}^{\bar\tau+\varepsilon}\int_\Omega\mathbf{B}[v,[v]_{\bar h}]\,\mathrm{d}x\mathrm{d}\tau\\&\le\frac{C}{S}\iint_{\Omega_S}\mathbf{B}[v,[v]_{\bar h}]\,\mathrm{d}x\mathrm{d}\tau\\&\quad-\int_0^S\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{\Phi_p}{|x-y|^{n+sp}}\chi_\varepsilon(\tau)\psi(\tau)\Big[([v]_{\bar h}(x,\tau)-v(x,\tau))-([v]_{\bar h}(y,\tau)-v(y,\tau))\Big]\,\mathrm{d}x\mathrm{d}y\mathrm{d}\tau.\end{aligned}$$

Since $\mathbf{B}[u,[u]_{\bar h}]$ is integrable, sending $\varepsilon\downarrow 0$ in the above display in turn implies that

$$\begin{aligned}&\int_\Omega\mathbf{B}[v(\bar\tau),[v]_{\bar h}(\bar\tau)]\,\mathrm{d}x\\&\le\frac{C}{S}\iint_{\Omega_S}\mathbf{B}[v,[v]_{\bar h}]\,\mathrm{d}x\mathrm{d}\tau\\&\quad-\int_0^S\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{\Phi_p}{|x-y|^{n+sp}}\psi(\tau)\Big[([v]_{\bar h}(x,\tau)-v(x,\tau))-([v]_{\bar h}(y,\tau)-v(y,\tau))\Big]\,\mathrm{d}x\mathrm{d}y\mathrm{d}\tau\end{aligned}\tag{3.10}$$

whenever $\bar\tau\in(0,S/2)\setminus N_h$, where N_h is a null set with respect to one-dimensional Lebesgue's measure. We preliminary take a sequence $\{h_j\}_{j\in\mathbb{N}}$ so that $h_j\downarrow 0$ as $j\to\infty$, and set $\mathcal{N}:=\bigcup_{j\in\mathbb{N}}N_{h_j}$; therefore, $\mathcal{N}$ is negligible as well. By the same procedure leading to (3.8) and the convergence $[v]_{\bar h_j}\to v$ in $L^{q+1}(\Omega_S)$, guaranteed by Lemma 2.5-(i), the right-hand side of (3.10) converges to zero in the limit $j\to\infty$. As a consequence, letting $j\to\infty$ in (3.10) with $[v]_{\bar h}\equiv[v]_{\bar h_j}$ renders

$$\limsup_{j\to\infty}\left(\sup_{\bar\tau\in[0,S/2]\setminus\mathcal{N}}\int_\Omega\mathbf{B}[v(\bar\tau),[v]_{\bar h_j}(\bar\tau)]\,\mathrm{d}x\right)\le 0,$$

which together with (2.3) in Lemma 2.2 implies that

$$\lim_{j\to\infty}\left(\sup_{\bar\tau\in[0,S/2]\setminus\mathcal{N}}\int_\Omega\left||v|^{\frac{q-1}{2}}v(\bar\tau)-|[v]_{\bar h_j}|^{\frac{q-1}{2}}[v]_{\bar h_j}(\bar\tau)\right|^2\mathrm{d}x\right)=0.\tag{3.11}$$

We now distinguish the cases between $q\ge 1$ and $0<q<1$. When considering $q\ge 1$, Lemma 2.3 with $\alpha=\frac{q+1}{2}$ renders that

$$\left|v(\bar\tau)-[v]_{\bar h_j}(\bar\tau)\right|^{\frac{q+1}{2}}\le C\left||v|^{\frac{q-1}{2}}v(\bar\tau)-|[v]_{\bar h_j}|^{\frac{q-1}{2}}[v]_{\bar h_j}(\bar\tau)\right|$$

for a constant $C \equiv C(q)$ and therefore, we have

$$\lim_{j\to\infty}\left(\sup_{\bar\tau\in[0,S/2]\setminus\mathcal{N}}\int_\Omega \left|v(\bar\tau)-[v]_{\bar h_j}(\bar\tau)\right|^{q+1}\,\mathrm{d}x\right)=0. \tag{3.12}$$

In the opposite case $0<q<1$, (2.3) with $\alpha=\frac{q+3}{2}$ gives

$$\left(|v(\bar\tau)|+|[v]_{\bar h_j}(\bar\tau)|\right)^{\frac{q-1}{2}}\left|v(\bar\tau)-[v]_{\bar h_j}(\bar\tau)\right| \le C\left||v|^{\frac{q-1}{2}}v(\bar\tau)-|[v]_{\bar h_j}|^{\frac{q-1}{2}}[v]_{\bar h_j}(\bar\tau)\right|.$$

Using this and appealing to Hölder's inequality with a pair of exponents $\left(\frac{2}{q+1},\frac{2}{1-q}\right)$, we have

$$\begin{aligned}
&\sup_{\bar\tau\in[0,S/2]\setminus\mathcal{N}}\int_\Omega \left|v(\bar\tau)-[v]_{\bar h_j}(\bar\tau)\right|^{q+1}\,\mathrm{d}x\\
&\le C\left(\sup_{\bar\tau\in[0,S/2]\setminus\mathcal{N}}\int_\Omega \left(|v(\bar\tau)|+|[v]_{\bar h_j}(\bar\tau)|\right)^{q-1}\left|v(\bar\tau)-[v]_{\bar h_j}(\bar\tau)\right|^2\,\mathrm{d}x\right)^{\frac{q+1}{2}}\\
&\qquad\cdot\left(\sup_{\bar\tau\in[0,S/2]\setminus\mathcal{N}}\int_\Omega \left[|v(\bar\tau)|+[v]_{\bar h_j}(\bar\tau)\right]^{q+1}\,\mathrm{d}x\right)^{\frac{1-q}{2}}\\
&\le C\left(\sup_{\bar\tau\in[0,S/2]\setminus\mathcal{N}}\int_\Omega \left||v|^{\frac{q-1}{2}}v(\bar\tau)-|[v]_{\bar h_j}|^{\frac{q-1}{2}}[v]_{\bar h_j}(\bar\tau)\right|^2\,\mathrm{d}x\right)^{\frac{q+1}{2}}\\
&\qquad\cdot\left(\sup_{\bar\tau\in[0,S/2]\setminus\mathcal{N}}\int_\Omega \left[|v(\bar\tau)|^{q+1}+|[v]_{\bar h_j}(\bar\tau)|^{q+1}\right]\mathrm{d}x\right)^{\frac{1-q}{2}}\\
&\to 0
\end{aligned}$$

as $h\downarrow 0$. Here, in the last line, we used (3.11) and Lemma 2.5-(i). Thus, we gain (3.12) in the case $0<q<1$ as well.

Thanks to Lemma 2.5-(vi), $[v]_{\bar h_j}(\tau)$ is continuous in $L^{q+1}(\Omega)$ for any $\tau\in[0,S]$ and so, $v(\tau)$ is also continuous on $[0,S/2]\setminus\mathcal{N}$ in $L^{q+1}(\Omega)$ since v is a uniform limit representation of $[v]_{\bar h_j}$ by (3.12). We stress that this representation is independent of the choice of sequences $\{h_j\}_{j\in\mathbb{N}}$. By means of (3.12) and the continuity of $[v]_{\bar h_j}(\tau)$ on $[0,S]$, we can find a continuous function $v^*(\tau):[0,S]\to L^{q+1}(\Omega)$ being equals to $v(\tau)$ in $[0,S/2]\setminus\mathcal{N}$ and therefore, $v^*(\tau)=v(\tau)$ holds for everywhere on $[0,S/2]$. In an analogous way as done above, letting $w\equiv[v]_h$ and $\zeta=\bar\chi_\varepsilon(\tau)\bar\psi(\tau)$ in Lemma 3.4 yields that $v(\tau)$ is also continuous in $L^{q+1}(\Omega)$ on $[S/2,S]$, where $\bar\chi_\varepsilon(\tau)$ and $\bar\psi(\tau)$ are reflexions with respect to $\tau=S/2$ of $\chi_\varepsilon(\tau)$ and $\psi(\tau)$, respectively. [*] As a result, v certainly belongs to $C([0,S];L^{q+1}(\Omega))$, finishing the proof. □

[*]More precisely, one could have proceeded as follows: Fix an arbitrary $\varepsilon>0$. In view of (3.12), there exists $j_*\in\mathbb{N}$ so that

$$\sup_{\tau\in[0,S/2]\setminus\mathcal{N}}\left\|v(\tau)-[v]_{\bar h_{j_*}}(\tau)\right\|_{L^{q+1}(\Omega)}<\varepsilon/3. \tag{3.13}$$

Lemma 2.5-(iii) yields that $[v]_{\bar h_j}(\tau)$ is uniformly continuous in $L^{q+1}(\Omega)$ on $\tau\in[0,S/2]$ and therefore, there exists $\delta>0$ such that

$$\forall\tau,\tau'\in[0,S/2]\ \text{ with }\ |\tau-\tau'|<\delta \quad\Longrightarrow\quad \left\|[v]_{\bar h_{j_*}}(\tau)-[v]_{\bar h_{j_*}}(\tau')\right\|_{L^{q+1}(\Omega)}<\varepsilon/3. \tag{3.14}$$

Theorem 3.5 allows to deduce the following energy identity:

Proposition 3.6. *With* $p > 1$, $q > 0$ *and* $s \in (0,1)$, *let* v *be a weak solution to* (1.6) *in the sense of Definition 2. Then the following quantitative estimate*

$$\frac{q}{q+1}\int_\Omega |v(\tau_2)|^{q+1}\,\mathrm{d}x + \int_{\tau_1}^{\tau_2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|v(x,\tau)-v(y,\tau)|^p}{|x-y|^{n+sp}}\,\mathrm{d}x\mathrm{d}y\mathrm{d}\tau = \frac{q}{q+1}\int_\Omega |v(\tau_1)|^{q+1}\,\mathrm{d}x \tag{3.16}$$

holds whenever $0 \le \tau_1 < \tau_2 \le S$*; in particular, the function*

$$\tau \mapsto \int_\Omega |v(\tau)|^{q+1}\,\mathrm{d}x$$

is Lipschitz continuous on $[0,S]$.

Proof. We shall split the proof into several step.

Step 1: First, a similar argument to [35, Lemma 2.10] yields the following mollified versions of weak formulation (3.1):

$$\begin{aligned}&\iint_{\Omega_S}\partial_\tau[|v|^{q-1}v]_h\,\varphi\,\mathrm{d}x\mathrm{d}\tau\\ &\qquad+\int_0^S\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{[\Phi_p]_h}{|x-y|^{n+sp}}\left(\varphi(x,\tau)-\varphi(y,\tau)\right)\,\mathrm{d}x\mathrm{d}y\mathrm{d}\tau\\ &=\int_\Omega |v|^{q-1}v(0)\left(\frac{1}{h}\int_0^S e^{-\frac{\vartheta}{h}}\varphi(x,\vartheta)\,\mathrm{d}\vartheta\right)\mathrm{d}x\end{aligned}\tag{3.17}$$

and

$$\begin{aligned}&\iint_{\Omega_S}\partial_\tau[|v|^{q-1}v]_{\bar h}\,\varphi\,\mathrm{d}x\mathrm{d}\tau\\ &\qquad+\int_0^S\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{[\Phi_p]_{\bar h}}{|x-y|^{n+sp}}\left(\varphi(x,\tau)-\varphi(y,\tau)\right)\,\mathrm{d}x\mathrm{d}y\mathrm{d}\tau\\ &=-\int_\Omega |v|^{q-1}v(S)\left(\frac{1}{h}\int_0^S e^{\frac{\vartheta-S}{h}}\varphi(x,\vartheta)\,\mathrm{d}\vartheta\right)\mathrm{d}x\end{aligned}\tag{3.18}$$

Combining two displays (3.13) and (3.14), it readily follows that

$$\forall\tau,\tau'\in[0,S/2]\setminus\mathcal{N}\ \text{ with }\ |\tau-\tau'|<\delta \quad\Longrightarrow\quad \|v(\tau)-v(\tau')\|_{L^{q+1}(\Omega)}<\varepsilon. \tag{3.15}$$

On the other hand, since $\mathcal{N}$ is negligible with respect to the one-dimensional Lebesgue measure, for any $\tau\in[0,S/2]$ there exists a sequence $\{\tau_i\}_{i\in\mathbb{N}}\subset[0,S/2]\setminus\mathcal{N}$ so that $\tau_i\to\tau$ as $i\to\infty$. Take a natural number N so that $|\tau_i-\tau_k|<\delta$ for all $i,k\ge N$. By means of (3.15),

$$\forall i,k\ge N \quad\Longrightarrow\quad \|v(\tau_i)-v(\tau_k)\|_{L^{q+1}(\Omega)}<\varepsilon,$$

that is, $\{v(\tau_i)\}_{i\in\mathbb{N}}$ is a Cauchy sequence in $L^{q+1}(\Omega)$, therefore the limiting function $v^*(\tau):=\lim_{i\to\infty}v(\tau_i)$ exists. A simple observation shows that this limiting function is uniquely determined, which is independent of the choice of sequences in $[0,S/2]\setminus\mathcal{N}$. Thanks to this uniqueness together with the fact that $v(\tau)$ is continuous on $[0,S/2]\setminus\mathcal{N}$ in $L^{q+1}(\Omega)$, $v(\tau)=\lim_{i\to\infty}v(\tau_i)=v_1^*(\tau)$ in turn follows for every $\tau\in[0,S/2]\setminus\mathcal{N}$. Finally, we shall show the validity of $v_1^*\in C\big([0,S/2]\,;L^{q+1}(\Omega)\big)$. For this, given $\varepsilon>0$, we first take $\delta>0$ fulfilling (3.15). For every $\tau,\tau'\in[0,S/2]$ with $|\tau-\tau'|<\delta/3$ we take sequences $\{\tau_i\}_{\mathbb{N}},\{\tau_i'\}_{i\in\mathbb{N}}\subset[0,S/2]\setminus\mathcal{N}$ so that $\tau_i\to\tau$, $\tau_i'\to\tau'$, respectively. By this construction, $|\tau_i-\tau_i'|<\delta$ holds for $i\in\mathbb{N}$ large enough and therefore, in view of (3.15), it is

$$\forall\tau_i,\tau_i'\in[0,S/2]\setminus\mathcal{N}\ \text{ with }\ |\tau_i-\tau_i'|<\delta \quad\Longrightarrow\quad \left\|v(\tau_i)-v(\tau_i')\right\|_{L^{q+1}(\Omega)}<\varepsilon.$$

Finally, sending $i\to\infty$ in the last inequality renders $\|v^*(\tau)-v^*(\tau')\|_{L^{q+1}(\Omega)}<\varepsilon$ for every $\tau,\tau'\in[0,S/2]$, as claimed. In a similar fashion with $w\equiv[v]_h$ and with $\bar\chi_\varepsilon(\tau)$ and $\bar\psi(\tau)$ mirrored on the whole interval $[0,S]$ under transformation $\tau\mapsto S-\tau$, we find $v_2^*\in C\big([0,S/2]\,;L^{q+1}(\Omega)\big)$. Finally, we let $v^*:=v_1^*$ on $[0,S/2]$ and $:=v_2^*$ on $[S/2,S]$, leading to the conclusion.

holds whenever $\varphi \in \mathscr{T}_S$, where we again shortened $\Phi_p \equiv \Phi_p\left(v(x,\tau) - v(y,\tau)\right)$. Indeed, (3.17) is verified by testing $\varphi \equiv [\varphi]_{\bar{h}}\eta_\varepsilon$ in (3.1), which is admissible in $\mathscr{T}_S$, where

$$\eta_\varepsilon(\tau) := \begin{cases} \tau/\varepsilon & \text{for} \quad 0 \le \tau \le \varepsilon, \\ 1 & \text{for} \quad \varepsilon \le \tau \le S, \end{cases}$$

whereas (3.18) is confirmed by choosing $\varphi \equiv [\varphi]_h \bar{\eta}_\varepsilon$ in (3.1), where the Lipschitz function $\bar{\eta}_\varepsilon$ is defined as

$$\bar{\eta}_\varepsilon(\tau) := \begin{cases} 1 & \text{for} \quad 0 \le \tau \le S - \varepsilon, \\ (S-\tau)/\varepsilon & \text{for} \quad S - \varepsilon \le \tau \le S, \\ 0 & \text{for} \quad \tau \ge S. \end{cases}$$

Here, in these procedures we used the Fubini theorem for the double integral and Theorem 3.5.

Moreover, for every $0 \le \tau_1 < \tau_2 \le S$ and $\varepsilon > 0$ small enough, let us define the following Lipschitz cut-off function shaped like a trapezoid (see Figure 1):

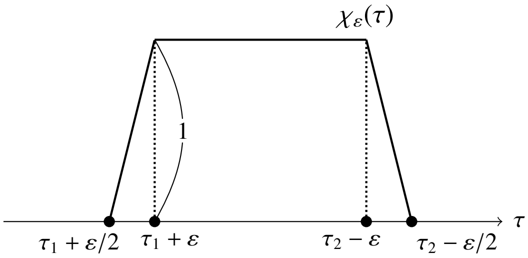

Figure 1. Graphs of $\chi_\varepsilon(\tau)$.

$$\chi = \chi_\varepsilon(\tau) := \begin{cases} 0, & \tau \in [0, \tau_1 + \varepsilon/2), \\ \frac{2}{\varepsilon}(\tau - \tau_1 - \frac{\varepsilon}{2}), & \tau \in [\tau_1 + \varepsilon/2, \tau_1 + \varepsilon), \\ 1, & \tau \in [\tau_1 + \varepsilon, \tau_2 - \varepsilon], \\ -\frac{2}{\varepsilon}(\tau - \tau_2 - \frac{\varepsilon}{2}), & \tau \in (\tau_2 - \varepsilon, \tau_2 - \varepsilon/2], \\ 0, & \tau \in (\tau_2 - \varepsilon/2, S]. \end{cases}$$

Choosing the testing function in (3.17) as

$$\varphi(x,\tau) = \chi_\varepsilon(t) v(x,\tau)$$

gives the estimates below respectively: The evolutional term is split into two terms

$$\begin{aligned} \iint_{\Omega_S} \partial_\tau [|v|^{q-1}v]_h \chi_\varepsilon v \,\mathrm{d}x\mathrm{d}\tau &= \iint_{\Omega_S} \chi_\varepsilon \partial_\tau [|v|^{q-1}v]_h \left(v - \left|[|v|^{q-1}v]_h\right|^{\frac{1}{q}-1} [|v|^{q-1}v]_h\right) \mathrm{d}x\mathrm{d}\tau \\ &\quad + \iint_{\Omega_S} \chi_\varepsilon \partial_\tau [|v|^{q-1}v]_h \left|[|v|^{q-1}v]_h\right|^{\frac{1}{q}-1} [|v|^{q-1}v]_h \,\mathrm{d}x\mathrm{d}\tau \\ &=: (\mathrm{I})_{\mathrm{i}} + (\mathrm{I})_{\mathrm{ii}}. \end{aligned}$$

We abbreviate

$$
\begin{aligned}
f_h &:= \begin{cases} \left|[|v|^{q-1}v]_h\right|^{\frac{1}{q}-1}[|v|^{q-1}v]_h & \text{if } v \neq 0 \\ 0 & \text{if } v = 0 \end{cases} \\
&\iff |f_h|^{q-1}f_h = [|v|^{q-1}v]_h
\end{aligned}
$$

and apply Lemma 2.5-(v) and Lemma 2.3 to infer that

$$
(\mathrm{I})_{\mathrm{i}} = \frac{1}{h}\iint_{\Omega_S} \chi_\varepsilon \left(|v|^{q-1}v - |f_h|^{q-1}f_h\right)(v - f_h)\, \mathrm{d}x\mathrm{d}\tau \geq 0.
$$

In view of integration by parts, we get

$$
\begin{aligned}
(\mathrm{I})_{\mathrm{ii}} &= \iint_{\Omega_S} \chi_\varepsilon \partial_\tau \left(\tfrac{q}{q+1}\left|[|v|^{q-1}u]_h\right|^{\frac{q+1}{q}}\right) \mathrm{d}x\mathrm{d}\tau \\
&= -\iint_{\Omega_S} \chi_\varepsilon' \tfrac{q}{q+1}\left|[|v|^{q-1}v]_h\right|^{\frac{q+1}{q}}\, \mathrm{d}x\mathrm{d}\tau \\
&= -\fint_{\tau_1+\varepsilon/2}^{\tau_1+\varepsilon}\int_\Omega \frac{q}{q+1}\left|[|v|^{q-1}v]_h\right|^{\frac{q+1}{q}}\, \mathrm{d}x\mathrm{d}\tau + \fint_{\tau_2-\varepsilon}^{\tau_2-\varepsilon/2}\int_\Omega \frac{q}{q+1}\left|[|v|^{q-1}v]_h\right|^{\frac{q+1}{q}}\, \mathrm{d}x\mathrm{d}\tau.
\end{aligned}
$$

Since by definition $|v|^{q-1}v \in L^{\frac{q+1}{q}}(\Omega_S)$, Lemma 2.5-(i) yields that

$$
\begin{cases} \left\|[|v|^{q-1}v]_h\right\|_{L^{\frac{q+1}{q}}(\Omega_S)} \leq \|v\|_{L^{q+1}(\Omega_S)}, \\ [|v|^{q-1}v]_h \to_{h\downarrow 0} |v|^{q-1}v \quad \text{in } L^{\frac{q+1}{q}}(\Omega_S). \end{cases} \tag{3.19}
$$

By means of (3.19), the fundamental theorem of calculus and Hölder's inequality, it is

$$
\begin{aligned}
&\iint_{\Omega_S} \left|\left|[|v|^{q-1}v]_h\right|^{\frac{q+1}{q}} - |v|^{q+1}\right| \mathrm{d}x\mathrm{d}\tau \\
\leq\ & \iint_{\Omega_S}\left(\int_0^1 \tfrac{q+1}{q}\left|\vartheta[|v|^{q-1}v]_h + (1-\vartheta)|v|^{q-1}v\right|^{\frac{q+1}{q}-1}\, \mathrm{d}\vartheta\right)\left|[|v|^{q-1}v]_h - |v|^{q-1}v\right|\, \mathrm{d}x\mathrm{d}\tau \\
\leq\ & C(q)\left(\iint_{\Omega_S}\left[\left|[|v|^{q-1}v]_h\right|^{\frac{q+1}{q}} + |v|^{q+1}\right]\mathrm{d}x\mathrm{d}\tau\right)^{\frac{1}{q+1}} \\
&\qquad \cdot\left(\iint_{\Omega_S}\left|[|v|^{q-1}v]_h - |v|^{q-1}v\right|^{\frac{q+1}{q}}\, \mathrm{d}x\mathrm{d}\tau\right)^{\frac{q}{q+1}} \\
\overset{(3.19)}{\to}\ & 0,
\end{aligned}
$$

that is,

$$
\left|[|v|^{q-1}v]_h\right|^{\frac{q+1}{q}} \to |v|^{q+1} \quad \text{in} \quad L^1(\Omega_S)
$$

as $h \downarrow 0$. Combining the preceding estimates above and passing to the limit as $h \downarrow 0$ yield

$$\begin{aligned}
&\liminf_{h\downarrow 0} \iint_{\Omega_S} \partial_\tau[|v|^{q-1}v]_h \chi_\varepsilon v \,\mathrm{d}x\mathrm{d}\tau \\
&= \liminf_{h\downarrow 0} \Big[(\mathrm{I})_{\mathrm{i}} + (\mathrm{I})_{\mathrm{ii}}\Big] \\
&\geq -\fint_{\tau_1+\varepsilon/2}^{\tau_1+\varepsilon} \int_\Omega \tfrac{q}{q+1}\, |v|^{q+1} \,\mathrm{d}x\mathrm{d}\tau + \fint_{\tau_2-\varepsilon}^{\tau_2-\varepsilon/2} \int_\Omega \tfrac{q}{q+1}\, |v|^{q+1} \,\mathrm{d}x\mathrm{d}\tau.
\end{aligned} \tag{3.20}$$

We now focus on the fractional term. Since $\operatorname{supp}(\chi_\varepsilon) \subseteq [\tau_1 + \varepsilon/2, \tau_2 - \varepsilon/2]$, Lemma 2.5-(i) (see the argument as in [36, Appendix A, Step 2] for details) yields that

$$\begin{aligned}
&\lim_{h\downarrow 0} \int_0^S \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{[\Phi_p]_h}{|x-y|^{n+sp}} \chi_\varepsilon(\tau)\,(v(x,\tau) - v(y,\tau)) \,\mathrm{d}x\mathrm{d}y\mathrm{d}\tau \\
&= \int_{\tau_1+\varepsilon/2}^{\tau_2-\varepsilon/2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{\Phi_p}{|x-y|^{n+sp}} \chi_\varepsilon(\tau)\,(v(x,\tau) - v(y,\tau)) \,\mathrm{d}x\mathrm{d}y\mathrm{d}\tau \\
&= \int_{\tau_1+\varepsilon/2}^{\tau_2-\varepsilon/2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|v(x,\tau) - v(y,\tau)|^p}{|x-y|^{n+sp}} \chi_\varepsilon(\tau) \,\mathrm{d}x\mathrm{d}y\mathrm{d}\tau.
\end{aligned} \tag{3.21}$$

We conclude this step by observing the right-hand side of (3.17) as $h \downarrow 0$. Under such a choice of $\varphi = \chi_\varepsilon(\tau)v$, by $\operatorname{supp}(\chi_\varepsilon) \subseteq [\tau_1 + \varepsilon/2, \tau_2 - \varepsilon/2]$ again, we have that

$$\int_0^S e^{-\frac{\vartheta}{h}} \varphi(x,\vartheta)\,\mathrm{d}\vartheta = \int_{\tau_1+\varepsilon/2}^S e^{-\frac{\vartheta}{h}} \chi_\varepsilon(\vartheta) v(x,\vartheta)\,\mathrm{d}\vartheta$$

and therefore, for any $\tau_1 \geq 0$ and fixed $\varepsilon > 0$,

$$\left| \int_\Omega |v|^{q-1} v(0) \frac{1}{h} \int_0^S e^{-\frac{\vartheta}{h}} \varphi(x,\vartheta)\,\mathrm{d}x \right| \leq \frac{e^{-\frac{\tau_1+\varepsilon/2}{h}}}{h} \int_\Omega |v(x,0)|^q \int_{\tau_1+\varepsilon/2}^S |v(x,\vartheta)|\,\mathrm{d}x \to 0 \tag{3.22}$$

in the limit $h \downarrow 0$. Combining this with (3.20) and (3.21) concludes that

$$\begin{aligned}
&-\fint_{\tau_1+\varepsilon/2}^{\tau_1+\varepsilon} \int_\Omega \tfrac{q}{q+1}\, |v|^{q+1} \,\mathrm{d}x\mathrm{d}\tau + \fint_{\tau_2-\varepsilon}^{\tau_2-\varepsilon/2} \int_\Omega \tfrac{q}{q+1}\, |v|^{q+1} \,\mathrm{d}x\mathrm{d}\tau \\
&\quad + \int_{\tau_1+\varepsilon/2}^{\tau_2-\varepsilon/2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|v(x,\tau) - v(y,\tau)|^p}{|x-y|^{n+sp}} \chi_\varepsilon(\tau) \,\mathrm{d}x\mathrm{d}y\mathrm{d}\tau \leq 0.
\end{aligned} \tag{3.23}$$

Step 2: We select the test function $\varphi = \chi_\varepsilon(\tau)v$ in the weak formulation (3.18). A completely same argument as Step 1 leads to

$$\begin{aligned}
&\limsup_{h\downarrow 0} \iint_{\Omega_S} \partial_\tau[|v|^{q-1}v]_{\bar{h}} \chi_\varepsilon v \,\mathrm{d}x\mathrm{d}\tau \\
&\leq -\fint_{\tau_1+\varepsilon/2}^{\tau_1+\varepsilon} \int_\Omega \tfrac{q}{q+1} |v|^{q+1} \,\mathrm{d}x\mathrm{d}\tau + \fint_{\tau_2-\varepsilon}^{\tau_2-\varepsilon/2} \int_\Omega \tfrac{q}{q+1} |v|^{q+1} \,\mathrm{d}x\mathrm{d}\tau,
\end{aligned}$$

where we have used the second identity in Lemma 2.5-(ii) and Lemma 2.3 and

$$\begin{aligned}
&\lim_{h\downarrow 0}\int_0^S\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{[\Phi_p]_{\bar h}}{|x-y|^{n+sp}}\chi_\varepsilon(\tau)\,(v(x,\tau)-v(y,\tau))\,\mathrm{d}x\mathrm{d}y\mathrm{d}\tau\\
&=\int_{\tau_1+\varepsilon/2}^{\tau_2-\varepsilon/2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|v(x,\tau)-v(y,\tau)|^p}{|x-y|^{n+sp}}\chi_\varepsilon(\tau)\,\mathrm{d}x\mathrm{d}y\mathrm{d}\tau.
\end{aligned}$$

A similar observation leading to (3.22) yields

$$\lim_{h\downarrow 0}-\int_\Omega|v|^{q-1}v(S)\left(\frac{1}{h}\int_0^S e^{\frac{\vartheta-S}{h}}\varphi(x,\vartheta)\,\mathrm{d}\vartheta\right)\mathrm{d}x=0.$$

Merging the content of the preceding displays, we obtain

$$\begin{aligned}
&-\fint_{\tau_1+\varepsilon/2}^{\tau_1+\varepsilon}\int_\Omega\tfrac{q}{q+1}\,|u|^{q+1}\,\mathrm{d}x\mathrm{d}\tau+\fint_{\tau_2-\varepsilon}^{\tau_2-\varepsilon/2}\int_\Omega\tfrac{q}{q+1}\,|u|^{q+1}\,\mathrm{d}x\mathrm{d}\tau\\
&\quad+\int_{\tau_1+\varepsilon/2}^{\tau_2-\varepsilon/2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|v(x,\tau)-v(y,\tau)|^p}{|x-y|^{n+sp}}\chi_\varepsilon(\tau)\,\mathrm{d}x\mathrm{d}y\mathrm{d}\tau\geq 0
\end{aligned}\tag{3.24}$$

in the limit $h\downarrow 0$. Hence, the above displays (3.23) and (3.24) eventually yield the identity

$$\begin{aligned}
&-\fint_{\tau_1+\varepsilon/2}^{\tau_1+\varepsilon}\int_\Omega\tfrac{q}{q+1}\,|v|^{q+1}\,\mathrm{d}x\mathrm{d}\tau+\fint_{\tau_2-\varepsilon}^{\tau_2-\varepsilon/2}\int_\Omega\tfrac{q}{q+1}\,|v|^{q+1}\,\mathrm{d}x\mathrm{d}\tau\\
&\quad+\int_{\tau_1+\varepsilon/2}^{\tau_2-\varepsilon/2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|v(x,\tau)-v(y,\tau)|^p}{|x-y|^{n+sp}}\chi_\varepsilon(\tau)\,\mathrm{d}x\mathrm{d}y\mathrm{d}\tau=0.
\end{aligned}$$

Since $v\in C([0,S]\,;L^{q+1}(\Omega))$ by Theorem 3.5, we can apply Lebesgue's differentiation theorem and the dominated convergence theorem to get

$$\begin{aligned}
&-\int_\Omega\tfrac{q}{q+1}\,|v(\tau_1)|^{q+1}\,\mathrm{d}x+\int_\Omega\tfrac{q}{q+1}\,|v(\tau_2)|^{q+1}\,\mathrm{d}x\\
&\quad+\int_{\tau_1}^{\tau_2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|v(x,\tau)-v(y,\tau)|^p}{|x-y|^{n+sp}}\,\mathrm{d}x\mathrm{d}y\mathrm{d}\tau=0
\end{aligned}$$

as $\varepsilon\downarrow 0$, proving the desired identity (3.16). Finally, this together with (3.3) leads to

$$\left|\int_\Omega|v(\tau_2)|^{q+1}\,\mathrm{d}x-\int_\Omega|v(\tau_1)|^{q+1}\,\mathrm{d}x\right|\leq C\left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|v_0(x)-v_0(y)|^p}{|x-y|^{n+sp}}\,\mathrm{d}x\mathrm{d}y\right)(\tau_2-\tau_1)$$

with a constant $C\equiv C(p,q)$, as claimed. □

4. Proof of Theorem 1.1

We report the proof of Theorem 1.1.

Proof. For the reader's convenience we will demonstrate a formal manipulation and reveal the relevance of intrinsic scaling above to the nonlocal p-Sobolev flow (1.1). In a completely similar way of [28, Appendix C], the argument will be guaranteed rigorously.

Fix powers $p > 1$, $q > 0$ and $s \in (0, 1)$ such that $q + 1 = p_s^*$ and $sp < n$ and use shorthand notation

$$\gamma(t) := \left(\int_{\mathbb{R}^n} v^{q+1}(x, \tau(t))\, \mathrm{d}x\right)^{\frac{1}{q+1}}.$$

The proof now goes in three steps.

Step 1: Firstly, by observation that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Lambda(g(t)) &= \Lambda'(g(t))g'(t)\\ &= (S^*)^{-1}e^{\Lambda(g(t))}\left(\int_{\mathbb{R}^n} v^{q+1}(x, \tau(t))\,\mathrm{d}x\right)^{\frac{sp}{n}}\\ &= (S^*)^{-1}e^{\Lambda(g(t))}\gamma(t)^{(q+1)\frac{sp}{n}}\end{aligned}$$

we gain

$$\tau_t \equiv \frac{\mathrm{d}\tau}{\mathrm{d}t} = S^*e^{-\Lambda(g(t))}\frac{\mathrm{d}}{\mathrm{d}t}\Lambda(g(t)) = \gamma(t)^{(q+1)\frac{sp}{n}}. \tag{4.1}$$

Since by Proposition 3.6, $\tau \mapsto \left(\int_{\mathbb{R}^n} v^{q+1}(x,\tau)\,dx\right)^{\frac{1}{q+1}}$ is differentiable almost everywhere, we observe that

$$\begin{aligned}\partial_t u^q &= \partial_\tau v^q \tau_t \gamma^{-q} + v^q(-q)\gamma^{-q-1}\gamma'(t)\\ &= \partial_\tau v^q\gamma^{1-p} - qu^q\gamma^{-1}\gamma'(t),\end{aligned} \tag{4.2}$$

where, in the last line, we have manipulated that $-q + (q + 1)\frac{sp}{n} = 1 - p$. Recalling identity (3.5) and (3.16) in Proposition 3.6, we deduce that

$$\begin{aligned}\gamma'(t) &= \frac{1}{q+1}\left(\int_{\mathbb{R}^n} v^{q+1}(x,\tau(t))\,\mathrm{d}x\right)^{\frac{1}{q+1}-1}\frac{\mathrm{d}}{\mathrm{d}\tau}\int_{\mathbb{R}^n} v^{q+1}(\tau)\,\mathrm{d}x\bigg|_{\tau=\tau(t)}\tau_t\\ &\overset{(3.5)}{=} -\frac{1}{q}\gamma^{1-p}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|v(x,\tau(t)) - v(y,\tau(t))|^p}{|x-y|^{n+sp}}\,\mathrm{d}x\mathrm{d}y\\ &= -\frac{1}{q}\gamma\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|u(x,t)-u(y,t)|^p}{|x-y|^{n+sp}}\,\mathrm{d}x\mathrm{d}y,\end{aligned} \tag{4.3}$$

where in the penultimate line we have used $-q + (q + 1)\frac{sp}{n} = 1 - p$ again, and

$$(-\Delta)_p^s u = \gamma^{1-p}(-\Delta)_p^s v. \tag{4.4}$$

We merge two displays (4.2) and (4.3) to get

$$\partial_t u^q = \partial_s v^q \gamma^{1-p} + \left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|u(x,t)-u(y,t)|^p}{|x-y|^{n+sp}}\,\mathrm{d}x\mathrm{d}y\right)u^q.$$

This together with (4.4) and $(1.6)_1$ yields that

$$\partial_t u^q + (-\Delta)_p^s u = \left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x,t)-u(y,t)|^p}{|x-y|^{n+sp}}\,\mathrm{d}x\mathrm{d}y\right) u^q,$$

therefore $(1.1)_1$ is confirmed.

Step 2: By definition (1.7) in the statement of Theorem 1.1, the composite function u satisfies $\int_\Omega u(t)^{q+1}\,\mathrm{d}x = 1$ for any $t \in [0,\infty)$, that is the volume constraint $(1.1)_2$.

We now show that u is indeed in the demanded class. Let $T < \infty$ be any positive number and set $S = S^*(1-e^{-\Lambda(g(T))})$. Notice that $T \uparrow \infty \iff S \uparrow S^*$. Thanks to Proposition 3.6, there exists a positive number $\gamma_{\min}$ so that

$$\gamma_{\min} := \min_{0\le t\le T} \gamma(t) = \min_{0\le \tau\le S} \|v(\tau)\|_{L^{q+1}(\mathbb{R}^n)} > 0.$$

By using $v = 0$ in $(\mathbb{R}^n \setminus \Omega) \times (0,S)$, $\gamma(t) \ge \gamma_{\min} > 0$, (3.2) and (3.3) with $v_0 = u_0$, and appealing to the Hölder inequality, we get

$$\begin{aligned}\int_{\mathbb{R}^n} |u(t)|^p\,\mathrm{d}x = \frac{1}{\gamma(t)^p}\int_\Omega |v(\tau(t))|^p\,\mathrm{d}x &\le \gamma_{\min}^{-p}|\Omega|^{(p_s^*-p)/p_s^*}\left(\int_\Omega |v(\tau(t))|^{p_s^*}\,\mathrm{d}x\right)^{p/p_s^*}\\ &\overset{(3.2)}{\le} \gamma_{\min}^{-p}|\Omega|^{(p_s^*-p)/p_s^*} < \infty\end{aligned}$$

and

$$\begin{aligned}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x,t)-u(y,t)|^p}{|x-y|^{n+sp}}\,\mathrm{d}x\mathrm{d}y &= \frac{1}{\gamma(t)^p}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|v(x,\tau(t))-v(y,\tau(t))|^p}{|x-y|^{n+sp}}\,\mathrm{d}x\mathrm{d}y\\ &\overset{(3.3)}{\le} \gamma_{\min}^{-p}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u_0(x)-u_0(y)|^p}{|x-y|^{n+sp}}\,\mathrm{d}x\mathrm{d}y < \infty,\end{aligned}$$

therefore $u \in L^\infty(0,T\,;\,W^{s,p}(\mathbb{R}^n))$, as desired.

Step 3: Finally, we show the function u satisfies the initial boundary condition $(1.1)_{3,4}$. Since $v(\tau) \in W_0^{s,p}(\Omega)$ for almost every $\tau \in (0,S)$, $u(t) = v(\tau(t))/\gamma(t) \in W_0^{s,p}(\Omega)$ for almost every $t \in [0,T]$, which confirms $(1.1)_3$. By adding and subtracting $\gamma(0) = \|u_0\|_{L^{q+1}(\Omega)} = 1$, we have

$$\begin{aligned}\|u(t)-u_0\|_{W^{s,p}(\mathbb{R}^n)} &= \left\|\frac{v(\tau(t))}{\gamma(t)} - u_0\right\|_{W^{s,p}(\mathbb{R}^n)}\\ &\le \frac{1}{\gamma(t)}\Big[\|v(\tau(t)) - u_0\|_{W^{s,p}(\mathbb{R}^n)} + \|u_0\|_{W^{s,p}(\mathbb{R}^n)}|\gamma(t)-\gamma(0)|\Big].\end{aligned}$$

Since by the monotone increasing property of $\Lambda(\tau)$ and $g(t)$ with $\Lambda(0) = g(0) = 0$, there holds that $\tau(t) = S^*(1-e^{-\Lambda(g(t))}) \downarrow 0$ if and only if $t \downarrow 0$. Therefore, thanks to the continuity of a map $[0,\infty) \ni w \mapsto w^{q+1}$ and Proposition 3.6, it verifies that

$$|\gamma(t) - \gamma(0)| \to 0$$

in the limit $t \downarrow 0$. Bearing in mind this and the condition $\|v(\tau) - u_0\|_{W^{s,p}(\mathbb{R}^n)} \to 0$ as $\tau \downarrow 0$, we gain

$$\|u(t) - u_0\|_{W^{s,p}(\mathbb{R}^n)} \to 0$$

as $t \downarrow 0$ and therefore, the proof is finally complete. □

5. Proof of Theorem 1.2

This final section is devoted to the proof of Theorem 1.2.

Proof of Theorem 1.2. The global existence result in turn follows from Theorem 3.1 and the nonlinear intrinsic scaling in Theorem 1.1. We now prove the demanded regularity. For every positive number $T < \infty$, let us take $S = S^*(1 - e^{-\Lambda(g(T))})$. By (4.1) and (4.3) we have

$$
\begin{aligned}
\partial_t\Big(|u|^{\frac{q-1}{2}} u\Big) &= \partial_t\Big(\gamma(t)^{-\frac{q+1}{2}} |v|^{\frac{q-1}{2}} v(\tau)\Big) \\
&\overset{(4.1),(4.3)_2}{=} -\frac{q+1}{2}\gamma(t)^{-\frac{q+1}{2}-1}\left(-\frac{1}{q}\gamma(t)^{1-p}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|v(x,\tau)-v(y,\tau)|^p}{|x-y|^{n+sp}}\,\mathrm{d}x\mathrm{d}y\right)|v|^{\frac{q-1}{2}} v \\
&\qquad + \gamma(t)^{-\frac{q+1}{2}+(q+1)\frac{sp}{n}}\partial_\tau\Big(|v|^{\frac{q-1}{2}} v\Big) \\
&= \frac{q+1}{2q}\left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|v(x,\tau)-v(y,\tau)|^p}{|x-y|^{n+sp}}\,\mathrm{d}x\mathrm{d}y\right)\gamma(t)^{-p}|u|^{\frac{q-1}{2}} u \\
&\qquad + \gamma(t)^{-\frac{q+1}{2}+(q+1)\frac{sp}{n}}\partial_\tau\Big(|v|^{\frac{q-1}{2}} v\Big).
\end{aligned}
$$

This observation together with three displays $(1.1)_2$, (4.1), $(4.3)_2$ and estimates (3.3) and (3.4) with $v_0 = u_0$ validates that

$$
\begin{aligned}
&\iint_{\Omega_T}\left|\partial_t\Big(|u|^{\frac{q-1}{2}} u\Big)\right|^2 \mathrm{d}x\mathrm{d}t \\
&\overset{(3.3)}{\leq} C(q)[u_0]^p_{W^{s,p}(\mathbb{R}^n)}\int_0^T \gamma(t)^{-2p}\left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|v(x,\tau(t))-v(y,\tau(t))|^p}{|x-y|^{n+sp}}\,\mathrm{d}x\mathrm{d}y\right) \\
&\qquad\qquad \cdot \underbrace{\left(\int_\Omega |u(t)|^{q+1}\,\mathrm{d}x\right)}_{=1}\mathrm{d}t \\
&\quad + C(q)\iint_{\Omega_S}\gamma(t)^{-(q+1)+2(q+1)\frac{sp}{n}-(q+1)\frac{sp}{n}}\left|\partial_\tau\Big(|v|^{\frac{q-1}{2}} v\Big)\right|^2 \mathrm{d}x\mathrm{d}\tau \\
&\overset{(4.3)_2,(1.1)_2}{=} C(q)[u_0]^p_{W^{s,p}(\mathbb{R}^n)}\int_0^T \gamma(t)^{-2p}\big(-q\gamma'(t)\gamma(t)^{p-1}\big)\,\mathrm{d}t \\
&\quad + C(q)\iint_{\Omega_S}\gamma(t)^{-p}\left|\partial_\tau\Big(|v|^{\frac{q-1}{2}} v\Big)\right|^2 \mathrm{d}x\mathrm{d}\tau \\
&\overset{(3.4)}{\leq} C(q)[u_0]^p_{W^{s,p}(\mathbb{R}^n)}\left[\frac{q}{p}\gamma(t)^{-p}\right]_{t=0}^T + C(q)\gamma_{\min}^{-p}[u_0]^p_{W^{s,p}(\mathbb{R}^n)} \\
&\leq C(q)[u_0]^p_{W^{s,p}(\mathbb{R}^n)}\gamma_{\min}^{-p},
\end{aligned}
$$

therefore the assertion is actually verified. □

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

All authors declare that they have no conflicts of interest.

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