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Research article

On parabolic Adams's, the Chiarenza-Frasca theorems, and some other results related to parabolic Morrey spaces†

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Abstract: We present several results of embedding type for parabolic Morrey and L_p spaces with or without mixed norms. Some other interpolation results for parabolic Morrey spaces are also given. The main object of investigation is the term $b^i D_i u$ and the ways to estimate it in various Morrey and L_p spaces in order to be able to treat it as a perturbation term in the parabolic equations.

Keywords: parabolic Morrey spaces; embedding theorems; Adams theorem; Chiarenza-Frasca theorem

1. Introduction

Let $\mathbb{R}^d$, $d \geq 1$, be a Euclidean space of points $x = (x^1, ..., x^d)$. In 1975 D. Adams [1] among many other things proved that, if $d \geq 2$ and we are given $u \in C_0^\infty = C_0^\infty(\mathbb{R}^d)$ with its gradient $Du = (D_1 u, ..., D_d u)$, $D_i = \partial/\partial x^i$, satisfying

$$\int_{|x-y|<\rho} |Du(y)|^q \, dy \leq \rho^{d-\beta q}, \tag{1.1}$$

with $q > 1$, $1 < \beta \leq d/q$, and any $\rho \in (0, \infty)$ and $x \in \mathbb{R}^d$, then for all $\rho \in (0, \infty)$ and $x \in \mathbb{R}^d$ we have

$$\int_{|x-y|<\rho} |u(y)|^r \, dy \leq N\rho^{d-(\beta-1)r} \tag{1.2}$$

with a constant N independent of u and r satisfying $(\beta - 1)r = \beta q$.

This fact played a crucial role in [10] where the author investigated the solvability of elliptic equations

$$a^{ij}D_{ij}u + b^iD_iu + u = f \quad (D_{ij} = D_iD_j) \tag{1.3}$$

with $b \notin L_{d,\mathrm{loc}}$ but rather satisfying for a sufficiently small $\hat{b}$, all sufficiently small ρ and all balls B of radius ρ

$$\int_B |b|^{d_0}\,dx \leq \hat{b}\rho^{d-d_0}$$

with certain $d_0 \in (d/2, d)$.

Our goal in this paper is to prepare necessary tools for developing a similar theory for parabolic equations. In Section 2 we prove an analog of Adams's intermediate estimate, which is the main starting point. Section 3 contains the parabolic analog of the embedding theorem mentioned in the beginning of the article. It also contains "local" interpolation inequalities in Morrey spaces allowing one to deal with Morrey's norms of expressions like b^iD_iu in domains when b is bounded. Section 4 is devoted to the parabolic analog of a Chiarenza-Frasca theorem allowing to estimate the L_p-norm rather than Morrey's norm of b^iD_iu. In Section 5 we treat parabolic Morrey spaces with mixed norms. The main object of investigation is the term b^iD_iu and the ways to estimate it in various Morrey and L_p spaces in order to be able to treat it as a perturbation term in the parabolic analog of (1.3).

We finish the introduction with some notation and a remark. Define $B_\rho(x) = \{y \in \mathbb{R}^d : |x-y| < \rho\}$, $\mathbb{R}^{d+1} := \{z = (t,x) : t \in \mathbb{R}, x \in \mathbb{R}^d\}$,

$$C_\rho(t,x) = \{(s,y) \in \times\mathbb{R}^{d+1} : |x-y| < \rho, t \leq s < t+\rho^2\}, \quad C_\rho = C_\rho(0)$$

and let $\mathbb{C}_\rho$ be the collection of $C_\rho(z)$, $z \in \mathbb{R}^{d+1}$, $\mathbb{C} = \{\mathbb{C}_\rho, \rho > 0\}$. For measurable $\Gamma \subset \mathbb{R}^{d+1}$ set $|\Gamma|$ to be its Lebesgue measure and when it makes sense set

$$f_\Gamma = \fint_\Gamma f\,dz = \frac{1}{|\Gamma|}\int_\Gamma f\,dz.$$

Similar notation is used for $f = f(x)$.

Remark 1.1. Formally, Adams proved (1.2) assuming that $d \geq 2$. However, it is also true if $d = 1$. To show this it suffices to take u depending only on one coordinate. The reader may wonder how the restriction $\beta \leq d/q$ will become $\beta \leq 1/q$. The point is that if $d = 1$ and $\beta > 1/q$, we have $d - \beta q < 0$ and condition (1.1) becomes only possible if $u = 0$.

2. Preliminary estimates

An important quantity characterizing $L_p = L_p(\mathbb{R}^{d+1})$ is what we call the index which is the exponent of ρ in the expression

$$\|I_{C_\rho}\|_{L_p} \quad \text{that is} \quad \frac{d+2}{p}.$$

For domains $Q \subset \mathbb{R}^{d+1}$, $p \in [1,\infty)$, and $\beta \in (0, (d+2)/p]$, introduce Morrey's space $E_{p,\beta}(Q)$ as the set of g such that

$$\|g\|_{E_{p,\beta}(Q)} := \sup_{\rho<\infty,(t,x)\in Q} \rho^\beta \,\|\!|gI_Q\|_{L_p(C_\rho(t,x))} < \infty, \tag{2.1}$$

where

$$\text{⨍}\!\!\|g\|_{L_p(\Gamma)} = \Big(\fint_{\Gamma} |g|^p \, dz \Big)^{1/p}.$$

We abbreviate $E_{p,\beta} = E_{p,\beta}(\mathbb{R}^{d+1})$. Observe that if $Q = C_R$ one can restrict ρ in (2.1) to $\rho \le R$ since $\beta \le (d+2)/p$. Also in that case one can allow (t,x) to be arbitrary, because, if $|x| \ge R$, then $B_R \cap B_\rho(x) \subset B_R \cap B_\rho(Rx/|x|)$. It is also useful to observe that, in case $Q = C_R$, one gets an equivalent norm by adding to the restrictions $\rho < \infty, (t,x) \in C_R$, the requirement that the geometric center of $C_\rho(t,x)$ be in C_R. This follows from the fact that the $L_p(C_\rho(t,x))$-norm of gI_{C_R} will only increase if we pull $C_\rho(t,x)$ down the t axis to $\{t = 0\}$ (if $\rho^2 > 2R^2$) or to the moment that the shifted $C_\rho(t,x)$ has its geometric center inside C_R.

There are many different notations for the norms in Morrey spaces. The convenience of the above notation is well illustrated by Theorem 3.1 and Corollary 5.7.

We will often, always tacitly, use the following formulas in which $u(t,x) = v(t/R^2, x/R)$:

$$\text{⨍}\!\!\|u\|_{L_p(C_R)} = \text{⨍}\!\!\|v\|_{L_p(C_1)}, \quad \|u\|_{E_{p,\beta}(Q)} = R^\beta \|v\|_{E_{p,\beta}(Q_R)},$$

where $Q_R = \{(t,x) : (R^2 t, Rx) \in Q\}$,

$$\|Du\|_{E_{p,\beta}(C_R)} = R^{\beta-1}\|Dv\|_{E_{p,\beta}(C_1)}, \quad \|D^2 u\|_{E_{p,\beta}(C_R)} = R^{\beta-2}\|v\|_{E_{p,\beta}(C_1)}.$$

For $s, r > 0, \alpha > 0$, and appropriate $f(t,x)$'s on $\mathbb{R}^{d+1}$ define

$$p_\alpha(s,r) = \frac{1}{s^{(d+2-\alpha)/2}} e^{-r^2/s} I_{s>0},$$

$$P_\alpha f(t,x) = \int_{\mathbb{R}^{d+1}} p_\alpha(s,|y|) f(t+s, x+y) \, dy ds.$$

Observe that, if f is independent of t, then

$$P_\alpha f(t,x) = P_\alpha f(x) = N(\alpha) \int_{\mathbb{R}^d} \frac{1}{|y|^{d-\alpha}} f(x+y)\, dy = N I_\alpha f(x),$$

where I_α is the Riesz potential. Therefore, one can get the Adams estimate found in the proof of Proposition 3.1 of [1] from (2.3) below. In our investigation the most important values of α are 1 and 2. Set

$$M_\beta f(t,x) = \sup_{\rho>0} \rho^\beta \fint_{C_\rho(t,x)} |f(z)| \, dz, \quad 0 \le \beta \le d+2,$$

$$Mf = M_0 f.$$

The following lemma is obtained by integrating by parts.

Lemma 2.1. *Let $\beta > 0$ be a finite number, $f(t) \ge 0$ be a function on $[0,\infty)$ such that*

$$t^{-\beta} \int_0^t f(s)\, ds \to 0$$

as $t \to \infty$. Then, for any $S \ge 0$,

$$\int_S^\infty t^{-\beta} f(t)\, dt \le \beta \int_S^\infty t^{-\beta-1} \Big(\int_S^t f(s)\, ds \Big) dt.$$

Lemma 2.2. *For any $\alpha \in (0,\beta), \beta \in (0, d+2]$ there exist constants N ($<\infty$) such that for any $f \geq 0$ and $\rho \in (0,\infty)$ we have*

$$P_\alpha(I_{C_\rho} f)(0) \leq N\rho^\alpha M f(0), \quad P_\alpha(I_{C_\rho^c} f)(0) \leq N\rho^{\alpha-\beta} M_\beta f(0), \tag{2.2}$$

$$P_\alpha f \leq N(M_\beta f)^{\alpha/\beta}(Mf)^{1-\alpha/\beta}. \tag{2.3}$$

In particular (by Hölder's inequality), for any $p \in [1,\infty]$, $q \in (1,\infty]$, and measurable Γ

$$\|P_\alpha f\|_{L_r(\Gamma)} \leq N \|M_\beta f\|^{\alpha/\beta}_{L_p(\Gamma)} \|f\|^{1-\alpha/\beta}_{L_q}, \tag{2.4}$$

provided that

$$\frac{1}{r} = \frac{\alpha}{\beta} \cdot \frac{1}{p} + \Big(1 - \frac{\alpha}{\beta}\Big)\frac{1}{q}.$$

Proof. We basically mimic the proof of Proposition 3.1 of [1]. Observe that (2.3) at the origin is easily obtained from summing up the inequalities in (2.2) and minimizing with respect to ρ. At any other point it is obtained by changing the origin. Furthermore clearly, we may assume that f is bounded with compact support. Set $Q_1 = \{(s,y) : |y| \geq \sqrt{s}\}$, $Q_2 = \{(s,y) : |y| \leq \sqrt{s}\}$. Dealing with $P_\alpha(fI_{Q_1})$ we observe that $p_\alpha(s,r) \leq Nr^{-(d+2-\alpha)}$ if $r \geq \sqrt{s}$. Therefore,

$$P_\alpha(fI_{Q_1 \cap C_\rho^c})(0) \leq N \int_\rho^\infty \frac{1}{r^{d+2-\alpha}} \int_0^{r^2} \Big(\int_{|y|=r} f(s,y)\, d\sigma_r \Big) ds dr,$$

where $d\sigma_r$ is the element of the surface area on $|y| = r$. By Lemma 2.1 ($\alpha < d+2$)

$$\begin{aligned} P_\alpha(fI_{Q_1 \cap C_\rho^c})(0) &\leq N \int_\rho^\infty \frac{1}{r^{d+3-\alpha}} \int_\rho^r \Big(\int_0^{\rho^2} \Big(\int_{|y|=\rho} f(s,y)\, d\sigma_\rho \Big) ds \Big) d\rho dr \\ &\leq N \int_\rho^\infty \frac{1}{r^{d+3-\alpha}} \int_0^r \Big(\int_0^{r^2} \Big(\int_{|y|=\rho} f(s,y)\, d\sigma_\rho \Big) ds \Big) d\rho dr \\ &= N \int_\rho^\infty \frac{1}{r^{d+3-\alpha}} I(r)\, dr, \end{aligned}$$

where

$$I(r) = \int_{C_r} f(s,y)\, dyds.$$

We use that $I(r) \leq Nr^{d+2-\beta} M_\beta f(0)$ and that $\alpha < \beta$. Then we see that

$$P_\alpha(fI_{Q_1 \cap C_\rho^c})(0) \leq N\rho^{\alpha-\beta} M_\beta f(0). \tag{2.5}$$

Next, by using Lemma 2.1 we obtain that

$$\begin{aligned} P_\alpha(fI_{Q_2 \cap C_\rho^c})(0) &\leq \int_{\rho^2}^\infty \frac{1}{s^{(d+2-\alpha)/2}} \int_{|y| \leq \sqrt{s}} f(s,y)\, dyds \\ &\leq N \int_{\rho^2}^\infty \frac{1}{s^{(d+4-\alpha)/2}} I(\sqrt{s})\, ds = N \int_\rho^\infty \frac{1}{r^{d+3-\alpha}} I(r)\, dr. \end{aligned}$$

This along with (2.5) prove the second inequality in (2.2).

As long as the first inequality is concerned, observe that similarly to Lemma 2.1 using that $\alpha > 0$ we have

$$P_\alpha(fI_{Q_1\cap C_\rho})(0) \le N\int_0^\rho \frac{1}{r^{d+2-\alpha}}\int_0^{r^2}\Big(\int_{|y|=r} f(s,y)\,d\sigma_r\Big)\,dsdr$$

$$= N\int_0^\rho \frac{1}{r^{d+2-\alpha}}\Big(\frac{\partial}{\partial r}\int_0^r\Big(\int_0^{\tau^2}\int_{|y|=\tau} f(s,y)\,d\sigma_\tau\,ds\Big)\,d\tau\Big)dr$$

$$= J_1 + N\int_0^\rho \frac{1}{r^{d+3-\alpha}}\int_0^r\Big(\int_0^{\tau^2}\Big(\int_{|y|=\tau} f(s,y)\,d\sigma_\tau\Big)\,ds\Big)\,d\tau dr$$

$$\le J_1 + N\int_0^\rho \frac{1}{r^{d+3-\alpha}} I(r)\,dr,$$

where

$$J_1 = N\frac{1}{\rho^{d+2-\alpha}}\int_0^\rho\Big(\int_0^{\tau^2}\int_{|y|=\tau} f(s,y)\,d\sigma_\tau\,ds\Big)\,d\tau \le N\frac{1}{\rho^{d+2-\alpha}} I(\rho)$$

Here $I(r) \le Nr^{d+2}Mf(0)$ and $\alpha > 0$, so that

$$P_\alpha(fI_{Q_1\cap C_\rho})(0) \le N\rho^\alpha Mf(0). \tag{2.6}$$

Furthermore,

$$P_\alpha(fI_{Q_2\cap C_\rho})(0) \le N\int_0^{\rho^2} \frac{1}{s^{(d+2-\alpha)/2}}\int_{|y|\le\sqrt{s}} f(s,y)\,dyds$$

$$\le J_2 + N\int_0^{\rho^2}\frac{1}{s^{(d+4-\alpha)/2}} I(\sqrt{s})\,ds = J_2 + N\int_0^\rho \frac{1}{r^{d+3-\alpha}} I(r)\,dr,$$

where

$$J_2 = N\frac{1}{\rho^{d+2-\alpha}}\int_0^{\rho^2}\int_{|y|\le\sqrt{\tau}} f(\tau,y)\,dyd\tau \le N\frac{1}{\rho^{d+2-\alpha}} I(\rho).$$

This and (2.6) prove the first inequality in (2.2). The lemma is proved.

Remark 2.3. If $d = \alpha = 1$ and f is independent of t, the inequalities (2.2) and (2.3) are useless, because the first one in (2.2) follows by definition and the second one and (2.3) are trivial because $M_\beta f = \infty$ $(\beta > \alpha = 1)$ unless $f = 0$.

If α is strictly less than the index of L_q, we have the following.

Corollary 2.4. *If* $\alpha \in (0, (d+2)/q)$, $q \in (1,\infty)$*, then there exists a constant* N *such that for any* $f \ge 0$ *we have*

$$\|P_\alpha f\|_{L_r} \le N\|f\|_{L_q}$$

as long as

$$\frac{d+2}{q} - \alpha = \frac{d+2}{r}.$$

In particular, (a classical embedding result) if $1 < q < d+2$ *and* $u \in C_0^\infty = C_0^\infty(\mathbb{R}^{d+1})$*, then*

$$\|Du\|_{L_r} \le N\|\partial_t u + \Delta u\|_{L_q} \quad (\partial_t = \partial/\partial t)$$

as long as

$$\frac{d+2}{q} - 1 = \frac{d+2}{r}.$$

Indeed, the first assertion follows from Hölder's inequality and (2.4) with $p = \infty$ and $\beta = (d+2)/q$ ($> \alpha$). The second assertion follows from the first one with $\alpha = 1$ ($< \beta$) and the fact that for $f = \partial_t u + \Delta u$ we have

$$Du(t,x) = c \int_{\mathbb{R}^{d+1}_+} \frac{y}{s^{(d+2)/2}} e^{-|y|^2/(4s)} f(t+s, x+y)\, dyds,$$

where c is a constant and $(|y|/s^{1/2})e^{-|y|^2/(4s)} \leq Ne^{-|y|^2/(8s)}$.

Remark 2.5. After Corollary 2.4 a natural question arises as to what power of summability $b = (b^i)$ will be sufficient for the term $b^i D_i u$ to be considered as a perturbation term in $\partial_t u + \Delta u + b^i D_i u$ in the framework of the L_q-theory. Observe that, in the notation of Corollary 2.4

$$\|b^i D_i u\|_{L_q} \leq \|b\|_{L_{d+2}} \|Du\|_{L_r} \leq N\|b\|_{L_{d+2}} \|\partial_t u + \Delta u\|_{L_q}. \tag{2.7}$$

It follows that b should be of class L_{d+2} and $q < d+2$. Of course, if b contains just bounded part, this part in $b^i D_i u$ is taken care of by interpolation inequalities.

In the next section we will also need the following result.

Corollary 2.6. *For any $\alpha \in (0, \beta), \beta \in (0, d+2]$ there exists a constant N such that for any $g \geq 0$, $\rho \in (0, \infty)$, and $(t,x) \in C_\rho$ we have*

$$P_\alpha(I_{C^c_{2\rho}} g)(t,x) \leq N\rho^{\alpha-\beta} M_\beta g(t,x).$$

Indeed, since

$$\{t+s \geq 4\rho^2 \quad \text{or} \quad |x+y| \geq 2\rho\} \subset \{s \geq \rho^2 \quad \text{or} \quad |y| \geq \rho\}$$

for $f = g(t+\cdot, x+\cdot)$ we have

$$P_\alpha(I_{C^c_{2\rho}} g)(t,x) \leq \int_{\mathbb{R}^{d+1}} I_{C^c_\rho}(s,y) p_\alpha(s,y) g(t+s, x+y)\, dyds = P_\alpha(I_{C^c_\rho} f)(0)$$

$$\leq N\rho^{\alpha-\beta} M_\beta f(0) = N\rho^{\alpha-\beta} M_\beta g(t,x).$$

3. A parabolic analog of the Adams Theorem 3.1 of [1]

Theorem 3.1. *For any $\alpha \in (0, \beta), \beta \in (0, (d+2)/q]$, $q \in (1, \infty)$, and r such that*

$$r(\beta - \alpha) = q\beta,$$

there is a constant N such that for any $f \geq 0$ we have

$$\|P_\alpha f\|_{E_{r,\beta-\alpha}} \leq N\|f\|_{E_{q,\beta}}. \tag{3.1}$$

Proof. It suffices to prove that for any $\rho > 0$

$$\rho^{\beta-\alpha}\Big(\fint_{C_\rho} |P_\alpha f|^r \, dz\Big)^{1/r} \le N\|f\|_{E_{q,\beta}},$$

that is

$$\rho^{\beta-\alpha-(d+2)/r}\Big(\int_{C_\rho} |P_\alpha f|^r \, dz\Big)^{1/r} \le N\|f\|_{E_{q,\beta}}, \tag{3.2}$$

Observe that by Hölder's inequality $M_\beta f \le N\|f\|_{E_{q,\beta}}$ and by definition

$$\Big(\int_{\mathbb{R}^{d+1}} I_{C_{2\rho}} f^q \, dz\Big)^{1/q} \le N\rho^{(d+2)/q-\beta}\|f\|_{E_{q,\beta}}.$$

It follows from Lemma 2.2 with $p = \infty$ that

$$\Big(\int_{C_\rho} |P_\alpha(I_{C_{2\rho}} f)|^r dz\Big)^{1/r} \le N\rho^{((d+2)/q-\beta)(1-\alpha/\beta)}\|f\|_{E_{q,\beta}}$$

$$= N\rho^{(d+2)/r-\beta+\alpha}\|f\|_{E_{q,\beta}}.$$

Furthermore, by Corollary 2.6

$$\Big(\int_{C_\rho} |P_\alpha(I_{C^c_{2\rho}} f)|^r dz\Big)^{1/r} \le N\rho^{(d+2)/r} \sup_{C_\rho} P_\alpha(I_{C^c_{2\rho}} f)$$

$$\le N\rho^{(d+2)/r+\alpha-\beta} E_{q,\beta} f.$$

By combining these estimates we come to (3.2) and the theorem is proved.

Remark 3.2. We did not explicitly used that $\beta \le (d + 2)/q$ and formally the proof is valid for any $\beta \in (0, \infty)$ if in Definition 2.1 we allow any $\beta > 0$. However, if $\beta > (d + 2)/q$ and $f \ne 0$, the right-hand side of (3.1) is infinite. Therefore, to make Theorem 3.1 nontrivial one requires $\beta \le (d + 2)/q$.

Remark 3.3. There is a simple relation of $P_{\alpha_1}P_{\alpha_2}$ to $P_{\alpha_1+\alpha_2}$, which, in light of Theorem 3.1, implies that, if $\beta > \alpha_2 \ge \alpha_1 > 0$, $q_1, q_2 \in (1, \infty)$, $q_1(\beta-\alpha_1) = q_2(\beta-\alpha_2) \le d+2$, then $\|P_{\alpha_2} f\|_{E_{q_2,\beta-\alpha_2}} \le N\|P_{\alpha_1} f\|_{E_{q_1,\beta-\alpha_1}}$. We leave details of the proof to the reader and we do not use this fact in what follows.

The following, obtained similarly to Corollary 2.4, was communicated to the author by Hongjie Dong.

Corollary 3.4. *If* $1 < q < d + 2$, $\beta \in (1, (d + 2)/q]$, *and* $u \in C_0^\infty$, *then*

$$\|Du\|_{E_{r,\beta-1}} \le N\|\partial_t u + \Delta u\|_{E_{q,\beta}}$$

as long as

$$r(\beta - 1) = q\beta, \quad \text{that is} \quad \frac{1}{r} = \frac{1}{q} - \frac{1}{\beta q}. \tag{3.3}$$

Remark 3.5. For $\beta = (d + 2)/q$ Corollary 3.4 yields the second part of Corollary 2.4 once more. This is because $E_{q,(d+2)/q} = L_q$.

Remark 3.6. In the framework of the Morrey spaces Corollary 3.4 opens up the possibility to treat the terms like $b^i D_i u$ as perturbation terms in operators like $\partial_t u + \Delta u + b^i D_i u$ even with rather low summability properties of $b = (b^i)$. To show this, observe that for $f, g \geq 0$ in the notation of Corollary 3.4

$$\rho^{\beta} \rlap{\,--}\|fgI_{C_\rho}\|_{L_q} \leq \rho \rlap{\,--}\|fI_{C_\rho}\|_{L_{\beta q}} \cdot \rho^{\beta-1} \rlap{\,--}\|gI_{C_\rho}\|_{L_r}.$$

It follows that

$$\|b^i D_i u\|_{E_{q,\beta}} \leq \|b\|_{E_{\beta q,1}} \|Du\|_{E_{r,\beta-1}} \leq N\|b\|_{E_{\beta q,1}} \|\partial_t u + \Delta u\|_{E_{q,\beta}}. \tag{3.4}$$

For $\beta = (d+2)/q$ estimate (3.4) coincide with (2.7), but for $\beta < (d+2)/q$ in the framework of Morrey spaces we allow b to be summable to the power $\beta q < d+2$ in contrast with Remark 2.5. However, we need $\|b\|_{E_{\beta q,1}} < \infty$ and, if we ask ourselves what r should be in order for $b \in L_r$ to have $\|b\|_{E_{\beta q,1}} < \infty$, the answer is $r = d+2$ at least. Still we gain the possibility to have higher singularities of b than functions from L_{d+2}. Elliptic versions of (3.4) for usual or generalized Morrey spaces are found in many papers, see, for instance, [5] and the references therein.

Next we move to deriving "local" versions of the above results. A statement somewhat weaker than Corollary 3.4 can be obtained from the following general result by taking (S, T) to be large enough and then sending $S \to -\infty, T \to \infty$.

Theorem 3.7. *Let* $1 < q < d+2$, $\beta \in (1, (d+2)/q]$ *and let* (3.3) *hold. Then there is a constant N such that for any* $u \in C_0^\infty$, $-\infty < S < T < \infty$, *and* $Q_{S,T} = (S,T) \times \mathbb{R}^d$

$$\|Du\|_{E_{r,\beta-1}(Q_{S,T})} \leq N\| |\partial_t u| + |\Delta u| \|_{E_{q,\beta}(Q_{S,T})} + N(T-S)^{-1}\|u\|_{E_{q,\beta}(Q_{S,T})}. \tag{3.5}$$

Proof. Shifting and changing the scales in $\mathbb{R}^{d+1}$ allow us to assume that $S = -1 = -T$. In that case consider the mapping $\Phi : [-3/2, 3/2] \to [-1, 1]$, $\Phi(t) = t(2/(|t| \vee 1) - 1)$ that preserves $[-1, 1]$, is Lipschitz continuous and has Lipschitz continuous inverse if restricted to $[-3/2, 3/2] \setminus (-1, 1)$. Then, obviously, for $w(t, x) = v(\Phi(t), x)$ we have

$$\|wI_{Q_{-3/2,3/2}}\|_{E_{q,\beta}} \leq N\|v\|_{E_{q,\beta}(Q_{-1,1})}, \tag{3.6}$$

where $N = N(q)$.

Now take $(t, x) \in Q_{-1,1}$, $\rho \in (0, \infty)$, and take $\zeta \in C_0^\infty(\mathbb{R})$ such that $\zeta = 1$ on $(-1, 1)$, $\zeta = 0$ outside $(-3/2, 3/2)$, and $|\zeta| + |\zeta'| \leq 4$.

Although the function ζw, where $w(s, y) = u(\Phi(s), y)$, is not as smooth as required in Corollary 3.4 the argument leading to it applies to $\zeta(s)w(s, y)$ (we have a general Remark 5.14 to that effect) and since $r(\beta - 1) = q\beta$ we have

$$\rho^{\beta-1} \rlap{\,--}\|DuI_{Q_{-1,1}}\|_{L_r(C_\rho(t,x))} \leq N\rho^{\beta-1} \rlap{\,--}\|D(\zeta w)\|_{L_r(C_\rho(t,x))}$$

$$\leq N\|I_{Q_{-3/2,3/2}}(|\partial_t(\zeta w)| + |\zeta \Delta w|)\|_{E_{q,\beta}}.$$

It only remains to note that the last expression is less than the right-hand side of (3.5) in light of (3.6). The theorem is proved.

To prove an interpolation theorem in C_R we need two lemmas.

Lemma 3.8. *Let* $0 < R_1 < 1 < R_2 < \infty$, $1 \le q < \infty$, $\beta \in (0, (d+2)/q]$. *Define* $\Gamma_1 = \bar{B}_1 \setminus B_{R_1}$, $\Gamma_2 = \bar{B}_{R_2} \setminus B_1$ *and let* $\Phi : \Gamma_2 \to \Gamma_1$ *be a smooth one-to-one mapping with* $|D\Phi|, |D\Phi^{-1}| \le K$, *where* K *is a constant. Let* $v(t,x) \ge 0$ *be zero outside* $G_2 := (0,1) \times \Gamma_2$ *and set* $u(t,x) = v(t, \Phi^{-1}(x)) I_{\Gamma_1}(x)$. *Then*

$$\|v\|_{E_{q,\beta}((0,1)\times B_{R_2})} \le N(d,q,\beta,K)\|u\|_{E_{q,\beta}(C_1)}. \tag{3.7}$$

Proof. Take $(t,x) \in (0,1) \times B_{R_2}$ and $\rho > 0$. Then

$$\rho^\beta\Big(\frac{1}{\rho^{d+2}}\int_{C_\rho(t,x)} I_{(0,1)\times B_{R_2}} v^q\,dyds\Big)^{1/q}$$

$$\le N\rho^\beta\Big(\frac{1}{\rho^{d+2}}\int_{\Psi(C_\rho(t,x)\cap G_2)} I_{C_1} u^q\,dyds\Big)^{1/q} =: I,$$

where $\Psi(s,y) = (s, \Phi(y))$. Observe that, if $C_\rho(t,x) \cap G_2 \ne \emptyset$, then $|y_1 - y_2| \le 2\rho$ for any $y_1, y_2 \in C_\rho(t,x) \cap G_2$. It follows that $\Phi(C_\rho(t,x) \cap G_2) \subset B$, where B is a ball of radius $2K\rho$ with center in B_1, and

$$I \le N(2K\rho)^\beta\Big(\frac{1}{(2K\rho)^{d+2}}\int_{(t,t+(2K\rho)^2)\times B} I_{C_1} u^q\,dyds\Big)^{1/q} \le N\|u\|_{E_{q,\beta}(C_1)}.$$

This proves the lemma.

The following lemma about the interpolation inequality (3.9) is quite natural and obviously useful, but its elliptic counterpart was proved only rather late in [10]. One of its goals is to be able to treat $b^i D_i u$, when b is bounded, as a perturbation term.

Lemma 3.9. *Let* $p \in (1,\infty)$, $0 < \beta \le (d+2)/p$. *Then there is a constant* N *such that, for any* $R \in (0,\infty)$, $\rho \le 2R$, $C \in \mathbb{C}_\rho$ *with its geometric center in* C_R, $\varepsilon \in (0,1]$, *and* $u \in C_0^\infty$, *we have*

$$\rho^\beta \;⫲ I_{C_R} Du\|_{L_p(C)} \le N\varepsilon R \sup_{\rho\le s\le 2R} s^\beta \;⫲ I_{C_R}(|\partial_t u| + |D^2 u|)\|_{L_p(C(s))}$$

$$+ N\varepsilon^{-1}R^{-1} \sup_{\rho\le s\le 2R} s^\beta \;⫲ I_{C_R}(u - c)\|_{L_p(C(s))}, \tag{3.8}$$

where c *is any constant and* $C(s) \in \mathbb{C}_s$ *with the geometric center the same as* C. *In particular,*

$$\|Du\|_{E_{p,\beta}(C_R)} \le N\varepsilon R\||\partial_t u| + |D^2 u|\,\|_{E_{p,\beta}(C_R)} + N\varepsilon^{-1}R^{-1}\|u\|_{E_{p,\beta}(C_R)}. \tag{3.9}$$

Proof. Changing scales shows that we may assume that $R = 1$. Obviously we may also assume that $c = 0$. Then denote $v = Du$, $w = |\partial_t u| + |D^2 u|$, $G_s = C(s) \cap C_1$,

$$U = \sup_{\rho\le s\le 2} s^\beta \;⫲ u\|_{L_p(G_s)}, \quad W = \sup_{\rho\le s\le 2} s^\beta \;⫲ (|\partial_t u| + |D^2 u|)\|_{L_p(G_s)},$$

By Poincaré's inequality (see, for instance, Lemma 5.9), for $\rho \le s \le 2$,

$$⫲ v - v_{G_s}\|_{L_p(G_s)} \le N(d,p)s \;⫲ w\|_{L_p(G_s)} \le N s^{1-\beta} W.$$

Also by interpolation inequalities, there exists a constant $N = N(d,p)$ such that, for $\varepsilon \in (0,1]$ and $\varepsilon \le s \le 2$,

$$⫲ v - v_{G_s}\|_{L_p(G_s)} \le 2 \;⫲ v\|_{L_p(G_s)} \le N \;⫲ w\|^{1/2}_{L_p(G_s)} \;⫲ u\|^{1/2}_{L_p(G_s)}$$

$$+ Ns^{-1} \text{--}\!\!\!\!\!\!\|u\|_{L_p(G_s)} \leq N \text{--}\!\!\!\!\!\!\|w\|^{1/2}_{L_p(G_s)} \text{--}\!\!\!\!\!\!\|u\|^{1/2}_{L_p(G_s)} + N\varepsilon^{-1} \text{--}\!\!\!\!\!\!\|u\|_{L_p(G_s)}, \tag{3.10}$$

which for $2 \geq s \geq \varepsilon \vee \rho$ yields

$$s^{\beta} \text{--}\!\!\!\!\!\!\|v - v_{G_s}\|_{L_p(G_s)} \leq NW^{1/2}U^{1/2} + N\varepsilon^{-1}U.$$

Hence, for any $\varepsilon \in (0, 1]$ and $\rho \leq s \leq 2$

$$s^{\beta} \text{--}\!\!\!\!\!\!\|v - v_{G_s}\|_{L_p(G_s)} \leq N_1\varepsilon W + N_2\varepsilon^{-1}U,$$

where $N_1 = N_1(d, p)$, $N_2 = N_2(d, p)$.

Following Campanato, one can transform this result to estimate v_{G_s} going along ρ, 2ρ,... and, since $\beta \in (0, (d + 2)/p]$, by Campanato's results (cf. for instance, Proposition 5.4 in [8]) one gets that

$$\rho^{\beta} \text{--}\!\!\!\!\!\!\|v\|_{L_p(G_\rho)} \leq N_3(N_1\varepsilon W + N_2\varepsilon^{-1}U) + N_3 \text{--}\!\!\!\!\!\!\|v\|_{L_p(G_2)},$$

where $N_3 = N_3(d, p, \beta)$. We estimate the last term as in (3.10) and come to what implies (3.8). The lemma is proved.

The following is a local version of Corollary 3.4. It allows us to draw the same conclusions as in Remark 3.6 in bounded domains.

Theorem 3.10. *Let $1 < q < d + 2$, $\beta \in (1, (d + 2)/q]$ and let $r(\beta - 1) = q\beta$. Then there is a constant N such that for any $R \in (0, \infty]$, $u \in C_0^\infty$,*

$$\|Du\|_{E_{r,\beta-1}(C_R)} \leq N\| |\partial_t u| + |D^2u| \|_{E_{q,\beta}(C_R)} + NR^{-2}\|u\|_{E_{q,\beta}(C_R)}. \tag{3.11}$$

Proof. The case of $R = \infty$ is obtained by passing to the limit. In case $R < \infty$, as usual, we may assume that $R = 1$. In that case, mimicking the Hestenes formula, for $1 \leq |x| \leq 6/5$ define

$$v(t, x) = 6u(t, x(2/|x| - 1)) - 8u(t, x(3/|x| - 2)) + 3u(t, x(4/|x| - 3))$$

$$=: 6v_1 - 8v_2 + 3v_3$$

and for $|x| \leq 1$ set $v(t, x) = u(t, x)$. One can easily check that $v \in C^{1,2}([0, 1] \times B_{6/5})$. In light of Lemmas 3.8 and 3.9, for instance,

$$\begin{aligned}\|D^2v\|_{E_{q,\beta}((0,1)\times B_{6/5})} &\leq \|D^2u\|_{E_{q,\beta}(C_1)} + N\|I_{B_{6/5}\setminus B_1}D^2v_1\|_{E_{q,\beta}((0,1)\times B_{6/5})} + ...\\ +N\|I_{B_{6/5}\setminus B_1}D^2v_3\|_{E_{q,\beta}((0,1)\times B_{6/5})} &\leq N\|D^2u\|_{E_{q,\beta}(C_1)} + N\|Du\|_{E_{q,\beta}(C_1)}\\ &\leq N\|D^2u\|_{E_{q,\beta}(C_1)} + N\|u\|_{E_{q,\beta}(C_1)}.\end{aligned} \tag{3.12}$$

Now take $(t, x) \in C_1$, $\rho \in (0, \infty)$, and take $\zeta \in C_0^\infty(\mathbb{R}^d)$ such that $\zeta = 1$ on B_1, $\zeta = 0$ outside $B_{6/5}$, and $|\zeta| + |D\zeta| + |D^2\zeta| \leq N = N(d)$.

By using Theorem 3.7 we get

$$\rho^{\beta-1} \text{--}\!\!\!\!\!\!\|DuI_{C_1}\|_{L_r(C_\rho(t,x))} \leq N\rho^{\beta-1} \text{--}\!\!\!\!\!\!\|I_{Q_{0,1}}D(\zeta v)\|_{L_r(C_\rho(t,x))}$$

$$\leq N\|D(\zeta v)\|_{E_{r,\beta-1}(Q_{0,1})} \leq N\| |\partial_t(\zeta v)| + |\Delta(\zeta v)| \|_{E_{q,\beta}(Q_{0,1})} + N\|\zeta v\|_{E_{q,\beta}(Q_{0,1})}$$

$$\leq N\| |\partial_t(\zeta v)| + |\Delta(\zeta v)| \|_{E_{q,\beta}((0,1)\times B_{6/5})} + N\|v\|_{E_{q,\beta}((0,1)\times B_{6/5})}$$

It only remains to note that the last expression is less than the right-hand side of (3.11) as is well seen from (3.12). The theorem is proved.

Remark 3.11. By considering functions depending only on x we naturally obtain "elliptic" analogs of our results. For instance, for $G \subset \mathbb{R}^d$ by defining

$$\|g\|_{E_{p,\beta}(G)} = \sup_{\rho<\infty, x\in G} \rho^{\beta} \, \text{⨍} gI_G\|_{L_p(B_\rho(x))},$$

we get from (3.11) for $u \in C_0^\infty(\mathbb{R}^d)$ that

$$\|Du\|_{E_{r,\beta-1}(B_R)} \leq N\| |D^2u| \|_{E_{q,\beta}(B_R)} + NR^{-2}\|u\|_{E_{q,\beta}(B_R)}, \tag{3.13}$$

whenever $1 < q < d, \beta \in (1, d/q]$ and $r(\beta - 1) = q\beta$. Actually, formally, one gets (3.13) even for $\beta \leq (d+2)/q$, but for $\beta > d/q$, both sides of (3.13) are infinite unless $u = 0$.

After that arguing as in (3.4) we see that for $1 < q < d, \beta \in (1, d/q]$

$$\|b^i D_i u\|_{E_{q,\beta}(B_1)} \leq N\|b\|_{E_{\beta q,1}(B_1)}\|\Delta u\|_{E_{q,\beta}(B_1)} + N\|u\|_{E_{q,\beta}(B_1)}. \tag{3.14}$$

From the point of view of the theory of elliptic equations the most desirable version of (3.14) would be

$$\|b^i D_i u\|_{E_{q,\beta}(B_1)} \leq \varepsilon\|\Delta u\|_{E_{q,\beta}(B_1)} + N(\varepsilon)\|u\|_{E_{q,\beta}(B_1)} \tag{3.15}$$

for any $\varepsilon > 0$ with $N(\varepsilon)$ independent of u. This fact is, actually, claimed in Theorem 5.4 of [5]. We will show that (3.15) cannot hold if ε is small enough.

Let $h(t)$ be a smooth nondecreasing function on $\mathbb{R}$ such that $h(t) = 0$ for $t \leq 0$, $h(t) = t$ for $t \geq 1$ and for $\delta > 0$ set $u_\delta(x) = h(\ln(\delta/|x|))$. Let $1 < q < d/2, \beta = 2, b(x) = 1/|x|$.

Then

$$\|u\|_{E_{q,\beta}(B_1)} \leq N(d)\|u_\delta\|_{L_d(B_1)} \to 0$$

as $\delta \downarrow 0$. At the same time

$$D_i u_\delta = -\frac{x_i}{|x|^2}h', \quad D_{ij}u_\delta = \frac{1}{|x|^2}\Big(2\frac{x_i x_j}{|x|^2} - \delta_{ij}\Big)h' + \frac{1}{|x|^2}\frac{x_i x_j}{|x|^2}h''.$$

It is seen that $|D^2 u_\delta| \leq N(d)/|x|^2$ and, since $q < d/2$, the $E_{q,\beta}(B_1)$-norm of $D^2 u_\delta$ is bounded as $\delta \downarrow 0$. Also, for $|x| \leq \delta/e$, we have $b|Du_\delta| = 1/|x|^2$, so that for $r \leq \delta/e$

$$\Big(\fint_{|x|\leq r} b^q |Du_\delta|^q \, dx\Big)^{1/q} = N(d,p)r^{-2}.$$

It follows that the $E_{q,\beta}(B_1)$-norm of $b|Du_\delta|$ is bounded away from zero as $\delta \downarrow 0$ and this shows that (3.15) cannot hold for all $\delta > 0$ if ε is small enough.

4. A parabolic version of Chiarenza–Frasca result [4]

In Remark 3.6 we have shown how to estimate a Morrey norm of $|b|\,|Du|$ in terms of a Morrey norm of b. Here, following [4], we show how to estimate an L_p-norm of the same quantity through the L_p-norms of $\partial_t u$ and $D^2 u$.

Theorem 4.1. *Let $d+2 \geq q > p > 1$, $b \in E_{q,1}$. Then for any $f \geq 0$ we have*

$$I := \int_{\mathbb{R}^{d+1}} |b|^p (P_1 f)^p \, dz \leq N \|b\|^p_{E_{q,1}} \|f\|_{L_p}, \tag{4.1}$$

where N depends only on d, p, q. In particular (see the proof of Corollary 2.4), for any $u \in C_0^\infty$

$$\int_{\mathbb{R}^{d+1}} |b|^p |Du|^p \, dz \leq N \|b\|^p_{E_{q,1}} K, \tag{4.2}$$

where $K = \|D^2 u, \partial_t u\|^p_{L_p}$ and N depends only on d, p, q.

Observe that we already know this result if $q = d+2$ from Remarks 2.5 or 3.6.

In the proof we are going to use "parabolic" versions of some results from Real Analysis associated with balls and cubes. These versions are obtained by easy adaptation of the corresponding arguments by replacing balls with parabolic cylinders and cubes with parabolic boxes. To make the adaptation more natural we introduce the "symmetric" maximal parabolic function operator by

$$\hat{M} f(t,x) = \sup_{\substack{C \in \mathbb{C}, \\ C \ni (t,x)}} \fint_C |f| \, dz,$$

where (recall that) $\mathbb{C}$ is the set of $C_r(z)$, $r > 0$, $z \in \mathbb{R}^{d+1}$. To prove the theorem we need the following.

Lemma 4.2. *a) For $r \in (0,\infty)$ define $D_r = \{|t| \leq r^2, |x| \leq r\}$. Then*

$$\hat{M} I_{D_r}(t,x) \leq I_{D_{2r}} + N I_{D^c_{2r}} \frac{r^{d+2}}{|t|^{(d+2)/2} \vee |x|^{d+2}} \leq N^2 \hat{M} I_{D_r}(t,x), \tag{4.3}$$

where $N = N(d)$.

b) For any nonnegative $g(t,x)$, $q \in [1,\infty)$, $\beta \in (0, d+2]$, $\alpha > 0$, $\alpha > 1 - q\beta/(d+2)$, and $r \in (0,\infty)$

$$\int_{\mathbb{R}^d} g^q (\hat{M} I_{D_r})^\alpha \, dz \leq N(d,q,\alpha,\beta) r^{d+2-q\beta} \|g\|^q_{E_{q,\beta}}. \tag{4.4}$$

Proof. Assertion a) is proved by elementary means. To prove b), we use a) and split D^c_{2r} into two parts $D^c_{2r} \cap \{|x|^2 \geq |t|\}$ and $D^c_{2r} \cap \{|x|^2 < |t|\}$ and, taking into account obvious symmetries, we see that it suffices to show that

$$I_1 := \int_{4r^2}^\infty \int_{B_{\sqrt{t}}} \frac{g^q(t,x)}{t^{\alpha(d+2)/2}} \, dxdt \leq N r^{(d+2)(1-\alpha)-q\beta} \|g\|^q_{E_{q,\beta}},$$

$$I_2 := \int_{|x| \geq 2r} \int_0^{|x|^2} \frac{g^q(t,x)}{|x|^{\alpha(d+2)}} \, dtdx \leq N r^{(d+2)(1-\alpha)-q\beta} \|g\|^q_{E_{q,\beta}}.$$

By observing that

$$\frac{1}{t^{\alpha(d+2)/2}} \int_{4r^2}^t \Big(\int_{B_{\sqrt{s}}} g^q(s,x) \, dx \Big) ds \leq \frac{t^{(d+2)/2 - q\beta/2}}{t^{\alpha(d+2)/2}} \|g\|^q_{E_{q,\beta}} \to 0$$

as $t \to \infty$, we have

$$I_1 = \int_{4r^2}^\infty \frac{1}{t^{\alpha(d+2)/2}} \frac{d}{dt} \Big(\int_{4r^2}^t \Big(\int_{B_{\sqrt{s}}} g^q(s,x) \, dx \Big) ds \Big) dt$$

$$= N\int_{4r^2}^{\infty}\frac{1}{t^{\alpha(d+2)/2+1}}\Big(\int_{4r^2}^{t}\int_{B_{\sqrt{s}}} g^q(s,x)\,dxds\Big)\,dt$$

$$\le N\|g\|^q_{E_{q,\beta}}\int_{4r^2}^{\infty}\frac{t^{(d+2)/2-q\beta/2}}{t^{\alpha(d+2)/2+1}}\,dt = Nr^{(d+2)(1-\alpha)-q\beta}\|g\|^q_{E_{q,\beta}}.$$

Also as is easy to see

$$I_2 = N\int_{2r}^{\infty}\frac{1}{\rho^{\alpha(d+2)}}\int_0^{\rho^2}\Big(\int_{|x|=\rho} g^p(t,x)\,d\sigma_\rho\Big)\,dtd\rho$$

$$\le N\int_{2r}^{\infty}\frac{1}{\rho^{\alpha(d+2)}}\frac{\partial}{\partial\rho}\Big(\int_{4r^2}^{\rho^2}\Big(\int_{|x|\le\rho} g^p(t,x)\,dx\Big)\,dt\Big)d\rho$$

$$\le N\|g\|^q_{E_{q,\beta}}\int_{2r}^{\infty}\frac{\rho^{d+2-q\beta}}{\rho^{\alpha(d+2)+1}}\,d\rho = Nr^{(d+2)(1-\alpha)-q\beta}\|g\|^q_{E_{q,\beta}}.$$

This proves the lemma.

Proof of Theorem 4.1. We follow some arguments in [4] and may assume that $b \ge 0$. First set $r_0 = (p+q)/2$ and assume that there is a constant N_0 such that $\hat{M}(|b|^{r_0}) \le N_0|b|^{r_0}$, that is, $|b|^{r_0}$ is in the class A_1 of Muckenhoupt. Observe that by Hölder's inequality $\|b\|_{E_{r_0,1}} \le \|b\|_{E_{q,1}}$. It is convenient to prove the following version of (4.1) (notice r_0 in place of q)

$$I \le N\|b\|^p_{E_{r_0,1}}\|f\|_{L_p}, \tag{4.5}$$

Then assume that $b \ge 0$, set $u = P_1 f$, and write

$$I = \int_{\mathbb{R}^{d+2}}(b^p u^{p-1})P_1 f\,dz = \int_{\mathbb{R}^{d+2}} P_1^*(b^p u^{p-1})f\,dz \le \|f\|_{L_p}\big\|P_1^*(b^p u^{p-1})\big\|_{L_{p'}}, \tag{4.6}$$

where $p' = p/(p-1)$ and P_1^* is the conjugate operator for P_1, namely, for any $g \ge 0$,

$$(P_1^* g)(s,x) = (P_1(g(-\cdot,-\cdot))(-s,-x). \tag{4.7}$$

Next, take $\gamma > 0$, such that $(1+\gamma)p \le r_0$, $1+\gamma p' \le r_0$, and $p \ge 1+\gamma$. Note that

$$P_1^*(b^p u^{p-1}) = P_1^*(b^{1+\gamma}(b^{p-1-\gamma}u^{p-1})) \le \Big(P_1^*(b^{(1+\gamma)p})\Big)^{1/p}\Big(P_1^*(b^{p-\gamma p'}u^p\Big)^{(p-1)/p}.$$

It follows that

$$\big\|P_1^*(b^p u^{p-1})\big\|_{L_{p'}} \le \Big(\int_{\mathbb{R}^d} b^{p-\gamma p'}u^p P_1\Big[\Big(P_1^*(b^{(1+\gamma)p})\Big)^{1/(p-1)}\Big]\,dz\Big)^{(p-1)/p}.$$

Now in light of (4.6) we see that, to prove (4.5) in our particular case, it only remains to show that

$$P_1\Big[\Big(P_1^*(b^{(1+\gamma)p})\Big)^{1/(p-1)}\Big] \le Nb^{\gamma p'}\|b\|^{p'}_{E_{r_0,1}}. \tag{4.8}$$

For $\alpha = 1$ and $\beta = (1+\gamma)p$ $(> \alpha)$ it follows from (2.3) and (4.7) that

$$P_1^*(b^{(1+\gamma)p}) \le N\|b\|_{E_{\beta,1}}(\hat{M}(b^{(1+\gamma)p}))^{1-1/(p+\gamma p)}.$$

where the last factor by assumption (and Hölder's inequality) is dominated by $Nb^{(1+\gamma)p-1}$ and $\|b\|_{E_{\beta,1}} \leq \|b\|_{E_{r_0,1}}$. After that to obtain (4.8) it suffices to use again (2.3) with $\alpha = 1$ and $\beta = 1 + \gamma p'$ to see that

$$P_1(b^{1+\gamma p'}) \leq N\|b\|_{E_{1+\gamma p',1}}(M(b^{1+\gamma p'}))^{1-1/(1+\gamma p')} \leq N\|b\|_{E_{r_0,1}} b^{\gamma p'}.$$

We now get rid of the assumption that $\hat{M}(|b|^{r_0}) \leq N_0|b|^{r_0}$ as in [4].

For $r_1 = (r_0 + q)/2$ we have $|b|^{r_0} \leq (\hat{M}(|b|^{r_1}))^{r_0/r_1} := \tilde{b}^{r_0}$ and since $r_0/r_1 < 1$, $\tilde{b}^{r_0}$ is an A_1-weight with $N_0 = N_0(r_0/r_1)$ (see, for instance, [7] p. 158). Therefore, (4.5) holds with $\tilde{b}$ in place of b and it only remains to show that

$$\|\tilde{b}\|_{E_{r_0,1}} \leq N\|b\|_{E_{q,1}},$$

that is, for any t, x, ρ,

$$\int_{C_\rho(t,x)} \tilde{b}^{r_0}\,dz \leq N\rho^{d+2-r_0}\|b\|^{r_0}_{E_{q,1}}. \tag{4.9}$$

Of course, we may assume that $t = 0, x = 0$. Then by Hölder's inequality we see that the left-hand side of (4.9) is less than

$$N\rho^{(d+2)(q-r_0)/q}\Big(\int_{\mathbb{R}^{d+1}} (\hat{M}(|b|^{r_1}))^{q/r_1} I_{C_\rho}\,dz\Big)^{r_0/q},$$

where the integral by a Fefferman-Stein Lemma 1, p. 111 of [6] and the fact that $q/r_1 > 1$ is dominated by

$$N\int_{\mathbb{R}^{d+1}} |b|^q \hat{M} I_{C_\rho}\,dz \leq N\rho^{d+2-q}\|b\|^q_{E_{q,1}},$$

where we used Lemma 4.2 b) for $\alpha = \beta = 1$. Hence,

$$\int_{C_r} \tilde{b}^{r_0}\,dz \leq N\rho^{(d+2)(q-r_0)/q+(d+2-q)r_0/q}\|b\|^{r_0}_{E_{q,1}},$$

which is (4.9).

An alternative way to get the result is to follow the proof of Theorem 3 of [3]. We have

$$\int_{\mathbb{R}^{d+1}} (\hat{M}(|b|^{r_1}))^{q/r_1} I_{C_\rho}\,dz \leq \int_{\mathbb{R}^{d+1}} (\hat{M}(|b|^{r_0}))^{q/r_0}(\hat{M} I_{C_\rho})^\alpha\,dz =: J,$$

where $\alpha \in (0, 1)$. An easy exercise leads to the well-known result that $(\hat{M} I_{C_\rho})^\alpha$ is an A_1-weight, and, hence, an A_{q/r_0}-weight. By the Muckenhoupt theorem

$$J \leq N\int_{\mathbb{R}^{d+1}} |b|^q(\hat{M} I_{C_\rho})^\alpha\,dz$$

and it only remains to use Lemma 4.2 b) again with $\beta = 1$ and any appropriate α. The theorem is proved.

Remark 4.3. In the above proof we tacitly assumed that $I < \infty$. One can easily avoid it by taking f with compact support, replacing $|b|$ with $|b| \wedge b_n$, where $n^{-1}b_n = 1 \wedge (\sqrt{|t|} + |x|)^{-1}$, observe that $b_n^{r_0} \in A_1$, and while checking that the new I is finite use Hölder's inequality and Corollary 2.4.

5. Mixed-norm estimates

For $q_1, q_2 \in [1, \infty]$ and measurable f and $\Gamma \subset \mathbb{R}^{d+1}$ introduce

$$\|f\|_{L_{q_1,q_2}} = \Big(\int_{\mathbb{R}} \Big(\int_{\mathbb{R}^d} |f(t,x)|^{q_1} \, dx \Big)^{q_2/q_1} dt \Big)^{1/q_2},$$

$$⨍\!\!\|f\|_{L_{q_1,q_2}(\Gamma)} = \|I_\Gamma\|^{-1}_{L_{q_1,q_2}} \|f I_\Gamma\|_{L_{q_1,q_2}}.$$

Here the index of L_{q_1,q_2} which is the exponent of ρ in the expression

$$\|I_{C_\rho}\|_{L_{q_1,q_2}} \quad \text{is} \quad \frac{d}{q_1} + \frac{2}{q_2}.$$

If in addition $0 < \beta \le d/q_1 + 2/q_2$, set

$$\|f\|_{E_{q_1,q_2,\beta}(Q)} = \sup_{\rho<\infty,(t,x)\in Q} \rho^\beta \, ⨍\!\!\|I_Q f\|_{L_{q_1,q_2}(C_\rho(t,x))}.$$

We also introduce the spaces $L_{q_1,q_2}(Q)$ and $E_{q_1,q_2,\beta}(Q)$ as the spaces of functions whose respective norms are finite. We abbreviate $L_{q_1,q_2} = L_{q_1,q_2}(\mathbb{R}^{d+1})$, $E_{q_1,q_2,\beta} = E_{q_1,q_2,\beta}(\mathbb{R}^{d+1})$.

The following is certainly well known.

Lemma 5.1. *Let f be a nonnegative function on $\mathbb{R}^{d+1}$, $p, q \in (1, \infty)$. Then for any $w_x(x), w_t(t)$ which are A_p Muckenhoupt weights on $\mathbb{R}^d$ and $\mathbb{R}$, respectively, we have*

$$\int_{\mathbb{R}^{d+1}} |\hat{M} f|^p w_x w_t \, dxdt \le N \int_{\mathbb{R}^{d+1}} |f|^p w_x w_t \, dxdt, \tag{5.1}$$

where N depends only on d, p, and the A_p-constants of w_x, w_t. Furthermore,

$$\int_{-\infty}^{\infty} \Big(\int_{\mathbb{R}^d} |\hat{M} f|^p \, dx \Big)^{q/p} dt \le N \int_{-\infty}^{\infty} \Big(\int_{\mathbb{R}^d} |f|^p \, dx \Big)^{q/p} dt, \tag{5.2}$$

where N depends only on d, p, q.

Proof. Estimate (5.1) follows by application of the Muckenhoupt theorem to $w_x w_t$, which is an A_p-weight on $\mathbb{R}^{d+1}$. Then observe that in the particular case that $w_x \equiv 1$, (5.1) means that

$$\int_{-\infty}^{\infty} \Big[\Big(\int_{\mathbb{R}^d} |\hat{M} f|^p \, dx \Big)^{1/p} \Big]^p w_t \, dt \le N \int_{-\infty}^{\infty} \Big[\Big(\int_{\mathbb{R}^d} |f|^p \, dx \Big)^{1/p} \Big]^p w_t \, dt$$

for any A_p-weight w_t, which implies (5.2) by the Rubio de Francia extrapolation theorem. The lemma is proved.

This lemma, (2.3), and Hölder's inequality immediately yield the following.

Lemma 5.2. *For any $\alpha \in (0, \beta), \beta \in (0, d+2]$, $p \in [1, \infty]$, $q_1, q_2 \in (1, \infty]$, there exists a constant N such that for any $f \ge 0$ and measurable Γ we have*

$$\|P_\alpha f\|_{L_{r_1,r_2}(\Gamma)} \le N \|M_\beta f\|^{\alpha/\beta}_{L_p(\Gamma)} \|f\|^{1-\alpha/\beta}_{L_{q_1,q_2}}, \tag{5.3}$$

provided that

$$\frac{1}{r_i} = \frac{\alpha}{\beta} \cdot \frac{1}{p} + \Big(1 - \frac{\alpha}{\beta}\Big) \frac{1}{q_i}, \quad i = 1, 2.$$

Similarly to Corollary 2.4 we have

Corollary 5.3. *Let* $q_1, q_2 \in (1, \infty]$,

$$\beta := \frac{d}{q_1} + \frac{2}{q_2} > 0,$$

$\alpha \in (0, \beta)$. *Then for any* $f \geq 0$ *we have*

$$\|P_\alpha f\|_{L_{r_1,r_2}} \leq N\|f\|_{L_{q_1,q_2}}$$

as long as $q_i\beta = r_i(\beta - \alpha)$, $i = 1, 2$.

In particular, (almost follows from Theorem 10.2 of [2]) if $\beta > 1$, *and* $u \in C_0^\infty$, *then*

$$\|Du\|_{L_{r_1,r_2}} \leq N\|\partial_t u + \Delta u\|_{L_{q_1,q_2}} \tag{5.4}$$

as long as $q_i\beta = r_i(\beta - 1)$, $i = 1, 2$.

Corollary 5.4. *Under the assumptions of Corollary 5.3, if* $\beta > 1$, *there is a constant N such that, for any* $b = (b^i) \in L_{\beta q_1, \beta q_2}$ *and* $u \in C_0^\infty$,

$$\|b^i D_i u\|_{L_{q_1,q_2}} \leq N\|b\|_{L_{\beta q_1,\beta q_2}}\|\partial_t u + \Delta u\|_{L_{q_1,q_2}}. \tag{5.5}$$

Indeed, by Hölder's inequality

$$\|b^i D_i u\|_{L_{q_1,q_2}} \leq \|b\|_{L_{\beta q_1,\beta q_2}}\|Du\|_{L_{r_1,r_2}}.$$

Remark 5.5. It is instructive to compare this result with Remark 2.5. Now we can treat $b \in L_{s_1,s_2}$ with $s_i \in (1, \infty]$ satisfying $d/s_1 + 2/s_2 = 1$.

Since $E_{q_1,q_2,\beta} = L_{q_1,q_2}$ if $\beta = d/q_1 + 2/q_2$, the following is a generalization of Corollary 5.3.

Theorem 5.6. *Let* $q_1, q_2 \in (1, \infty]$,

$$\frac{d}{q_1} + \frac{2}{q_2} \geq \beta > 0,$$

$\alpha \in (0, \beta)$. *Then there is a constant N such that for any* $f \geq 0$ *we have*

$$\|P_\alpha f\|_{E_{r_1,r_2,\beta-\alpha}} \leq N\|f\|_{E_{q_1,q_2,\beta}}, \tag{5.6}$$

where $r_i(\beta - \alpha) = q_i\beta$, $i = 1, 2$.

Proof. It suffices to prove that for any $\rho > 0$

$$\rho^{\beta-\alpha}\Big(\fint_0^{\rho^2}\Big(\fint_{B_\rho}|P_\alpha f|^{r_1}\,dy\Big)^{r_2/r_1}ds\Big)^{1/r_2} \leq N\|f\|_{E_{q_1,q_2,\beta}},$$

that is

$$\rho^{\beta-\alpha-(d/r_1+2/r_2)}\Big(\int_0^{\rho^2}\Big(\int_{B_\rho}|P_\alpha f|^{r_1}\,dy\Big)^{r_2/r_1}ds\Big)^{1/r_2} \leq N\|f\|_{E_{q_1,q_2,\beta}}. \tag{5.7}$$

Observe that by Hölder's inequality $M_\beta f \le N\|f\|_{E_{q_1,q_2,\beta}}$ and by definition

$$\|I_{C_{2\rho}}f\|_{L_{q_1,q_2}} = N\rho^{d/q_1+2/q_2}\Big(\fint_0^{4\rho^2}\Big(\fint_{B_{2\rho}}|f|^{q_1}\,dy\Big)^{q_2/q_1}ds\Big)^{1/q_2}$$

$$\le N\rho^{d/q_1+2/q_2-\beta}\|f\|_{E_{q_1,q_2,\beta}} = N\rho^{(d/r_1+2/r_2)\beta/(\beta-\alpha)-\beta}\|f\|_{E_{q_1,q_2,\beta}}.$$

It follows from Lemma 5.2 with $p=\infty$ that (5.7) holds with $I_{C_{2\rho}}f$ in place of f on the left. Furthermore, by Corollary 2.6 we have $|P_\alpha(I_{C^c_{2\rho}}f)| \le N\rho^{\alpha-\beta}M_\beta f$ in C_ρ. Therefore,

$$\rho^{\beta-\alpha}\Big(\fint_0^{\rho^2}\Big(\fint_{B_\rho}|P_\alpha(I_{C^c_{2\rho}}f)|^{r_1}\,dy\Big)^{r_2/r_1}ds\Big)^{1/r_2} \le N\sup M_\beta f \le N\|f\|_{E_{q_1,q_2,\beta}}.$$

By combining these results we come to (3.2) and the theorem is proved.

Corollary 5.7. *Under the assumptions of Theorem 5.6, if $\beta > 1$, for any $u \in C_0^\infty$, we have*

$$\|Du\|_{E_{r_1,r_2,\beta-1}} \le N\|\partial_t u + \Delta u\|_{E_{q_1,q_2,\beta}},$$

where $r_i(\beta-1) = q_i\beta$, $i = 1,2$. This coincides with (5.4) *if β is equal to the index of L_{q_1,q_2}.*

Remark 5.8. Corollary 5.7 opens up the possibility to treat the terms like b^iD_iu as perturbation terms in operators like $\partial_t u + \Delta u + b^iD_iu$ with even lower summability properties of $b = (b^i)$ than in Remark 5.5. To show this observe that for q_1, q_2, β as in Theorem 5.6 with $\beta > 1$ and $s_i = \beta q_i \in (1,\infty]$, $i = 1,2$, we have

$$\rho^\beta \;⫲ I_{C_\rho}b^iD_iu\|_{L_{q_1,q_2}} \le \rho \;⫲ bI_{C_\rho}\|_{L_{s_1,s_2}} \cdot \rho^{\beta-1} \;⫲ I_{C_\rho}Du\|_{L_{r_1,r_2}}$$

implying that

$$\|b^iD_iu\|_{E_{q_1,q_2,\beta}} \le \|b\|_{E_{s_1,s_2,1}}\|Du\|_{E_{r_1,r_2,\beta-1}} \le N\|b\|_{E_{s_1,s_2,1}}\|\partial_t u + \Delta u\|_{E_{q_1,q_2,\beta}}, \tag{5.8}$$

where $d/s_1 + 2/s_2 \ge 1$.

However, note that we also need

$$\rho \;⫲ bI_{C_\rho(t,x)}\|_{L_{s_1,s_2}}$$

to be bounded as a function of ρ, t, x. If we ask ourselves what $\tau > 0$ should be to guarantee this boundedness if $b \in L_{\tau s_1,\tau s_2}$, if $d/s_1 + 2/s_2 > 1$, the slightly disappointing answer is that $\tau = d/s_1 + 2/s_2$, so that $d/(\tau s_1) + 2/(\tau s_2) = 1$. Still functions in $E_{s_1,s_2,1}$ may have higher singularities than those in $L_{\tau s_1,\tau s_2}$.

Another advantage of (5.8) in comparison with (5.5) is seen when b depends only on t or $|b(t,x)| \le \hat b(t)$. In that case (5.8) becomes

$$\|b^iD_iu\|_{E_{q_1,q_2,\beta}} \le N\|\hat b\|_{E_{\beta q_2,1/2}(\mathbb{R})}\|\partial_t u + \Delta u\|_{E_{q_1,q_2,\beta}},$$

and if $\beta q_2 = 2$, then

$$\|\hat b\|_{E_{\beta q_2,1/2}(\mathbb{R})} = \|\hat b\|_{L_2(\mathbb{R})}.$$

Thus for any $q_1 \in (1,\infty]$ and $q_2 \in (1,2)$

$$\|b^iD_iu\|_{E_{q_1,q_2,2/q_2}} \le N\|\hat b\|_{L_2(\mathbb{R})}\|\partial_t u + \Delta u\|_{E_{q_1,q_2,2/q_2}}.$$

In case $q_1 \in (1, d)$, $q_2 \in (1, \infty]$, $1 < \beta \leq d/q_1$, and $\|b(\cdot, t)\|_{E_{\beta q_1,1}(\mathbb{R}^d)} \leq \hat{b} < \infty$ for any t, we also have

$$\|b^i D_i u\|_{E_{q_1,q_2,\beta}} \leq N\hat{b}\|\partial_t u + \Delta u\|_{E_{q_1,q_2,\beta}}.$$

An application of the last inequality in case u, b are independent of t, $\beta = d/q_1$, $q_1 \in (1, d)$, and $q_2 = \infty$, yields the well-known estimate

$$\|b^i D_i u\|_{L_{q_1}(\mathbb{R}^d)} \leq N\|b\|_{L_d(\mathbb{R}^d)}\|\Delta u\|_{L_{q_1}(\mathbb{R}^d)}.$$

To extend the embedding and interpolation results to Morrey spaces with mixed norms we need the following result very useful also in other circumstances.

Lemma 5.9 (Poincaré's inequality)**.** *Let* $1 \leq r_1, r_2 < \infty$, $u \in C_0^\infty$, $\rho \in (0, \infty)$. *Then*

$$⨍\!\!⨍ Du - (Du)_{C_\rho}\|^{r_2}_{L_{r_1,r_2}(C_\rho)} \leq N(d, r_1, r_2)\rho^{r_2} ⨍\!\!⨍ |\partial_t u| + |D^2 u|\|^{r_2}_{L_{r_1,r_2}(C_\rho)}. \tag{5.9}$$

Proof. We follow the usual way (see, for instance, Lemma 4.2.2 of [9]). First, due to self-similar transformations, we may take $\rho = 1$. In that case, for a $\zeta \in C_0^\infty(B_1)$ with unit integral, introduce

$$v(t) = \int_{B_1} \zeta(y) Du(t, y)\, dy.$$

Then by the usual Poincaré inequality

$$\int_{B_1} |Du(t, x) - v(t)|^{r_1}\, dx = \int_{B_1} \Big| \int_{B_1} [Du(t, x) - Du(t, y)]\zeta(y)\, dy\Big|^{r_1} dx$$

$$\leq N \int_{B_1}\int_{B_1} |Du(t, x) - Du(t, y)|^{r_1}\, dxdy \leq N \int_{B_1} |D^2 u(t, x)|^{r_1}\, dx. \tag{5.10}$$

Next, observe that for any constant vector v the left-hand side of (5.9) is less than a constant times (recall that $\rho = 1$)

$$\int_0^1 \Big(\int_{B_1} |Du(t, x) - v|^{r_1}\, dx\Big)^{r_2/r_1} dt$$

$$\leq N \int_0^1 \Big(\int_{B_1} |Du(t, x) - v(t)|^{r_1}\, dx\Big)^{r_2/r_1} dt + N \int_0^1 |v(t) - v|^{r_2}\, dt.$$

By (5.10) the first term on the right is less than the right-hand side of (5.9). To estimate the second term, take

$$v = \int_0^1 v(t)\, dt.$$

Then by Poincaré's inequality

$$\int_0^1 |v(t) - v|^{r_2}\, dt \leq N \int_0^1 \Big|\int_{B_1} \zeta \partial_t Du\, dx\Big|^{r_2} dt = N \int_0^1 \Big|\int_{B_1} (D\zeta)\partial_t u\, dx\Big|^{r_2} dt$$

and to finish the proof it only remains to use Hölder's inequality. The lemma is proved.

The usual Poincaré inequality was used in the proof of Lemma 3.9. Also observe that mixed-norms estimates like (3.10) are available in [2] (see Theorem 9.5 there). Therefore, by using Lemma 5.9 and following very closely the proofs of Lemmas 3.8, 3.9, and Theorems 3.10 we arrive at the following results about interpolation and embedding for Morrey spaces with mixed norms.

Lemma 5.10. *Let $q_1, q_2 \in (1, \infty)$, $0 < \beta \le d/q_1 + 2/q_2$. Then there is a constant N such that, for any $R \in (0, \infty)$, $\varepsilon \in (0, 1]$, and $u \in C_0^\infty$,*

$$\|Du\|_{E_{q_1,q_2,\beta}(C_R)} \le N\varepsilon R \| |\partial_t u| + |D^2u| \|_{E_{q_1,q_2,\beta}(C_R)} + N\varepsilon^{-1}R^{-1}\|u\|_{E_{q_1,q_2,\beta}(C_R)}. \tag{5.11}$$

Theorem 5.11. *Let $q_1, q_2 \in (1, \infty)$, $1 < \beta \le d/q_1 + 2/q_2$ and let $r_i(\beta - 1) = q_i\beta$, $i = 1, 2$. Then there is a constant N such that for any $R \in (0, \infty]$, $u \in C_0^\infty$ we have*

$$\|Du\|_{E_{r_1,r_2,\beta-1}(C_R)} \le N\| |\partial_t u| + |D^2u| \|_{E_{q_1,q_2,\beta}(C_R)} + NR^{-2}\|u\|_{E_{q_1,q_2,\beta}(C_R)}. \tag{5.12}$$

Remark 5.12. By taking u depending only on x we recover from Lemma 5.10 and Theorem 5.11 their "elliptic" counterpart stated as Lemmas 4.4 and 4.7 in [10], respectively.

Remark 5.13. Theorem 5.11 is the most general results of the paper containing as particular cases our previous results on embeddings. Thus, Corollary 5.7 (in an obvious rougher form) follows from Theorem 5.11 when $R = \infty$ and contains embedding results for Lebesgue spaces with mixed norms as $\beta = d/q_1 + 2/q_2$ and for L_q-spaces as $q = q_1 = q_2$.

Remark 5.14. We stated our results only for $u \in C_0^\infty$ just for convenience. Let us show why, for instance, Theorem 5.11 is valid as long as $\partial_t u, Du, D^2u \in E_{q_1,q_2,\beta}(C_R)$. For that, it suffices to prove that for any $R' < R$, $\rho > 0$, $(t, x) \in C_{R'}$ the quantity

$$I := \rho^\beta \;⫲ I_{C_{R'}} Du\|_{L_{r_1,r_2}(C_\rho(t,x))}$$

is less than the right-hand side of (5.12) with $(R')^{-2}$ in place of R^{-2}. For $\varepsilon > 0$ define $u^{(\varepsilon)} = (I_{C_R}u) * \zeta_\varepsilon$, where $\zeta_\varepsilon(x) = \varepsilon^{-d-1}\zeta(t/\varepsilon, x/\varepsilon)$, nonnegative $\zeta \in C_0^\infty$ has integral one and $\zeta(t, x) = 0$ for $t \ge 0$. Also introduce I^ε by replacing u in the definition of I with $u^{(\varepsilon)}$. Of course, $I^\varepsilon \to I$ as $\varepsilon \downarrow 0$ and by Theorem 5.11

$$I^\varepsilon \le N\| |\partial_t u^{(\varepsilon)}| + |D^2u^{(\varepsilon)}| \|_{E_{q_1,q_2,\beta}(C_{R'})} + N(R')^{-2}\|u^{(\varepsilon)}\|_{E_{q_1,q_2,\beta}(C_{R'})} =: J^\varepsilon.$$

Observe that if ε is small enough and $(s, y) \in C_{R'}$, then $\partial_t u^{(\varepsilon)}(s, y) = (I_{C_R}\partial_t u) * \zeta_\varepsilon(s, y)$. Similar formulas are valid for $D^2u^{(\varepsilon)}$ and by Minkowski's inequality (the norm of a sum is less then the sum of norms) we have

$$\begin{aligned} J^\varepsilon \le \int_{\mathbb{R}^{d+1}} \zeta(s, y)\Big(N\|I_{C_R}(|\partial_t u| + |D^2u|)(\cdot - \varepsilon(s, y)\|_{E_{q_1,q_2,\beta}(C_{R'})} \\ +N(R')^{-2}\|I_{C_R}u(\cdot - \varepsilon(s, y))\|_{E_{q_1,q_2,\beta}(C_{R'})}\Big)\, dyds \\ = \int_{\mathbb{R}^{d+1}} \zeta(s, y)\Big(N\|I_{C_R}(|\partial_t u| + |D^2u|)\|_{E_{q_1,q_2,\beta}(C_{R'}-\varepsilon(s,y))} \\ +N(R')^{-2}\|I_{C_R}u\|_{E_{q_1,q_2,\beta}(C_{R'}-\varepsilon(s,y))}\Big)\, dyds. \end{aligned}$$

Since in the last integral $C_{R'} - \varepsilon(s, y) \subset C_R$ if ε is small enough, it follows that for small ε

$$J^\varepsilon \le N\|I_{C_R}(|\partial_t u| + |D^2u|)\|_{E_{q_1,q_2,\beta}(C_R)} + N(R')^{-2}\|I_{C_R}u\|_{E_{q_1,q_2,\beta}(C_R)}$$

which yields the desired result.

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Conflict of interest

The author declares no conflict of interest.

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