Research article

Uniqueness of entire solutions to quasilinear equations of p-Laplace type[†]

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Abstract: We prove the uniqueness property for a class of entire solutions to the equation

$$\begin{cases} -\operatorname{div} \mathcal{A}(x, \nabla u) = \sigma, \quad u \geq 0 \quad \text{in } \mathbb{R}^n, \\ \liminf_{|x| \to \infty} u = 0, \end{cases}$$

where σ is a nonnegative locally finite measure in $\mathbb{R}^n$, absolutely continuous with respect to the p-capacity, and $\operatorname{div} \mathcal{A}(x, \nabla u)$ is the $\mathcal{A}$-Laplace operator, under standard growth and monotonicity assumptions of order p ($1 < p < \infty$) on $\mathcal{A}(x, \xi)$ ($x, \xi \in \mathbb{R}^n$); the model case $\mathcal{A}(x, \xi) = \xi |\xi|^{p-2}$ corresponds to the p-Laplace operator Δ_p on $\mathbb{R}^n$. Our main results establish uniqueness of solutions to a similar problem,

$$\begin{cases} -\operatorname{div} \mathcal{A}(x, \nabla u) = \sigma u^q + \mu, \quad u \geq 0 \quad \text{in } \mathbb{R}^n, \\ \liminf_{|x| \to \infty} u = 0, \end{cases}$$

in the sub-natural growth case $0 < q < p - 1$, where μ, σ are nonnegative locally finite measures in $\mathbb{R}^n$, absolutely continuous with respect to the p-capacity, and $\mathcal{A}(x, \xi)$ satisfies an additional homogeneity condition, which holds in particular for the p-Laplace operator.

1. Introduction

We prove the uniqueness property for a class of *reachable* solutions to the equation

$$\begin{cases} -\Delta_p u = \sigma, \quad u \geq 0 \quad \text{in } \mathbb{R}^n, \\ \liminf_{|x|\to\infty} u = 0, \end{cases} \tag{1.1}$$

where $\sigma \geq 0$ is a *locally finite* Borel measure in $\mathbb{R}^n$ absolutely continuous with respect to the p-capacity, and $\Delta_p u = \text{div}(\nabla u |\nabla u|^{p-2})$ $(1 < p < \infty)$ is the p-Laplace operator.

More general $\mathcal{A}$-Laplace operators $\text{div}\,\mathcal{A}(x, \nabla u)$ in place of Δ_p, under standard growth and monotonicity assumptions of order p on $\mathcal{A}(x, \xi)$ $(x, \xi \in \mathbb{R}^n)$, are treated as well (see Section 2). All solutions u of (1.1) are understood to be $\mathcal{A}$-superharmonic (or, equivalently, locally renormalized) solutions in $\mathbb{R}^n$ (see [19] and Section 3 below).

We often use bilateral global pointwise estimates of solutions to (1.1) obtained by Kilpeläinen and Malý [20, 21] in terms of the Havin–Maz'ya–Wolff potentials (often called Wolff potentials) $\mathbf{W}_{1,p}\sigma$. Criteria of existence of solutions to (1.1), which ensure that $\mathbf{W}_{1,p}\sigma \not\equiv \infty$, can be found in [33] (see also Section 3 below).

We remark that existence and uniqueness results are known for certain classes of solutions to quasilinear equations with $\mathcal{A}$-Laplace operators similar to (1.1) in arbitrary domains $\Omega \subseteq \mathbb{R}^n$ (not necessarily bounded), but with various additional restrictions on $\mathcal{A}(x, \xi)$ and data σ. We refer to [2] for $\sigma \in L^1(\Omega)$, and [25] for measures σ with *finite* total variation in Ω. Notice that in general only local analogues of the Kilpeläinen and Malý pointwise estimates are known for solutions in domains $\Omega \subsetneq \mathbb{R}^n$. For our purposes, we need global pointwise estimates, which at the moment are available only for $\Omega = \mathbb{R}^n$, or in the case $p = 2$ for *linear* operators $\text{div}\,(\mathcal{A}(x)\nabla u)$ in terms of positive Green's functions in domains $\Omega \subsetneq \mathbb{R}^n$.

In Section 4, we prove uniqueness of nontrivial *reachable* solutions to the problem

$$\begin{cases} -\Delta_p u = \sigma u^q + \mu, \quad u \geq 0 \quad \text{in } \mathbb{R}^n, \\ \liminf_{|x|\to\infty} u = 0, \end{cases} \tag{1.2}$$

in the sub-natural growth case $0 < q < p - 1$, where μ, σ are nonnegative locally finite measures in $\mathbb{R}^n$ absolutely continuous with respect to the p-capacity. We observe that such a uniqueness property generally fails in the case $q \geq p - 1$.

When we treat the uniqueness problem for solutions of equations of type (1.2), but with more general $\mathcal{A}$-Laplace operators $\text{div}\,\mathcal{A}(x, \nabla \cdot)$ in place of Δ_p, we impose the additional homogeneity condition $\mathcal{A}(x, \lambda \xi) = \lambda^{p-1}\mathcal{A}(x, \xi)$, for all $\xi \in \mathbb{R}^n$ and $\lambda > 0$ (see Section 4). We emphasize that our proof of uniqueness for reachable solutions of such equations relies upon the homogeneity of order $p - 1$ of the $\mathcal{A}$-Laplacian, as well as homogeneity of order q of the term σu^q, for $0 < q < p - 1$. Our main tool in this proof is provided by bilateral pointwise estimates for all entire solutions obtained recently in [37], which do not require the homogeneity of the $\mathcal{A}$-Laplacian.

We observe that in the case $p = 2$ *all* superharmonic solutions of Eqs (1.1) and (1.2) are reachable, and hence unique. An analogue of this fact is true for more general equations with the linear uniformly elliptic $\mathcal{A}$-Laplace operator $\text{div}\,(\mathcal{A}(x)\nabla u)$, with bounded measurable coefficients $\mathcal{A} \in L^\infty(\mathbf{R}^n)^{n\times n}$, in place of Δ. In other words, all entire $\mathcal{A}$-superharmonic solutions to such equations are unique. For

similar problems in domains $\Omega \subseteq \mathbb{R}^n$ and linear operators with positive Green's function satisfying some additional properties (in particular, in uniform domains) the uniqueness property was obtained recently in [38].

The uniqueness of nontrivial *bounded* (superharmonic) solutions for (1.2) in the case $p = 2$ was proved earlier by Brezis and Kamin [7]. For solutions $u \in C(\overline{\Omega})$ in bounded smooth domains $\Omega \subset \mathbb{R}^n$ and $\mu, \sigma \in C(\overline{\Omega})$, along with some more general equations involving monotone increasing, concave nonlinearities on the right-hand side, the uniqueness property was originally established by Krasnoselskii [23, Theorem 7.14].

As shown below, for $p \neq 2$, *all* p-superharmonic solutions u to (1.1) or (1.2) are reachable, and hence unique, if, for instance, the condition $\liminf_{|x|\to\infty} u = 0$ in (1.1) or (1.2), respectively, is replaced with $\lim_{|x|\to\infty} u = 0$. See Sections 3 and 4, where we discuss this and other conditions that ensure that all solutions are reachable.

Existence criteria and bilateral pointwise estimates for all $\mathcal{A}$-superharmonic solutions to (1.2) were obtained in [37]. (See also earlier results in [9] involving *minimal* solutions in the case $\mu = 0$.) In particular, it is known that the measure σ is necessarily absolutely continuous with respect to the p-capacity provided there exists a nontrivial $u \geq 0$ such that $-\Delta_p u \geq \sigma u^q$ ([9], Lemma 3.6).

We remark that the proofs of the main existence results in [9, 37] for (1.2) in the case $\mu = 0$ used a version of the comparison principle ([9], Lemma 5.2) that contained some inaccuracies. A corrected form of this comparison principle is provided in Lemma 4.1 below. The other parts of [9, 37] are unaffected by this correction.

With regards to the existence problem, we prove additionally that we can always construct a reachable solution to either (1.1) or (1.2), whenever a solution to the corresponding equation exists (see Theorem 3.10 and Remark 4.7 below).

2. $\mathcal{A}$-superharmonic functions

Let $\Omega \subseteq \mathbb{R}^n$, $n \geq 2$, be an open set. By $\mathcal{M}^+(\Omega)$ we denote the cone of nonnegative locally finite Borel measures in Ω, and by $\mathcal{M}_b^+(\Omega)$ the subcone of finite measures in $\mathcal{M}^+(\Omega)$. For $\mu \in \mathcal{M}^+(\Omega)$, we set $\|\mu\|_{\mathcal{M}^+(\Omega)} = \mu(\Omega)$ even if $\mu(\Omega) = +\infty$. The space of finite signed Borel measures in Ω is denoted by $\mathcal{M}_b(\Omega)$. By $\|\mu\|_{\mathcal{M}_b(\Omega)}$ we denote the total variation of $\mu \in \mathcal{M}_b(\Omega)$.

Let $\mathcal{A}: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ be a Carathéodory function in the sense that the map $x \to \mathcal{A}(x, \xi)$ is measurable for all $\xi \in \mathbb{R}^n$, and the map $\xi \to \mathcal{A}(x, \xi)$ is continuous for a.e. $x \in \mathbb{R}^n$. Throughout the paper, we assume that there are constants $0 < \alpha \leq \beta < \infty$ and $1 < p < n$ such that for a.e. x in $\mathbb{R}^n$,

$$
\begin{aligned}
\mathcal{A}(x, \xi) \cdot \xi \geq \alpha|\xi|^p, \quad |\mathcal{A}(x, \xi)| \leq \beta|\xi|^{p-1}, \quad \forall \xi \in \mathbb{R}^n, \\
[\mathcal{A}(x, \xi_1) - \mathcal{A}(x, \xi_2)] \cdot (\xi_1 - \xi_2) > 0, \quad \forall \xi_1, \xi_2 \in \mathbb{R}^n, \ \xi_1 \neq \xi_2.
\end{aligned}
\tag{2.1}
$$

In the uniqueness results of Sec. 4, we assume additionally the homogeneity condition

$$
\mathcal{A}(x, \lambda\xi) = \lambda^{p-1}\mathcal{A}(x, \xi), \qquad \forall \xi \in \mathbb{R}^n, \lambda > 0.
\tag{2.2}
$$

Such homogeneity conditions are often used in the literature (see [18, 21]).

For an open set $\Omega \subset \mathbb{R}^n$, it is well known that every weak solution $u \in W_{\text{loc}}^{1,p}(\Omega)$ to the equation

$$
-\text{div}\mathcal{A}(x, \nabla u) = 0 \qquad \text{in } \Omega
\tag{2.3}
$$

has a continuous representative. Such continuous solutions are said to be $\mathcal{A}$-*harmonic* in Ω. If $u \in W^{1,p}_{\text{loc}}(\Omega)$ and

$$\int_\Omega \mathcal{A}(x, \nabla u) \cdot \nabla \varphi \, dx \geq 0,$$

for all nonnegative $\varphi \in C^\infty_0(\Omega)$, i.e., $-\text{div}\mathcal{A}(x, \nabla u) \geq 0$ in the distributional sense, then u is called a *supersolution* to (2.3) in Ω.

A function $u\colon \Omega \to (-\infty, \infty]$ is called $\mathcal{A}$-*superharmonic* if u is not identically infinite in each connected component of Ω, u is lower semicontinuous, and for all open sets D such that $\overline{D} \subset \Omega$, and all functions $h \in C(\overline{D})$, $\mathcal{A}$-harmonic in D, it follows that $h \leq u$ on ∂D implies $h \leq u$ in D.

A typical example of $\mathcal{A}(x, \xi)$ is given by $\mathcal{A}(x, \xi) = |\xi|^{p-2}\xi$, which gives rise to the p-Laplacian $\Delta_p u = \text{div}\,(|\nabla u|^{p-2}\nabla u)$. In this case, $\mathcal{A}$-superharmonic functions will be called p-superharmonic functions.

We recall here the fundamental connection between supersolutions of (2.3) and $\mathcal{A}$-superharmonic functions discussed in [18].

Proposition 2.1 ([18]). (i) *If $u \in W^{1,p}_{\text{loc}}(\Omega)$ is such that*

$$-\text{div}\mathcal{A}(x, \nabla u) \geq 0 \qquad \text{in } \Omega,$$

then there is an $\mathcal{A}$-superharmonic function v such that $u = v$ a.e. Moreover,

$$v(x) = \text{ess} \liminf_{y \to x} v(y), \qquad x \in \Omega. \tag{2.4}$$

(ii) *If v is $\mathcal{A}$-superharmonic, then (2.4) holds. Moreover, if $v \in W^{1,p}_{\text{loc}}(\Omega)$, then*

$$-\text{div}\mathcal{A}(x, \nabla v) \geq 0 \qquad \text{in } \Omega.$$

(iii) *If v is $\mathcal{A}$-superharmonic and locally bounded, then $v \in W^{1,p}_{\text{loc}}(\Omega)$, and*

$$-\text{div}\mathcal{A}(x, \nabla v) \geq 0 \qquad \text{in } \Omega.$$

Note that if u is $\mathcal{A}$-superharmonic, then the gradient of u may not exist in the sense of distributions in the case $1 < p \leq 2 - 1/n$. On the other hand, if u is an $\mathcal{A}$-superharmonic function, then its truncation $u_k = \min\{u, k\}$ is $\mathcal{A}$-superharmonic as well, for any $k > 0$. Moreover, by Proposition 2.1(iii) we have $u_k \in W^{1,p}_{\text{loc}}(\Omega)$. Using this we define the very weak gradient

$$Du := \lim_{k \to \infty} \nabla[\min\{u, k\}] \qquad \text{a.e. in } \Omega.$$

If either $u \in L^\infty(\Omega)$ or $u \in W^{1,1}_{\text{loc}}(\Omega)$, then Du coincides with the regular distributional gradient of u. In general we have the following gradient estimates [20] (see also [18]).

Proposition 2.2 ([20]). *Suppose u is $\mathcal{A}$-superharmonic in Ω and $1 \leq q < \frac{n}{n-1}$. Then both $|Du|^{p-1}$ and $\mathcal{A}(\cdot, Du)$ belong to $L^q_{\text{loc}}(\Omega)$. Moreover, if $p > 2 - \frac{1}{n}$, then Du coincides with the distributional gradient of u.*

Note that by Proposition 2.2 and the dominated convergence theorem, we have

$$-\text{div}\mathcal{A}(x, \nabla u)(\varphi) := \int_\Omega \mathcal{A}(x, Du) \cdot \nabla\varphi \, dx$$

$$= \lim_{k\to\infty} \int_\Omega \mathcal{A}(x, \nabla \min\{u, k\}) \cdot \nabla\varphi \, dx \geq 0,$$

whenever $\varphi \in C_0^\infty(\Omega)$, $\varphi \geq 0$, and u is $\mathcal{A}$-superharmonic in Ω. It follows from Riesz's representation theorem (see [18, Theorem 21.2]) that there exists a unique measure $\mu[u] \in \mathcal{M}^+(\Omega)$ called the Riesz measure of u such that

$$-\text{div}\mathcal{A}(x, \nabla u) = \mu[u] \qquad \text{in } \Omega.$$

3. Quasilinear equations with locally finite measure data in the entire space

In this section, we investigate the problems of existence and uniqueness of $\mathcal{A}$-superharmonic solutions in the entire space $\mathbb{R}^n$ to the equation

$$\begin{cases} -\text{div }\mathcal{A}(x, \nabla u) = \sigma, & u \geq 0 \quad \text{in } \mathbb{R}^n, \\ \liminf_{|x|\to\infty} u = 0, \end{cases} \tag{3.1}$$

with measures $\sigma \in \mathcal{M}^+(\mathbb{R}^n)$ (not necessarily finite).

There has been a lot of work addressing the existence and uniqueness problem for quasilinear equations of the form

$$\begin{cases} -\text{div }\mathcal{A}(x, \nabla u) = \sigma & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.2}$$

in a *bounded* domain $\Omega \subset \mathbb{R}^n$, where $\sigma \in L^1(\Omega)$, or, more generally, $\sigma \in \mathcal{M}_b(\Omega)$; see, e.g., [2, 4–6, 10–12, 22, 25]. For arbitrary domains, including $\mathbb{R}^n$, we refer to the papers [2] (for L^1 data) and [25] (for data in $\mathcal{M}_b(\Omega)$). In these papers one can find the notions of *entropy solutions* (see [2, 6, 22]), *SOLA* (solutions obtained as limits of approximations) for L^1 data (see [11]), *reachable solutions* (see [10]), and *renormalized solutions* (see [12]).

The current state of the art on the uniqueness problem for (3.2) is that most results require that $\sigma << \text{cap}_p$, i.e., σ is absolutely continuous with respect to the p-capacity in the sense that $\sigma(K) = 0$ for any compact set $K \subset \Omega$ such that $\text{cap}_p(K) = 0$. The p-capacity $\text{cap}_p(\cdot)$ is a natural capacity associated with the p-Laplacian defined by

$$\text{cap}_p(K) := \inf\left\{ \int_\Omega |\nabla h|^p dx \colon \ h \in C_0^\infty(\Omega), \ h \geq 1 \text{ on } K \right\},$$

for any compact set $K \subset \Omega$.

For later use, we now recall the following equivalent definitions of a (global) renormalized solution to Eq (3.2) (see [12]). For our purposes, we shall restrict ourselves to the case $\sigma \in \mathcal{M}_b^+(\Omega)$ and nonnegative solutions. Recall that we may use the decomposition $\sigma = \sigma_0 + \sigma_s$, where both σ_0 and σ_s are nonnegative measures such that $\sigma_0 << \text{cap}_0$, and σ_s is concentrated on a set of zero p-capacity.

Definition 3.1. *Let $\sigma \in \mathcal{M}_b^+(\Omega)$, where $\Omega \subset \mathbb{R}^n$ is a bounded open set. Then $u \geq 0$ is said to be a renormalized solution of* (3.2) *if the following conditions hold:*
(a) *The function u is measurable and finite almost everywhere, and $T_k(u)$ belongs to $W_0^{1,p}(\Omega)$ for every $k > 0$, where $T_k(s) := \min\{k, s\}$, $s \geq 0$.*
(b) *The gradient Du of u satisfies $|Du|^{p-1} \in L^q(\Omega)$ for all $q < \frac{n}{n-1}$.*
(c) *For any $h \in W^{1,\infty}(\mathbb{R})$ with compact support, and any $\varphi \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ such that $h(u)\varphi \in W_0^{1,p}(\Omega)$,*

$$\int_\Omega \mathcal{A}(x, Du) \cdot \nabla(h(u)\varphi)\, dx = \int_\Omega h(u)\varphi\, d\sigma_0,$$

and for any $\varphi \in C_b^0(\Omega)$ (the space of bounded and continuous functions in Ω),

$$\lim_{m \to \infty} \frac{1}{m} \int_{\{m \leq u \leq 2m\}} \mathcal{A}(x, Du) \cdot Du\, \varphi\, dx = \int_\Omega \varphi\, d\sigma_s.$$

Definition 3.2. *Let $\sigma \in \mathcal{M}_b^+(\Omega)$, where $\Omega \subset \mathbb{R}^n$ is a bounded open set. Then $u \geq 0$ is said to be a renormalized solution of* (3.2) *if u satisfies* (a) *and* (b) *in Definition 3.1, and if the following condition holds:*
(c) *For any $h \in W^{1,\infty}(\mathbb{R})$ with such that h' has compact support, and any $\varphi \in W^{1,r}(\Omega) \cap L^\infty(\Omega)$, $r > n$, such that $h(u)\varphi \in W_0^{1,p}(\Omega)$,*

$$\int_\Omega \mathcal{A}(x, Du) \cdot \nabla(h(u)\varphi)\, dx = \int_\Omega h(u)\varphi\, d\sigma_0 + h(+\infty) \int_\Omega \varphi\, d\sigma_s.$$

Here $h(+\infty) := \lim_{s \to +\infty} h(s)$.

Definition 3.3. *Let $\sigma \in \mathcal{M}_b^+(\Omega)$, where $\Omega \subset \mathbb{R}^n$ is a bounded open set. Then $u \geq 0$ is said to be a renormalized solution of* (3.2) *if u satisfies* (a) *and* (b) *in Definition 3.1, and if the following conditions hold:*
(c) *For every $k > 0$, there exists $\lambda_k \in \mathcal{M}_b^+(\Omega)$ concentrated on the set $\{u = k\}$ such that $\lambda_k \ll \mathrm{cap}_p$, and $\lambda_k \to \sigma_s$ in the narrow topology of measures in Ω as $k \to \infty$, i.e.,*

$$\lim_{k \to \infty} \int_\Omega \varphi\, d\lambda_k = \int_\Omega \varphi\, d\sigma_s, \quad \forall \varphi \in C_b^0(\Omega).$$

(d) *For every $k > 0$,*

$$\int_{\{u < k\}} \mathcal{A}(x, Du) \cdot \nabla\varphi\, dx = \int_{\{u < k\}} \varphi\, d\sigma_0 + \int_\Omega \varphi\, d\lambda_k$$

for all $\varphi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$.

We shall also need the notion of a *local* renormalized (nonnegative) solution on a general open set $\Omega \subseteq \mathbb{R}^n$ (not necessarily bounded) associated with a measure $\sigma \in \mathcal{M}^+(\Omega)$ (not necessarily finite). We recall the following equivalent definitions (see [3]), adapted to the case of nonnegative solutions.

Definition 3.4. *Let $\sigma \in \mathcal{M}^+(\Omega)$, where $\Omega \subseteq \mathbb{R}^n$ is an open set. Then a nonnegative function u is said to be a local renormalized solution of the equation $-\mathrm{div}\mathcal{A}(x, \nabla u) = \sigma$, if the following conditions hold:*
(a) *The function u is measurable and finite almost everywhere, and $T_k(u)$ belongs to $W_{\mathrm{loc}}^{1,p}(\Omega)$, for every $k > 0$, where $T_k(s) := \min\{k, s\}$, $s \geq 0$.*

(b) *The gradient Du of u satisfies $|Du|^{p-1} \in L^q_{\text{loc}}(\Omega)$ for all $0 < q < \frac{n}{n-1}$, and $u^{p-1} \in L^s_{\text{loc}}(\Omega)$ for all $0 < s < \frac{n}{n-p}$.*

(c) *For any $h \in W^{1,\infty}(\mathbb{R})$ with compact support, and any $\varphi \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ with compact support in Ω such that $h(u)\varphi \in W^{1,p}(\Omega)$,*

$$\int_\Omega \mathcal{A}(x, Du) \cdot \nabla(h(u)\varphi)\, dx = \int_\Omega h(u)\varphi\, d\sigma_0,$$

and for any $\varphi \in C^0_b(\Omega)$ with compact support in Ω,

$$\lim_{m \to \infty} \frac{1}{m} \int_{\{m \le u \le 2m\}} \mathcal{A}(x, Du) \cdot Du\, \varphi\, dx = \int_\Omega \varphi\, d\sigma_s.$$

Definition 3.5. *Let $\sigma \in \mathcal{M}^+(\Omega)$, where $\Omega \subseteq \mathbb{R}^n$ is an open set. Then a nonnegative function u is said to be a local renormalized solution of the equation $-\operatorname{div}\mathcal{A}(x, \nabla u) = \sigma$, if u satisfies* (a) *and* (b) *in Definition 3.4, and if the following conditions hold:*

(c) *For every $k > 0$, there exists a nonnegative measure $\lambda_k \ll \operatorname{cap}_p$, concentrated on the sets $\{u = k\}$, such that $\lambda_k \to \sigma_s$ weakly as measures in Ω as $k \to \infty$, i.e.,*

$$\lim_{k \to \infty} \int_\Omega \varphi\, d\lambda_k = \int_\Omega \varphi\, d\sigma_s,$$

for all $\varphi \in C^0_b(\Omega)$ with compact support in Ω.

(d) *For every $k > 0$,*

$$\int_{\{u < k\}} \mathcal{A}(x, Du) \cdot \nabla\varphi\, dx = \int_{\{u < k\}} \varphi\, d\sigma_0 + \int_\Omega \varphi\, d\lambda_k$$

for all φ in $W^{1,p}_0(\Omega) \cap L^\infty(\Omega)$ with compact support in Ω.

We now discuss solutions of (3.1) for general measures $\sigma \in \mathcal{M}^+(\mathbb{R}^n)$. It is known that a necessary and sufficient condition for (3.1) to admit an $\mathcal{A}$-superharmonic solution is the finiteness condition

$$\int_1^\infty \left(\frac{\sigma(B(0,\rho))}{\rho^{n-p}} \right)^{\frac{1}{p-1}} \frac{d\rho}{\rho} < +\infty; \tag{3.3}$$

(see, e.g., [33, 34]). Thus, it is possible to solve (3.1) for a wide and optimal class of measures σ satisfying (3.3) that are not necessarily finite.

We mention that (3.3) is equivalent to the condition $\mathbf{W}_{1,p}\sigma(x) < +\infty$ for some $x \in \mathbb{R}^n$ (or equivalently quasi-everywhere in $\mathbb{R}^n$ with respect to the p-capacity), where

$$\mathbf{W}_{1,p}\sigma(x) := \int_0^\infty \left(\frac{\sigma(B(x,\rho))}{\rho^{n-p}} \right)^{\frac{1}{p-1}} \frac{d\rho}{\rho}$$

is the Havin–Maz'ya–Wolff potential of σ (often called the Wolff potential); see [17, 26].

By the fundamental result of Kilpeläinen and Malý [20, 21], any $\mathcal{A}$-superharmonic solution u to Eq (3.1) satisfies the following global pointwise estimates,

$$\frac{1}{K}\mathbf{W}_{1,p}\sigma(x) \le u(x) \le K\mathbf{W}_{1,p}\sigma(x), \qquad \forall x \in \mathbb{R}^n, \tag{3.4}$$

where $K > 0$ is a constant depending only on n, p and the structural constants α and β in (2.1).

Our main goal here is to introduce a new notion of a solution to (3.1) so that existence is obtained under the natural growth condition (3.3) for σ, and uniqueness is guaranteed as long as $\sigma \ll \mathrm{cap}_p$ (see Definition 3.8 below).

We begin with the following result on the existence of a minimal solution to (3.1) in case the measure σ is continuous with respect to the p-capacity.

Theorem 3.6. *Let $\sigma \in \mathcal{M}^+(\mathbb{R}^n)$, where $\sigma \ll \mathrm{cap}_p$. Suppose that (3.3) holds. Then there exists a minimal $\mathcal{A}$-superharmonic solution to Eq (3.1).*

Proof. Condition (3.3) implies that

$$\int_1^\infty \left(\frac{\sigma(B(x,\rho))}{\rho^{n-p}} \right)^{\frac{1}{p-1}} \frac{d\rho}{\rho} < +\infty$$

for all $x \in \mathbb{R}^n$. Thus,

$$\{\mathbf{W}_{1,p}\sigma = \infty\} = \left\{ x \in \mathbb{R}^n : \mathbf{W}_{1,p}^1 \sigma := \int_0^1 \left(\frac{\sigma(B(x,\rho))}{\rho^{n-p}} \right)^{\frac{1}{p-1}} \frac{d\rho}{\rho} = \infty \right\}.$$

This yields

$$\mathrm{cap}_p(\{\mathbf{W}_{1,p}\sigma = \infty\}) = \lim_{j\to\infty} \mathrm{cap}_p(\{\mathbf{W}_{1,p}\sigma = \infty\} \cap B_j(0))$$

$$= \lim_{j\to\infty} \mathrm{cap}_p(\{x \in B_j(0) : \mathbf{W}_{1,p}^1(\sigma|_{B_{j+1}(0)}) = \infty\})$$

$$= 0.$$

Here we used the fact that, if $\mu \in \mathcal{M}_b^+(\mathbb{R}^n)$, then $\mathrm{cap}_p(\{\mathbf{W}_{1,p}\mu = \infty\}) = 0$ (see [1, Proposition 6.3.12]). It follows that $\sigma(\{\mathbf{W}_{1,p}\sigma = \infty\}) = 0$, since $\sigma \ll \mathrm{cap}_p$.

Let σ_k ($k = 1, 2, \dots$) be the restriction of σ to the set $B_k(0) \cap \{\mathbf{W}_{1,p}\sigma < k\}$. We then have that σ_k weakly converges to σ, and

$$\int_{\mathbb{R}^n} \mathbf{W}_{1,p}\sigma_k \, d\sigma_k \le k\sigma(B_k(0)) < +\infty.$$

Hence, $\sigma_k \in W^{-1,p'}(B_k(0))$ ($1/p + 1/p' = 1$), and for each $k > 0$, there exists a unique nonnegative solution $u_k \in W_0^{1,p}(B_k(0))$ to the problem

$$\begin{cases} -\mathrm{div}\,\mathcal{A}(x, \nabla u_k) &= \sigma_k &\text{in } B_k(0), \\ u_k &= 0 &\text{on } \partial B_k(0). \end{cases} \tag{3.5}$$

If we set $u_k = 0$ in $\mathbb{R}^n \setminus B_k(0)$, then the sequence $\{u_k\}$ is non-decreasing, and by [33, Theorem 2.1],

$$u_k \le K\,\mathbf{W}_{1,p}\sigma < \infty \quad d\sigma - \text{a.e.}$$

By [20, Theorem 1.17], it follows that the function $u := \lim_{k\to\infty} u_k$ is $\mathcal{A}$-superharmonic in $\mathbb{R}^n$. Moreover, $u \le K\,\mathbf{W}_{1,p}\sigma$, and consequently

$$\liminf_{|x|\to\infty} u(x) \le K \liminf_{|x|\to\infty} \mathbf{W}_{1,p}\sigma(x) = 0.$$

Thus, u is an $\mathcal{A}$-superharmonic solution of (3.1).

To show the minimality of u, let v be another $\mathcal{A}$-superharmonic solution of (3.1). From the construction of u, it is enough to show that $u_k \leq v$ for any $k \geq 1$. To this end, let v_j, $j = 1, 2, \ldots$, be the Riesz measure of $\min\{v, j\}$. Since v is $\mathcal{A}$-superharmonic, it is also a local renormalized solution to $-\operatorname{div} \mathcal{A}(x, \nabla v) = \sigma$ in $\mathbb{R}^n$ (see [19]). Hence, by a result of [3, 12] and the fact that $\sigma << \operatorname{cap}_p$, we obtain

$$v_j = \sigma|_{\{v < j\}} + \alpha_j$$

for $\alpha_j \in \mathcal{M}^+(\mathbb{R}^n)$ concentrated in the set $\{v = j\}$.

Using the estimate $v \leq K\mathbf{W}_{1,p}\sigma$, we deduce

$$v_j \geq \sigma|_{\{v < j\}} \geq \sigma|_{\{K\mathbf{W}_{1,p}\sigma < j\}} \geq \sigma|_{\{\mathbf{W}_{1,p}\sigma < k\}} \geq \sigma_k,$$

provided $j/K > k$. Since $u_k \in W_0^{1,p}(B_k(0))$ and $\min\{v, j\} \in W^{1,p}(B_k(0))$, by the comparison principle (see [9, Lemma 5.1]), we estimate

$$u_k \leq \min\{v, j\} \leq v,$$

provided $j \geq Kk$. Thus, $u = \lim_{k \to \infty} u_k \leq v$. This completes the proof of the theorem. □

The proof of the minimality of u above can be modified to obtain the following comparison principle.

Theorem 3.7 (Comparison Principle). *Let* $\sigma, \tilde{\sigma} \in \mathcal{M}^+(\mathbb{R}^n)$, *where* $\sigma \leq \tilde{\sigma}$ *and* $\sigma << \operatorname{cap}_p$, $1 < p < n$. *Then* $u \leq \tilde{u}$, *where* u *is the minimal* $\mathcal{A}$-*superharmonic solution of* (3.1) *and* $\tilde{u}$ *is any* $\mathcal{A}$-*superharmonic solution of* (3.1) *with datum* $\tilde{\sigma}$ *in place of* σ.

Proof. Let σ_k', $k = 1, 2, \ldots$, be the restriction of σ to the set $B_k(0) \cap \{\mathbf{W}_{1,p}\tilde{\sigma} < k\}$. Since $\sigma << \operatorname{cap}_p$ we have that σ_k' weakly converges to σ. Moreover, as $\mathbf{W}_{1,p}\sigma_k' \leq \mathbf{W}_{1,p}\sigma \leq \mathbf{W}_{1,p}\tilde{\sigma} < k$ on the set $\{\mathbf{W}_{1,p}\tilde{\sigma} < k\}$, it follows that

$$\int_{\mathbb{R}^n} \mathbf{W}_{1,p}\sigma_k' \, d\sigma_k' \leq k\,\sigma(B_k(0)) < +\infty.$$

Hence, $\sigma_k' \in W^{-1,\frac{p}{p-1}}(B_k(0))$, and for each $k > 0$ there exists a unique nonnegative solution $u_k' \in W_0^{1,p}(B_k(0))$ to the problem

$$\begin{cases} -\operatorname{div} \mathcal{A}(x, \nabla u_k') &= \sigma_k' &\text{in } B_k(0), \\ u_k' &= 0 &\text{on } \partial B_k(0). \end{cases}$$

Letting $u_k' = 0$ in $\mathbb{R}^n \setminus B_k(0)$, we have that the sequence $\{u_k'\}$ is non-decreasing, and by [33, Theorem 2.1],

$$u_k' \leq K\mathbf{W}_{1,p}\sigma.$$

Then u_k' converges pointwise to an $\mathcal{A}$-superharmonic solution u' of (3.1) by [20, Theorem 1.17]. On the other hand, by the comparison principle of [9, Lemma 5.1], we have

$$u_k' \leq u_k, \qquad \forall k \geq 1,$$

where u_k is defined in (3.5). Hence, letting $k \to \infty$, we get $u' \leq u$, which yields $u' = u$ by the minimality of u.

We now let $\tilde{\sigma}_j$ ($j = 1, 2, \dots$) be the Riesz measure of $\min\{\tilde{u}, j\}$. Recall that the Riesz measure $\tilde{\sigma}$ of $\tilde{u}$ can be decomposed as

$$\tilde{\sigma} = \tilde{\sigma}_0 + \tilde{\sigma}_s,$$

where $\tilde{\sigma}_0 \in \mathcal{M}^+(\mathbb{R}^n)$, $\tilde{\sigma}_0 << \mathrm{cap}_p$, and $\tilde{\sigma}_s \in \mathcal{M}^+(\mathbb{R}^n)$ is concentrated on a set of zero p-capacity. Then by a result of [3, 12], we have

$$\tilde{\sigma}_j = \tilde{\sigma}_0|_{\{\tilde{u} < j\}} + \tilde{\alpha}_j, \tag{3.6}$$

where $\tilde{\alpha}_j \in \mathcal{M}^+(\mathbb{R}^n)$ is concentrated in the set $\{\tilde{u} = j\}$. On the other hand, since $\tilde{\sigma}_s(\{\tilde{u} < \infty\}) = 0$ (see [19, Lemma 2.9]), we can rewrite (3.6) as

$$\tilde{\sigma}_j = \tilde{\sigma}|_{\{\tilde{u} < j\}} + \tilde{\alpha}_j. \tag{3.7}$$

Now using the estimate $\tilde{u} \leq K\mathbf{W}_{1,p}\tilde{\sigma}$ and (3.7), we have

$$\tilde{\sigma}_j \geq \tilde{\sigma}|_{\{\tilde{u} < j\}} \geq \sigma|_{\{K\mathbf{W}_{1,p}\tilde{\sigma} < j\}} \geq \sigma|_{\{\mathbf{W}_{1,p}\tilde{\sigma} < k\}} \geq \sigma'_k,$$

provided $j/K > k$.

Since $u'_k \in W_0^{1,p}(B_k(0))$ and $\min\{\tilde{u}, j\} \in W^{1,p}(B_k(0))$, by the comparison principle of [9, Lemma 5.1] we find

$$u'_k \leq \min\{\tilde{u}, j\} \leq \tilde{u},$$

provided we choose a j such that $j \geq Kk$. Letting $k \to \infty$, we obtain $u \leq \tilde{u}$ as desired. $\qquad\square$

Theorem 3.6 justifies the existence (and hence uniqueness) of the minimal $\mathcal{A}$-superharmonic solution to (3.1) provided condition (3.3) holds and $\sigma << \mathrm{cap}_p$. It is not known if condition (3.3) alone is enough for the existence of the minimal solution. It is also not known if under condition (3.3) and $\sigma << \mathrm{cap}_p$ all $\mathcal{A}$-superharmonic solutions to (3.1) coincide with the minimal solution. For a partial result in this direction, see Theorem 3.12 below.

We now introduce a new notion of a solution so that uniqueness is guaranteed for all nonnegative locally finite measures σ such that $\sigma << \mathrm{cap}_p$. Our definition is an adaptation of the notion of the reachable solution of [10, Definition 2.3].

Definition 3.8. *Let $\sigma \in \mathcal{M}^+(\mathbb{R}^n)$. We say that a function $u : \mathbb{R}^n \to [0, +\infty]$ is an $\mathcal{A}$-superharmonic reachable solution to Eq (3.1) if u is an $\mathcal{A}$-superharmonic solution of (3.1), and there exist two sequences $\{u_i\}$ and $\{\sigma_i\}$, $i = 1, 2, \dots$, such that*
(i) Each $\sigma_i \in \mathcal{M}^+(\mathbb{R}^n)$ is compactly supported in $\mathbb{R}^n$, and $\sigma_i \leq \sigma$;
(ii) Each u_i is an $\mathcal{A}$-superharmonic solution of (3.1) with datum σ_i in place of σ;
(iii) $u_i \to u$ a.e. in $\mathbb{R}^n$.

Remark 3.9. *The notion of reachable solution was introduced in [10] for equations over bounded domains with finite measure data. It is also related to the notion of SOLA (Solution Obtained as Limit of Approximations) of [11] for L^1 data over bounded domains. By* (iii) *and the weak continuity result of [36], we see that $\sigma_i \to \sigma$ weakly as measures in $\mathbb{R}^n$. The extra requirement $\sigma_i \leq \sigma$ in our definition plays an important role in the proof of uniqueness in the case when the datum σ is absolutely continuous with respect to cap_p.*

Theorem 3.10. *Suppose $\sigma \in \mathcal{M}^+(\mathbb{R}^n)$, and suppose (3.3) holds. Then there exists an $\mathcal{A}$-superharmonic reachable solution to (3.1). Moreover, if additionally $\sigma << \mathrm{cap}_p$, then any $\mathcal{A}$-superharmonic reachable solution is unique and coincides with the minimal solution.*

Proof. **Existence:** Suppose that (3.3) holds. Then $\mathbf{W}_{1,p}\sigma < +\infty$ quasi-everywhere and hence almost everywhere. For each $i = 1, 2, \ldots$, let u_i^j be an $\mathcal{A}$-superharmonic renormalized solution (see [12]) to

$$\begin{cases} -\mathrm{div}\,\mathcal{A}(x, \nabla u_i^j) &= \sigma|_{B_i(0)} \quad \text{in } B_j(0), \\ u_i^j &= 0 \qquad\quad\;\; \text{on } \partial B_j(0). \end{cases}$$

Note that $\sigma|_{B_i(0)} \leq \sigma$ and $\sigma|_{B_i(0)} \to \sigma$ weakly as measures in $\mathbb{R}^n$. Also, by [33], we have

$$u_i^j \leq K\,\mathbf{W}_{1,p}(\sigma|_{B_i(0)}).$$

Hence, by [20, Theorem 1.17], there exist an $\mathcal{A}$-superharmonic function u_i in $\mathbb{R}^n$ with

$$u_i \leq K\,\mathbf{W}_{1,p}(\sigma|_{B_i(0)}) \leq K\,\mathbf{W}_{1,p}\sigma < +\infty \text{ a.e.,} \tag{3.8}$$

and a subsequence $\{u_i^{j_k}\}_k$ such that $u_i^{j_k} \to u$ and $Du_i^{j_k} \to Du_i$ a.e. as $k \to \infty$. These estimates yield that the Riesz measure of u_i is $\sigma|_{B_i(0)}$ and

$$\liminf_{|x|\to\infty} u_i = 0.$$

Using again [20, Theorem 1.17] and (3.8), we find a subsequence of $\{u_i\}$ that converges a.e. to an $\mathcal{A}$-superharmonic reachable solution u of (3.1).

Uniqueness: We now assume further that $\sigma << \mathrm{cap}_p$. Let u be an $\mathcal{A}$-superharmonic reachable solution in the sense of Definition 3.8 with approximating sequences $\{u_i\}$ and $\{\sigma_i\}$. Let us fix an $i \in \{1, 2, \ldots\}$. Then there exists a positive integer $N = N(i)$ such that $\mathrm{supp}(\sigma_i) \subset B_N(0)$. Let v be the minimal $\mathcal{A}$-superharmonic solution to (3.1). Also, let v_N be the minimal $\mathcal{A}$-superharmonic solution to (3.1) with datum $\sigma|_{B_N(0)}$ in place of σ. We have, by Theorem 3.7,

$$u \geq v \geq v_N.$$

Thus, as $u_j \to u$ a.e., it is enough to show that

$$v_N \geq u_i. \tag{3.9}$$

Note that since $\sigma_i \leq \sigma$ and $\mathrm{supp}(\sigma_i) \subset B_N(0)$ we have that

$$\sigma_i \leq \sigma|_{B_N(0)}. \tag{3.10}$$

For $R > 0$, let $0 \leq \Theta = \Theta_R \leq 1$ be a cutoff function such that

$$\Theta \in C_0^\infty(B_R(0)), \qquad \Theta \equiv 1 \text{ on } B_{R/2}(0), \quad \text{and} \quad |\nabla\Theta| \leq C/R.$$

For any $k > 0$, we set

$$T_k^+(t) = \begin{cases} t & \text{if } 0 \leq t \leq k, \\ k & \text{if } t > k, \\ 0 & \text{if } t < 0. \end{cases}$$

Also, for any $m > 0$, we define the following Lipschitz function with compact support on $\mathbb{R}$:

$$h_m(t) = \begin{cases} 1 & \text{if } 0 \le |t| \le m, \\ 0 & \text{if } |t| \ge 2m, \\ -\frac{t}{m} + 2 & \text{if } m < t < 2m, \\ \frac{t}{m} + 2 & \text{if } -2m < t < -m. \end{cases}$$

As u_i and v_N are both local renormalized solutions (see [3, 19]), we may use

$$h_m(u_i)h_m(v_n)T_k^+(u_i - v_N)\Theta, \qquad m, k > 0,$$

as test functions and thus obtaining

$$\int_{\mathbb{R}^n} \mathcal{A}(x, Du_i) \cdot \nabla \left[h_m(u_i)h_m(v_N)T_k^+(u_i - v_N)\Theta \right] dx$$

$$= \int_{\mathbb{R}^n} h_m(u_i)h_m(v_n)T_k^+(u_i - v_N)\Theta d\sigma_i,$$

and

$$\int_{\mathbb{R}^n} \mathcal{A}(x, Dv_N) \cdot \nabla \left[h_m(u_i)h_m(v_N)T_k^+(u_i - v_N)\Theta \right] dx$$

$$= \int_{\mathbb{R}^n} h_m(u_i)h_m(v_n)T_k^+(u_i - v_N)\Theta d\sigma|_{B_N(0)}.$$

Let

$$I = \int_{\mathbb{R}^n} \mathcal{A}(x, Du_i) \cdot \nabla \left[h_m(u_i)h_m(v_N)T_k^+(u_i - v_N)\Theta \right] dx,$$

and

$$II = \int_{\mathbb{R}^n} \mathcal{A}(x, Dv_N) \cdot \nabla \left[h_m(u_i)h_m(v_N)T_k^+(u_i - v_N)\Theta \right] dx.$$

Then by (3.10) we have

$$I - II \le 0. \tag{3.11}$$

On the other hand, we can write

$$I - II$$
$$= \int_{\mathbb{R}^n} [\mathcal{A}(x, Du_i) - \mathcal{A}(x, Dv_N)] \cdot \nabla T_k^+(u_i - v_N) \, h_m(u_i) \, h_m(v_N) \, \Theta \, dx$$
$$+ \int_{\mathbb{R}^n} [\mathcal{A}(x, Du_i) - \mathcal{A}(x, Dv_N)] \cdot Du_i \, h_m'(u_i) \, T_k^+(u_i - v_N) \, h_m(v_N) \, \Theta \, dx$$
$$+ \int_{\mathbb{R}^n} [\mathcal{A}(x, Du_i) - \mathcal{A}(x, Dv_N)] \cdot Dv_N \, h_m'(v_N) \, T_k^+(u_i - v_N) \, h_m(u_i) \, \Theta \, dx$$
$$+ \int_{\mathbb{R}^n} [\mathcal{A}(x, Du_i) - \mathcal{A}(x, Dv_N)] \cdot \nabla\Theta \, T_k^+(u_i - v_N) \, h_m(u_i) \, h_m(v_N) \, dx.$$

Thus, in view of (3.11), it follows that

$$\int_{\{0 < u_i - v_N < k\}} [\mathcal{A}(x, Du_i) - \mathcal{A}(x, Dv_N)] \cdot (Du_i - Dv_N) \, h_m(u_i) \, h_m(v_N) \, \Theta \, dx$$

$$\leq - \int_{\mathbb{R}^n} [\mathscr{A}(x, Du_i) - \mathscr{A}(x, Dv_N)] \cdot Du_i \, h'_m(u_i) \, T_k^+(u_i - v_N) \, h_m(v_N) \, \Theta \, dx$$

$$- \int_{\mathbb{R}^n} [\mathscr{A}(x, Du_i) - \mathscr{A}(x, Dv_N)] \cdot Dv_N \, h'_m(v_N) \, T_k^+(u_i - v_N) \, h_m(u_i) \, \Theta \, dx$$

$$- \int_{\mathbb{R}^n} [\mathscr{A}(x, Du_i) - \mathscr{A}(x, Dv_N)] \cdot \nabla\Theta \, T_k^+(u_i - v_N) \, h_m(u_i) \, h_m(v_N) \, dx$$

$$=: A_m + B_m + C_m.$$

To estimate $|A_m|$, we observe that $|h_m(t)| \leq 1$ and $|h'_m(t)| \leq 1/m$. Hence,

$$|A_m| \leq \beta \frac{k}{m} \int_{\{m < u_i < 2m, \, 0 < v_N < 2m\}} \left[|Du_i|^{p-1} + |Dv_N|^{p-1} \right] |Du_i| \, \Theta \, dx$$

$$\leq C \frac{k}{m} \int_{\{0 < u_i < 2m, \, 0 < v_N < 2m\}} [|Du_i|^p + |Dv_N|^p] \, \Theta \, dx.$$

On the other hand, using $T_{2m}^+(u_i) \, \Theta$ as a test function for the equation of u_i and invoking condition (2.1), we estimate

$$\alpha \int_{0 < u_i < 2m} |Du_i|^p \, \Theta \, dx \leq \int_{\mathbb{R}^n} T_{2m}^+(u_i) \, \Theta \, d\sigma_i$$

$$+ \beta \int_{\mathbb{R}^n} |Du_i|^{p-1} T_{2m}^+(u_i) \, |\nabla\Theta| \, dx.$$

Since $T_{2m}^+(u_i)/m \leq 2$, and $T_{2m}^+(u_i)/m$ converges to zero quasi-everywhere, we deduce

$$\lim_{m\to\infty} \frac{1}{m} \int_{0 < u_i < 2m} |Du_i|^p \, \Theta \, dx = 0.$$

Similarly,

$$\lim_{m\to\infty} \frac{1}{m} \int_{0 < v_N < 2m} |Dv_N|^p \, \Theta \, dx = 0.$$

Hence,

$$\lim_{m\to\infty} |A_m| = 0. \tag{3.12}$$

A similar argument gives

$$\lim_{m\to\infty} |B_m| = 0.$$

To estimate $|C_m|$, we first use the pointwise bound (3.4) to obtain

$$|C_m| \leq \frac{c}{R} \int_{A_R} \left[|Du_i|^{p-1} + |Dv_N|^{p-1} \right] \min\{\mathbf{W}_{1,p}(\sigma|_{B_N(0)}), k\} \, dx,$$

where A_R is the annulus

$$A_R = \{R/2 < |x| < R\}.$$

Note that for $R > 4N$ we have

$$\mathbf{W}_{1,p}(\sigma|_{B_N(0)})(x) = \int_{R/4}^{\infty} \left[\frac{\sigma(B_t(x) \cap B_N(0))}{t^{n-p}} \right]^{\frac{1}{p-1}} \frac{dt}{t} \approx R^{\frac{p-n}{p-1}}$$

for all $x \in A_R$. Thus,

$$
\begin{aligned}
|C_m| &\leq c \, R^{\frac{p-n}{p-1}} R^{-1} \int_{A_R} \left[|Du_i|^{p-1} + |Dv_N|^{p-1} \right] dx \\
&\leq c \, R^{\frac{p-n}{p-1}} R^{-1} R^{n-p+1} \left[(\inf_{A_R} u_i)^{p-1} + (\inf_{A_R} v_N)^{p-1} \right] \\
&\leq c \, R^{\frac{p-n}{p-1}} R^{-1} R^{n-p+1} R^{p-n} = c \, R^{\frac{p-n}{p-1}},
\end{aligned}
$$

where we used the Caccioppoli inequality and the weak Harnack inequality in the second bound. This gives

$$\lim_{R \to \infty} \limsup_{m \to \infty} |C_m| = 0. \tag{3.13}$$

Since $h_m(u_i) \, h_m(v_N) \to 1$ a.e. as $m \to \infty$, and $\Theta(x) \to 1$ everywhere as $R \to \infty$, it follows from (3.12)–(3.13) and Fatou's lemma that

$$\int_{\{0 < u_i - v_N < k\}} [\mathcal{A}(x, Du_i) - \mathcal{A}(x, Dv_N)] \cdot (Du_i - Dv_N) \, dx \leq 0.$$

Letting $k \to \infty$, we deduce

$$\int_{\{u_i - v_N > 0\}} [\mathcal{A}(x, Du_i) - \mathcal{A}(x, Dv_N)] \cdot (Du_i - Dv_N) \, dx \leq 0.$$

Since the integrand is strictly positive whenever $Du_i \neq Dv_N$, we infer that $Du_i = Dv_N$ a.e. on the set $\{u_i - v_N > 0\}$.

We next claim that the function $T_k^+(u_i - v_N)$ belongs to $W^{1,p}_{\text{loc}}(\mathbb{R}^n)$ for any $k > 0$. To see this, for any $m > k$, we compute

$$
\begin{aligned}
&\nabla T_k^+(T_m^+(u_i) - T_m^+(v_N)) \tag{3.14} \\
&= [\nabla T_m^+(u_i) - \nabla T_m^+(v_N)] \chi_{\{0 < T_m^+(u_i) - T_m^+(v_N) < k\}} \\
&= [Du_i \chi_{\{0 < u_i < m\}} - Dv_N \chi_{\{0 < v_N < m\}}] \chi_{\{0 < T_m^+(u_i) - T_m^+(v_N) < k\}} \\
&= [Du_i \chi_{\{0 < u_i < m\}} - Dv_N \chi_{\{0 < v_N < m\}}] \chi_{\{0 < m - v_N < k, u_i \geq m, v_N < m\}} \\
&= -Dv_N \chi_{\{0 < v_N < m\}} \chi_{\{0 < m - v_N < k, u \geq m, v < m\}},
\end{aligned}
$$

where χ_A is the characteristic function of a set A. Thus,

$$\int_{\mathbb{R}^n} |\nabla T_k^+(T_m^+(u_i) - T_m^+(v_N))|^p \, \Theta \, dx \leq \int_{\{m-k < v_N < m\}} |Dv_N|^p \, \Theta \, dx.$$

On the other hand, using $H_{m,k}(v_N)\Theta$ as a test function for the equation of v_N, where

$$H_{m,k}(t) = \begin{cases} 1 & \text{if } 0 \le |t| \le m - k, \\ 0 & \text{if } |t| \ge m, \\ -\frac{t}{k} + \frac{m}{k} & \text{if } m - k < t < m, \\ \frac{t}{k} + \frac{m}{k} & \text{if } -m < t < -(m-k), \end{cases}$$

we have

$$\frac{\alpha}{k} \int_{\{m-k<v_N<m\}} |Dv_N|^p \, \Theta \, dx \le \int_{\mathbb{R}^n} \Theta \, d\sigma|_{B_N(0)} + \beta \int_{\mathbb{R}^n} |Dv_N|^{p-1} \, |\nabla\Theta| \, dx.$$

Thus, for each fixed $k > 0$, the sequence $\{T_k^+(T_m^+(u_i) - T_m^+(v_N))\}_m$ is uniformly bounded in $W_{\text{loc}}^{1,p}(\mathbb{R}^n)$. Since $T_k^+(T_m^+(u_i) - T_m^+(v_N)) \to T_k^+(u_i - v_N)$ a.e. as $m \to \infty$, we see that $T_k^+(u_i - v_N) \in W_{\text{loc}}^{1,p}(\mathbb{R}^n)$.

We are now ready to complete the proof of the theorem. Since $T_k^+(u_i - v_N) = T_k^+(T_m^+(u_i) - T_m^+(v_N))$ a.e. on the set $\{u_i < m, v_N < m\}$ and the two functions belong to $W_{\text{loc}}^{1,p}(\mathbb{R}^n)$, by (3.14) we have

$$\nabla T_k^+(u_i - v_N) = \nabla T_k^+(T_m^+(u_i) - T_m^+(v_N)) = 0$$

a.e. on the set $\{u_i < m, v_N < m\}$ for any $m > 0$. Thus, $\nabla T_k^+(u_i - v_N) = 0$ a.e. in $\mathbb{R}^n$, which implies the existence of a constant $\kappa \ge 0$ such that

$$\max\{u_i - v_N, 0\} = \kappa$$

a.e. in the entire space $\mathbb{R}^n$. Note that if $\kappa \ne 0$, then $u_i = v_N + \kappa$ in $\mathbb{R}^n$, which violates the condition at infinity, $\liminf_{|x|\to\infty} u_i(x) = 0$. It follows that $\kappa = 0$, which yields (3.9), as desired. $\qquad\square$

The following version of the comparison principle in $\mathbb{R}^n$ is an immediate consequence of Theorems 3.7 and 3.10.

Corollary 3.11. *Let $\sigma, \tilde{\sigma} \in \mathcal{M}^+(\mathbb{R}^n)$, where $\sigma \le \tilde{\sigma}$ and $\sigma << \text{cap}_p$, $1 < p < n$. Let u be an $\mathcal{A}$-superharmonic reachable solution of* (3.1)*, and $\tilde{u}$ any $\mathcal{A}$-superharmonic solution of* (3.1) *with datum $\tilde{\sigma}$ in place of σ. Then $u \le \tilde{u}$ in $\mathbb{R}^n$.*

For $\sigma \in \mathcal{M}^+(\mathbb{R}^n)$ such that that $\sigma << \text{cap}_p$, sometimes it is desirable to know when an $\mathcal{A}$-superharmonic solution to (3.1) is also the $\mathcal{A}$-superharmonic reachable solution to (3.1), and hence also the minimal $\mathcal{A}$-superharmonic solution to (3.1). The following theorem provides some sufficient conditions in terms of the weak integrability of the gradient of the solution, or in terms of the finiteness of the datum σ.

Theorem 3.12. *Let $\sigma \in \mathcal{M}^+(\mathbb{R}^n)$, where $\sigma << \text{cap}_p$. Suppose that any one of the following conditions holds:*
(i) $|Du| \in L^{\gamma,\infty}(\mathbb{R}^n)$ for some $(p-1)n/(n-1) \le \gamma < p$, where $L^{\gamma,\infty}(\mathbb{R}^n)$ is the weak L^γ space in $\mathbb{R}^n$;
(ii) $|Du| \in L^p(\mathbb{R}^n)$;
(iii) $\sigma \in \mathcal{M}_b^+(\mathbb{R}^n)$.
Then any $\mathcal{A}$-superharmonic solution u to the Eq (3.1) coincides with the minimal $\mathcal{A}$-superharmonic solution.

Proof. Let v be the minimal $\mathcal{A}$-superharmonic solution of (3.1). Our goal is to show that $u \leq v$ a.e. Let $\Theta(x) = \Theta_R(x), R > 0, T_k^+(t), k > 0$, and $h_m(t), m > 0$ be as in the proof of Theorem 3.10. Then arguing as in the proof of Theorem 3.10, with u in place of u_i and v in place of v_N, we have

$$\int_{\{0 < u_i - v_N < k\}} [\mathcal{A}(x, Du) - \mathcal{A}(x, Dv)] \cdot (Du - Dv) \, h_m(u) \, h_m(v) \, \Theta \, dx$$
$$\leq A_m + B_m + C_m,$$

where now

$$A_m = -\int_{\mathbb{R}^n} [\mathcal{A}(x, Du) - \mathcal{A}(x, Dv)] \cdot Du \, h'_m(u) \, T_k^+(u - v) \, h_m(v) \, \Theta \, dx,$$

$$B_m = -\int_{\mathbb{R}^n} [\mathcal{A}(x, Du) - \mathcal{A}(x, Dv)] \cdot Dv \, h'_m(v) \, T_k^+(u - v) \, h_m(u) \, \Theta \, dx,$$

and

$$C_m = -\int_{\mathbb{R}^n} [\mathcal{A}(x, Du) - \mathcal{A}(x, Dv)] \cdot \nabla\Theta \, T_k^+(u - v) \, h_m(u) \, h_m(v) \, dx.$$

As in the proof of Theorem 3.10, we have

$$\lim_{m \to \infty} (|A_m| + |B_m|) = 0. \tag{3.15}$$

As for C_m, we have

$$|C_m| \leq \frac{c}{R} \int_{A_R} \left[|Du|^{p-1} + |Dv|^{p-1} \right] \min\{\mathbf{W}_{1,p}\sigma, k\} \, dx,$$

where, as above, A_R is the annulus
$$A_R = \{R/2 < |x| < R\}.$$

Suppose now that condition (i) holds. Then $|Du|^{p-1} \in L^{\frac{q}{q-1}, \infty}(\mathbb{R}^n)$ for some $q \in (p, n]$. Set

$$m = \frac{n(p-1)q}{n(q-1) - (p-1)q} > 0,$$

and note that $\mathbf{W}_{1,p}\sigma \in L^{m,\infty}(\mathbb{R}^n)$. A proof of this fact in the 'sublinear' case $(p-1)q/(q-1) \leq 1$ can be found in [32].

We have that either $m \leq q$ or $m > q$. In the case $m \leq q$, for any $\epsilon > 0$ we find

$$\frac{1}{R} \int_{A_R} |Du|^{p-1} \min\{\mathbf{W}_{1,p}\sigma, k\} dx$$
$$\leq \frac{1}{R} \left\| |Du|^{p-1} \right\|_{L^{\frac{q}{q-1}, \infty}(A_R)} \left\| \min\{\mathbf{W}_{1,p}\sigma, k\} \right\|_{L^{q,1}(A_R)}$$
$$\leq \frac{1}{R} \left\| |Du|^{p-1} \right\|_{L^{\frac{q}{q-1}, \infty}(\mathbb{R}^n)} \left\| \min\{\mathbf{W}_{1,p}\sigma, k\} \right\|_{L^{q+\epsilon, \infty}(A_R)} |A_R|^{\frac{\epsilon}{q(q+\epsilon)}}$$
$$\leq Ck^{\frac{q+\epsilon-m}{q+\epsilon}} \left\| |Du|^{p-1} \right\|_{L^{\frac{q}{q-1}, \infty}(\mathbb{R}^n)} \left\| \mathbf{W}_{1,p}\sigma \right\|_{L^{m,\infty}(\mathbb{R}^n)}^{\frac{m}{q+\epsilon}} R^{\frac{n\epsilon}{q(q+\epsilon)} - 1}.$$

Here we shall choose $\epsilon > 0$ such that
$$\frac{n\epsilon}{q(q+\epsilon)} - 1 < 0.$$

In the case $m > q$, we have

$$\frac{1}{R}\int_{A_R}|Du|^{p-1}\min\{\mathbf{W}_{1,p}\sigma, k\}dx \tag{3.16}$$

$$\leq \frac{1}{R}\left\||Du|^{p-1}\right\|_{L^{\frac{q}{q-1},\infty}(A_R)}\left\|\min\{\mathbf{W}_{1,p}\sigma, k\}\right\|_{L^{q,1}(A_R)}$$

$$\leq \frac{1}{R}\left\||Du|^{p-1}\right\|_{L^{\frac{q}{q-1},\infty}(\mathbb{R}^n)}\left\|\mathbf{W}_{1,p}\sigma\right\|_{L^{m,\infty}(\mathbb{R}^n)}|A_R|^{\frac{m-q}{qm}}$$

$$\leq C\left\||Du|^{p-1}\right\|_{L^{\frac{q}{q-1},\infty}(\mathbb{R}^n)}\left\|\mathbf{W}_{1,p}\sigma\right\|_{L^{m,\infty}(\mathbb{R}^n)}R^{\frac{(m-q)n}{qm}-1}.$$

Note that, since $q < p$,

$$\frac{(m-q)n}{mq} - 1 = \frac{n(p-1)-n(q-1)+(p-1)q}{(p-1)q} - 1 < 0.$$

Hence, in both cases we have, for any fixed $k > 0$,

$$\lim_{R\to\infty}\frac{1}{R}\int_{A_R}|Du|^{p-1}\min\{\mathbf{W}_{1,p}\sigma, k\}dx = 0, \tag{3.17}$$

and likewise,

$$\lim_{R\to\infty}\frac{1}{R}\int_{A_R}|Dv|^{p-1}\min\{\mathbf{W}_{1,p}\sigma, k\}dx = 0. \tag{3.18}$$

On the other hand, suppose now that condition (ii) holds, i.e., $|Du| \in L^p(\mathbb{R}^n)$. Then $\mathbf{W}_{1,p}\sigma \in L^{\frac{np}{n-p}}(\mathbb{R}^n)$, and as in (3.16) we have

$$\frac{1}{R}\int_{A_R}|Du|^{p-1}\min\{\mathbf{W}_{1,p}\sigma, k\}dx$$

$$\leq C\left\||Du|^{p-1}\right\|_{L^{\frac{p}{p-1}}(A_R)}\left\|\mathbf{W}_{1,p}\sigma\right\|_{L^{\frac{np}{n-p}}(A_R)},$$

and likewise for v. Thus (3.17) and (3.18) also hold under condition (ii).

Finally, suppose that (iii) holds. For any $1 < r < \frac{n}{n-1}$ and $\epsilon \in (0,1)$ such that $\epsilon\frac{r}{r-1} < \frac{n(p-1)}{n-p}$, we have

$$\frac{1}{R}\int_{A_R}|Du|^{p-1}\min\{\mathbf{W}_{1,p}\sigma, k\}dx$$

$$\leq k^{1-\epsilon}\frac{1}{R}\left(\int_{B_R(0)}|Du|^{(p-1)r}dx\right)^{\frac{1}{r}}\left(\int_{B_R(0)}(\mathbf{W}_{1,p}\sigma)^{\frac{\epsilon r}{r-1}}dx\right)^{\frac{r-1}{r}}$$

$$\leq Ck^{1-\epsilon}R^{\frac{n}{r}-1}\left(\frac{\inf_{B_R(0)}u}{R}\right)^{p-1}R^{n\frac{r-1}{r}}\left(\inf_{B_R(0)}u\right)^{\epsilon},$$

where we used the Caccioppoli inequality and the weak Harnack inequality in the last bound (see [18, Theorem 7.46]).

Hence, using [9, Lemma 3.1] we get

$$\frac{1}{R} \int_{A_R} |Du|^{p-1} \min\{\mathbf{W}_{1,p}\sigma, k\} dx \tag{3.19}$$

$$\leq C k^{1-\epsilon} R^{n-p} \left(\int_R^\infty \left(\frac{\sigma(B_t(0))}{t^{n-p}} \right)^{\frac{1}{p-1}} \frac{dt}{t} \right)^{p-1} \left(\inf_{B_R(0)} u \right)^\epsilon$$

$$\leq C k^{1-\epsilon} \sigma(\mathbb{R}^n) \left(\inf_{B_R(0)} u \right)^\epsilon.$$

A similar inequality holds for v in place of u. Thus, we see that (3.17) and (3.18) hold under condition (iii) as well.

Now (3.17) and (3.18) yield that, for any $k > 0$, we have

$$\lim_{R \to \infty} \limsup_{m \to \infty} |C_m| = 0. \tag{3.20}$$

Using (3.15) and (3.20), we deduce

$$\int_{\{0<u-v<k\}} [\mathcal{A}(x, Du) - \mathcal{A}(x, Dv)] \cdot (Du - Dv) dx \leq 0$$

for any $k > 0$. This implies $Du = Dv$ a.e. on the set $\{u - v > 0\}$ and, as in the proof of Theorem 3.10, in view of the condition at infinity, we deduce $u \leq v$ a.e. as desired. $\square$

We now provide a criterion for reachability by requiring only the finiteness of the approximating measures σ_i.

Corollary 3.13. *Let u be an $\mathcal{A}$-superharmonic solution of* (3.1), *where $\sigma \in \mathcal{M}^+(\mathbb{R}^n)$, and $\sigma << \mathrm{cap}_p$. Suppose that there exist two sequences $\{u_i\}$ and $\{\sigma_i\}$, $i = 1, 2, \ldots$, such that the following conditions hold:*
(i) *each $\sigma_i \in \mathcal{M}_b^+(\mathbb{R}^n)$, and $\sigma_i \leq \sigma$;*
(ii) *each u_i is an $\mathcal{A}$-superharmonic solution of* (3.1) *with datum σ_i in place of σ;*
(iii) *$u_i \to u$ a.e. in $\mathbb{R}^n$.*
Then u is an $\mathcal{A}$-superharmonic reachable solution of (3.1), *and thus coincides with the minimal solution.*

Proof. By Theorem 3.12, each u_i is a reachable solution. Thus by a diagonal process argument, we see that u is also a reachable solution. Alternatively, this can also be proved by modifying the proof of the uniqueness part in Theorem 3.10, taking into account estimates of the form (3.19). $\square$

Theorem 3.12 formally holds under the condition $|Du| \in L^{\gamma,\infty}(\mathbb{R}^n)$ for $0 < \gamma < (p-1)n/(n-1)$ as in this case $\sigma = 0$. The proof of this fact, especially in the case $0 < \gamma \leq p - 1$, requires some results obtained recently in [31].

Theorem 3.14. *If u is an $\mathcal{A}$-superharmonic function in $\mathbb{R}^n$ such that $|Du| \in L^{\gamma,\infty}(\mathbb{R}^n)$ for some $0 < \gamma < (p-1)n/(n-1)$, then $\sigma = 0$ where σ is the Riesz measure of u.*

Proof. Let $Q_r(x)$, $r > 0$, denote the open cube $Q_r(x) := x + (-r, r)^n$ with center $x \in \mathbb{R}^n$ and side-length $2r$. Using $\Phi \in C_0^\infty(Q_r(x))$, $\Phi \geq 0$, $\Phi = 1$ on $Q_{r/2}(x)$, and $|\nabla\Phi| \leq C/r$, as a test function we have

$$\sigma(Q_{r/2}(x)) \leq \frac{C}{r} \int_{Q_r(x)} |Du|^{p-1} dy. \tag{3.21}$$

Thus if $\gamma \in (p-1, (p-1)n/(n-1))$, for any $R > 0$ we use Hölder's inequality to get

$$\sigma(Q_{R/2}(0)) \leq \frac{C}{R} \|Du\|_{L^{\gamma,\infty}(\mathbb{R}^n)}^{p-1} R^{n\frac{\gamma-p+1}{\gamma}}.$$

Note that $n\frac{\gamma-p+1}{\gamma} < 1$ and thus letting $R \to \infty$ we get $\sigma = 0$.

We now consider the case $0 < \gamma \leq p - 1$. Let γ_1 be a fixed number in $(p-1), (p-1)n/(n-1))$. By [31, Lemma 2.3], for any cube $Q_\rho(x) \subset \mathbb{R}^n$, we have

$$\left(\fint_{Q_\rho(x)} |Du|^{\gamma_1} dy\right)^{\frac{1}{\gamma_1}} \lesssim \left[\frac{\sigma(Q_{3\rho/2}(x))}{\rho^{n-1}}\right]^{\frac{1}{p-1}}$$
$$+ \frac{1}{\rho} \inf_{q \in \mathbb{R}} \left(\fint_{Q_{3\rho/2}(x)} |u-q|^{p-1} dy\right)^{\frac{1}{p-1}}.$$

On the other hand, by [31, Corollary 1.3] we find

$$\frac{1}{\rho} \inf_{q \in \mathbb{R}} \left(\fint_{Q_{3\rho/2}(x)} |u-q|^{p-1} dy\right)^{\frac{1}{p-1}} \lesssim \left[\frac{\sigma(Q_{2\rho}(x))}{\rho^{n-1}}\right]^{\frac{1}{p-1}}$$
$$+ \left(\fint_{Q_{2\rho}(x)} |Du|^{p-1} dy\right)^{\frac{1}{p-1}}.$$

Note that [31, Corollary 1.3] is stated for $1 < p < 3/2$ but the argument there also works for all $1 < p \leq n$ after taking into account the comparison estimates of [14, 27, 28].

Hence, it follows that

$$\left(\fint_{Q_\rho(x)} |Du|^{\gamma_1} dy\right)^{\frac{1}{\gamma_1}} \lesssim \left[\frac{\sigma(Q_{2\rho}(x))}{\rho^{n-1}}\right]^{\frac{1}{p-1}} + \left(\fint_{Q_{2\rho}(x)} |Du|^{p-1} dy\right)^{\frac{1}{p-1}}$$
$$\lesssim \left(\fint_{Q_{4\rho}(x)} |Du|^{p-1} dy\right)^{\frac{1}{p-1}},$$

where we used (3.21) with $r = 4\rho$ in the last inequality. This allows us to employ a covering/iteration argument as in [16, Remark 6.12] to obtain that

$$\left(\fint_{Q_\rho(x)} |Du|^{\gamma_1} dy\right)^{\frac{1}{\gamma_1}} \lesssim \left(\fint_{Q_{4\rho}(x)} |Du|^{\epsilon} dy\right)^{\frac{1}{\epsilon}} \tag{3.22}$$

for any $\epsilon > 0$.

Thus, if $0 < \gamma < p - 1$, in view of (3.21), (3.22), and Hölder's inequality, we get

$$\sigma(Q_{R/2}(0)) \leq \frac{C}{R} \|Du\|_{L^{\gamma,\infty}(\mathbb{R}^n)}^{\frac{p-1}{\gamma}} R^{n-n\frac{p-1}{\gamma}} \to 0,$$

as $R \to \infty$. Hence, $\sigma = 0$. The case $\gamma = p - 1$ is treated similarly, starting with the inequality

$$\sigma(Q_{R/2}(0)) \leq \frac{C}{R} \left(\int_{Q_R(0)} |Du|^{(p-1)(1+\epsilon)} dx \right)^{\frac{1}{1+\epsilon}} R^{\frac{n\epsilon}{1+\epsilon}}$$

for a sufficiently small $\epsilon > 0$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Due to the results of [13, 14, 24, 30] (see also [15, 29]), under some additional regularity conditions on the nonlinearity $\mathcal{A}(x, \xi)$, one has

$$|Du(x)| \leq C \left[\mathbf{I}_1 \sigma(x) \right]^{\frac{1}{p-1}} \quad \text{a.e. } x \in \mathbb{R}^n,$$

provided u is an $\mathcal{A}$-superharmonic solution to the Eq (3.1). This gradient estimate holds in particular for $\mathcal{A}(x, \xi) = |\xi|^{p-2}\xi$, i.e., the p-Laplacian Δ_p, which yields the following corollary.

Corollary 3.15. *Let $\sigma \in \mathcal{M}^+(\mathbb{R}^n)$. Suppose that one of the following conditions holds:*
(i) $\sigma << \mathrm{cap}_p$ and $\mathbf{I}_1\sigma \in L^{s,\infty}(\mathbb{R}^n)$ for some $n/(n - 1) < s < p/(p - 1)$. This holds in particular if $\sigma = f \in L^{t,\infty}(\mathbb{R}^n)$ for some $1 < t < np/(np - n + p)$;
(ii) $\mathbf{I}_1\sigma \in L^{p/(p-1)}(\mathbb{R}^n)$, i.e., σ is of finite energy.
Then any p-superharmonic solution u to the equation

$$\begin{cases} -\Delta_p u = \sigma, \quad u \geq 0 \quad \text{in } \mathbb{R}^n, \\ \liminf_{|x| \to \infty} u = 0, \end{cases}$$

coincides with the minimal p-superharmonic solution.

Finally, we show that if the condition at infinity, $\liminf_{|x| \to \infty} u = 0$ in (3.1), is replaced with the stronger one $\lim_{|x| \to \infty} u = 0$, then all $\mathcal{A}$-superharmonic solutions are indeed reachable.

Theorem 3.16. *Suppose that u is an $\mathcal{A}$-superharmonic solution of the equation*

$$\begin{cases} -\mathrm{div}\, \mathcal{A}(x, \nabla u) = \sigma, \quad u \geq 0 \quad \text{in } \mathbb{R}^n, \\ \lim_{|x| \to \infty} u = 0, \end{cases} \tag{3.23}$$

where $\sigma \in \mathcal{M}^+(\mathbb{R}^n)$, and $\sigma << \mathrm{cap}_p$. Then u is the unique $\mathcal{A}$-superharmonic solution of (3.23), which coincides with the minimal $\mathcal{A}$-superharmonic reachable solution of (3.1).

Proof. First notice that the condition $\lim_{|x| \to \infty} u = 0$ in (3.23) yields, in view of (3.4),

$$\lim_{|x| \to \infty} \mathbf{W}_{1,p}\sigma(x) = 0.$$

For any $\epsilon > 0$, let

$$\Omega_\epsilon := \{x \in \mathbb{R}^n : u(x) > \epsilon\},$$

and

$$u_\epsilon := \max\{u, \epsilon\} - \epsilon.$$

Clearly, Ω_ϵ is a bounded open set, $u_\epsilon = u - \epsilon$ on Ω_ϵ, and $u_\epsilon = 0$ in $\mathbb{R}^n \setminus \Omega_\epsilon$.

Let v be the minimal solution of (3.1), which is also the minimal solution of (3.23), since $v \leq K\mathbf{W}_{1,p}\sigma$, and hence $\lim_{|x|\to\infty} v = 0$. It is enough to show that

$$u_\epsilon \leq v \tag{3.24}$$

in Ω_ϵ, as this will yield that $u \leq v$ in $\mathbb{R}^n$ after letting $\epsilon \to 0^+$.

Now by Lemma 4.1 below, to verify (3.24), it suffices to show that u_ϵ is a renormalized solution of

$$\begin{cases} -\operatorname{div}\mathcal{A}(x, \nabla u_\epsilon) = \sigma & \text{in } \Omega_\epsilon, \\ u_\epsilon = 0 & \text{on } \partial\Omega_\epsilon. \end{cases}$$

Note that, for any $k > 0$, $T_k(u_\epsilon) = T_{k+\epsilon}(u) - \epsilon$. We have $T_k(u_\epsilon) \in W^{1,p}_{\text{loc}}(\mathbb{R}^n)$, $T_k(u_\epsilon)$ is quasi-continuous in $\mathbb{R}^n$, and $T_k(u_\epsilon) = 0$ everywhere in $\mathbb{R}^n \setminus \Omega_\epsilon$. Thus $T_k(u_\epsilon) \in W^{1,p}_0(\Omega_\epsilon)$ (see [18, Theorem 4.5]).

As u is a local renormalized solution in $\mathbb{R}^n$, for every $k > 0$ there exists a nonnegative measure $\lambda_{k+\epsilon} << \text{cap}_p$, concentrated on the sets $\{u = k + \epsilon\}$, such that $\lambda_{k+\epsilon} \to 0$ weakly as measures in $\mathbb{R}^n$ as $k \to \infty$. Since Ω_ϵ is bounded, this implies that $\lambda_{k+\epsilon} \to 0$ in the narrow topology of measures in Ω_ϵ.

Moreover, for $k > 0$,

$$\int_{\{u < k+\epsilon\}} \mathcal{A}(x, Du) \cdot \nabla\varphi \, dx = \int_{\{u < k+\epsilon\}} \varphi \, d\sigma_0 + \int_{\mathbb{R}^n} \varphi \, d\lambda_{k+\epsilon},$$

for every $\varphi \in W^{1,p}_0(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ with compact support in $\mathbb{R}^n$. In particular, we have

$$\int_{\{u_\epsilon < k\} \cap \Omega_\epsilon} \mathcal{A}(x, Du) \cdot \nabla\varphi \, dx = \int_{\{u_\epsilon < k\} \cap \Omega_\epsilon} \varphi \, d\sigma_0 + \int_{\{u_\epsilon = k\} \cap \Omega_\epsilon} \varphi \, d\lambda_{k+\epsilon}$$

for every $\varphi \in W^{1,p}_0(\Omega_\epsilon) \cap L^\infty(\Omega_\epsilon)$.

Thus, we conclude that u_ϵ is a renormalized solution in Ω_ϵ, as desired. $\qquad\square$

4. Quasilinear equations with a sub-natural growth term in $\mathbb{R}^n$

In this section, we study solutions to the equation

$$\begin{cases} -\operatorname{div}\mathcal{A}(x, \nabla u) = \sigma u^q + \mu, & u \geq 0 \quad \text{in } \mathbb{R}^n, \\ \liminf_{|x|\to\infty} u = 0, \end{cases} \tag{4.1}$$

in the sub-natural growth case $0 < q < p - 1$, with $\mu, \sigma \in \mathcal{M}^+(\mathbb{R}^n)$.

We consider nontrivial $\mathcal{A}$-superharmonic solutions to (4.1) such that $0 < u < \infty$ $d\sigma$-a.e., which implies $u \in L^q_{\text{loc}}(\mathbb{R}^n, \sigma)$, so that $\sigma u^q + \mu \in \mathcal{M}^+(\mathbb{R}^n)$ (see [37]).

As was noted in the Introduction, $\sigma << \text{cap}_p$ whenever there exists a nontrivial solution u to (4.1), for any μ (in particular, $\mu = 0$).

The existence and uniqueness of nontrivial *reachable* $\mathcal{A}$-superharmonic solutions to (4.1), under the additional assumption $\mu << \text{cap}_p$, are proved below. Without this restriction on μ, the existence of nontrivial solutions, not necessarily reachable, was obtained recently in [37], along with bilateral pointwise estimates of solutions in terms of nonlinear potentials.

We use this opportunity to make a correction in the proof of the existence property for (4.1) in the case $\mu = 0$ given in [9, Theorem 1.1], which used a version of the comparison principle ([9], Lemma 5.2). It was invoked in the proof of [37, Theorem 1.1] as well. Some inaccuracies in the statement of this comparison principle and its proof are fixed in the following lemma. The rest of the proofs of [9, Theorem 1.1] and [37, Theorem 1.1] remains valid with this correction. (See the proof of Theorem 4.2 below.)

Lemma 4.1. *Let Ω be a bounded open set in $\mathbb{R}^n$. Suppose that $\mu, \nu \in \mathcal{M}_b^+(\Omega)$, where $\mu \leq \nu$ and $\mu << \mathrm{cap}_p$. If $u \geq 0$ is a renormalized solution of*

$$\begin{cases} -\mathrm{div}\,\mathcal{A}(x, \nabla u) = \mu & in\ \Omega, \\ u = 0 & on\ \partial\Omega, \end{cases} \tag{4.2}$$

and if $v \geq 0$ is an $\mathcal{A}$-superharmonic function in Ω with Riesz measure ν such that $\min\{v, k\} \in W^{1,p}(\Omega)$ for any $k > 0$, then $u \leq v$ a.e.

Proof. Let ν_j, $j > 0$, be the Riesz measure of $\min\{v, j\}$. Since $\min\{v, j\} \in W^{1,p}(\Omega)$ we see that ν_j belongs to the dual of $W_0^{1,p}(\Omega)$ (see [18, Theorem 21.6]). As in (3.7), we have

$$\nu_j = \nu|_{\{v < j\}} + \alpha_j$$

for a measure $\alpha_j \in \mathcal{M}_b^+(\Omega)$ concentrated in the set $\{v = j\}$. Thus the measure $\mu_j := \mu|_{\{v < j\}} \leq \nu|_{\{v < j\}} \leq \nu_j$ for any $j > 0$. This implies that μ_j also belongs to the dual of $W_0^{1,p}(\Omega)$, and hence there exists a unique solution u_j to the equation

$$-\mathrm{div}\,\mathcal{A}(x, \nabla u_j) = \mu_j, \qquad u_j \in W_0^{1,p}(\Omega). \tag{4.3}$$

Then by the comparison principle (see [9, Lemma 5.1]) we find

$$0 \leq u_1 \leq u_2 \leq \cdots \leq u_j \leq \min\{v, j\}$$

for any integer $j > 0$. Thus there is a function $\tilde{u}$ on Ω such that $0 \leq \tilde{u} \leq v$ a.e. and $u_j \to \tilde{u}$ as $j \to \infty$. We now claim that $\tilde{u}$ is also a renormalized solution to Eq (4.2). If this is verified then, as $\mu << \mathrm{cap}_p$, we must have that $\tilde{u} = u$ a.e. (see [12, 25]) and thus $u \leq v$ a.e. as desired.

To show that $\tilde{u}$ is the renormalized solution of (4.2), we first use $T_k^+(u_j)$, $k > 0$, as a test function for (4.3) to obtain

$$\alpha \int_\Omega |\nabla T_k^+(u_j)|^p dx \leq k\mu_j(\Omega) \leq k\mu(\Omega). \tag{4.4}$$

Since $T_k^+(u_j) \to T_k^+(\tilde{u})$ a.e. as $j \to \infty$, we see that $T_k^+(\tilde{u}) \in W_0^{1,p}(\Omega)$ and

$$\alpha \int_\Omega |\nabla T_k^+(\tilde{u})|^p dx \leq k\mu(\Omega)$$

for any $k > 0$. By [2, Lemmas 4.1 and 4.2], this yields

$$\tilde{u} \in L^{\frac{n(p-1)}{n-p}, \infty}(\Omega) \quad \text{and} \quad D\tilde{u} \in L^{\frac{n(p-1)}{n-1}, \infty}(\Omega).$$

Moreover, arguing as in Step 4 of the proof of Theorem 3.4 in [12], we see that $\{\nabla u_j\}$ is a Cauchy sequence in measure which converges to $D\tilde{u}$ a.e. in Ω. There is no need to take a subsequence here as the limit is independent of any subsequence.

Moreover, for any Lipschitz function $h : \mathbb{R} \to \mathbb{R}$ such that h' has compact support and for any function $\varphi \in W^{1,r}(\Omega) \cap L^\infty(\Omega)$, $r > n$, such that $h(\tilde{u})\varphi \in W_0^{1,p}(\Omega)$, we have

$$\int_\Omega \mathcal{A}(x, \nabla u_j) \cdot D\tilde{u}\, h'(\tilde{u})\,\varphi dx + \int_\Omega \mathcal{A}(x, \nabla u_j) \cdot \nabla\varphi\, h(\tilde{u})dx = \int_\Omega h(\tilde{u})\,\varphi d\mu_j.$$

Thus if the support of h' is in $[-M, M]$, $M > 0$, then, using $0 \le u_j \le \tilde{u}$, we can rewrite the above equality as

$$\int_\Omega \mathcal{A}(x, \nabla T_M^+(u_j)) \cdot \nabla T_M^+(\tilde{u})\, h'(\tilde{u})\,\varphi dx + \int_\Omega \mathcal{A}(x, \nabla u_j) \cdot \nabla\varphi\, h(\tilde{u})dx$$
$$= \int_\Omega h(\tilde{u})\,\varphi d\mu_j = \int_{\{0 \le v < j\}} h(\tilde{u})\,\varphi d\mu.$$

Note that by (4.4) and [2, Lemma 4.2], we have that ∇u_j is uniformly bounded in $L^{\frac{n(p-1)}{n-1}, \infty}(\Omega)$ and $\nabla T_M^+(u_j)$ is uniformly bounded in $L^p(\Omega)$. Thus by the Vitali Convergence Theorem, the left-hand side of the above equality converges to

$$\int_\Omega \mathcal{A}(x, \nabla T_M^+(\tilde{u})) \cdot \nabla T_M^+(\tilde{u})\, h'(\tilde{u})\,\varphi dx + \int_\Omega \mathcal{A}(x, D\tilde{u}) \cdot \nabla\varphi\, h(\tilde{u})dx$$
$$= \int_\Omega \mathcal{A}(x, D\tilde{u}) \cdot D\tilde{u}\, h'(\tilde{u})\,\varphi dx + \int_\Omega \mathcal{A}(x, D\tilde{u}) \cdot \nabla\varphi\, h(\tilde{u})dx.$$

On the other hand, by the Lebesgue Dominated Convergence Theorem we have

$$\lim_{j\to\infty} \int_{\{0 \le v < j\}} h(\tilde{u})\,\varphi d\mu = \int_\Omega h(\tilde{u})\,\varphi d\mu.$$

Thus, we get

$$\int_\Omega \mathcal{A}(x, D\tilde{u}) \cdot D\tilde{u}\, h'(\tilde{u})\,\varphi dx + \int_\Omega \mathcal{A}(x, D\tilde{u}) \cdot \nabla\varphi\, h(\tilde{u})dx = \int_\Omega h(\tilde{u})\,\varphi d\mu,$$

which yields that $\tilde{u}$ is the renormalized solution of (4.2) (see Definition 3.2). $\qquad\square$

We recall that by $\varkappa = \varkappa(\mathbb{R}^n)$ we denote the least constant in the weighted norm inequality (see [9,37])

$$\left(\int_{\mathbb{R}^n} |\varphi|^q\, d\sigma \right)^{\frac{1}{q}} \le \varkappa \,\| - \operatorname{div} \mathcal{A}(x, \nabla\varphi) \|_{\mathcal{M}^+(\mathbb{R}^n)}^{\frac{1}{p-1}}, \tag{4.5}$$

for all $\mathcal{A}$-superharmonic functions $\varphi \ge 0$ in $\mathbb{R}^n$ such that $\liminf\limits_{|x|\to\infty} \varphi(x) = 0$. Notice that by estimates (3.4), $K^{-1}\varphi \le \mathbf{W}_{1,p}\mu \le K\varphi$, where $\mu = -\operatorname{div} \mathcal{A}(x, \nabla\varphi) \in \mathcal{M}^+(\mathbb{R}^n)$. Here we may assume without loss of generality that $\mu \in \mathcal{M}_b^+(\mathbb{R}^n)$, so that $\mathbf{W}_{1,p}\mu \not\equiv \infty$. Consequently, (4.5) is equivalent to the inequality

$$\left(\int_{\mathbb{R}^n} (\mathbf{W}_{1,p}\mu)^q\, d\sigma \right)^{\frac{1}{q}} \le \kappa \,\|\mu\|_{\mathcal{M}^+(\mathbb{R}^n)}^{\frac{1}{p-1}} \qquad \text{for all } \mu \in \mathcal{M}_b^+(\mathbb{R}^n),$$

where $K^{-1} \varkappa \leq \kappa \leq K \varkappa$. In particular, one can replace $\operatorname{div} \mathcal{A}(x, \nabla \varphi)$ in (4.5) by Δ_p, up to a constant which depends only on K.

By $\varkappa(B)$, where B is a ball in $\mathbb{R}^n$, we denote the least constant in a similar localized weighted norm inequality with the measure σ_B in place of σ, where $\sigma_B = \chi_B \sigma$ is the restriction of σ to B.

The so-called *intrinsic* nonlinear potential $\mathbf{K}_{p,q}\sigma$, introduced in [9], is defined by

$$\mathbf{K}_{p,q}\sigma(x) = \int_0^\infty \left(\frac{\varkappa(B(x,t))^{\frac{q(p-1)}{p-1-q}}}{t^{n-p}} \right)^{\frac{1}{p-1}} \frac{dt}{t}, \qquad x \in \mathbb{R}^n.$$

Here $B = B(x, t)$ is a ball in $\mathbb{R}^n$ of radius $t > 0$ centered at x. As was noticed in [9], $\mathbf{K}_{p,q}\sigma \not\equiv +\infty$ if and only if

$$\int_1^\infty \left(\frac{\varkappa(B(0,t))^{\frac{q(p-1)}{p-1-q}}}{t^{n-p}} \right)^{\frac{1}{p-1}} \frac{dt}{t} < \infty. \tag{4.6}$$

By [9, Theorem 1.1], there exists a nontrivial $\mathcal{A}$-superharmonic solution to the homogeneous equation (4.1) in the case $\mu = 0$ if and only if $\mathbf{W}_{1,p}\sigma \not\equiv +\infty$ and $\mathbf{K}_{p,q}\sigma \not\equiv +\infty$, i.e., conditions (3.3) and (4.6) hold. The next theorem shows that this solution is actually reachable.

Theorem 4.2. *Let $0 < q < p - 1$, and let $\sigma \in \mathcal{M}^+(\mathbb{R}^n)$. Then the nontrivial minimal $\mathcal{A}$-superharmonic solution u of*

$$\begin{cases} -\operatorname{div} \mathcal{A}(x, \nabla u) = \sigma u^q, & u \geq 0 \quad in \ \mathbb{R}^n, \\ \liminf_{|x| \to \infty} u = 0, \end{cases} \tag{4.7}$$

constructed in the proof of [9, Theorem 1.1] under the conditions (3.3) and (4.6), is an $\mathcal{A}$-superharmonic reachable solution.

Proof. We start with the same construction as in the proof of [9, Theorem 1.1] for the minimal $\mathcal{A}$-superharmonic solution u, but with datum $\sigma|_{B_m(0)}$ in place of σ ($m = 1, 2, \dots$).

For a fixed m, let v_m be the minimal $\mathcal{A}$-superharmonic solution to the equation

$$\begin{cases} -\operatorname{div} \mathcal{A}(x, \nabla v_m) = \sigma|_{B_m(0)} v_m^q, & v_m \geq 0, \quad in \ \mathbb{R}^n, \\ \liminf_{|x| \to \infty} v_m = 0, \end{cases}$$

We recall from the construction in [9] that

$$v_m = \lim_{j \to \infty}(\lim_{k \to \infty} v_{j,m}^k),$$

where $v_{1,m}^k$ ($k = 0, 1, 2, \dots$) is the $\mathcal{A}$-superharmonic renormalized solution of

$$\begin{cases} -\operatorname{div} \mathcal{A}(x, \nabla v_{1,m}^k) = \sigma|_{B_m(0) \cap B_{2^k}(0)} w_{0,m}^q & in \ B_{2^k}(0), \\ v_{1,m}^k = 0 \quad on \ \partial B_{2^k}(0), \end{cases}$$

with $w_{0,m} = c_0(\mathbf{W}_{1,p}(\sigma|_{B_m(0)}))^{\frac{p-1}{p-1-q}}$, and $v_{j,m}^k$ ($k = 0, 1, 2, \dots, j = 2, 3, \dots$) is the $\mathcal{A}$-superharmonic renormalized solution of

$$\begin{cases} -\operatorname{div} \mathcal{A}(x, \nabla v_{j,m}^k) = \sigma|_{B_m(0) \cap B_{2^k}(0)} (\lim_{i \to \infty} v_{j-1,m}^i)^q & in \ B_{2^k}(0), \\ v_{j,m}^k = 0 \quad on \ \partial B_{2^k}(0). \end{cases}$$

Here c_0 is a fixed constant such that

$$0 < c_0 \leq \min\left\{\left(\mathfrak{c}^{\frac{q}{p-1-q}}K^{-1}\right)^{\frac{p-1}{p-1-q}}, CK^{\frac{1-p}{p-1-q}}\right\}, \qquad (4.8)$$

where C is the constant in (3.9) of [9], and $\mathfrak{c}$ is the constant in (3.10) of [9] with $\alpha = 1$.

We also recall from [9] that

$$u = \lim_{j\to\infty}(\lim_{k\to\infty} u_j^k),$$

where u_j^k are the $\mathcal{A}$-superharmonic renormalized solutions of the corresponding problems in $B_{2^k}(0)$ with σ in place of $\sigma|_{B_m(0)}$. In particular, $\min(u_j^k, l) \in W_0^{1,p}(B_{2^k}(0))$ and $\min(v_{j,m}^k, l) \in W_0^{1,p}(B_{2^k}(0))$ for all $l > 0$.

Thus, by the above version of the comparison principle (Lemma 4.1) we see that

$$v_{j,m_1}^k \leq v_{j,m_2}^k \leq u_j^k \quad \text{in } B_{2^k}(0),$$

whenever $m_1 \leq m_2$.

This yields

$$0 \leq v_1 \leq v_2 \leq \cdots \leq v_m \leq u \quad \text{in } \mathbb{R}^n.$$

Letting now $m \to \infty$, we obtain an $\mathcal{A}$-superharmonic reachable solution

$$v := \lim_{m\to\infty} v_m$$

to (4.7) such that $v \leq u$ in $\mathbb{R}^n$. As u is the minimal $\mathcal{A}$-superharmonic solution of (4.7), we see on the other hand that $u \leq v$, and thus $u = v$, which completes the proof. $\qquad\square$

Remark. In the proof of [9, Theorem 1.1], there is a misprint in the exponent in inequality (4.8) above for the constant c_0. This choice of c_0 ensures the minimality of the solution u of (4.7) constructed in [9].

We recall that, by [37, Theorem 1.1] and [37, Remark 4.3], a nontrivial $\mathcal{A}$-superharmonic solution of (4.1) exists if and only if $\mathbf{W}_{1,p}\sigma \not\equiv \infty$, $\mathbf{K}_{p,q}\sigma \not\equiv \infty$, and $\mathbf{W}_{1,p}\mu \not\equiv \infty$, i.e., the following three conditions hold:

$$\int_1^\infty \left(\frac{\sigma(B(0,\rho))}{\rho^{n-p}}\right)^{\frac{1}{p-1}} \frac{d\rho}{\rho} < +\infty, \qquad (4.9)$$

$$\int_1^\infty \left(\frac{\varkappa(B(0,\rho))^{\frac{q(p-1)}{p-1-q}}}{t^{n-p}}\right)^{\frac{1}{p-1}} \frac{d\rho}{\rho} < \infty, \qquad (4.10)$$

$$\int_1^\infty \left(\frac{\mu(B(0,\rho))}{\rho^{n-p}}\right)^{\frac{1}{p-1}} \frac{d\rho}{\rho} < +\infty. \qquad (4.11)$$

Theorem 4.3. *Let $0 < q < p - 1$, and let $\mu, \sigma \in \mathcal{M}^+(\mathbb{R}^n)$, where $\mu << \mathrm{cap}_p$. Then, under the conditions (4.9), (4.10), and (4.11), there exists a nontrivial minimal reachable $\mathcal{A}$-superharmonic solution of (4.1).*

Proof. Since the case $\mu = 0$ was treated in Theorem 4.2 above, without loss of generality we may assume that $\mu \neq 0$. We recall that in the proof of [37, Theorem 1.1], a nontrivial $\mathcal{A}$-superharmonic solution u of (4.1), was constructed using the following iteration process. We set $u_0 = 0$, and for $j = 0, 1, 2, \ldots$ construct the iterations

$$\begin{cases} -\text{div}\,\mathcal{A}(x, \nabla u_{j+1}) = \sigma u_j^q + \mu & \text{in } \mathbb{R}^n, \\ \liminf_{|x|\to\infty} u_{j+1} = 0, \end{cases} \qquad (4.12)$$

where $u_j \in L^q_{\text{loc}}(\mathbb{R}^n, d\sigma)$. We observe that, for each j, the solution u_{j+1} was chosen in [37] so that $u_j \leq u_{j+1}$ $(j = 0, 1, 2, \ldots)$ by a version of the comparison principle (see [34, Lemma 3.7 and Lemma 3.9]). Then $u := \lim_{j\to\infty} u_j$ is a nontrivial $\mathcal{A}$-superharmonic solution of (4.1).

We now modify this argument as follows to obtain a *minimal* nontrivial $\mathcal{A}$-superharmonic solution of (4.1). Notice that $\mu << \text{cap}_p$ by assumption, and, as mentioned above, $\sigma << \text{cap}_p$, since a solution exists. Hence, clearly the measure $\sigma u_j^q + \mu << \text{cap}_p$ as well. By Theorem 3.6, u_{j+1} can be chosen as the *minimal* $\mathcal{A}$-superharmonic solution to (4.12).

It follows by induction that $u_j \leq u_{j+1}$ $(j = 0, 1, 2, \ldots)$. Indeed, this is trivial when $j = 0$, and then by the inductive step,

$$\sigma u_{j-1}^q + \mu \leq \sigma u_j^q + \mu, \quad j = 1, 2, \ldots,$$

which is obvious when $j = 1$. From this, using Theorem 3.7 we deduce $u_j \leq u_{j+1}$ for all $j = 1, 2, \ldots$.

Similarly, if $\tilde{u}$ is any $\mathcal{A}$-superharmonic solution of (4.1), then again arguing by induction and using Theorem 3.7, we deduce that $u_{j+1} \leq \tilde{u}$ $(j = 0, 1, 2, \ldots)$, since

$$\sigma u_{j-1}^q + \mu \leq \sigma \tilde{u}^q + \mu, \quad j = 1, 2, \ldots.$$

Consequently, $u \leq \tilde{u}$, i.e., u is the minimal $\mathcal{A}$-superharmonic solution of (4.1).

We next show that u is a reachable solution. Using a similar iteration process with $\sigma|_{B_m(0)}$ in place of σ and $\mu|_{B_m(0)}$ in place of μ $(m = 1, 2, \ldots)$, we set $v_{0,m} = 0$ and define $v_{j,m}$ to be the minimal $\mathcal{A}$-superharmonic solution to the equation

$$\begin{cases} -\text{div}\,\mathcal{A}(x, \nabla v_{j+1,m}) = \sigma|_{B_m(0)} v_{j,m}^q + \mu|_{B_m(0)}, & v_{j,m} \geq 0, \quad \text{in } \mathbb{R}^n, \\ \liminf_{|x|\to\infty} v_{j,m} = 0, \end{cases}$$

where $v_{j,m} \leq v_{j+1,m}$ for each $m = 1, 2, \ldots$.

As above, arguing by induction and using Theorem 3.7, we deduce

$$v_{j,m_1} \leq v_{j,m_2} \leq u_j, \quad j = 1, 2, \ldots,$$

whenever $m_1 \leq m_2$. It follows that $v_m := \lim_{j\to\infty} v_{j,m} \leq u$ $(m = 1, 2, \ldots)$ is an $\mathcal{A}$-superharmonic solution of the equation

$$\begin{cases} -\text{div}\,\mathcal{A}(x, \nabla v_m) = \sigma|_{B_m(0)} v_m^q + \mu|_{B_m(0)}, & v_m \geq 0 \quad \text{in } \mathbb{R}^n, \\ \liminf_{|x|\to\infty} v_m = 0, \end{cases}$$

where $v_{m_1} \leq v_{m_2} \leq u$ if $m_1 \leq m_2$.

Thus, letting $m \to \infty$, we obtain an $\mathcal{A}$-superharmonic *reachable* solution $v := \lim_{m\to\infty} v_m$ to (4.7) such that $v \leq u$. Since u is the minimal $\mathcal{A}$-superharmonic solution of (4.7), we see that $u = v$, which completes the proof. $\qquad\square$

We now prove the uniqueness property for reachable solutions of (4.1).

Theorem 4.4. *Let $0 < q < p − 1$, and let $\mu, \sigma \in \mathcal{M}^+(\mathbb{R}^n)$, where $\mu << \mathrm{cap}_p$. Suppose $\mathcal{A}$ satisfies conditions* (2.1) *and* (2.2)*. Then nontrivial $\mathcal{A}$-superharmonic reachable solutions of* (4.1) *are unique.*

Proof. Let u, v be two nontrivial $\mathcal{A}$-superharmonic solutions of (4.1) in $\mathbb{R}^n$. Then by [37, Theorem 1.1] and [37, Remark 4.3], there exists a constant $C \geq 1$, depending only on p, q and n, such that

$$C^{-1} u \leq v \leq C u \quad \text{in } \mathbb{R}^n.$$

Hence, clearly,

$$-\mathrm{div}\, \mathcal{A}(x, \nabla v) = \sigma v^q + \mu \leq C^q (\sigma u^q + \mu) = -\mathrm{div}\, \mathcal{A}(x, \nabla(C^{\frac{q}{p-1}} u)).$$

Notice that here by definition $u, v \in L^q_{\mathrm{loc}}(\mathbb{R}^n, \sigma)$. Suppose that v is a reachable solution of (4.1) in $\mathbb{R}^n$. Then by Corollary 3.11 with $\sigma v^q + \mu$ in place of σ, and $\tilde{\sigma} = C^q (\sigma u^q + \mu)$, it follows that $v \leq C^{\frac{q}{p-1}} u$.

By iterating this argument, we deduce

$$v \leq C^{(\frac{q}{p-1})^j} u \quad \text{in } \mathbb{R}^n, \qquad j = 1, 2, \dots.$$

Since $0 < q < p − 1$, letting $j \to \infty$ in the preceding inequality, we obtain $v \leq u$ in $\mathbb{R}^n$. Interchanging the roles of u and v, we see that actually $u = v$ in $\mathbb{R}^n$. □

Corollary 4.5. *Nontrivial $\mathcal{A}$-superharmonic solutions u of* (4.1) *are unique under the assumptions of Theorem 4.4, provided any one of the following conditions holds:*
(i) $u \in L^q(\mathbb{R}^n, d\sigma)$ and $\mu \in \mathcal{M}^+_b(\mathbb{R}^n)$, or equivalently $\varkappa(\mathbb{R}^n) < \infty$ and $\mu \in \mathcal{M}^+_b(\mathbb{R}^n)$;
(ii) $\lim_{|x| \to \infty} u(x) = 0$;
(iii) $|Du| \in L^p(\mathbb{R}^n)$, or $|Du| \in L^{\gamma,\infty}(\mathbb{R}^n)$ for some $(p − 1)n/(n − 1) \leq \gamma < p$.

Proof. Suppose first that (i) holds. By [9, Theorem 4.4], $\varkappa(\mathbb{R}^n) < \infty$ if and only if there exists a nontrivial $\mathcal{A}$-superharmonic solution $u \in L^q(\mathbb{R}^n, d\sigma)$ of (4.7). In particular, since by [37, Theorem 4.1],

$$u \geq C \left[\mathbf{W}_{1,p}\sigma + \left(\mathbf{K}_{p,q}\sigma \right)^{\frac{p-1}{p-1-q}} \right],$$

it follows that

$$\mathbf{W}_{1,p}\sigma \in L^{\frac{q(p-1)}{p-1-q}}(\mathbb{R}^n, d\sigma) \quad \text{and} \quad \mathbf{K}_{p,q}\sigma \in L^q(\mathbb{R}^n, d\sigma).$$

Next, we denote by φ an $\mathcal{A}$-superharmonic solution to the equation

$$\begin{cases} -\mathrm{div}\, \mathcal{A}(x, \nabla\varphi) = \mu, & \varphi \geq 0 \quad \text{in } \mathbb{R}^n, \\ \liminf_{|x| \to \infty} \varphi = 0, \end{cases}$$

where μ is the Riesz measure of φ. Notice that $\varphi \geq K^{-1} \mathbf{W}_{1,p}\mu$ by the lower bound in inequality (3.4). Since $\mu \in \mathcal{M}_b(\mathbb{R}^n)$ and $\varkappa(\mathbb{R}^n) < \infty$, using φ as a test function in inequality (4.5) yields $\mathbf{W}_{1,p}\mu \in L^q(\mathbb{R}^n, d\sigma)$.

Hence, by [37, Theorem 1.1] and [37, Remark 4.3], we deduce that there exists a nontrivial $\mathcal{A}$-superharmonic solution of (4.1) $u \in L^q(\mathbb{R}^n, d\sigma)$, and, for any such a solution, $\sigma u^q + \mu \in \mathcal{M}_b(\mathbb{R}^n)$. It

follows that u is a reachable $\mathcal{A}$-superharmonic solution of (4.1) by Theorem 3.10 and Theorem 3.12 (iii).

In case (ii), by Theorem 3.16 u is a reachable solution of (4.1).

In case (iii), u is a reachable $\mathcal{A}$-superharmonic solution of (4.1) by Theorem 3.10 and Theorem 3.12 (i), (ii).

In all these cases, reachable $\mathcal{A}$-superharmonic solutions are unique by Theorem 4.4. $\qquad\square$

Remark 4.6. *Uniqueness of finite energy solutions u to (4.1) such that $|Du| \in L^p(\mathbb{R}^n)$ in Corollary 4.5(iii) was established in [35, Theorem 6.1] in the special case of the p-Laplace operator using a different method. (See also an earlier result [8, Theorem 5.1] in the case $\mu = 0$.) Solutions of finite energy to (4.1) exist if and only if $\mathbf{W}_{1,p}\sigma \in L^{\frac{(1+q)(p-1)}{p-1-q}}(\mathbb{R}^n, d\sigma)$ and $\mathbf{W}_{1,p}\mu \in L^1(\mathbb{R}^n, d\mu)$ ([35, Theorem 1.1]).*

Remark 4.7. *Under the assumptions of Theorem 4.3, but without the restriction $\mu << \mathrm{cap}_p$, it is still possible to prove the existence of an $\mathcal{A}$-superharmonic reachable solution (not necessarily minimal) of (4.1). The construction of such a solution makes use of an extension of [33, Lemma 6.9] proved below.*

Proof. To prove this claim, we shall construct first a nondecreasing sequence $\{u_m\}_{m\geq 1}$ of $\mathcal{A}$-superharmonic solutions of

$$
\begin{cases}
-\mathrm{div}\,\mathcal{A}(x, \nabla u_m) = \sigma|_{B_m(0)} u_m^q + \mu|_{B_m(0)}, & u_m \geq 0, \quad \text{in } \mathbb{R}^n, \\
\liminf_{|x|\to\infty} u_m = 0.
\end{cases}
$$

Then by [37, Theorem 1.1 and Remark 4.3],

$$
u_m(x) \leq C\left(\mathbf{W}_{1,p}\mu(x) + \mathbf{K}_{p,q}\sigma(x) + [\mathbf{W}_{1,p}\sigma(x)]^{\frac{p-1}{p-1-q}}\right), \quad x \in \mathbb{R}^n.
$$

It follows from [20, Theorem 1.17] that $u_m \to u$ pointwise everywhere, where u is an $\mathcal{A}$-superharmonic reachable solution of (4.1).

The construction of $\{u_m\}_{m\geq 1}$ can be done as follows. It suffices to demonstrate only how to construct u_1 and u_2 such that $u_2 \geq u_1$, since the construction of u_m for $m \geq 3$ is completely analogous. Let v_1 be an $\mathcal{A}$-superharmonic solution of

$$
\begin{cases}
-\mathrm{div}\,\mathcal{A}(x, \nabla v_1) = \mu|_{B_1(0)}, & v_1 \geq 0, \quad \text{in } \mathbb{R}^n, \\
\liminf_{|x|\to\infty} v_1 = 0.
\end{cases}
$$

Here as above v_1 is an a.e. pointwise limit of a subsequence of $\{v_1^{(k)}\}_{k\geq 1}$, where each $v_1^{(k)}$ is a nonnegative $\mathcal{A}$-superharmonic renormalized solution of

$$
\begin{cases}
-\mathrm{div}\,\mathcal{A}(x, \nabla v_1^{(k)}) = \mu|_{B_1(0)} & \text{in } B_k(0), \\
v_1^{(k)} = 0 & \text{on } \partial B_k(0).
\end{cases}
$$

Next, for any $j \geq 1$, let v_{j+1} be an $\mathcal{A}$-superharmonic solution of

$$
\begin{cases}
-\mathrm{div}\,\mathcal{A}(x, \nabla v_{j+1}) = \sigma|_{B_1(0)} v_j^q + \mu|_{B_1(0)}, & v_{j+1} \geq 0, \quad \text{in } \mathbb{R}^n, \\
\liminf_{|x|\to\infty} v_{j+1} = 0.
\end{cases}
$$

Notice that v_{j+1} is an a.e. pointwise limit of a subsequence of $\{v_{j+1}^{(k)}\}_{k\geq 1}$, where each $v_{j+1}^{(k)}$ is a nonnegative $\mathcal{A}$-superharmonic renormalized solution of

$$\begin{cases} -\operatorname{div} \mathcal{A}(x, \nabla v_{j+1}^{(k)}) = \sigma|_{B_1(0)} v_j^q + \mu|_{B_1(0)} & \text{in } B_k(0), \\ v_{j+1}^{(k)} = 0 & \text{on } \partial B_k(0). \end{cases}$$

By [33, Lemma 6.9] we may assume that $v_2^{(k)} \geq v_1^{(k)}$ for all $k \geq 1$, and hence $v_2 \geq v_1$. In the same way, by induction we deduce that $v_{j+1}^{(k)} \geq v_j^{(k)}$ for all $j, k \geq 1$. It follows that $v_{j+1} \geq v_j$, and

$$v_{j+1} \leq C \, \mathbf{W}_{1,p}(\sigma v_{j+1}^q) + C \, \mathbf{W}_{1,p}(\mu).$$

Then by [37, Theorem 4.1], for any $j \geq 1$, we obtain the bound

$$v_{j+1}(x) \leq C \left(\mathbf{W}_{1,p}\mu(x) + \mathbf{K}_{p,q}\sigma(x) + [\mathbf{W}_{1,p}\sigma(x)]^{\frac{p-1}{p-1-q}} \right), \quad x \in \mathbb{R}^n. \tag{4.13}$$

Thus, the nondecreasing sequence $\{v_j\}_{j\geq 1}$ converges to an $\mathcal{A}$-superharmonic solution u_1 of

$$\begin{cases} -\operatorname{div} \mathcal{A}(x, \nabla u_1) = \sigma|_{B_1(0)} u_1^q + \mu|_{B_1(0)}, & u_1 \geq 0, \quad \text{in } \mathbb{R}^n, \\ \displaystyle\liminf_{|x|\to\infty} u_1 = 0. \end{cases}$$

To construct u_2 such that $u_2 \geq u_1$, let w_1 be an $\mathcal{A}$-superharmonic solution of

$$\begin{cases} -\operatorname{div} \mathcal{A}(x, \nabla w_1) = \mu|_{B_2(0)}, & w_1 \geq 0, \quad \text{in } \mathbb{R}^n, \\ \displaystyle\liminf_{|x|\to\infty} w_1 = 0. \end{cases}$$

Notice that w_1 is an a.e. pointwise limit of a subsequence of $\{w_1^{(k)}\}_{k\geq 1}$, where each $w_1^{(k)}$ is a nonnegative $\mathcal{A}$-superharmonic renormalized solution of

$$\begin{cases} -\operatorname{div} \mathcal{A}(x, \nabla w_1^{(k)}) = \mu|_{B_2(0)} & \text{in } B_k(0), \\ w_1^{(k)} = 0 & \text{on } \partial B_k(0). \end{cases}$$

Again, by [33, Lemma 6.9] we may assume that $w_1^{(k)} \geq v_1^{(k)}$ for all $k \geq 1$, and hence $w_1 \geq v_1$.

Next, for any $j \geq 1$, let w_{j+1} be an $\mathcal{A}$-superharmonic solution of

$$\begin{cases} -\operatorname{div} \mathcal{A}(x, \nabla w_{j+1}) = \sigma|_{B_2(0)} w_j^q + \mu|_{B_2(0)}, & w_{j+1} \geq 0, \quad \text{in } \mathbb{R}^n, \\ \displaystyle\liminf_{|x|\to\infty} w_{j+1} = 0. \end{cases}$$

Notice that w_{j+1} is an a.e. pointwise limit of a subsequence of $\{w_{j+1}^{(k)}\}_{k\geq 1}$, where each $w_{j+1}^{(k)}$ is a nonnegative $\mathcal{A}$-superharmonic renormalized solution of

$$\begin{cases} -\operatorname{div} \mathcal{A}(x, \nabla w_{j+1}^{(k)}) = \sigma|_{B_2(0)} w_j^q + \mu|_{B_2(0)} & \text{in } B_k(0), \\ w_{j+1}^{(k)} = 0 & \text{on } \partial B_k(0). \end{cases}$$

We can ensure here that $w_{j+1}^{(k)} \geq \max\{v_{j+1}^{(k)}, w_j^{(k)}\}$ for all $j, k \geq 1$. Indeed, since $w_1 \geq v_1$ and $w_1 \geq 0$, by Lemma 4.8 below we may assume that $w_2^{(k)} \geq \max\{v_2^{(k)}, w_1^{(k)}\}$ for all $k \geq 1$, and hence $w_2 \geq \max\{v_2, w_1\}$. Repeating this argument by induction we obtain $w_{j+1}^{(k)} \geq \max\{v_{j+1}^{(k)}, w_j^{(k)}\}$ for all $j, k \geq 1$.

It follows that $w_{j+1} \geq \max\{v_{j+1}, w_j\}$ for all $j \geq 1$. As in (4.13) we have

$$w_{j+1}(x) \leq C\left(\mathbf{W}_{1,p}\mu(x) + \mathbf{K}_{p,q}\sigma(x) + [\mathbf{W}_{1,p}\sigma(x)]^{\frac{p-1}{p-1-q}}\right), \quad x \in \mathbb{R}^n,$$

and hence the nondecreasing sequence $\{w_j\}$ converges to an $\mathcal{A}$-superharmonic solution u_2 of

$$\begin{cases} -\operatorname{div}\mathcal{A}(x, \nabla u_2) = \sigma|_{B_2(0)}u_2^q + \mu|_{B_2(0)}, & u_2 \geq 0, \quad \text{in } \mathbb{R}^n, \\ \liminf_{|x|\to\infty} u_2 = 0 \end{cases}$$

such that $u_2 \geq u_1$, as desired. $\qquad\square$

The following lemma invoked in the argument presented above is an extension of [33, Lemma 6.9].

Lemma 4.8. *Let Ω be a bounded open set in $\mathbb{R}^n$ and let $\mu_1, \mu_2 \in \mathcal{M}_b^+(\Omega)$. Suppose that u_i $(i = 1, 2)$ is a renormalized solution of*

$$\begin{cases} -\operatorname{div}\mathcal{A}(x, \nabla u_i) = \mu_i & \text{in } \Omega, \\ u_i = 0 & \text{on } \partial\Omega. \end{cases}$$

Then for any measure $\nu \in \mathcal{M}_b^+(\Omega)$ such that $\nu \geq \mu_1$ and $\nu \geq \mu_2$, there is a renormalized solution ν of

$$\begin{cases} -\operatorname{div}\mathcal{A}(x, \nabla v) = \nu & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases}$$

such that $v \geq u_1$ and $v \geq u_2$ a.e.

Proof. For $i = 1, 2$, let $u_{i,k} = \min\{u_i, k\}$ $(k = 1, 2, \dots)$. Then $u_{i,k}$ is the bounded renormalized solution of

$$\begin{cases} -\operatorname{div}\mathcal{A}(x, \nabla u_{i,k}) = \mu_{i0}|_{\{u_i < k\}} + \lambda_{i,k} & \text{in } \Omega, \\ u_{i,k} = 0 & \text{on } \partial\Omega. \end{cases}$$

Here $\mu_i = \mu_{i0} + \mu_{is}$ $(i = 1, 2)$ is the decomposition of μ_i used in Section 3 above, where $\mu_{i0}, \mu_{is} \in \mathcal{M}_b^+(\Omega)$, $\mu_{i0} \ll \operatorname{cap}_p$, and μ_{is} is concentrated on a set of zero p-capacity. Moreover, $\lambda_{i,k} \in \mathcal{M}_b^+(\Omega)$ and $\lambda_{i,k} \to \mu_{is}$ in the narrow topology of measures as $k \to \infty$ (see Definition 3.3).

Now let v_k $(k = 1, 2, \dots)$ be a renormalized solution of

$$\begin{cases} -\operatorname{div}\mathcal{A}(x, \nabla v_k) = \sum_{i=1}^2 (\mu_{i0} + \lambda_{i,k}) + (\nu - \mu_1) + (\nu - \mu_2) & \text{in } \Omega, \\ v_k = 0 & \text{on } \partial\Omega. \end{cases}$$

Then by [33, Lemma 6.8] we deduce $v_k \geq \max\{u_{1,k}, u_{2,k}\}$ for all $k \geq 1$. Finally, we use the stability results of [12] to find a subsequence of $\{v_k\}$ that converges a.e. to a desired function v. $\qquad\square$

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Conflict of interest

The authors declare no conflict of interest.

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