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Research article

Interior curvature bounds for a type of mixed Hessian quotient equations†

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Abstract: We derive interior curvature bounds for admissible solutions of a class of mixed Hessian curvature equations subject to affine Dirichlet data. As an application, we study a Plateau type problem for locally convex Weingarten hypersurfaces.

Keywords: prescribed curvature problems; mixed Hessian quotient equations; curvature estimates

Dedicated to Professor Neil S. Trudinger on occasion of his 80th birthday.

1. Introduction

One classical problem in convex geometry is the Minkowski problem, which is to find convex hypersurfaces in $\mathbb{R}^{n+1}$ whose Gaussian curvature is prescribed as a function defined on $\mathbb{S}^n$ in terms of the inverse Gauss map. It has been settled by the works of Minkowski [23], Alexandrov [1], Fenchel and Jessen [28], Nirenberg [25], Pogorelov [26], Cheng and Yau [3], etc.. In smooth catagory, the Minkowski problem is equivalent to solve following Monge-Ampère equation

$$\det(\nabla^2 u + u g_{\mathbb{S}^n}) = f \quad \text{on } \mathbb{S}^n,$$

where u is the support function of the convex hypersurface, $\nabla^2 u + u g_{\mathbb{S}^n}$ the spherical Hessian matrix of the function u. If we take an orthonormal frame on $\mathbb{S}^n$, the spherical Hessian of u is $W_u(x) := u_{ij}(x) + u(x)\delta_{ij}$, whose eigenvalues are actually the principal radii of the hypersurface.

The general problem of finding a convex hypersurface, whose k-th symmetric function of the principal radii is the prescribed function on its outer normals for $1 \leq k < n$, is often called the Christoffel-Minkowski problem. It corresponds to finding convex solutions of the nonlinear Hessian equation

$$\sigma_k(W_u) = f \quad \text{on } \mathbb{S}^n.$$

This problem was settled by Guan et al [14, 15]. In [16], Guan and Zhang considered a mixed Hessian equation as follows

$$\sigma_k(W_u(x)) + \alpha(x)\sigma_{k-1}(W_u(x)) = \sum_{l=0}^{k-2} \alpha_l(x)\sigma_l(W_u(x)), \quad x \in \mathbb{S}^n, \tag{1.1}$$

where $\alpha(x), \alpha_l(x)\,(0 \le l \le k-1)$ are some functions on $\mathbb{S}^n$. By imposing some group-invariant conditions on those coefficient's functions as in [11], the authors proved the existence of solutions.

Let M be a hypersurface of Euclidean space $\mathbb{R}^{n+1}$ and $M = \operatorname{graph} u$ in a neighbourhood of some point at which we calculate. Let A be the second fundamental form of M, $\lambda(A) = (\lambda_1, \cdots, \lambda_n) \in \mathbb{R}^n$ the eigenvalues of A with respect to the induced metric of $M \subset \mathbb{R}^{n+1}$, i.e., the principle curvatures of M, and $\sigma_k(\lambda)$ the k-th elementary symmetric function, $\sigma_0(\lambda) = 1$. It is natural to study the prescribing curvature problems on this aspect. In 1980s, Caffarelli, Nirenberg and Spruck studied the prescribing Weingarten curvature problem. The problem is equivalent to solve the following equation

$$\sigma_k(\lambda)(X) = f(X), \quad X \in \mathcal{M}.$$

When $k = n$, the problem is just the Minkowski problem; when $k = 1$, it is the prescribing mean curvature problem, c.f. [30, 33]. The prescribing Weingarten curvature problem has been studied by many authors, we refer to [2, 9, 11–13, 29, 37] and references therein for related works. Recently, Zhou [36] generalised above mixed prescribed Weingarten curvature equation. He obtained interior gradient estimates for

$$\sigma_k(A) + \alpha(x)\sigma_{k-1}(A) = \sum_{l=0}^{k-2} \alpha_l(x)\sigma_l(A), \quad x \in B_r(0) \subset \mathbb{R}^n \tag{1.2}$$

where $\sigma_k(A) := \sigma_k(\lambda(A))$, and the coefficients satisfy $\alpha_{k-2} > 0$ and $\alpha_l \ge 0$ for $0 \le l \le k-3$.

Mixed Hessian type of equations arise naturally from many important geometric problems. One example is the so-called Fu-Yau equation arising from the study of the Hull-Strominger system in theoretical physics, which is an equation that can be written as the linear combination of the first and the second elementary symmetric functions

$$\sigma_1(i\partial\bar{\partial}(e^u + \alpha' e^{-u})) + \alpha'\sigma_2(i\partial\bar{\partial}u) = \phi \tag{1.3}$$

on n-dimensional compact Kähler manifolds. There are a lot of works related to this equation recently, see [6, 7, 27] for example. Another important example is the special Lagrangian equations introduced by Harvey and Lawson [18], which can be written as the alternative combinations of elementary symmetric functions

$$\sin\theta\Big(\sum_{k=0}^{[\frac{n}{2}]}(-1)^k\sigma_{2k}(D^2u)\Big) + \cos\theta\Big(\sum_{k=0}^{[\frac{n-1}{2}]}(-1)^k\sigma_{2k+1}(D^2u)\Big) = 0.$$

This equation is equivalent to

$$F(D^2u) := \arctan\lambda_1 + \cdots + \arctan\lambda_n = \theta$$

where λ_i's are the eigenvalues of D^2u. It is called supercritical if $\theta \in (\frac{(n-2)\pi}{2}, \frac{n\pi}{2})$ and hypercritical if $\theta \in (\frac{(n-1)\pi}{2}, \frac{n\pi}{2})$. The Lagrangian phase operator F is concave for the hypercritical case and has convex level sets for the supercritical case, while in general F fails to be concave. For subcritical case, i.e., $0 \le \theta < \frac{(n-2)\pi}{2}$, solutions of the special Lagrangian equation can fail to have interior estimates [24, 35]. Jacob-Yau [20] initiated to study the deformed Hermitian Yang-Mills (dHYM) equation on a compact Kähler manifold (M, ω):

$$\mathrm{Re}(\chi_u + \sqrt{-1}\omega)^n = \cot\theta_0 \mathrm{Im}(\chi_u + \sqrt{-1}\omega)^n,$$

where χ is a closed real $(1,1)$-form, $\chi_u = \chi + \sqrt{-1}\partial\bar{\partial}u$, and θ_0 is the angles of the complex number $\int_M(\chi + \sqrt{-1}\omega)^n$, u is the unknown real smooth function on M. Jacob-Yau showed that dHYM equation has an equivalent form of special Lagrangian equation. Collins-Jacob-Yau [5] solved the dHYM equation by continuity method and Fu-Zhang [8] gave an alternative approach by dHYM flow, both of which considered in the supercritical case. For more results concerning about dHYM equation and special Lagrangian equation, one can consult Han-Jin [17], Chu-Lee [4] and the references therein. Note that for $n = 3$ and hypercritical $\theta \in (\pi, \frac{3\pi}{2})$, the special Lagrangian equation (1.3) is

$$\sigma_3(D^2u) + \tan\theta\sigma_2(D^2u) = \sigma_1(D^2u) + \tan\theta\sigma_0(D^2u)$$

which is included in (1.1).

In this paper we derive interior curvature bounds for admissible solutions of a class of curvature equations subject to affine Dirichlet data. Let Ω be a bounded domain in $\mathbb{R}^n$, and let $u \in C^4(\Omega) \cap C^{0,1}(\bar{\Omega})$ be an admissible solution of

$$\begin{cases} \sigma_k(\lambda) + g(x,u)\sigma_{k-1}(\lambda) = \sum_{l=0}^{k-2} \alpha_l(x,u)\sigma_l(\lambda) & \text{in } \Omega, \\ u = \phi & \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

where $g(x,u)$ and $\alpha_l(x,u) > 0$, $l = 0, 1, \cdots, k-2$, are given smooth functions on $\bar{\Omega} \times \mathbb{R}$ and ϕ is affine, $\lambda = (\lambda_1, \cdots, \lambda_n)$ is the vector of the principal curvatures of graph u. u is the *admissible solution* in the sense that $\lambda \in \Gamma_k$ for points on the graph of u, with

$$\Gamma_k = \{\lambda \in \mathbb{R}^n | \sigma_1(\lambda) > 0, \cdots, \sigma_k(\lambda) > 0\}.$$

For simplicity we denote $F = G_k - \sum_{l=0}^{k-2} \alpha_l G_l$ and $G_l = \sigma_l(\lambda)/\sigma_{k-1}(\lambda)$ for $l = 0, 1, \cdots, k-2, k$. The ellipticity and concavity properties of the operator F have been proved in [16]. Our main result is as follows.

Theorem 1.1. *Assume that for every l $(0 \le l \le k-2)$, $\alpha_l, g \in C^{1,1}(\bar{\Omega} \times \mathbb{R})$, $\alpha_l > 0$, and $g > 0$ or $g < 0$. ϕ is affine in* (1.4)*. For any fixed $\beta > 0$, if $u \in C^4(\Omega) \cap C^{0,1}(\bar{\Omega})$ is an admissible solution of* (1.4)*, then there exists a constant C, depending only on $n, k, \beta, \|u\|_{C^1(\bar{\Omega})}, \alpha_l, g$ and their first and second derivatives, such that the second fundamental form $\mathbf{A}$ of graph u satisfies*

$$|\mathbf{A}| \le \frac{C}{(\phi - u)^\beta}.$$

Remark 1.1. *Comparing with [16], here we require $g > 0$ or $g < 0$ additionally. Also our curvature estimates still hold if $\alpha_l \equiv 0$ for some $0 \le l \le k-2$. More over, if $\alpha_l \equiv 0$ for all $l = 0, 1, \cdots, k-2$, Eq* (1.4) *becomes the Hessian quotient equation and the results can be followed from [29].*

To see that this is an interior curvature estimate, we need to verify that $\phi - u > 0$ on Ω. We apply the strong maximum principle for the minimal graph equation. Since ϕ is affine, it satisfies the following minimal graph equation

$$Qu := (1+|Du|^2)\triangle u - u_i u_j u_{ij} = nH(1+|Du|^2)^{\frac{3}{2}} = 0 \quad \text{on} \quad \Omega.$$

Since u is k-admissible solution, and $n \ge k \ge 2$, graph of u is mean-convex and $Qu > Q\phi = 0$. By the comparison principle for quasilinear equations (Theorem 10.1 in [10]), we then have $\phi > u$ on Ω.

The main application of the curvature bound of Theorem 1.1 is to extend various existence results for the Dirichlet problem for curvature equations of mixed Hessian type.

Theorem 1.2. *Let Ω be a bounded domain in $\mathbb{R}^n$, let $\alpha_l, g \in C^{1,1}(\bar{\Omega}\times\mathbb{R})$ satisfying* $\inf |g| > 0$, *$\partial_u g(x,u) \le 0$, $\alpha_l > 0$ and $\partial_u \alpha_l(x,u) \ge 0$. Suppose there is an admissible function $\underline{u} \in C^2(\Omega) \cap C^{0,1}(\bar{\Omega})$ satisfying*

$$F[\underline{u}] \ge -g(x,\underline{u}) \quad in\ \Omega, \qquad \underline{u} = 0 \quad on\, \partial\Omega. \tag{1.5}$$

Then the problem

$$F[u] = -g(x,u) \quad in\ \Omega, \qquad u = 0 \quad on\ \partial\Omega. \tag{1.6}$$

has a unique admissible solution $u \in C^{3,\alpha}(\Omega) \cap C^{0,1}(\bar{\Omega})$ for all $\alpha \in (0,1)$.

Remark 1.2. *$\partial_u g \le 0$, $\partial_u \alpha_l(x,u) \ge 0$ and the existence of sub-solutions are required in the C^0 estimate. The C^1 interior estimate is a slightly modification of the result in Theorem 5.1.1 [36] since the coefficients g, α_l of* (1.2) *are independent of u. We use conditions $\partial_u g \le 0$ and $\partial_u \alpha_l(x,u) \ge 0$ again to eliminate extra terms in the C^1 estimate.*

As a further application of the a priori curvature estimate we also consider a Plateau-type problem for locally convex Weingarten hypersurfaces. Let Σ be a finite collection of disjoint, smooth, closed, codimension 2 submanifolds of $\mathbb{R}^{n+1}$. Suppose Σ bounds a locally uniformly convex hypersurface $\mathcal{M}_0$ with

$$f_{(n)}(\lambda^0) := \frac{\sigma_n}{\sigma_{n-1}}(\lambda^0) - \sum_{l=0}^{n-2} \alpha_l \frac{\sigma_l}{\sigma_{n-1}}(\lambda^0) \ge c,$$

where $\lambda^0 = (\lambda^0_1, \cdots, \lambda^0_n)$ are the principal curvatures of $\mathcal{M}_0$ and α_l's are positive constants, $c \ne 0$ is a constant. Is there a locally convex hypersurface $\mathcal{M}$ with boundary Σ and $f_{(n)}(\lambda) = c$, where $\lambda = (\lambda_1, \cdots, \lambda_n)$ are the principal curvatures of $\mathcal{M}$?

Theorem 1.3. *Let Σ, $f_{(n)}(\lambda)$ be as above. If Σ bounds a locally uniformly convex hypersurface $\mathcal{M}_0$ with $f_{(n)}(\lambda^0) \ge c$ at each point of $\mathcal{M}_0$. Then Σ bounds a smooth, locally convex hypersurface $\mathcal{M}$ with $f_{(n)}(\lambda) = c$ at each point of $\mathcal{M}$.*

2. Proof of the curvature bound

We compute using a local orthonormal frame field $\hat{\mathbf{e}}_1, \cdots, \hat{\mathbf{e}}_n$ defined on $\mathcal{M}$ = graph u in a neighbourhood of the point at which we are computing. The standard basis of $\mathbb{R}^{n+1}$ is denoted by $\mathbf{e}_1, \cdots, \mathbf{e}_{n+1}$. Covariant differentiation on $\mathcal{M}$ in the direction $\hat{\mathbf{e}}_i$ is denoted by ∇_i. The components of the second fundamental form $\mathbf{A}$ of $\mathcal{M}$ in the basis $\hat{\mathbf{e}}_1, \cdots, \hat{\mathbf{e}}_n$ are denoted by (h_{ij}). Thus

$$h_{ij} = \langle D_{\hat{\mathbf{e}}_i}\hat{\mathbf{e}}_j, \nu\rangle,$$

where D and $\langle\cdot,\cdot\rangle$ denote the usual connection and inner product on $\mathbb{R}^{n+1}$, and ν denotes the upward unit normal

$$\nu = \frac{(-Du, 1)}{\sqrt{1+|Du|^2}}.$$

The differential equation in (1.4) can then be expressed as

$$F(\mathbf{A}, X) = -g(X). \tag{2.1}$$

As usual we denote first and second partial derivatives of F with respect to h_{ij} by F^{ij} and $F^{ij,rs}$. We assume summation from 1 to n over repeated Latin indices unless otherwise indicated. Following two lemmas are similar to the ones in [29] with minor changes, so we omit the proof.

Lemma 2.1. *The second fundamental form h_{ab} satisfies*

$$\begin{aligned} F^{ij}\nabla_i\nabla_j h_{ab} = &- F^{ij,rs}\nabla_a h_{ij}\nabla_b h_{rs} + F^{ij}h_{ij}h_{ap}h_{pb} \\ &- F^{ij}h_{ip}h_{pj}h_{ab} - \nabla_a\nabla_b g + \sum_{l=0}^{k-2}(\nabla_a\alpha_l\nabla_b G_l + \nabla_b\alpha_l\nabla_a G_l) \\ &+ \sum_{l=0}^{k-2}\nabla_a\nabla_b\alpha_l\cdot G_l. \end{aligned}$$

Lemma 2.2. *For any $\alpha = 1, \cdots, n+1$, we have*

$$F^{ij}\nabla_i\nabla_j\nu_\alpha + F^{ij}h_{ip}h_{pj}\nu_\alpha = \langle\nabla g, \mathrm{e}_\alpha\rangle - \sum_{l=0}^{k-2}\langle\nabla\alpha_l, \mathrm{e}_\alpha\rangle G_l.$$

Lemma 2.3. *There is a constant $C > 0$, depending only on $n, k, \inf\alpha_l, |g|_{C^0}$, so that for any $l = 0, 1, \cdots, k-2$,*

$$|G_l| \le C.$$

Proof. Proof by contradiction. If the result is not true, then for any integer i, there is an admissible solution $u_{(i)}$, a point $x_{(i)} \in \Omega$ and an index $0 \le l_{(i)} \le k-2$, so that

$$\frac{\sigma_{l_{(i)}}}{\sigma_{k-1}}(\lambda[u_{(i)}]) > i \quad \text{at } x_{(i)}.$$

By passing to a subsequence, we may assume $l_{(i)} \to l_\infty$ and $x_{(i)} \to x_\infty \in \bar{\Omega}$ as $i \to +\infty$. Therefore

$$\lim_{i\to+\infty}\frac{\sigma_{l_\infty}}{\sigma_{k-1}}(\lambda[u_{(i)}])(x_{(i)}) = +\infty,$$

or we may simply write $\frac{\sigma_{l_\infty}}{\sigma_{k-1}} \to +\infty$ if no ambiguilty arises. Since $\alpha_{l_\infty} > 0$, and g is bounded, by (1.4) we have $\frac{\sigma_k}{\sigma_{k-1}} \to +\infty$. For i large enough, $\sigma_k > 0$. By Newton-MacLaurin inequalities, we have

$$\frac{\sigma_{l_\infty}}{\sigma_{k-1}} = \frac{\sigma_{l_\infty}}{\sigma_{l_\infty+1}} \cdots \frac{\sigma_{k-2}}{\sigma_{k-1}} \le C(\frac{\sigma_{k-1}}{\sigma_k})^{k-1-l_\infty} \to 0.$$

We therefore get a contradiction. □

Proof of Theorem 1.1. Here the argument comes from [29]. Let $\eta = \phi - u$. $\eta > 0$ in Ω. For a function Φ to be chosen and a constant $\beta > 0$ fixed, we consider the function

$$\tilde{W}(X, \xi) = \eta^{\beta}(\exp\Phi(\nu_{n+1}))h_{\xi\xi}$$

for all $X \in \mathcal{M}$ and all unit vector $\xi \in \mathrm{T}_X\mathcal{M}$. Then $\tilde{W}$ attains its maximum at an interior point $X_0 \in \mathcal{M}$, in a direction $\xi_0 \in \mathrm{T}_{X_0}\mathcal{M}$ which we may take to be $\hat{\mathbf{e}}_1$. We may assume that (h_{ij}) is diagonal at X_0 with eigenvalues $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n$. Without loss of generality we may assume that the $\hat{\mathbf{e}}_1, \cdots, \hat{\mathbf{e}}_n$ has been chosen so that $\nabla_i\hat{\mathbf{e}}_j = 0$ at X_0 for all $i, j = 1, \cdots, n$. Let $\tau = \hat{\mathbf{e}}_1$. Then $W(X) = \tilde{W}(X, \tau)$ is defined near X_0 and has an interior maximum at X_0. Let $Z := h_{ab}\tau_a\tau_b$. By the special choice of frame and the fact that h_{ij} is diagonal at X_0 in this frame, we can see that

$$\nabla_i Z = \nabla_i h_{11} \qquad \text{and} \qquad \nabla_i\nabla_j Z = \nabla_i\nabla_j h_{11} \quad \text{at } X_0$$

Therefore the scalar function Z satisfies the same equation as the component h_{11} of the tensor h_{ij}. Thus at X_0, we have

$$\frac{\nabla_i W}{W} = \beta\frac{\nabla_i\eta}{\eta} + \Phi'\nabla_i\nu_{n+1} + \frac{\nabla_i h_{11}}{h_{11}} = 0 \tag{2.2}$$

and

$$\begin{aligned}\frac{\nabla_i\nabla_j W}{W} - \frac{\nabla_i W\nabla_j W}{W^2} =&\beta\Big(\frac{\nabla_i\nabla_j\eta}{\eta} - \frac{\nabla_i\eta\nabla_j\eta}{\eta^2}\Big)\\ &+ \Phi''\nabla_i\nu_{n+1}\nabla_j\nu_{n+1} + \Phi'\nabla_i\nabla_j\nu_{n+1}\\ &+ \frac{\nabla_i\nabla_j h_{11}}{h_{11}} - \frac{\nabla_i h_{11}\nabla_j h_{11}}{h_{11}^2}\end{aligned} \tag{2.3}$$

is nonpositive in the sense of matrices at X_0. By Lemmas 2.1 and 2.2, we have, at X_0,

$$\begin{aligned}0 \ge&\beta F^{ij}\Big(\frac{\nabla_i\nabla_j\eta}{\eta} - \frac{\nabla_i\eta\nabla_j\eta}{\eta^2}\Big) + \Phi'' F^{ij}\nabla_i\nu_{n+1}\nabla_j\nu_{n+1}\\ &- (\Phi'\nu_{n+1} + 1)F^{ij}h_{ip}h_{pj} + F^{ij}h_{ij}h_{11} - \frac{\nabla_1\nabla_1 g}{h_{11}}\\ &+ \Phi'\langle\nabla g, \mathbf{e}_{n+1}\rangle - \frac{1}{h_{11}}F^{ij,rs}\nabla_1 h_{ij}\nabla_1 h_{rs} - F^{ij}\frac{\nabla_i h_{11}\nabla_j h_{11}}{h_{11}^2}\\ &- \sum_{l=0}^{k-2}\Phi'\langle\nabla\alpha_l, \mathbf{e}_{n+1}\rangle\frac{\sigma_l}{\sigma_{k-1}} + \sum_{l=0}^{k-2}\frac{1}{h_{11}}\Big(2\nabla_1\alpha_l\cdot\nabla_1\frac{\sigma_l}{\sigma_{k-1}} + \nabla_1\nabla_1\alpha_l\cdot\frac{\sigma_l}{\sigma_{k-1}}\Big).\end{aligned} \tag{2.4}$$

Using Gauss's formula

$$\nabla_i\nabla_j X_\alpha = h_{ij}\nu_\alpha,$$

we have

$$\nabla_1\nabla_1 g(X) = \sum_{\alpha=1}^{n+1}\frac{\partial g}{\partial X_\alpha}\nabla_1\nabla_1 X_\alpha + \sum_{\alpha,\beta=1}^{n+1}\frac{\partial^2 g}{\partial X_\alpha \partial X_\beta}\nabla_1 X_\alpha \nabla_1 X_\beta$$
$$= \sum_{\alpha=1}^{n+1}\frac{\partial g}{\partial X_\alpha}\nu_\alpha h_{11} + \sum_{\alpha,\beta=1}^{n+1}\frac{\partial^2 g}{\partial X_\alpha \partial X_\beta}\nabla_1 X_\alpha \nabla_1 X_\beta.$$

Consequently,

$$|\frac{\nabla_1\nabla_1 g}{h_{11}}| \leq C.$$

For the same reason, we have for all $l = 0, \cdots, k-2$,

$$|\frac{\nabla_1\nabla_1 \alpha_l}{h_{11}}| \leq C.$$

Taking Lemma 2.3 into count, we estimate the two terms in the last line of (2.4) as

$$-\sum_{l=0}^{k-2}\Phi'\langle\nabla\alpha_l, \mathbf{e}_{n+1}\rangle\frac{\sigma_l}{\sigma_{k-1}} + \sum_{l=0}^{k-2}\frac{1}{h_{11}}\nabla_1\nabla_1\alpha_l\cdot\frac{\sigma_l}{\sigma_{k-1}} \geq -C|\Phi'| - C.$$

Recall that $F = G_k - \sum\alpha_l G_l$ and it is well-known that the operator $(\frac{\sigma_{k-1}}{\sigma_l})^{\frac{1}{k-1-l}}$ is concave for $0 \leq l \leq k-2$. It follows that

$$(\frac{1}{G_l})^{\frac{1}{k-1-l}} \text{ is a concave operator for } \forall l = 0, 1, \cdots, k-2.$$

For any symmetric matrix $(B_{ij}) \in \mathbb{R}^{n\times n}$, we have

$$\left\{(\frac{1}{G_l})^{\frac{1}{k-1-l}}\right\}^{ij,rs} B_{ij}B_{rs} \leq 0.$$

Direct computation shows that

$$G_l^{ij,rs}B_{ij}B_{rs} \geq \frac{1}{G_l}\cdot\frac{k-l}{k-1-l}\cdot(G_l^{ij}B_{ij})^2.$$

Note that G_k is also a concave operator.

$$-\frac{1}{h_{11}}F^{ij,rs}\nabla_1 h_{ij}\nabla_1 h_{rs} + \sum_{l=0}^{k-2}\frac{2}{h_{11}}\nabla_1\alpha_l\cdot\nabla_1\frac{\sigma_l}{\sigma_{k-1}}$$
$$= -\frac{1}{h_{11}}G_k^{ij,rs}\nabla_1 h_{ij}\nabla_1 h_{rs} + \sum_{l=0}^{k-2}\frac{\alpha_l}{h_{11}}G_l^{ij,rs}\nabla_1 h_{ij}\nabla_1 h_{rs} + \sum_{l=0}^{k-2}\frac{2}{h_{11}}\nabla_1\alpha_l\cdot\nabla_1\frac{\sigma_l}{\sigma_{k-1}}$$
$$\geq \frac{1}{h_{11}}\sum_{l=0}^{k-2}G_l^{-1}\alpha_l C_l(\nabla_1 G_l + \frac{\nabla_1\alpha_l}{C_l\alpha_l}G_l)^2 - \frac{1}{h_{11}}\sum_{l=0}^{k-2}\frac{(\nabla_1\alpha_l)^2}{C_l\alpha_l}G_l$$
$$\geq -\frac{C}{h_{11}}$$

where $C_l = \frac{k-l}{k-1-l}$. By the homogeneity of G_l's, we see that

$$F^{ij}h_{ij} = G_k + \sum_{l=0}^{k-2} \alpha_l (k-1-l) G_l \geq G_k + \sum_{l=0}^{k-2} \alpha_l \frac{\sigma_l}{\sigma_{k-1}} \geq \inf |g| > 0.$$

Using Lemma 2.3 again, we have

$$F^{ij}h_{ij} \leq C.$$

Next we assume that ϕ has been extended to be constant in the $\mathbf{e}_{n+1}$ direction.

$$\begin{aligned}\nabla_i \nabla_j \eta &= \sum_{\alpha,\beta=1}^{n} \frac{\partial^2 \phi}{\partial X_\alpha \partial X_\beta} \nabla_i X_\alpha \nabla_j X_\beta + \sum_{\alpha=1}^{n} \frac{\partial \phi}{\partial X_\alpha} \nabla_i \nabla_j X_\alpha - \nabla_i \nabla_j X_{n+1} \\ &= \sum_{\alpha=1}^{n} \frac{\partial \phi}{\partial X_\alpha} \nu_\alpha h_{ij} - h_{ij} \nu_{n+1}.\end{aligned}$$

Consequently,

$$F^{ij} \nabla_i \nabla_j \eta = (\sum_{\alpha=1}^{n} \frac{\partial \phi}{\partial X_\alpha} \nu_\alpha - \nu_{n+1}) F^{ij} h_{ij}.$$

Using above estimates in (2.4), we have, at X_0,

$$\begin{aligned}0 \geq &- \frac{C\beta}{\eta} - \beta F^{ij} \frac{\nabla_i \eta \nabla_j \eta}{\eta^2} + \Phi'' F^{ij} \nabla_i \nu_{n+1} \nabla_j \nu_{n+1} - F^{ij} \frac{\nabla_i h_{11} \nabla_j h_{11}}{h_{11}^2} \\ &- (\Phi' \nu_{n+1} + 1) F^{ij} h_{ip} h_{pj} + \inf |g| h_{11} - C(1 + |\Phi'|). \end{aligned} \tag{2.5}$$

Next, using (2.2), we have

$$\begin{aligned}F^{ij} \frac{\nabla_i h_{11} \nabla_j h_{11}}{h_{11}^2} &= F^{ij} \Big(\beta \frac{\nabla_i \eta}{\eta} + \Phi' \nabla_i \nu_{n+1}\Big)\Big(\beta \frac{\nabla_j \eta}{\eta} + \Phi' \nabla_j \nu_{n+1}\Big) \\ &\leq (1 + \gamma^{-1}) \beta^2 F^{ij} \frac{\nabla_i \eta \nabla_j \eta}{\eta^2} + (1 + \gamma)(\Phi')^2 F^{ij} \nabla_i \nu_{n+1} \nabla_j \nu_{n+1}\end{aligned}$$

for any $\gamma > 0$. Therefore at X_0 we have, since $|\nabla \eta| \leq C$,

$$\begin{aligned}0 \geq &- \frac{C\beta}{\eta} - C[\beta + (1 + \gamma^{-1})\beta^2] \frac{\sum_{i=1}^{n} F^{ii}}{\eta^2} \\ &+ [\Phi'' - (1 + \gamma)(\Phi')^2] F^{ij} \nabla_i \nu_{n+1} \nabla_j \nu_{n+1} \\ &- [\Phi' \nu_{n+1} + 1] F^{ij} h_{ip} h_{pj} + \inf |g| h_{11} - C(1 + |\Phi'|). \end{aligned} \tag{2.6}$$

We choose a positive constant a, so that

$$a \leq \frac{1}{2} \nu_{n+1} = \frac{1}{2\sqrt{1 + |Du|^2}}$$

which depends only on $\sup_\Omega |Du|$. Therefore

$$\frac{1}{\nu_{n+1} - a} \leq \frac{1}{a} \leq C.$$

We now choose

$$\Phi(t) = -\log(t-a).$$

Then

$$\Phi'(t) = \frac{-1}{t-a}, \quad \Phi''(t) = \frac{1}{(t-a)^2},$$

and

$$\begin{aligned} -(\Phi' t + 1) &= \frac{a}{t-a}, \\ \Phi'' - (1+\gamma)(\Phi')^2 &= -\frac{\gamma}{(t-a)^2}. \end{aligned}$$

By direct computation, we have $\nabla_i \nu_{n+1} = -h_{ip}\langle \hat{\mathbf{e}}_p, \mathbf{e}_{n+1}\rangle$, and therefore

$$F^{ij}\nabla_i \nu_{n+1}\nabla_j \nu_{n+1} = F^{ij} h_{ip} h_{jq} \langle \hat{\mathbf{e}}_p, \mathbf{e}_{n+1}\rangle \langle \hat{\mathbf{e}}_q, \mathbf{e}_{n+1}\rangle \leq F^{ij} h_{ip} h_{pj}.$$

Next we choose $0 < \gamma \leq \frac{a^2}{2}$, then we have

$$-(\Phi' t + 1) + [\Phi'' - (1+\gamma)(\Phi')^2] = \frac{a}{t-a} - \frac{\gamma}{(t-a)^2} \geq \frac{\frac{1}{2}a^2}{(t-a)^2} > 0.$$

Thus we have

$$0 \geq -\frac{C\beta}{\eta} - C(\beta, a)\eta^{-2}(\sum_{i=1}^{n} F^{ii}) + \inf |g| h_{11} - C(a). \tag{2.7}$$

In the following we show that $\sum_{i=1}^{n} F^{ii} \leq C$. By the definition of operator F and Lemma 2.3, we have

$$\begin{aligned} \sum_{i=1}^{n} F^{ii} &= \sum_{i=1}^{n} (\frac{\sigma_k}{\sigma_{k-1}})^{ii} - \sum_{i=1}^{n}\sum_{l=0}^{k-2} \alpha_l (\frac{\sigma_l}{\sigma_{k-1}})^{ii} \\ &= \sum_{i=1}^{n} \frac{\sigma_{k-1}(\lambda|i)}{\sigma_{k-1}} - \frac{\sigma_k}{\sigma_{k-1}^2}\sum_{i=1}^{n}\sigma_{k-2}(\lambda|i) + \frac{\alpha_0}{\sigma_{k-1}^2}\sum_{i=1}^{n}\sigma_{k-2}(\lambda|i) \\ &\quad + \sum_{l=1}^{k-2}\alpha_l \frac{\sum_i \sigma_l \sigma_{k-2}(\lambda|i) - \sum_i \sigma_{k-1}\sigma_{l-1}(\lambda|i)}{\sigma_{k-1}^2} \\ &= n-k+1 - (n-k+2)\frac{\sigma_k \sigma_{k-2}}{\sigma_{k-1}^2} + (n-k+2)\alpha_0 \frac{\sigma_{k-2}}{\sigma_{k-1}^2} \\ &\quad + \sum_{l=1}^{k-2}\alpha_l \frac{(n-k+2)\sigma_l\sigma_{k-2} - (n-l+1)\sigma_{k-1}\sigma_{l-1}}{\sigma_{k-1}^2} \\ &\leq n-k+1 + (n-k+2)|\frac{\sigma_k}{\sigma_{k-1}} G_{k-2}| + (n-k+2)|\alpha_0|_{C^0}|G_{k-2}G_0| \\ &\quad + (n-k+2)|G_{k-2}|\sum_{l=1}^{k-2}|\alpha_l|_{C^0}|G_l|. \end{aligned}$$

From Eq (1.4) we have $|G_k| \le C$, therefore $\sum_i F^{ii} \le C$. At X_0, we get an upper bound

$$\lambda_1 \le \frac{C(\beta, a)}{\eta^2}.$$

Consequently, $W(X_0)$ satisfies an upper bound. Since $W(X) \le W(X_0)$, we get the required upper bound for the maximum principle curvature. Since $\lambda \in \Gamma_k$ and $n \ge k \ge 2$, u is at least mean-convex and

$$\sum_{i=1}^{n} \lambda_i > 0.$$

Therefore $\lambda_n \ge -(n-1)\lambda_1$ and

$$|\mathbf{A}| = \sqrt{\sum_{i=1}^{n} \lambda_i^2} \le C(n)\lambda_1 \le \frac{C}{(\phi - u)^\beta}.$$

□

3. The Dirichlet problem

In this section we prove Theorem 1.2. By comparison principle, we have $0 \ge u \ge \underline{u}$. For any $\Omega' \Subset \Omega$, $\inf_{\Omega'} \underline{u} \le u \le c(\Omega') < 0$. First we show the gradient bound of admissible solutions of (1.6). We need following lemmas to prove the gradient estimate.

Lemma 3.1. *Suppose $A = \{a_{ij}\}_{n\times n}$ satisfies $\lambda(A) \in \Gamma_{k-1}$, $a_{11} < 0$ and $\{a_{ij}\}_{2\le i,j\le n}$ is diagonal, then*

$$\sum_{i=2}^{n} \frac{\partial F}{\partial a_{1i}} a_{1i} \le 0. \tag{3.1}$$

Proof. Let

$$B = \begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix}, \quad C = \begin{pmatrix} 0 & a_{12} & \cdots & a_{1n} \\ a_{21} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & 0 & \cdots & 0 \end{pmatrix}.$$

$A(t) := B + tC$, $f(t) := F(A(t))$. Suppose $a_{1i} = a_{i1}$ for all $2 \le i \le n$. Directly we have

$$\sigma_k(A(t)) = \sigma_k(B) - t^2 \sum_{i=2}^{n} a_{1i}^2 \sigma_{k-2}(B|1i),$$

where $(B|ij)$ is the submatrix of B formed by deleting i-th, j-th rows and columns. Easily we see that for $t \in [-1, 1]$, $\lambda(A(t)) \in \Gamma_{k-1}$ and f is concave on $[-1, 1]$. $f(-1) = f(1) = F(A)$. So $f'(1) \le 0$. While

$$f'(1) = 2 \sum_{i=2}^{n} \frac{\partial F}{\partial a_{1i}} a_{1i}.$$

□

Remark 3.1. *By the concavity of $\frac{\sigma_k}{\sigma_{k-1}}$, we can prove following inequality with $\lambda(B)\in\Gamma_{k-1}$*

$$\sigma_{k-2}(B|1i)\sigma_{k-1}(B)-\sigma_{k-3}(B|1i)\sigma_k(B)\geq 0\quad \forall 2\leq i\leq n. \tag{3.2}$$

We let $f(t)=\frac{\sigma_k}{\sigma_{k-1}}(A(t))$.

$$f'(1)=\frac{-2(\sum_{i=2}^n a_{1i}\sigma_{k-2}(B|1i))}{\sigma_{k-1}(A)}-\frac{\sigma_k(A)(\sum_{i=2}^n a_{1i}^2\sigma_{k-3}(B|1i))}{\sigma_{k-1}^2(A)}\leq 0.$$

Equivalently,

$$\sigma_{k-1}(B)(\sum_{i=2}^n a_{1i}^2\sigma_{k-2}(B|1i))-\sigma_k(B)(\sum_{i=2}^n a_{1i}^2\sigma_{k-3}(B|1i))\geq 0. \tag{3.3}$$

We can choose $a_{1i}>0$ small enough and $a_{1j}=0$ for $j\neq i$ and $2\leq j\leq n$, so that $\lambda(A)\in\Gamma_{k-1}$. Then (3.3) implies (3.2).

Lemma 3.2. *Let $\alpha_{k-2}>0$ and $\alpha_l\geq 0$ for $0\leq l\leq k-3$. Suppose symmetric matrix $A=\{a_{ij}\}_{n\times n}$ satisfying*

$$\lambda(A)\in\Gamma_{k-1},\, a_{11}<0, \text{and } \{a_{ij}\}_{2\leq i,j\leq n} \text{ is diagonal.}$$

Then

$$\frac{\partial F}{\partial a_{11}}\geq C_0(\sum_{i=1}^n\frac{\partial F}{\partial a_{ii}}) \tag{3.4}$$

where C_0 depends on $n,k,|u|_{C^0},|g|_{C^0},\inf\alpha_{k-2}$.

Proof. Note that

$$\begin{aligned}\frac{\partial}{\partial a_{11}}(\frac{\sigma_l}{\sigma_{k-1}}(A)) =&\frac{\sigma_{l-1}(A|1)\sigma_{k-1}(A)-\sigma_l(A)\sigma_{k-2}(A|1)}{\sigma_{k-1}^2(A)}\\ =&\sum_{i=2}^n\frac{a_{1i}^2}{\sigma_{k-1}^2(A)}[\sigma_{l-2}(A|1i)\sigma_{k-2}(A|1)-\sigma_{l-1}(A|1)\sigma_{k-3}(A|1i)]\\ &+\sigma_{k-1}^{-2}(A)[\sigma_{l-1}(A|1)\sigma_{k-1}(A|1)-\sigma_l(A|1)\sigma_{k-2}(A|1)].\end{aligned}$$

For $0\leq l\leq k-2$,

$$\frac{\partial}{\partial a_{11}}(\frac{\sigma_l}{\sigma_{k-1}}(A))\leq -C_{n,l}\frac{\sigma_l(A|1)\sigma_{k-2}(A|1)}{\sigma_{k-1}^2(A)}.$$

As for $l=k$,

$$\frac{\partial}{\partial a_{11}}(\frac{\sigma_k}{\sigma_{k-1}}(A))\geq C_{n,k}\frac{\sigma_{k-1}^2(A|1)}{\sigma_{k-1}^2(A)}\geq C_{n,k}.$$

Therefore

$$\frac{\partial F}{\partial a_{11}}\geq C_{n,k}+C_{n,k}\inf\alpha_{k-2}\frac{\sigma_{k-2}^2(A|1)}{\sigma_{k-1}^2(A)}.$$

Next we compute $\sum_{i=1}^n\frac{\partial F}{\partial a_{ii}}$ as

$$\sum_{i=1}^n\frac{\partial F}{\partial a_{ii}} =n-k+1-(n-k+2)\frac{\sigma_k\sigma_{k-2}}{\sigma_{k-1}^2}(A)+(n-k+2)\alpha_0\frac{\sigma_{k-2}}{\sigma_{k-1}^2}(A)$$

$$
\begin{aligned}
&+\sum_{l=1}^{k-2}\alpha_l\frac{(n-k+2)\sigma_l(A)\sigma_{k-2}(A)-(n-l+1)\sigma_{k-1}(A)\sigma_{l-1}(A)}{\sigma_{k-1}^2(A)}\\
&\le n-k+1-(n-k+2)\frac{\sigma_{k-2}(A)}{\sigma_{k-1}(A)}(\frac{\sigma_k}{\sigma_{k-1}}(A)-\sum_{l=0}^{k-2}\alpha_l\frac{\sigma_l}{\sigma_{k-1}}(A))\\
&\le C_{n,k}+C_{n,k}|g|_{C^0}\frac{\sigma_{k-2}}{\sigma_{k-1}}(A)\\
&\le C(n,k,|g|_{C^0})+C(n,k,|g|_{C^0})(\frac{\sigma_{k-2}}{\sigma_{k-1}}(A))^2\\
&\le C(n,k,|g|_{C^0})+C(n,k,|g|_{C^0})\frac{\sigma_{k-2}^2(A|1)}{\sigma_{k-1}^2(A)}\\
&\le C(n,k,|g|_{C^0},\inf\alpha_{k-2})\frac{\partial F}{\partial a_{11}}.
\end{aligned}
$$

□

Lemma 3.3. *For any $\Omega' \Subset \Omega$, there is a constant C depending only on $\Omega', n, k, \alpha_l, g$ and their first derivatives, such that if u is an admissible solution of (1.6), then*

$$|Du| \le C$$

on Ω'.

Proof. Since we require that $\partial_u g \le 0$ and $\partial_u \alpha_l \ge 0$, we only need to modify the equation (5.42) in [36] (i.e., (A.6)), where extra terms $\sum_{l=0}^{k-2}\frac{(\alpha_l)_u}{\log u_1}\frac{\sigma_l(A)}{\sigma_{k-1}(A)} - \frac{g_u}{\log u_1}$ should be included. These terms are all good terms and Zhou's proof will also hold in our case. For reader's convenience, we sketch the proof in the appendix below. □

Now we give the proof of Theorem 1.2.

Proof of Thorem 1.2. The theorem can be proved by solving uniformlly elliptic approximating problems.

$$F_\epsilon[u_\epsilon] = -g_\epsilon(x, u_\epsilon) \quad \text{in}\,\Omega, \qquad u_\epsilon = 0 \quad \text{on}\,\partial\Omega,$$

for $\epsilon > 0$ small, and $\underline{u}$ is an admissible subsolution for each of the approximating problems. By the comparison principle and Theorem 1.1, the interior gradient estimates in [36](modified), we have uniform C^2 interior estimates for u_ϵ. Then Evans-Krylov's theory, together with Schauder theory, imply uniform estimates for $\|u_\epsilon\|_{C^{3,\alpha}(\Omega')}$ for any $\Omega' \Subset \Omega$. Theorem 2 then follows by extracting a suitable subsequence as $\epsilon \to 0$.

□

4. The Plateau problem

In this section we prove Theorem 1.3. The notion of locally convex hypersurface we use is the same as that in [29].

Definition 4.1. *A compact, connected, locally convex hypersurface $\mathcal{M}$ (possibly with boundary) in $\mathbb{R}^{n+1}$ is an immersion of an n-dimensional, compact, oriented and connected manifold $\mathcal{N}$ (possibly with boundary) in $\mathbb{R}^{n+1}$, that is, a mapping $T : \mathcal{N} \to \mathcal{M} \subset \mathbb{R}^{n+1}$, such that for any $p \in \mathcal{N}$ there is a neighbourhood $\omega_p \subset \mathcal{N}$ such that*

- *T is a homeomorphism from ω_p to $T(\omega_p)$;*
- *$T(\omega_p)$ is a convex graph;*
- *the convexity of $T(\omega_p)$ agrees with the orientation.*

Since $\mathcal{M}$ is immersed, a point $x \in \mathcal{M}$ may be the image of several points in $\mathcal{N}$. Since $\mathcal{M}$ and $\mathcal{N}$ are compact, $T^{-1}(x)$ consists of only finitely many points. Let $r > 0$ and $x \in \mathcal{M}$. For small enough r, $T^{-1}(\mathcal{M} \cap B_r^{n+1}(x))$ consists of several disjoint open sets $U_1, \cdots, U_s$ of $\mathcal{N}$ such that $T|_{U_i}$ is a homeomorphism of U_i onto $T(U_i)$ for each $i = 1, \cdots, s$. By an r-neighbourhood $\omega_r(x)$ of x in $\mathcal{M}$ we mean any one of the sets $T(U_i)$. We say that $\omega_r(x)$ is convex if $\omega_r(x)$ lies on the boundary of its convex hull.

We shall use following lemma (see [32] Theorem A) to prove Theorem 1.3.

Lemma 4.1. *Let $\mathcal{M}_0 \subset B_R(0)$ be a locally convex hypersurface with C^2-boundary $\partial\mathcal{M}_0$. Suppose that on $\partial\mathcal{M}_0$, the principal curvatures $\lambda_1^0, \cdots, \lambda_n^0$ of $\mathcal{M}_0$ satisfy*

$$C_0^{-1} \leq \lambda_i^0 \leq C_0, \quad i = 1, 2, \cdots, n,$$

for some $C_0 > 0$. Then there exist positive constants r and α, depending only on n, C_0, R and $\partial\mathcal{M}_0$, such that for any point $p \in \mathcal{M}_0$, each r-neighbourhood $\omega_r(p)$ of p is convex, and there is a closed cone $C_{p,\alpha}$ with vertex p and angle α such that $\omega_r(p) \cap C_{p,\alpha} = \{p\}$.

Note that for any point $p \in \mathcal{M}_0$, if one chooses the axial direction of the cone $C_{p,\alpha}$ as the x_{n+1}-axis, then each δ-neighbourhood of p can be represented as a graph,

$$x_{n+1} = u(x), \quad |x| \leq \delta,$$

for any $\delta < r\sin(\alpha/2)$. The cone condition also implies

$$|Du(x)| \leq C, \quad |x| < \delta,$$

where $C > 0$ only depends on α. Lemma 4.1 holds not just for $\mathcal{M}_0$, but also for a family of locally convex hypersurfaces, with uniform r and α.

For $2 \leq k \leq n$, denote

$$f_{(k)}(\lambda) = \frac{\sigma_k}{\sigma_{k-1}}(\lambda) - \sum_{l=0}^{n-2} \alpha_l \frac{\sigma_l}{\sigma_{k-1}}(\lambda).$$

α_l's are positive constants. With the aid of Lemma 4.1, we use the Perron method to obtain a viscosity solution of the Plateau problem for the curvature function $f_{(n)}$, using the following lemma.

Lemma 4.2. *Let Ω be a bounded domain in $\mathbb{R}^n$ with Lipschitz boundary. Let $\phi \in C^{0,1}(\bar{\Omega})$ be a k-convex viscosity subsolution of*

$$f_{(k)}(\lambda) = \frac{\sigma_k}{\sigma_{k-1}}(\lambda) - \sum_{l=0}^{k-2} \alpha_l \frac{\sigma_l}{\sigma_{k-1}}(\lambda) = c \quad \text{in}\, \Omega, \tag{4.1}$$

where $\alpha_l > 0$ and $c \neq 0$ are all constants. Then there is a viscosity solution u of (4.1) *such that $u = \phi$ on $\partial\Omega$.*

Proof. The proof uses the well-known Perron method. Let Ψ denote the set of k-convex subsolutions v of (4.1) with $v = \phi$ on $\partial\Omega$. Then Ψ is not empty and the required solution u is given by

$$u(x) = \sup\{v(x) : v \in \Psi\}.$$

It is a standard argument. The key ingredient that needs to be mentioned is the solvability of the Dirichlet problem

$$f_{(k)}(\lambda) = c \quad \text{in } B_r, \qquad u = u_0 \quad \text{on } \partial B_r, \tag{4.2}$$

in small enough balls $B_r \subset \mathbb{R}^n$, if u_0 is any Lipschitz viscosity subsolution of (4.2). This is a consequence of [31] Theorem 6.2 with slight modification. □

Using Lemma 4.2 and the argument of [32], we conclude that there is a locally convex hypersurface $\mathcal{M}$ with boundary Σ which satisfies the equation $f_{(n)}(\lambda) = c$ in the viscosity sense; that is, for any point $p \in \mathcal{M}$, if $\mathcal{M}$ is locally represented as the graph of a convex function u (by Lemma 4.1), then u is a viscosity solution of $f_{(n)}(\lambda) = c$.

Following we discuss the regularity of $\mathcal{M}$. The interior regularity follows in the same way as [29].

Boundary regularity

The boundary regularity of $\mathcal{M}$ is a local property. The boundary estimates we need are contained in [19, 21]. However, they can not be applied directly to $\mathcal{M}$. Since we are working in a neighbourhood of a boundary point $p_0 \in \mathcal{M}$, which we may take to be the origin, we may assume that for a smooth bounded domain $\Omega \subset \mathbb{R}^n$ with $0 \in \partial\Omega$ and small enough $\rho > 0$ we have

$$\mathcal{M} \cap (B_\rho \times \mathbb{R}) = \operatorname{graph} u, \quad \mathcal{M}_0 \cap (B_\rho \times \mathbb{R}) = \operatorname{graph} u_0,$$

where $u \in C^\infty(\Omega_\rho) \cap C^{0,1}(\bar{\Omega}_\rho)$, and $u_0 \in C^\infty(\bar{\Omega}_\rho)$ are k-convex solutions of

$$f_{(k)}[u] = c \quad \text{in}\, \Omega_\rho, \qquad f_{(k)}[u_0] \geq c \quad \text{in}\, \Omega_\rho,$$

with

$$u \geq u_0 \quad \text{in}\, \Omega_\rho, \qquad u = u_0 \quad \text{on}\, \partial\Omega \cap B_\rho.$$

We may choose the coordinate system in $\mathbb{R}^n$ in such a way that Ω is uniformly convex, and moreover, so that for some $\epsilon_0 > 0$ we have

$$\frac{\sigma_{k-1}(\kappa')}{\sigma_{k-2}(\kappa')} \geq \epsilon_0 > 0 \tag{4.3}$$

on $\partial\Omega \cap B_\rho$, where $\kappa' = (\kappa'_1, \cdots, \kappa'_{n-1})$ denotes the vector of principal curvatures of $\partial\Omega$. We recall that the principal curvatures of graph(u) are the eigenvalues of the matrix

$$(I - \frac{Du \otimes Du}{1 + |Du|^2})(\frac{D^2 u}{\sqrt{1 + |Du|^2}}).$$

We denote $\sigma_k(p, r)$ as the k-th elementary symmetric function of the eigenvalues of the matrix

$$(I - \frac{p \otimes p}{1 + |p|^2})r, \; p = (p_1, \cdots, p_n), \; r = (r_{ij})_{n\times n}.$$

Let $f_{(k)}(p,r) = \frac{\sigma_k}{\sigma_{k-1}}(p,r) - \sum_{l=0}^{k-2} \alpha_l (1+|p|^2)^{\frac{k-l}{2}} \frac{\sigma_l}{\sigma_{k-1}}(p,r)$. $\lambda(r)$ is the vector formed by eigenvalues of r. For any $p \in \mathbb{R}^n$ and symmetric matrices r, s with $\lambda(r), \lambda(s) \in \Gamma_k$, we have

$$\sum_{i,j} \frac{\partial f_{(k)}}{\partial r_{ij}}(p,r) s_{ij} \geq f_{(k)}(p,s) + \sum_{l=0}^{k-2} (k-l)\alpha_l (1+|p|^2)^{\frac{k-l}{2}} \frac{\sigma_l}{\sigma_{k-1}}(p,r). \tag{4.4}$$

For later purposes we note the simple estimate, if $r \geq 0$,

$$\frac{1}{1+|p|^2}\sigma_k(0,r) \leq \sigma_k(p,r) \leq \sigma_k(0,r),$$

and the development

$$\sigma_k(p,r) = \frac{1+|\tilde{p}|^2}{1+|p|^2} r_{nn}\sigma_{k-1}(\tilde{p},\tilde{r}) + O((|r_{st}|^k)_{(s,t)\neq(n,n)}),$$

where $p = (p_1, \cdots, p_n) \in \mathbb{R}^n$, $r = (r_{ij})_{n\times n}$, $\tilde{p} = (p_1, \cdots, p_{n-1}) \in \mathbb{R}^{n-1}$, $\tilde{r} = (r_{ij})_{i,j=1,\cdots,n-1}$.

We suppose that $\partial\Omega$ is the graph of $\omega : B_\rho^{n-1}(0) \subset \mathbb{R}^n \to \mathbb{R}$ and $u(\tilde{x}, \omega(\tilde{x})) = \varphi(\tilde{x})$. Furthermore, $\omega(0) = 0$, $D\omega(0) = 0$, $D\varphi(0) = 0$ and ω is a strictly convex function of $\tilde{x}$. The curvature equation is equivalent to

$$f_{(k)}(Du, D^2u) = c\sqrt{1+|Du|^2} \tag{4.5}$$

defined in some domain $\Omega \subset \mathbb{R}^n$. We have following boundary estimates for second derivatives of u.

Lemma 4.3. *Let $u \in C^3(\bar{\Omega})$ be a k-convex solution of* (4.5). *We assume* (4.3) *with $\epsilon > 0$. Then the estimate*

$$|D^2u(0)| \leq C(n,k,\alpha_l,c,\epsilon,\|\omega\|_{C^3},\|\varphi\|_{C^4},\|u\|_{C^1},\lambda_{\min}(D^2\omega(0))) \tag{4.6}$$

holds true where $\lambda_{\min}$ denotes the smallest eigenvalue.

Remark 4.1. *On $\partial\Omega$, we have for $i, j = 1, \cdots, n-1$,*

$$\begin{aligned} &u_i + u_n\omega_i = \varphi_i, \\ &u_{ij} + u_{in}\omega_j + u_{nj}\omega_i + u_{nn}\omega_i\omega_j + u_n\omega_{ij} = \varphi_{ij}. \end{aligned}$$

Therefore $|u_{ij}(0)| = |\varphi_{ij}(0) - u_n(0)\omega_{ij}(0)| \leq C$. It remains to show that $|u_{in}(0)| \leq C$ and $|u_{nn}(0)| \leq C$. We follow [19, 21] to obtain mixed second derivative boundary estimates and double normal second derivative boundary estimate.

Proof. Let

$$\Omega_{d,\kappa} = \{x(\tilde{x}, x_n) \in \Omega \,|\, |\tilde{x}| < d, \omega(\tilde{x}) < x_n < \tilde{\omega}(\tilde{x}) + \frac{\kappa}{2}d^2\}$$

where $0 < d < \rho$, $\tilde{\omega}(\tilde{x}) := \omega(\tilde{x}) - \frac{\kappa}{2}|\tilde{x}|^2$, and $\kappa > 0$ is chosen small enough such that $\tilde{\omega}$ is still strictly convex. We decompose $\partial\Omega_{d,\kappa} = \partial_1\Omega_{d,\kappa} \cup \partial_2\Omega_{d,\kappa} \cup \partial_3\Omega_{d,\kappa}$ with

$$\begin{aligned} &\partial_1\Omega_{d,\kappa} = \{x \in \partial\Omega_{d,\kappa} \,|\, x_n = \omega(\tilde{x})\}, \\ &\partial_2\Omega_{d,\kappa} = \{x \in \partial\Omega_{d,\kappa} \,|\, x_n = \omega(\tilde{x}) + \frac{\kappa}{2}d^2\}, \end{aligned}$$

$$\partial_3\Omega_{d,\kappa} = \{x \in \partial\Omega_{d,\kappa} \| \tilde{x}| = d\}.$$

Our lower barrier function v will be of the form

$$v(x) = \theta(\tilde{x}) + h(\rho(x)) \tag{4.7}$$

where $\theta(\tilde{x})$ is an arbitrary C^2-function, $h(\rho) = \exp\{B\rho\} - \exp\{\kappa Bd^2\}$ and $\rho(x) = \kappa d^2 + \tilde{\omega}(\tilde{x}) - x_n$. Denote $F^{ij} = \frac{\partial f_{(k)}(Du,D^2u)}{\partial u_{ij}}$.

Mixed second derivative boundary estimates

By (4.4) and Lemma 2.3, we have

$$F^{ij}v_{ij} \geq f_{(k)}(Du, D^2v) + C$$

where C depends only on n, k, α_l's,c, $\|Du\|_{C^0}$. We choose an orthonormal frame $\{b_i\}_{i=1}^n$ with $b_n = -\frac{D\rho}{|D\rho|}$ and denote $v_{(s)} = \frac{\partial v}{\partial b_s}$. Directly, we have

$$\begin{aligned}
&v_{(s)} = \theta_{(s)} + h'\rho_{(s)},\ (1 \leq s \leq n-1); \quad v_{(n)} = \theta_{(n)} - h'\sqrt{1 + |D\tilde{\omega}|^2};\\
&v_{(st)} = \theta_{(st)} + h'\tilde{\omega}_{(st)},\ (s,t) \neq (n,n);\\
&v_{(nn)} = \theta_{(nn)} + h'\tilde{\omega}_{(nn)} + h''(1 + |D\tilde{\omega}|^2).
\end{aligned}$$

We may choose d small so that $|Du|$ is also small. Note that $|D\tilde{\omega}|$ is small since we can choose d, κ small. By choosing large enough B, we caculate

$$\begin{aligned}
f_{(k)}(Du, D^2v) =& \frac{\sigma_k}{\sigma_{k-1}}(Du, D^2v) - \sum_{l=0}^{k-2} \alpha_l (1 + |Du|^2)^{\frac{k-l}{2}} \frac{\sigma_l}{\sigma_{k-1}}(Du, D^2v)\\
\geq& (1-\epsilon)\frac{\sigma_k}{\sigma_{k-1}}(0, D^2v) - 2\sum_{l=0}^{k-2} \alpha_l \frac{\sigma_l}{\sigma_{k-1}}(0, D^2v)\\
\geq& (1-\epsilon)^2 h' \frac{\sigma_{k-1}}{\sigma_{k-2}}(0, \tilde{\omega}_{(st)}) - 2\sum_{l=1}^{k-2} \alpha_l (h')^{l-k+1} \frac{\sigma_{l-1}}{\sigma_{k-2}}(0, \tilde{\omega}_{(st)}) - o(B^{-1})
\end{aligned}$$

where in the last line, $1 \leq s, t \leq n-1$. Finally, we see that for large enough B and small enough d and κ the estimate

$$(1-\delta)h' \leq |Dv| \leq (1+\delta)h'$$

is valid for small δ. Therefore

$$F^{ij}v_{ij} \geq (1-\epsilon)\frac{\sigma_{k-1}}{\sigma_{k-2}}(0, \tilde{\omega}_{(st)})|Dv| + C. \tag{4.8}$$

Let τ be a C^2-smooth vector field which is tangential along $\partial\Omega$. Following [19, 21] we then introduce the function

$$w = 1 - \exp(-a\tilde{w}) - b|x|^2$$

where $\tilde{w} = u_\tau - \frac{1}{2}\sum_{i=1}^{n-1} u_s^2$ and a, b are positive constants. Since on $\partial_1\Omega_{d,\kappa}$, $u = \varphi$, and

$$w|_{\partial_1\Omega_{d,\kappa}} \geq a\varphi_\tau - c|\tilde{x}|^2,\ w(0) = 0,\ w|_{\partial_2\Omega_{d,\kappa}\cup\partial_3\Omega_{d,\kappa}} \geq -M$$

for suitable constants c, M depending on $a, b, \|u\|_{C^1}$ and $\|\varphi\|_{C^1}$. By differentiation of Eq (4.5), we obtain

$$F^{ij}u_{ijp} + F^i u_{ip} = c\bar{v}_p$$

where $F^i := \frac{\partial f_{(k)}}{\partial u_i}$ and $\bar{v} := \sqrt{1+|Du|^2}$.

$$\begin{aligned} F^{ij}\tilde{w}_{ij} =& F^{ij}u_{ijp}\tau_p + F^{ij}(u_{pj}\tau_{pi} + u_{pi}\tau_{pj}) + F^{ij}\tau_{ijp}u_p - \sum_{s=1}^{n-1} F^{ij}(u_{is}u_{js} + u_{sij}u_s) \\ =& c(\bar{v}_p\tau_p - \sum_{s=1}^{n-1}\bar{v}_s u_s) - F^i u_{ip}\tau_p + \sum_{s=1}^{n-1} F^i u_{is}u_s + F^{ij}(u_{pj}\tau_{pi} + u_{pi}\tau_{pj}) \\ &+ F^{ij}\tau_{ijp}u_p - \sum_{s=1}^{n-1} F^{ij}u_{is}u_{js}. \end{aligned} \tag{4.9}$$

By the definition of $\tilde{w}$, we have

$$c(\bar{v}_p\tau_p - \sum_{s=1}^{n-1}\bar{v}_s u_s) = \frac{c}{\bar{v}}(\langle D\tilde{w}, Du\rangle - \mathrm{Hess}(\tau)(Du, Du)). \tag{4.10}$$

Then we compute F^i. Denote $b_{ij} = \delta_{ij} - \frac{u_i u_j}{\bar{v}^2}$ and $c_{ij} = b_{ip}u_{pj}$. $f_{(k)}$ can be rewritten as

$$f_{(k)} = f_{(k)}(c_{ij}, \bar{v}) = \frac{\sigma_k}{\sigma_{k-1}}(c_{ij}) - \sum_{l=0}^{k-2}\alpha_l \bar{v}^{k-l}\frac{\sigma_l}{\sigma_{k-1}}(c_{ij}).$$

Directly we have

$$\begin{aligned} F^i =& \frac{\partial f_{(k)}}{\partial u_i} = \frac{\partial f_{(k)}}{\partial c_{pq}}\frac{\partial c_{pq}}{\partial u_i} + \frac{\partial f_{(k)}}{\partial \bar{v}}\frac{\partial \bar{v}}{\partial u_i} \\ =& -\frac{1}{\bar{v}^2}f_{(k)}^{iq}u_{ql}u_l - \frac{1}{\bar{v}^2}f_{(k)}^{pq}u_{iq}u_p + \frac{2}{\bar{v}^3}f_{(k)}^{pq}u_p u_l u_{lq}u_i - \sum_{l=0}^{k-2}\alpha_l(k-l)\bar{v}^{k-l-2}\frac{\sigma_l}{\sigma_{k-1}}(c_{ij})u_i \end{aligned}$$

where $f_{(k)}^{pq} := \frac{\partial f_{(k)}}{\partial c_{pq}}$. Therefore

$$\begin{aligned} -F^i u_{ip}\tau_p + \sum_{s=1}^{n-1} F^i u_{is}u_s =& \Big(-\frac{1}{\bar{v}^2}f_{(k)}^{iq}u_{ql}u_l - \frac{1}{\bar{v}^2}f_{(k)}^{pq}u_{iq}u_p + \frac{2}{\bar{v}^3}f_{(k)}^{pq}u_p u_l u_{lq}u_i \\ & - \sum_{l=0}^{k-2}\alpha_l(k-l)\bar{v}^{k-l-2}\frac{\sigma_l}{\sigma_{k-1}}(c_{ij})u_i\Big)(-\tilde{w}_i + u_p\tau_{pi}). \end{aligned} \tag{4.11}$$

In order to derive the right hand side of (4.11), we use the same coordinate system as [21], which corresponds to the projection of principal curvature directions of the graph of u onto $\mathbb{R}^n \supset \Omega$. Fixing a point $y \in \Omega$, we choose a basis of eigenvectors $\hat{e}_1, \cdots, \hat{e}_n$ of the matrix (c_{ij}) at y, corresponding to the eigenvalues $\lambda_1, \cdots, \lambda_n$ and orthonormal with respect to the inner product given by the matrix $I + Du \otimes Du$. Using a subscript α to denote differentiation with respect to $\hat{e}_\alpha$, $\alpha = 1, \cdots, n$, so that

$$u_\alpha = \hat{e}^i_\alpha u_i = \langle Du, \hat{e}_\alpha \rangle, \quad u_{\alpha\alpha} = \lambda_\alpha = \hat{e}^i_\alpha \hat{e}^j_\alpha u_{ij}.$$

Then we obtain

$$\begin{aligned}\frac{1}{\bar{v}^2} f^{iq}_{(k)} u_{ql} u_l (\tilde{w}_i - u_p \tau_{pi}) =& \frac{1}{\bar{v}^2} \frac{\partial f_{(k)}}{\partial \lambda_\alpha} \lambda_\alpha u_\alpha (\tilde{\omega}_\alpha - \mathrm{Hess}(\tau)(Du, \hat{e}_\alpha)) \\ \leq& \delta \frac{\partial f_{(k)}}{\partial \lambda_\alpha} \lambda^2_\alpha + C(\delta) \frac{\partial f_{(k)}}{\partial \lambda_\alpha} \tilde{w}^2_\alpha + C(\delta) \sum_{\alpha=1}^n \frac{\partial f_{(k)}}{\partial \lambda_\alpha}.\end{aligned}$$

The second term of (4.11) can be estimated in the same way as above. As for the third term of (4.11), we calculate as

$$|f^{pq}_{(k)} u_p u_l u_{lq}| = |f^{pq}_{(k)} (u_{pq} - c_{pq})| \leq C|Du|^2.$$

Thus

$$- F^i u_{ip} \tau_p + \sum_{s=1}^{n-1} F^i u_{is} u_s \leq 2\delta \frac{\partial f_{(k)}}{\partial \lambda_\alpha} \lambda^2_\alpha + C(\delta) \frac{\partial f_{(k)}}{\partial \lambda_\alpha} \tilde{w}^2_\alpha + C(\delta) \sum_{\alpha=1}^n \frac{\partial f_{(k)}}{\partial \lambda_\alpha} + C|\tilde{w}_i u_i - \tau_{ij} u_i u_j|. \tag{4.12}$$

Let (η^α_i) denote the inverse matrix to $(\hat{e}^i_\alpha)$, we write

$$u_{s\alpha} = \hat{e}^i_\alpha u_{is} = \lambda_\alpha \eta^\alpha_s.$$

Furthermore,

$$\sum_{s=1}^{n-1} \frac{\partial f_{(k)}}{\partial \lambda_\alpha} u^2_{s\alpha} = \frac{\partial f_{(k)}}{\partial \lambda_\alpha} \lambda^2_\alpha \sum_{s=1}^{n-1} (\eta^\alpha_s)^2.$$

Now we reason similarly to [21]. If for all $\alpha = 1, \cdots, n$, we have

$$\sum_{s=1}^{n-1} (\eta^\alpha_s)^2 \geq \epsilon > 0 \tag{4.13}$$

where ϵ is a small postive number. Then we clearly have

$$\sum_{s=1}^{n-1} \frac{\partial f_{(k)}}{\partial \lambda_\alpha} u^2_{s\alpha} \geq \epsilon \frac{\partial f_{(k)}}{\partial \lambda_\alpha} \lambda^2_\alpha. \tag{4.14}$$

On the other hand, if (4.13) is not true, then

$$\sum_{s=1}^{n-1} (\eta^\gamma_s)^2 < \epsilon$$

for some γ, which implies

$$\sum_{s=1}^{n-1} (\eta^\alpha_s)^2 \geq \delta_0 > 0$$

for all $\alpha \neq \gamma$. Hence

$$\sum_{s=1}^{n-1} \frac{\partial f_{(k)}}{\partial \lambda_\alpha} u_{s\alpha}^2 \geq \delta_0 \sum_{\alpha \neq \gamma} \frac{\partial f_{(k)}}{\partial \lambda_\alpha} \lambda_\alpha^2. \tag{4.15}$$

Then we use Theorem 3,4 in [22] to deduce that

$$\begin{aligned}
&\sum_{\alpha\neq\gamma}(\frac{\sigma_k}{\sigma_{k-1}})_{,\alpha}\lambda_\alpha^2 \geq \frac{1}{C(n,k)}(\frac{\sigma_k}{\sigma_{k-1}})_{,\alpha}\lambda_\alpha^2,\\
&\sum_{\alpha\neq\gamma}(-\frac{\sigma_l}{\sigma_{k-1}})_{,\alpha}\lambda_\alpha^2 \geq \frac{1}{C(n,k,l)}(-\frac{\sigma_l}{\sigma_{k-1}})_{,\alpha}\lambda_\alpha^2,\\
&\sum_{\alpha\neq\gamma}(-\frac{1}{\sigma_{k-1}})_{,\alpha}\lambda_\alpha^2 \geq \frac{1}{C(n,k,0)}(-\frac{1}{\sigma_{k-1}})_{,\alpha}\lambda_\alpha^2 - \frac{1}{C(n,k,0)}\frac{\sigma_1}{\sigma_{k-1}}
\end{aligned}$$

where subscript '$,\alpha$' denotes differentiation with respect to λ_α. Therefore,

$$\sum_{s=1}^{n-1} \frac{\partial f_{(k)}}{\partial \lambda_\alpha} u_{s\alpha}^2 \geq \delta' \frac{\partial f_{(k)}}{\partial \lambda_\alpha} \lambda_\alpha^2 - C. \tag{4.16}$$

Combing (4.9), (4.10), (4.12), (4.16), we have

$$F^{ij}\tilde{w}_{ij} \leq C|\langle D\tilde{w}, Du\rangle| + CF^{ij}\tilde{w}_i\tilde{w}_j + C\sum_{i=1}^n F^{ii} \tag{4.17}$$

where we have chosen $\delta << \delta'$, so that $\frac{\partial f_{(k)}}{\partial \lambda_\alpha}\lambda_\alpha^2$ can be discarded. Note that in (4.17), we also have used the fact that $\sum_{i=1}^n F^{ii} \geq C_0 > 0$. By choosing a, b large, we conclude that

$$F^{ij}w_{ij} \leq C|\langle Dw, Du\rangle|. \tag{4.18}$$

From (4.8), (4.18), by comparison principle, we have at 0,

$$u_{\tau n}(0) = \frac{1}{a}w_n(0) \geq \frac{1}{a}v_n(0).$$

Since τ is an arbitrary tangential direction at $0 \in \partial\Omega$, if we replace τ by $-\tau$, we get an upper bound for $u_{\tau n}(0)$.

Double normal second derivative boundary estimate

We turn to estimate $|u_{nn}(0)|$. The idea is to estimate u_{nn} in a first step at some optimally chosen point y and in a second step conclude from this the estimate in the given point. We introduce a smooth moving orthonormal frame $\{b_1, \cdots, b_n\}$ with $b_n = (-\omega_{\tilde{x}}, 1)/\sqrt{1+|\omega_{\tilde{x}}|^2}$ being the upward normal to $\partial\Omega$. Here $\omega_{\tilde{x}}$ is the gradient of $\omega(\tilde{x})$. Let

$$G = \frac{\sigma_{k-1}}{\sigma_{k-2}}(u_{(\tilde{x})}, u_{(\tilde{x}\tilde{x})}) - \sum_{l=1}^{k-2}\alpha_l\sqrt{1+|Du|^2}^{k-l}\frac{\sigma_{l-1}}{\sigma_{k-2}}(u_{(\tilde{x})}, u_{(\tilde{x}\tilde{x})}) - c\sqrt{1+|Du|^2}$$

on $\partial\Omega$, where $u_{(\tilde{x})} = (\frac{\partial u}{\partial b_1}, \cdots, \frac{\partial u}{\partial b_{n-1}})$ and $u_{(\tilde{x}\tilde{x})} = (\frac{\partial^2 u}{\partial b_i \partial b_j})_{1\le i,j\le n-1}$. For simplicity, we denote $\tilde{p} = u_{(\tilde{x})}$, $\tilde{r} = u_{(\tilde{x}\tilde{x})}$, $\bar{v} = \sqrt{1+|Du|^2}$. First we observe that

$$f_{(k)}(p,r) < \lim_{r_{nn}\to+\infty} f_{(k)}(p,r) = \frac{\sigma_{k-1}}{\sigma_{k-2}}(\tilde{p},\tilde{r}) - \sum_{l=1}^{k-2} \alpha_l \bar{v}^{k-l} \frac{\sigma_{l-1}}{\sigma_{k-2}}(\tilde{p},\tilde{r})$$

from what we see that $G > 0$. Hence the function

$$\tilde{G} = G(x) + \frac{4|\tilde{x}|^2}{\bar{\rho}^2}\bar{G}$$

with $\bar{G} = \max\{G(x)|x \in \partial\Omega, |\tilde{x}| < \rho\}$ and $0 < \bar{\rho} < \rho$ attains its minimum over $\partial\Omega \cap B_\rho(0)$ at some point $y \in \partial\Omega \cap B_{\bar{\rho}/2}(0)$. If $|u_{nn}(y)| < C$, then $G(y) > C^{-1} > 0$.

$$G(0) = \tilde{G}(0) \ge \tilde{G}(y) > G(y) > C^{-1} > 0.$$

Therefore $G(0)$ is strictly positive and we have

$$|u_{nn}(0)| < +\infty.$$

To check that $|u_{nn}(y)| < +\infty$, we proceed in essentially the same way as in mixed second derivative estimates. The point y plays the role of the origin and the function $\tilde{w}$ is defined as

$$\tilde{w}(x) = -(u_n(x) - u_n(y)) - K|Du(x) - Du(y)|^2$$

where K is a sufficiently big constant. In order to apply the comparison principle, we need to obtain that

$$w(x) \ge \tilde{\theta}(\tilde{x}) - C|\tilde{x} - \tilde{y}|^2 (x \in \partial\Omega \cap B_\rho(0))$$

where $\tilde{\theta}$ is some C^2-smooth function. We reason similarly to Lemma 2.5 in [19]. The choice of the moving frame gives

$$u_{(s)} = \varphi_{(s)}, \quad u_{(st)} = \varphi_{(st)} - u_n\omega_{(st)} (s,t = 1,\cdots,n-1).$$

By the concavity of $\frac{\sigma_{k-1}}{\sigma_{k-2}}(\tilde{p},\tilde{r})$, $-\frac{\sigma_l}{\sigma_{k-2}}(\tilde{p},\tilde{r})(l = 0,\cdots,k-3)$ in $\tilde{r}$ and the convexity of $\sqrt{1+|\tilde{p}|^2}$ in $\tilde{p}$, we compute

$$0 \le \tilde{G}(x) - \tilde{G}(y) \le g(y,x)(u_n(y) - u_n(x)) + h(y,x) \tag{4.19}$$

with

$$\begin{aligned} g(y,x) =& (\frac{\sigma_{k-1}}{\sigma_{k-2}})^{st}(\tilde{p}(x),\tilde{r}(y))\omega_{(st)}(x) - \sum_{l=1}^{k-2}\alpha_l\bar{v}^{k-l}(x)(\frac{\sigma_{l-1}}{\sigma_{k-2}})^{st}(\tilde{p}(x),\tilde{r}(y))\omega_{(st)}(x) \\ &+ \Big(\sum_{l=1}^{k-2}\alpha_l(k-l)\bar{v}^{k-l-2}(y)\frac{\sigma_{l-1}}{\sigma_{k-2}}(\tilde{p}(y),\tilde{r}(y)) + c\bar{v}^{-1}(y)\Big)(u_n(y) - u_i(y)\omega_i(x)) \end{aligned}$$

and

$$h(y,x) = \frac{\sigma_{k-1}}{\sigma_{k-2}}(\varphi_{\tilde{x}}(x),\tilde{r}(y)) - \frac{\sigma_{k-1}}{\sigma_{k-2}}(\varphi_{\tilde{x}}(y),\tilde{r}(y)) + (\frac{\sigma_{k-1}}{\sigma_{k-2}})^{st}(\varphi_{\tilde{x}}(x),\tilde{r}(y))\Psi_{st}(y,x)$$

$$
\begin{aligned}
&- \sum_{l=1}^{k-2} \alpha_l \bar{v}^{k-l} \Big(\frac{\sigma_{l-1}}{\sigma_{k-2}}\Big)^{st} (\varphi_{\tilde{x}}(x), \tilde{r}(y)) \Psi_{st}(y, x) \\
&+ \sum_{l=1}^{k-2} \alpha_l \bar{v}^{k-l} \Big(\frac{\sigma_{l-1}}{\sigma_{k-2}} (\varphi_{\tilde{x}}(y), \tilde{r}(y)) - \frac{\sigma_{l-1}}{\sigma_{k-2}} (\varphi_{\tilde{x}}(x), \tilde{r}(y)) \Big) \\
&+ [c\bar{v}^{-1} - \sum_{l=1}^{k-2} \alpha_l (k-l) \bar{v}^{k-l-2} \frac{\sigma_{l-1}}{\sigma_{k-2}} (\tilde{p}(y), \tilde{r}(y))] \cdot A + \frac{4\bar{G}}{\bar{\rho}^2} (|\tilde{x}|^2 - |\tilde{y}|^2)
\end{aligned}
$$

where $\Psi_{st}(y, x) = \varphi_{(st)}(x) - \varphi_{(st)}(y) - u_n(y)(\omega_{(st)}(x) - \omega_{(st)}(y))$, $A = [\varphi_i(y) - \varphi_i(x) - u_n(y)(\omega_i(y) - \omega_i(x))]u_i(y)$. We may take $\tilde{\theta}(\tilde{x}) = -\frac{h}{g}(y, x)$ if we can show that $g(y, x) > 0$. This is true since $|Du|$ is small and $-(\frac{\sigma_{l-1}}{\sigma_{k-1}})^{st}$ is semi-positive definite, together with condition (4.3). This completes the proof of the boundary regularity. □

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Conflict of interest

The authors declare no conflict of interest.

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A. Appendix. Proof of Lemma 3.3

In this appendix, we sketch the proof of Lemma 3.3 for reader's convenience. For the original proof, see [36].

Without loss of generality, we assume $\Omega = B_r(0)$. Let $\rho = r^2 - |x|^2$, $M = osc_{B_r}u$, $\tilde{g}(u) = \frac{1}{M}(M + u - \inf_{B_r} u)$, $\phi(x,\xi) = \rho(x)\tilde{g}(u)\log(u_\xi(x))$. This auxiliary function ϕ comes from [34]. Suppose ϕ attains its maximum at (x_0, e_1). Furthermore, by rotating $e_2, \cdots, e_n$, we can assume that $\{u_{ij}(x_0)\}_{2\le i,j\le n}$ is diagonal. Thus $\varphi(x) = \log\rho(x) + \log\tilde{g}(u(x)) + \log\log u_1$ also attains a local maximum at $x_0 \in B_r(0)$. At x_0, we have

$$0 = \varphi_i = \frac{\rho_i}{\rho} + \frac{\tilde{g}_i}{\tilde{g}} + \frac{u_{1i}}{u_1\log u_1}, \tag{A.1}$$

$$0 \ge \varphi_{ij} = \frac{\rho_{ij}}{\rho} - \frac{\rho_i\rho_j}{\rho^2} + \frac{\tilde{g}_{ij}}{\tilde{g}} - \frac{\tilde{g}_i\tilde{g}_j}{\tilde{g}^2} + \frac{u_{1ij}}{u_1\log u_1} - (1 + \frac{1}{\log u_1})\frac{u_{1i}u_{1j}}{u_1^2\log u_1}. \tag{A.2}$$

Only in this proof we denote that $F^{ij} := \frac{\partial F}{\partial u_{ij}}$. F^{ij} is positive definite. Taking trace with φ_{ij} and using (A.1), we have

$$\begin{aligned}
0 &\ge F^{ij}\varphi_{ij} \\
&= F^{ij}\Big(\frac{\rho_{ij}}{\rho} + 2\frac{\rho_i}{\rho}\frac{\tilde{g}_j}{\tilde{g}} + \frac{\tilde{g}_{ij}}{\tilde{g}}\Big) + F^{ij}\Big(\frac{u_{1ij}}{u_1\log u_1} - (1 + \frac{2}{\log u_1})\frac{u_{1i}u_{1j}}{u_1^2\log u_1}\Big) \\
&:= \mathcal{A} + \mathcal{B}.
\end{aligned} \tag{A.3}$$

It is well-known that the principal curvatures of graph u are the eigenvalues of matrix $A = (a_{ij})_{n\times n}$:

$$a_{ij} = \frac{1}{W}\Big(u_{ij} - \frac{u_i u_l u_{lj}}{W(W+1)} - \frac{u_j u_l u_{li}}{W(W+1)} + \frac{u_i u_j u_p u_q u_{pq}}{W^2(W+1)^2}\Big)$$

where $W = \sqrt{1+|Du|^2}$. Next we compute F^{ij} at x_0.

$$\frac{\partial a_{ij}}{\partial u_{ij}} = \begin{cases} \frac{1}{W^3} & i = j = 1, \\ \frac{1}{W^2} & i = 1, j \geq 2 \text{ or } i \geq 2, j = 1, \\ \frac{1}{W} & i \geq 2, j \geq 2. \end{cases}$$

For two different sets $\{p, q\} \neq \{i, j\}$, $\frac{\partial a_{pq}}{\partial u_{ij}} = 0$. Therefore

$$F^{ij} = \frac{\partial F}{\partial a_{ij}}\frac{\partial a_{ij}}{\partial u_{ij}} = \begin{cases} \frac{1}{W^3}\frac{\partial F}{\partial a_{11}} & i = j = 1, \\ \frac{1}{W^2}\frac{\partial F}{\partial a_{ij}} & i = 1, j \geq 2 \text{ or } i \geq 2, j = 1, \\ \frac{1}{W}\frac{\partial F}{\partial a_{ij}} & i \geq 2, j \geq 2. \end{cases}$$

Direct computation shows that

$$\mathcal{A} = \frac{-2}{\rho}(\sum_{i=1}^{n}\frac{\partial F}{\partial u_{ii}}) + \frac{1}{M\tilde{g}}(\sum_{i,j=1}^{n}\frac{\partial F}{\partial u_{ij}}\cdot u_{ij}) + \frac{2u_1}{M\rho\tilde{g}}\sum_{i=1}^{n}\frac{\partial F}{\partial u_{1i}}\cdot\rho_i,$$

$$\sum_{i=1}^{n}\frac{\partial F}{\partial u_{ii}} = \frac{\partial F}{\partial a_{11}}\frac{1}{W^3} + \sum_{i=2}^{n}\frac{\partial F}{\partial a_{ii}}\frac{1}{W} \leq \frac{1}{W}\sum_{i=1}^{n}\frac{\partial F}{\partial a_{ii}},$$

$$\sum_{i,j=1}^{n}\frac{\partial F}{\partial u_{ij}}\cdot u_{ij} = \sum_{i,j=1}^{n}\frac{\partial F}{\partial a_{ij}}\cdot a_{ij} = \frac{\sigma_k}{\sigma_{k-1}}(A) - \sum_{l=0}^{k-2}\alpha_l(l-k+1)\frac{\sigma_l}{\sigma_{k-1}}(A).$$

By (A.1), suppose that $u_1 \gg 1$, then we have $u_{11} < 0$ and

$$\begin{aligned}
\frac{2u_1}{M\rho\tilde{g}}\sum_{i=1}^{n}\frac{\partial F}{\partial u_{1i}}\cdot\rho_i &= \frac{2u_1}{M\rho\tilde{g}}(\frac{\partial F}{\partial u_{11}}\rho_1 + \sum_{i=2}^{n}\frac{\partial F}{\partial u_{1i}}\rho_i) \\
&= \frac{2u_1}{M\rho\tilde{g}}(\frac{\partial F}{\partial a_{11}}\frac{\rho_1}{W^3} - \sum_{i=2}^{n}\frac{\partial F}{\partial a_{1i}}\frac{u_{1i}\rho}{W^2 u_1\log u_1}) \\
&\geq -\frac{4ru_1}{MW^3\rho\tilde{g}}\frac{\partial F}{\partial a_{11}} - \frac{2}{M\tilde{g}\log u_1}\sum_{i=2}^{n}\frac{\partial F}{\partial a_{1i}}a_{1i} \\
&\geq -\frac{4ru_1}{MW^3\rho\tilde{g}}\sum_{i=1}^{n}\frac{\partial F}{\partial a_{ii}}
\end{aligned}$$

where we have used (3.1). Therefore

$$\mathcal{A} \geq (-\frac{2}{W\rho} - C\frac{u_1}{W^3})(\sum_{i=1}^{n}\frac{\partial F}{\partial a_{ii}}) + \frac{1}{M\tilde{g}}(-g + \sum_{l=0}^{k-2}\alpha_l(k-l)\frac{\sigma_l}{\sigma_{k-1}}(A)). \tag{A.4}$$

In the following we turn to estimate $\mathcal{B}$. By the definition of a_{ij}, we have at x_0,

$$\frac{\partial a_{11}}{\partial x_1} = \frac{1}{W^3}u_{111} - \frac{3u_1}{W^5}u_{11}^2 - \frac{2u_1}{W^3(W+1)}\sum_{k=2}^{n} u_{k1}^2,$$

for $i \geq 2$,

$$\frac{\partial a_{1i}}{\partial x_1} = \frac{1}{W^2}u_{1i1} - \frac{2u_1}{W^4}u_{11}u_{1i} - \frac{u_1}{W^2(W+1)}u_{1i}u_{ii} - \frac{u_1}{W^3(W+1)}u_{11}u_{1i},$$

$$\frac{\partial a_{ii}}{\partial x_1} = \frac{1}{W}u_{ii1} - \frac{u_1}{W^3}u_{11}u_{ii} - \frac{2u_1}{W^2(W+1)}u_{1i}^2,$$

for $i \geq 2, j \geq 2, i \neq j$,

$$\frac{\partial a_{ij}}{\partial x_1} = \frac{1}{W}u_{ij1} - 2\frac{u_{i1}u_{j1}u_1}{W^2(W+1)}.$$

Taking derivatives with respect to x_1 on both sides of (1.4), we have

$$\sum_{i,j=1}^{n} \frac{\partial F}{\partial a_{ij}}\frac{\partial a_{ij}}{\partial x_1} - \sum_{l=0}^{k-2}(\alpha_l)_{,1}\frac{\sigma_l}{\sigma_{k-2}} = -g_{,1}.$$

For the first term of $\mathcal{B}$, we calculate as

$$\sum_{i,j=1}^{n} \frac{\partial F}{\partial u_{ij}}\frac{u_{ij1}}{u_1 \log u_1} = \frac{1}{u_1 \log u_1}\Big(\frac{\partial F}{\partial a_{11}}\frac{u_{111}}{W^3} + 2\sum_{i\geq 2}\frac{\partial F}{\partial a_{1i}}\frac{u_{1i1}}{W^2} + \sum_{i,j\geq 2}\frac{\partial F}{\partial a_{ij}}\frac{u_{ij1}}{W}\Big),$$

$$\begin{aligned}
F^{ij}u_{ij1} =& \frac{\partial F}{\partial a_{11}}\Big(\frac{\partial a_{11}}{\partial x_1} + \frac{3u_1}{W^5}u_{11}^2 + \frac{2u_1}{W^3(W+1)}\sum_{k\geq 2}u_{k1}^2\Big) + 2\sum_{i\geq 2}\frac{\partial F}{\partial a_{1i}}\Big(\frac{\partial a_{1i}}{\partial x_1} + \frac{2u_1}{W^4}u_{11}u_{1i} \\
&+ \frac{u_1u_{1i}u_{ii}}{W^2(W+1)} + \frac{u_1u_{11}u_{1i}}{W^3(W+1)}\Big) + \sum_{i\neq j\geq 2}\frac{\partial F}{\partial a_{ij}}\Big(\frac{\partial a_{ij}}{\partial x_1} + 2\frac{u_1u_{i1}u_{j1}}{W^2(W+1)}\Big) \\
&+ \sum_{i=2}^{n}\frac{\partial F}{\partial a_{ii}}\Big(\frac{\partial a_{ii}}{\partial x_1} + \frac{u_1u_{11}u_{ii}}{W^3} + \frac{2u_1u_{1i}^2}{W^2(W+1)}\Big) \\
=& - g_{,1} + \sum_{l=0}^{k-2}(\alpha_l)_{,1}\frac{\sigma_l}{\sigma_{k-1}}(A) + \frac{u_1u_{11}}{W^2}\Big(- g + \sum_{l=0}^{k-2}(k-l)\alpha_l\frac{\sigma_l}{\sigma_{k-1}}(A)\Big) \\
&+ \frac{\partial F}{\partial a_{11}}\Big(\frac{2u_1}{W^5}u_{11}^2 + \frac{2u_1}{W^3(W+1)}\sum_{k\geq 2}u_{k1}^2\Big) + 2\sum_{i\geq 2}\frac{\partial F}{\partial a_{1i}}\Big(\frac{u_1}{W^4}u_{11}u_{1i} + \frac{u_1u_{1i}u_{ii}}{W^2(W+1)} \\
&+ \frac{u_1u_{11}u_{1i}}{W^3(W+1)}\Big) + 2\sum_{i\neq j\geq 2}\frac{\partial F}{\partial a_{ij}}\frac{u_1u_{i1}u_{j1}}{W^2(W+1)} + 2\sum_{i=2}^{n}\frac{\partial F}{\partial a_{ii}}\frac{u_1u_{1i}^2}{W^2(W+1)}.
\end{aligned}$$

For the second term of $\mathcal{B}$, we calculate

$$\sum_{i,j=1}^{n}\frac{\partial F}{\partial u_{ij}}u_{1i}u_{1j} = \frac{\partial F}{\partial a_{11}}\frac{u_{11}^2}{W^3} + 2\sum_{i=2}^{n}\frac{\partial F}{\partial a_{1i}}\frac{u_{11}u_{1i}}{W^2} + \sum_{2\leq i,j\leq n}\frac{\partial F}{\partial a_{ij}}\frac{u_{1i}u_{1j}}{W}.$$

Therefore

$$
\begin{aligned}
\mathcal{B} =&\frac{1}{u_1\log u_1}\Big(-g_{,1}+\sum_{l=0}^{k-2}(\alpha_l)_{,1}\frac{\sigma_l}{\sigma_{k-1}}(A)\Big)+\frac{u_{11}}{W^2\log u_1}\Big(-g+\sum_{l=0}^{k-2}(k-l)\alpha_l\frac{\sigma_l}{\sigma_{k-1}}(A)\Big)\\
&+\Big(\frac{2}{W^5\log u_1}-(1+\frac{2}{\log u_1})\frac{1}{u_1^2\log u_1 W^3}\Big)\frac{\partial F}{\partial a_{11}}u_{11}^2+\frac{2}{W^3(W+1)\log u_1}\sum_{k\geq 2}\frac{\partial F}{\partial a_{11}}u_{k1}^2\\
&+\Big(\frac{2}{W^4\log u_1}+\frac{2}{W^3(W+1)\log u_1}-(1+\frac{2}{\log u_1})\frac{2}{W^2u_1^2\log u_1}\Big)\sum_{i=2}^{n}\frac{\partial F}{\partial a_{1i}}u_{11}u_{1i}\\
&+\frac{2}{W^2(W+1)\log u_1}\sum_{i=2}^{n}\frac{\partial F}{\partial a_{1i}}u_{1i}u_{ii}+\Big(\frac{2}{W^2(W+1)\log u_1}-\frac{1+2/\log u_1}{Wu_1^2\log u_1}\Big)\times\\
&\sum_{2\leq i,j\leq n}\frac{\partial F}{\partial a_{ij}}u_{1i}u_{1j}.
\end{aligned}
$$

Since $\{\frac{\partial F}{\partial a_{ij}}\}_{1\leq i,j\leq n}$ is positive definite, so is $\{\frac{\partial F}{\partial a_{ij}}\}_{2\leq i,j\leq n}$. $W=\sqrt{1+u_1^2}\approx u_1$. Therefore

$$
\begin{aligned}
\mathcal{B} \geq&\frac{1}{u_1\log u_1}\Big(-g_{,1}+\sum_{l=0}^{k-2}(\alpha_l)_{,1}\frac{\sigma_l}{\sigma_{k-1}}(A)\Big)+\frac{u_{11}}{W^2\log u_1}\Big(-g+\sum_{l=0}^{k-2}(k-l)\alpha_l\frac{\sigma_l}{\sigma_{k-1}}(A)\Big)\\
&+\frac{1-\delta}{W^5\log u_1}\frac{\partial F}{\partial a_{11}}u_{11}^2+\frac{2}{W^2(W+1)\log u_1}\Big(\sum_{i=2}^{n}\frac{\partial F}{\partial a_{1i}}u_{1i}u_{ii}\Big)
\end{aligned}
\tag{A.5}
$$

where $\delta>0$ is a small constant, depending only on u_1. By (A.3), (A.4), (A.5), we have

$$
\begin{aligned}
0\geq&(-\frac{2}{W\rho}-\frac{Cu_1}{W^3})(\sum_{i=1}^{n}\frac{\partial F}{\partial a_{ii}})+(\frac{1}{M\tilde{g}}+\frac{u_{11}}{W^2\log u_1})(-g+\sum_{l=0}^{n}\alpha_l(k-l)\frac{\sigma_l}{\sigma_{k-1}})\\
&+\frac{1}{u_1\log u_1}(-g_{,1}+\sum_{l=0}^{k-2}(\alpha_l)_{,1}\frac{\sigma_l}{\sigma_{k-1}})+\frac{1-\delta}{W^5\log u_1}\frac{\partial F}{\partial a_{11}}u_{11}^2\\
&+\frac{2}{W^2(W+1)\log u_1}(\sum_{i=2}^{n}\frac{\partial F}{\partial a_{1i}}u_{1i}u_{ii}).
\end{aligned}
\tag{A.6}
$$

Since we require that $g_u\leq 0$ and $(\alpha_l)_u\geq 0$,

$$
-g_{,1}+\sum_{l=0}^{k-2}(\alpha_l)_{,1}\frac{\sigma_l}{\sigma_{k-1}}\geq-\frac{\partial g}{\partial x_1}+\sum_{l=0}^{k-2}\frac{\partial\alpha_l}{\partial x_1}\frac{\sigma_l}{\sigma_{k-1}}.
$$

We claim that

$$
\sum_{i=2}^{n}\frac{\partial F}{\partial a_{1i}}u_{1i}u_{ii}\geq -C\frac{u_1^2\log^2 u_1}{W}\frac{|D\rho|^2}{\rho^2}\frac{\partial F}{\partial a_{11}}. \tag{A.7}
$$

We deter the proof of (A.7). By (A.1), we see that the leading term in (A.6) is $\frac{1-\delta}{W^5\log u_1}\frac{\partial F}{\partial a_{11}}u_{11}^2\approx\frac{\log u_1}{W}\frac{\partial F}{\partial a_{11}}>0$. Other terms have order at most $O(W^{-1})$, therefore

$$
\log u_1\leq C.
$$

The interior gradient estimate is proved after we check (A.7). Let $\Upsilon = \{2 \le j \le n | a_{jj} \ge 0\}$. Note that $a_{11} < 0$ and $\lambda(A) \in \Gamma_k$.

$$
\begin{aligned}
\sum_{i=2}^{n} \frac{\partial F}{\partial a_{1i}} u_{1i} u_{ii} =& - \sum_{i=2}^{n} \Bigg[\frac{\sigma_{k-2}(A|1i)\sigma_{k-1}(A) - \sigma_{k-3}(A|1i)\sigma_k(A)}{\sigma_{k-1}^2(A)} \\
&+ \sum_{l=1}^{k-2} \alpha_l \frac{\sigma_{k-3}(A|1i)\sigma_l(A) - \sigma_{l-2}(A|1i)\sigma_{k-1}(A)}{\sigma_{k-1}^2(A)} \Bigg] a_{i1} u_{1i} u_{ii} \\
\ge& - \sum_{i \in \Upsilon}^{n} \Bigg[\frac{\sigma_{k-2}(A|1i)\sigma_{k-1}(A) - \sigma_{k-3}(A|1i)\sigma_k(A)}{\sigma_{k-1}^2(A)} \\
&+ \sum_{l=1}^{k-2} \alpha_l \frac{\sigma_{k-3}(A|1i)\sigma_l(A) - \sigma_{l-2}(A|1i)\sigma_{k-1}(A)}{\sigma_{k-1}^2(A)} \Bigg] a_{ii} \frac{u_{1i}^2}{W} \\
\ge& - \sum_{i \in \Upsilon} \Bigg[\frac{a_{ii}\sigma_{k-2}(A|1i)\sigma_{k-1}(A)}{\sigma_{k-1}^2(A)} + \sum_{l=1}^{k-2} \alpha_l \frac{a_{ii}\sigma_{k-3}(A|1i)\sigma_l(A)}{\sigma_{k-1}^2(A)} \Bigg] \frac{u_{1i}^2}{W} \\
\ge& - \sum_{i \in \Upsilon} \Bigg[\frac{C_{n,k}\sigma_{k-1}(A|1)\sigma_{k-1}(A)}{\sigma_{k-1}^2(A)} + \sum_{l=1}^{k-2} \alpha_l \frac{C_{n,k}\sigma_{k-2}(A|1)\sigma_l(A)}{\sigma_{k-1}^2(A)} \Bigg] \frac{u_{1i}^2}{W} \\
\ge& - \Bigg[\frac{C_{n,k}\sigma_{k-1}(A|1)\sigma_{k-1}(A)}{\sigma_{k-1}^2(A)} + \sum_{l=1}^{k-2} \alpha_l \frac{C_{n,k}\sigma_{k-2}(A|1)\sigma_l(A)}{\sigma_{k-1}^2(A)} \Bigg] \sum_{i=2}^{n} \frac{u_{1i}^2}{W} \\
\ge& - \Bigg[C_{n,k} \frac{\sigma_{k-1}(A|1)\sigma_{k-1}(A) - \sigma_{k-2}(A|1)\sigma_k(A)}{\sigma_{k-1}^2(A)} \\
&+ \sum_{l=0}^{k-2} \alpha_l C_{n,k} \frac{\sigma_{k-2}(A|1)\sigma_l(A) - \sigma_{l-1}(A)\sigma_{k-1}(A)}{\sigma_{k-1}^2(A)} \Bigg] \sum_{i=2}^{n} \frac{u_{1i}^2}{W} \\
\ge& - C(n,k) \sum_{i=2}^{n} \frac{u_{1i}^2}{W} \frac{\partial F}{\partial a_{11}} \\
\ge& - C(n,k) \frac{u_1^2 \log^2 u_1}{W} \frac{|D\rho|^2}{\rho^2} \frac{\partial F}{\partial a_{11}}.
\end{aligned}
$$

Thus (A.7) holds and the gradient estimate is proved.