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Research article

A monotonicity approach to Pogorelov's Hessian estimates for Monge-Ampère equation†

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Abstract: We present an integral approach to Pogorelov's Hessian estimates for the Monge-Ampère equation, originally obtained via a pointwise argument.

Keywords: Monge-Ampère equation

Dedicated to Neil S. Trudinger on the occasion of his 80th birthday.

In this note, we present a mean value inequality approach to Pogorelov's Hessian estimates for the Monge-Ampère equation, derived via a pointwise argument [3].

Theorem 0.1. *Let u be a smooth convex solution to* $\det D^2u = 1$ *with* $Du(0) = 0$ *on* $D_\tau = \left\{x \in \mathbb{R}^n : x \cdot u_x \le \tau^2\right\}$. *Then*

$$\left|D^2u(0)\right| \le \left[2|B_1| \frac{\tau^n}{|D_\tau|} \frac{|\partial D_\tau|}{|D_\tau|} \|Du\|_{L^\infty(D_\tau)}\right]^{2n}. \tag{0.1}$$

The Hessian estimates for the (dual) potential equation of minimal Lagrangian surfaces, including the two dimensional Monge-Ampère equation $\det D^2u = 1$, obtained in recent years, originate in Trudinger's classic mean value inequality proof of the gradient estimates for the minimal hypersurface equation, by Bombieri-De Giorgi-Miranda [2].

0.1. Monotonicity on maximal surface

Taking the gradient of the both sides of the Monge-Ampère equation

$$\ln\det D^2u = 0, \tag{0.2}$$

we have

$$\sum_{i,j=1}^{n} g^{ij}\partial_{ij}\left(x, Du\left(x\right)\right) = 0, \tag{0.3}$$

where $\left(g^{ij}\right)$ is the inverse of the induced metric $g = \left(g_{ij}\right) = D^2u$ on the Lagrangian graph $M = (x, Du\left(x\right)) \subset (\mathbb{R}^n \times \mathbb{R}^n, 2dxdy)$ (for simplicity of notation, we drop the 2 in $g = 2D^2u$). Because of (0.2) and (0.3), the Laplace-Beltrami operator of the metric g also takes the non-divergence form $\triangle_g = \sum_{i,j=1}^{n} g^{ij}\partial_{ij}$. Denote the extrinsic distance of the position vector (x, Du) to the origin by

$$z = (x_1, \cdots, x_n) \cdot Du = x \cdot u_x \overset{u_x(0)=0}{=} x \cdot \left(u_x\left(x\right) - u_x\left(0\right)\right) \overset{u \text{ convex}}{\geq} 0.$$

Then

$$\begin{aligned}
\left|\nabla_g z\right|^2 &= \sum_{i,j=1}^{n} g^{ij}\partial_i z \partial_j z = \sum_{i,j,k=1}^{n} g^{ij}\left(u_i + x_k u_{ki}\right)\left(u_j + x_k u_{kj}\right) \\
&\overset{p}{=} \sum_{i=1}^{n} g^{ii}\left(u_i^2 + x_i^2 u_{ii}^2 + 2x_i u_i u_{ii}\right) \geq 4x \cdot u_x, \\
\triangle_g z = x \cdot \triangle_g u_x + u_x \cdot \triangle_g x + 2\left\langle \nabla_g x, \nabla_g u_x \right\rangle_g &= 2\sum_{i,j,k=1}^{n} g^{ij}\partial_i x_k \partial_j u_k \\
&\overset{p}{=} 2\sum_{i=1}^{n} g^{ii}u_{ii} = 2n \leq \frac{n}{2}\frac{\left|\nabla_g z\right|^2}{z},
\end{aligned}$$

where at any fixed point p, we assume that D^2u is diagonalized, and we use (0.3) for $\triangle_g z$. In terms of $s = \sqrt{z}$, we have

$$\left|\nabla_g s\right| \geq 1 \text{ and } \triangle_g s \leq (n-1)\left|\nabla_g s\right|^2 / s. \tag{0.4}$$

Following [2, p.392], set

$$\psi\left(s\right) \overset{\chi=\chi_{[0,1]}}{=} \int_s^\infty t\chi\left(t/\rho\right)dt = \begin{cases} \frac{1}{2}\left(\rho^2 - s^2\right) & 0 \leq s \leq \rho \\ 0 & s > \rho \end{cases},$$

actually in the following, χ is taken as a nonnegative smooth approximation of the characteristic function of $(-\infty, 1) \subset (-\infty, \infty)$ with support in $(-\infty, 1)$. We have

$$\begin{aligned}
\triangle_g \psi\left(s\right) &= \psi' \triangle_g s + \psi'' \left|\nabla_g s\right|^2 \\
&= -s\chi\left(s/\rho\right)\triangle_g s - \left[\chi\left(s/\rho\right) + \frac{s}{\rho}\chi'\left(s/\rho\right)\right]\left|\nabla_g s\right|^2 \\
&\geq -\left[n\chi\left(s/\rho\right) + \frac{s}{\rho}\chi'\left(s/\rho\right)\right]\left|\nabla_g s\right|^2 \\
&= \rho^{n+1}\frac{d}{d\rho}\left[\rho^{-n}\chi\left(s/\rho\right)\right]\left|\nabla_g s\right|^2,
\end{aligned}$$

where we use (0.4) in the above inequality. Multiply both sides by any nonnegative superharmonic quantity $q : q \geq 0$ and $\triangle_g q \leq 0$, then integrate over the whole maximal surface M, one has

$$0 \geq \int_M \psi \triangle_g q dv_g = \int_M q \triangle_g \psi dv_g \geq \rho^{n+1}\frac{d}{d\rho}\left[\int_M q\rho^{-n}\chi\left(s/\rho\right)\left|\nabla_g s\right|^2 dv_g\right].$$

Note $1 \leq \left|\nabla_g s\right| \overset{x\to 0}{\to} 1$ by tedious asymptotic analysis and $dv_g = dx$, after taking limit in the smooth approximation of the characteristic function, we obtain

$$|B_1|\, q(0) \geq \tau^{-n} \int_{D_\tau} q \left|\nabla_g s\right|^2 dv_g \geq \tau^{-n} \int_{D_\tau} q dx. \tag{0.5}$$

0.2. Superharmonic quantity

Lemma 0.1. *Suppose u is a smooth convex solution to* $\det D^2 u = 1$. *Then*

$$\triangle_g \ln\det\left[I + D^2 u(x)\right] \geq \frac{1}{2n} \left|\nabla_g \ln\det\left[I + D^2 u(x)\right]\right|^2, \tag{0.6}$$

or equivalently for $q(x) = \left\{\det\left[I + D^2 u(x)\right]\right\}^{\frac{-1}{2n}}$

$$\triangle_g q \leq 0. \tag{0.7}$$

To begin the proof of Lemma 0.1, we first denote $b(x) = \ln\det\left[I + D^2 u(x)\right]$ and rewrite $\triangle_g b$ only in terms of the second and third order derivatives of u, relying on the following equations for the first and second order derivatives of u :

$$0 = \partial_\alpha \ln\det D^2 u = \sum_{i,j=1}^n g^{ij}\partial_{ij}u_\alpha \overset{p}{=} \sum_{i=1}^n g^{ii} u_{ii\alpha}, \tag{0.8}$$

$$0 = \sum_{i,j=1}^n \partial_\beta \left(g^{ij}\partial_{ij}u_\alpha\right) = \sum_{i,j=1}^n g^{ij}\partial_{ij}u_{\alpha\beta} - \sum_{i,j,k,l=1}^n g^{ik}\partial_\beta g_{kl} g^{lj}\partial_{ij}u_\alpha,$$

$$\triangle_g u_{\alpha\beta} = \sum_{i,j=1}^n g^{ij}\partial_{ij}u_{\alpha\beta} \overset{p}{=} \sum_{k,l=1}^n g^{kk}g^{ll}u_{kl\alpha}u_{kl\beta}, \tag{0.9}$$

where at any fixed point p, we assume that $D^2 u$ is diagonalized. The first and second derivatives of b are

$$\begin{aligned}
\partial_\alpha b &= \sum_{i,j=1}^n (I+g)^{ij}\, u_{ij\alpha} \\
\partial_{\alpha\beta} b &= \sum_{i,j=1}^n (I+g)^{ij}\, \partial_{\alpha\beta}u_{ij} - \sum_{i,j,k,l=1}^n (I+g)^{ik}\, \partial_\beta\left(\delta_{kl} + g_{kl}\right)(I+g)^{lj}\, u_{ij\alpha} \\
&\overset{p}{=} \sum_{i=1}^n (1+u_{ii})^{-1}\, \partial_{\alpha\beta}u_{ii} - \sum_{k,l=1}^n (1+u_{kk})^{-1}(1+u_{ll})^{-1}\, u_{kl\alpha}u_{kl\beta},
\end{aligned}$$

where $\left((I+g)^{ij}\right) = (I+g)^{-1}$. Coupled with (0.9), we arrive at

$$\begin{aligned}
\triangle_g b &= \sum_{\alpha,\beta=1}^n g^{\alpha\beta}\partial_{\alpha\beta} b \overset{p}{=} \sum_{\alpha=1}^n g^{\alpha\alpha}\partial_{\alpha\alpha} b \\
&= \sum_{i=1}^n (1+u_{ii})^{-1}\, \triangle_g u_{ii} - \sum_{\alpha,k,l=1}^n g^{\alpha\alpha}(1+u_{kk})^{-1}(1+u_{ll})^{-1}\, u_{kl\alpha}^2 \\
&= \sum_{i,k,l=1}^n (1+\lambda_i)^{-1}\, g^{kk}g^{ll}u_{kli}^2 - \sum_{\alpha,k,l=1}^n g^{\alpha\alpha}(1+\lambda_k)^{-1}(1+\lambda_l)^{-1}\, u_{kl\alpha}^2
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i,j,k=1}^{n} \left[(1+\lambda_i)^{-1} g^{jj} g^{kk} u_{ijk}^2 - (1+\lambda_i)^{-1} (1+\lambda_k)^{-1} g^{jj} u_{ijk}^2 \right] \\
&= \sum_{i,j,k=1}^{n} \lambda_i (1+\lambda_i)^{-1} (1+\lambda_k)^{-1} g^{ii} g^{jj} g^{kk} u_{ijk}^2 \qquad (0.10)\\
&= \sum_{i,j,k=1}^{n} \lambda_i (1+\lambda_i)^{-1} (1+\lambda_k)^{-1} h_{ijk}^2,
\end{aligned}
$$

where we denote (the second fundamental form) $\sqrt{g^{ii}g^{jj}g^{kk}}u_{ijk}$ by h_{ijk}. Let $\mu_i = \frac{\lambda_i - 1}{\lambda_i + 1} \in (-1,1)$, and regrouping those terms h_{ijk} with three repeated indices, two repeated ones, and none, we have

$$
\begin{aligned}
\triangle_g b &= \frac{1}{4} \sum_{i,j,k=1}^{n} (1+\mu_i)(1-\mu_k) h_{ijk}^2 \\
&= \left\{ \begin{array}{c} \frac{1}{4} \sum_i \left[\left(1-\mu_i^2\right) h_{iii}^2 + \sum_{j \neq i} \left(3 - \mu_j^2 - 2\mu_i\mu_j\right) h_{ijj}^2 \right] \\ + \frac{1}{2} \sum_{i>j>k} \left(3 - \mu_i\mu_j - \mu_j\mu_k - \mu_k\mu_i\right) h_{ijk}^2 \end{array} \right\} \geq 0.
\end{aligned}
$$

Accordingly at p, we have

$$
\begin{aligned}
\left|\nabla_g b\right|^2 &= \sum_{\alpha,\beta=1}^{n} g^{\alpha\beta} \partial_\alpha b \partial_\beta b \overset{p}{=} \sum_{\alpha=1}^{n} g^{\alpha\alpha} \left[\sum_{j=1}^{n} \left(1+\lambda_j\right)^{-1} u_{jj\alpha} \right]^2 \\
&= \sum_{\alpha=1}^{n} \left[\sum_{j=1}^{n} \left(1+\lambda_j\right)^{-1} \lambda_j g^{jj} \sqrt{g^{\alpha\alpha}} u_{jj\alpha} \right]^2 \\
&= \frac{1}{4} \sum_{i=1}^{n} \left[\sum_{j=1}^{n} \left(1+\mu_j\right) h_{ijj} \right]^2 = \frac{1}{4} \sum_{i=1}^{n} \left[\sum_{j=1}^{n} \left(1-\mu_j\right) h_{ijj} \right]^2,
\end{aligned}
$$

where the last equality follows from (0.8) or $\sum_{j=1}^{n} h_{ijj} = 0$, and the corresponding expressions with $\left(1+\mu_j\right)$ and $\left(1-\mu_j\right)$ for each $\mu_i < 0$ and $\mu_i \geq 0$ respectively are used to justify the Jacobi inequality (0.6) in the following.

For each fixed i, case $\mu_i \geq 0$:

$$
\begin{aligned}
\frac{1}{2n} \left(\sum_{j=1}^{n} \left(1-\mu_j\right) h_{ijj} \right)^2 &\leq \frac{1}{2} (1-\mu_i)^2 h_{iii}^2 + \sum_{j \neq i} \frac{1}{2} \left(1-\mu_j\right)^2 h_{ijj}^2 \\
&\leq (1+\mu_i)(1-\mu_i) h_{iii}^2 + \sum_{j \neq i} \left[1 - \mu_j^2 + 2\left(1-\mu_i\mu_j\right) \right] h_{ijj}^2,
\end{aligned}
$$

where in the last inequality we used

$$
\frac{1}{2} \left(1-\mu_j\right)^2 \leq \left\{ \begin{array}{ll} 1-\mu_j^2 & \text{for } \mu_j \in [0,1) \\ 2\left(1-\mu_i\mu_j\right) & \text{for } \mu_j \in (-1,0) \text{ and } \mu_i \geq 0 \end{array} \right. ;
$$

case $\mu_i \in (-1,0)$: Symmetrically we have

$$
\frac{1}{2n} \left(\sum_{j=1}^{n} \left(1+\mu_j\right) h_{ijj} \right)^2 \leq (1+\mu_i)(1-\mu_i) h_{iii}^2 + \sum_{j \neq i} \left[1 - \mu_j^2 + 2\left(1-\mu_i\mu_j\right) \right] h_{ijj}^2.
$$

We have proved the Jacobi inequality (0.6) in Lemma 0.1.

0.3. Divergence of $\triangle u$

Plug in the superharmonic quantity from (0.7) to (0.5), we get

$$\left\{\det\left[I + D^2u(0)\right]\right\}^{\frac{1}{2n}} = q^{-1}(0) \leq |B_1|\,\tau^n \frac{1}{\int_{D_\tau} q dx}.$$

From

$$|D_\tau|^2 = \left(\int_{D_\tau} q^{1/2}q^{-1/2}dx\right)^2 \leq \int_{D_\tau} qdx \int_{D_\tau} q^{-1}dx,$$

we have

$$\frac{1}{\int_{D_\tau} qdx} \leq \frac{1}{|D_\tau|^2}\int_{D_\tau} q^{-1}dx.$$

Now

$$\begin{aligned}\int_{D_\tau} q^{-1}dx &= \int_{D_\tau} \left[(1+\lambda_1)\cdots(1+\lambda_n)\right]^{\frac{1}{2n}} dx < \int_{D_\tau} (1+\lambda_{\max})\,dx \qquad (0.11)\\ &\overset{1\leq\lambda_{\max}}{\leq} \int_{D_\tau} 2\lambda_{\max}dx \leq 2\int_{D_\tau} \triangle u dx = 2\int_{\partial D_\tau} u_\gamma dA \leq 2\,|\partial D_\tau|\,\|Du\|_{L^\infty(D_\tau)}.\end{aligned}$$

Therefore, we arrive at the claimed estimate in Theorem 0.1.

$$\left|D^2u(0)\right| < \det\left[I + D^2u(0)\right] \leq \left[2\,|B_1|\,\frac{\tau^n}{|D_\tau|}\frac{|\partial D_\tau|}{|D_\tau|}\,\|Du\|_{L^\infty(D_\tau)}\right]^{2n}.$$

Remark 0.1. Relying on a "rougher" superharmonic quantity $q = \lambda_{\max}^{-1/(n-1)}$ satisfying $\triangle_g q \leq 0$, repeat the above arguments, in particular, with $(1+\lambda_{\max})$ in (0.11) replaced by $\lambda_{\max}$, we have a sharper estimate

$$\left|D^2u(0)\right| = \lambda_{\max}(0) \leq \left[|B_1|\,\frac{\tau^n}{|D_\tau|}\frac{|\partial D_\tau|}{|D_\tau|}\,\|Du\|_{L^\infty(D_\tau)}\right]^{n-1}. \qquad (0.12)$$

Remark 0.2. In addition to the conditions in Theorem 0.1, assuming $u(0) = 0$, and the solution $u(x)$ exists on $\left\{x \in \mathbb{R}^n : u(x) \leq \tau^2\right\}$, then we have

$$\Gamma_\tau = \left\{x \in \mathbb{R}^n : u(x) \leq \varepsilon(n)\,\tau^2\right\} \subset D_\tau = \left\{x \in \mathbb{R}^n : x\cdot u_x \leq \tau^2\right\} \subset \Gamma_{\tau/\sqrt{\varepsilon(n)}}$$

for a small dimensional constant $\varepsilon(n)$, where the second inclusion follows from $0 \leq u_r = (ru_r)_r - ru_{rr} \leq (ru_r)_r$ for the convex function u; and the first inclusion follows from the fact that the gradient Du is small at low enough level set of u, which can be derived from the "separation" Corollary 1 in [1, p.40], of lower level set of the convex solution u from the boundary of the upper level set of u, combined with the invariance of the "extrinsic distance" $x\cdot u_x(x)$ and the equation $\det D^2u(x) = 1$ under affine transform $v(x) = u(Ax)$ with $\det A = 1 : x\cdot v_x(x) = Ax\cdot u_x(Ax)$, $\det D^2v(x) = 1$, and the invariance of the equation $\det D^2u(x)$ under scaling $v(x) = u(\tau x)/\tau^2 : \det D^2v(x) = 1$.

We claim

$$\left|D^2u(0)\right| = \lambda_{\max}(0) \leq \left[C(n)\,\frac{|\partial\Gamma_\tau|}{|\Gamma_\tau|}\,\|Du\|_{L^\infty(\Gamma_\tau)}\right]^{n-1}, \qquad (0.13)$$

or a weaker estimate

$$\left|D^2 u(0)\right| \le \left[C(n)\frac{|\partial\Gamma_\tau|}{|\Gamma_\tau|}\|Du\|_{L^\infty(\Gamma_\tau)}\right]^{2n}. \tag{0.14}$$

In fact, going with the sharper superharmonic quantity $q = \lambda_{\max}^{-1/(n-1)}$, (0.5) becomes

$$|B_1|\, q(0) \ge \tau^{-n}\int_{D_\tau} q dx \ge \tau^{-n}\int_{\Gamma_\tau} q dx.$$

Repeating Step 0.3 Divergence of $\triangle u$, with D_τ replaced by Γ_τ, we have

$$\left|D^2 u(0)\right| = \lambda_{\max}(0) \le \left[|B_1|\frac{\tau^n}{|\Gamma_\tau|}\frac{|\partial\Gamma_\tau|}{|\Gamma_\tau|}\|Du\|_{L^\infty(\Gamma_\tau)}\right]^{n-1}.$$

By John's lemma, there exists an ellipsoid E such that the convex set Γ_τ satisfies $E \subset \Gamma_\tau \subset nE$. Alexandrov estimate and simple barrier argument combined with the equation $\det D^2 u = 1$ on Γ_τ and E respectively, lead to $c(n)\tau^n \le |\Gamma_\tau| \le C(n)\tau^n$.

Consequently, we arrive at the sharper Hessian estimate (0.13) in terms of the level set of solution u.

For the weaker Hessian estimate (0.14) in terms of the level set u, just repeat the above argument with the weaker superharmonic quantity $q = \left\{\det\left[I + D^2 u\right]\right\}^{-\frac{1}{2n}}$.

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Conflict of interest

The author declares no conflict of interest.

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