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Research article

Some comparison results and a partial bang-bang property for two-phases problems in balls†

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Abstract: In this paper, we present two type of contributions to the study of two-phases problems. In such problems, the main focus is on optimising a diffusion function a under L^∞ and L^1 constraints, this function a appearing in a diffusive term of the form $-\nabla \cdot (a\nabla)$ in the model, in order to maximise a certain criterion. We provide a parabolic Talenti inequality and a partial bang-bang property in radial geometries for a general class of elliptic optimisation problems: namely, if a radial solution exists, then it must saturate, at almost every point, the L^∞ constraints defining the admissible class. This is done using an oscillatory method.

Keywords: optimisation; optimal control of PDEs; two-phases problems; Talenti inequality

1. Introduction

1.1. Scope of the paper, informal presentation of the problem

Scope of the paper In this paper, we aim at investigating several properties for a natural shape optimisation problem that arises in heterogeneous heat conduction: *what is the optimal way to design the properties of a material in order to optimise its performance?* This question has received a lot of attention from the mathematical community over the last decades [1, 10–16, 22, 30] and our goal in this paper is to offer some complementary qualitative results. Mathematically, these problems are often dubbed *two-phase problems* and write, in their most general form, as follows: considering that the piece consists of a basic material, with conductivity $\alpha > 0$, we try to find the best location ω for the inclusion of another material having conductivity $\beta > \alpha$. The resulting diffusive part of the equation under consideration writes

$$-\nabla \cdot \Big((\alpha + (\beta - \alpha)1_{\omega})) \nabla \cdot \Big). \tag{1.1}$$

This diffusive part is supplemented with a source term, and can be considered in elliptic or parabolic models. We study some aspects of both cases in the present paperin the case of radial geometries. To state the generic type of question we are interested in, we write down the typical equation in the elliptic case: for aball Ω, a source term $f \in L^2(\Omega)$ (the influence of which is also discussed) and an inclusion $\omega \subset \Omega$, let u^ω be the unique solution of

$$\begin{cases} -\nabla \cdot \Big((\alpha + (\beta - \alpha)1_{\omega}) \nabla u^\omega \Big) = f & \text{in } \Omega\,, \\ u^\omega = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

We consider a volume constraint, enforced by a parameter $V_1 \in (0; \mathrm{Vol}(\Omega))$, and we investigate the problem

$$\sup_{\omega \subset \Omega\,, \mathrm{Vol}(\omega) = V_1} \mathcal{J}(\omega) := \int_\Omega j(u_\omega),$$

for a certain non-linearity j. More specifically, we consider the set

$$\mathfrak{M}(\Omega) := \{a \in L^\infty(\Omega) : \; a = \alpha + (\beta - \alpha)1_\omega \text{ for some measurable } \omega \subset \Omega\,, \mathrm{Vol}(\omega) = V_1\}\,, \tag{1.3}$$

also called the set of *bang-bang* functions, as well as its natural compactification for the weak $L^\infty - *$ topology,

$$\mathcal{A}(\Omega) := \left\{ a \in L^\infty(\Omega)\,, \alpha \leqslant a \leqslant \beta\,, \int_\Omega a = V_0 := \alpha \mathrm{Vol}(\Omega) + (\beta - \alpha) V_1 \right\}.$$

For $a \in \mathcal{A}(\Omega)$, we define $u_{a,f}$ as the solution of (1.2) with $\alpha + (\beta - \alpha)1_\omega$ replaced with a. We will be interested in two formulations: the initial (unrelaxed) one

$$\sup_{a \in \mathfrak{M}(\Omega)} \int_\Omega j(u_a)$$

as well as the relaxed one

$$\sup_{a \in \mathcal{A}(\Omega)} \int_\Omega j(u_a).$$

It should be noted that we will also for some results have to optimise with respect to the source term f, but that the main difficulty usually lies in handling the term a. The two formulations of the problem have their interest, as it may be interesting to see when the two coincides. In other words, is a solution to the second problem a solution of the first one? Let us already underline several basic facts: first, as is customary in this type of optimisation problems (we detail the references later on and for the moment refer to [31]) we do not expect existence of solutions in all geometry, and the proper type of relaxation should rather be of the H-convergence type. Nevertheless, we offer some results about these two problems. Second, the type of problems we are considering are not energetic (in the sense that the criterion we aim at optimising can not a priori be derived from the natural energy associated with the PDE constraint). This leads to several difficulties, most notably in handling the adjoint of the optimisation problem and in the ensuing loss of natural convexity or concavity of the functional to optimise. Third, we distinguish between two types of results: the first type correspond to Talenti inequalities, where we rearrange both the coefficient a and the source term f. In the elliptic case, this follows from results of [6], and our contribution here is the application of these methods to the parabolic case. A second type of result, given in Theorem II, deals with a possible identification of the two formulations (i.e., if a solution to the relaxed problem exists then it is a solution of the unrelaxed one) in radial geometries, and we do not need for this second type of results to rearrange the source term f. This result is the main contribution of this article.

Informal statement of the results Our goal is thus threefold. For the sake of presentation we indicate to which case (i.e., optimisation with respect to a, f or both) each item corresponds. We write $u_{a,f}$ for the solution of the equation with diffusion a and source term f. We will need a comparison inequality provided in [6] and that we recall in Theorem A.

1) **Existence and partial characterisation in radial geometries (optimisation with respect to a and f)** This matter of existence and/or characterisation of optimal a in the case of radial geometries is the topic of the two first results. The Talenti inequality from [6] leads to a comparison principle, but leaves open the question of the existence of optimal shapes: if Ω is a centred ball, is there a radially symmetric solution a^* to the optimisation problem in Ω of the form
$$a^* = \alpha + (\beta - \alpha)1_{\omega^*}$$
for some measurable subset $\omega^* \subset \Omega$? We prove in Theorem I that it is the case when the function j is convex and we also allow ourselves to rearrange the source term f. We use the ideas contained in [15] to do so and prove this theorem for the sake of completeness; we highlight that the main contribution here is to prove that the methods of [15] work for non-energetic functionals.
2) **Weak bang-bang property under monotonicity assumption in radial geometries (optimisation with respect to a)** The second result of the "elliptic problem" part, is the main result of this paper, Theorem II. In it, we give a weak bang-bang property that does not require convexity assumptions on the function j (and so no clear convexity on $\mathcal{J}$). Namely, we prove that if, in a centred ball Ω, a solution a^* to the optimisation problem exists, and if j is increasing, then this solution has to be of bang-bang type. It is notable that, in this theorem, we do not require the term f to be rearranged as well and that we can handle non-energetic problem. This is proved by introducing, for two-phase problems, an oscillatory method reminiscent of the ideas of [25].
3) **Comparison results for parabolic models (optimisation with respect to time-dependent a and f)** We provide, in Theorem III, a parabolic Talenti inequality. The proof is an adaptation of a result of [29], combined with the methods of [6].

Plan of the paper This paper is organised as follows: in Section 1.2 we present the models, the optimisation problems and give some elements about the Schwarz rearrangement. In Section 1.3 we state our main results. Section 1.4 contains the bibliographical references. The rest of the paper is devoted to the proofs of the main results. Finally, in the Conclusion, we state several open problems that we deem interesting.

1.2. Mathematical model and preliminaries

1.2.1. Admissible sets

Henceforth, Ω is a centred ball in $\mathbb{R}^d$, and $V_0 \in (0; \mathrm{Vol}(\Omega))$ is a fixed parameter that serves as a volume constraint. As explained in the first paragraph, we are interested in both elliptic and parabolic models. This leads us to define two admissible classes: the first one, used for elliptic problems, is

$$\mathcal{A}(\Omega) := \left\{ a \in L^\infty(\Omega) : \alpha \leqslant a \leqslant \beta \text{ a.e. in } \Omega\,, \int_\Omega a = V_0 \right\} \tag{1.4}$$

while the second, defined for a certain time horizon $T > 0$, is

$$\mathcal{A}(\Omega;T) := \{ a \in L^\infty((0;T)\times\Omega) : \alpha \leqslant a \leqslant \beta \text{ a.e. in } (0;T)\times\Omega\,, \\ \text{for a.e. } t \in (0;T)\,, \int_\Omega a(t,\cdot) = V_0 \right\} \tag{1.5}$$

The set of admissible sources, on which we also place a volume constraint modelled via a constant $F_0 \in (0; \mathrm{Vol}(\Omega))$, is

$$\mathcal{F}(\Omega) := \left\{ f \in L^\infty(\Omega) : 0 \leqslant f \leqslant 1 \text{ a.e. in } \Omega\,, \int_\Omega f = F_0 \right\}. \tag{1.6}$$

Similarly, we define, in the parabolic case,

$$\mathcal{F}(\Omega;T) := \{ f \in L^\infty((0;T)\times\Omega) : 0 \leqslant f \leqslant 1 \text{ a.e. in } (0;T)\times\Omega\,, \\ \text{for a.e. } t \in (0;T) \int_\Omega f(t,\cdot) = F_0 \right\}. \tag{1.7}$$

1.2.2. Statement of the equations and of the optimisation problems

Main equation in the elliptic case In the elliptic case, the main equation reads as follows: for any $a \in \mathcal{A}(\Omega)$ and any $f \in \mathcal{F}(\Omega)$, $u_{\mathrm{ell},a,f}$ is the unique solution of

$$\begin{cases} -\nabla\cdot(a\nabla u_{\mathrm{ell},a,f}) = f & \text{in } \Omega\,, \\ u_{\mathrm{ell},a,f} = 0 & \text{in } \Omega. \end{cases} \tag{1.8}$$

The solution $u_{\mathrm{ell},a,f}$ is the unique minimiser in $W_0^{1,2}(\Omega)$ of the energy functional

$$\mathcal{E}_{a,f} : W_0^{1,2}(\Omega) \ni u \mapsto \frac{1}{2}\int_\Omega a|\nabla u|^2 - \int_\Omega fu. \tag{1.9}$$

Remark 1. *Although for the classes $\mathcal{A}(\Omega)$ and $\mathcal{A}(\Omega;T)$ the lower bounds $0 < \alpha \leqslant a$ ensure coercivity of the associated energy, it may be asked whether the non-negativity constraint on the sources can be relaxed. It may be difficult, as we need in our proofs the following crucial fact: when Ω is the ball, when f and a are radially symmetric functions of $\mathcal{A}(\Omega)$ and $\mathcal{F}(\Omega)$ respectively, the solution $u_{a,f}$ is radially non-increasing in Ω. This may not be the case, for instance when $f < 0$ close to the center of the ball. Thus we choose simplicity and assume $f \geqslant 0$ almost everywhere.*

In the elliptic case, the goal is to solve the following problem: let $j \in C^1(\mathbb{R})$ be a given non-linearity, then the problem is

$$\boxed{\sup_{a\in\mathcal{A}(\Omega)\,,f\in\mathcal{F}(\Omega)}\left\{\mathcal{J}_{\text{ell}}(a,f) := \int_\Omega j\left(u_{\text{ell},a,f}\right)\right\}.} \tag{$\mathbf{P}_{\text{ell},j}$}$$

In [6], a comparison result that we will use later on is proved; we recall it in Theorem A. This comparison result states roughly speaking, that if j is increasing, there exist two radially symmetric functions $\tilde{a}$ and f^* such that $\mathcal{J}_{\text{ell}}(a,f) \leqslant \mathcal{J}_{\text{ell}}(\tilde{a},f)$, with f^* still admissible; $\tilde{a}$, however, may violate some constraints. Here, our main contribution is Theorem I, in which we prove it is possible to choose a radially symmetric $\tilde{a}$ that satisfies the constraints if we assume that j is convex and C^2. This is done by adapting the methods of [15].

Second, in Theorem II, we are interested in the following alternative formulation: $f \in \mathcal{F}(\Omega)$ being fixed, solve

$$\boxed{\sup_{a\in\mathcal{A}(\Omega)}\left\{\mathcal{J}_{\text{ell}}(a) := \int_\Omega j\left(u_{\text{ell},a,f}\right)\right\}.} \tag{$\mathbf{P}_{\text{ell},j,f}$}$$

We prove, using an oscillatory technique that, if a solution a^* exists and if j is increasing, then we must have $a^* \in \mathfrak{M}(\Omega)$. We underline that this result does not require rearranging f.

Main equation in the parabolic case In the parabolic case, the main equation reads as follows: for any $a \in \mathcal{A}(\Omega;T)$, any $f \in \mathcal{F}(\Omega;T)$, $u_{\text{parab},a,f}$ is the unique solution of

$$\begin{cases}\frac{\partial u_{\text{parab},a,f}}{\partial t} - \nabla\cdot(a\nabla u_{\text{parab},a,f}) = f & \text{in } (0;T)\times\Omega\,,\\ u_{\text{parab},a,f} = 0 & \text{on } \partial\Omega\,,\\ u_{\text{parab},a,f}(0,\cdot) = 0 & \text{in } \Omega.\end{cases} \tag{1.10}$$

The parabolic optimisation problem assumes the following form: for two given non-linearities j_1 and j_2 in $C^1(\mathbb{R})$ we consider the optimisation problem

$$\boxed{\sup_{a\in\mathcal{A}(\Omega;T)\,,f\in\mathcal{F}(\Omega;T)}\left\{\mathcal{J}_{\text{parab}}(a,f) := \iint_{(0;T)\times\Omega} j_1\left(u_{\text{parab},a,f}\right) + \int_\Omega j_2\left(u_{\text{parab},a,f}(T)\right)\right\}.} \tag{$\mathbf{P}_{\text{parab},j_1,j_2}$}$$

The main result is Theorem III, in which a parabolic isoperimetric inequality (with respect to the coefficient a) is obtained: namely, it is better to have radially symmetric a and f.

1.2.3. Preliminaries on rearrangements

In this section we recall the key points about the Schwarz rearrangement,which will be used constantly throughout this paper, and about the rearrangement of Alvino and Trombetti [6, 7] that is crucial in dealing with two-phase isoperimetric problems.

Schwarz rearrangement: definitions, properties and order relations We refer to section 1.4 for further references, for instance for parabolic isoperimetric inequalities and for the time being we recall the basic definitions of the Schwarz rearrangement. We refer to [19,20,23] for a thorough introduction.

Definition 2 (Schwarz rearrangement of sets)**.** *For a given bounded connected open set* Ω_0 *, the Schwarz rearrangement* Ω_0^* *of* Ω_0 *is the unique centred ball* $\mathbb{B}_{\Omega_0} = \mathbb{B}(0; R_{\Omega_0})$ *such that*

$$\mathrm{Vol}(\mathbb{B}_{\Omega_0}) = \mathrm{Vol}(\Omega_0). \tag{1.11}$$

For rearrangements of functions, we use the distribution function: for any $p \in [1; +\infty)$, for any function $u \in L^p(\Omega)$, $u \geqslant 0$, its distribution function is

$$\mu_u : \mathbb{R}_+ \ni t \mapsto \mathrm{Vol}(\{u > t\}). \tag{1.12}$$

Definition 3 (Schwarz rearrangement of a function)**.** *For any function* $u \in L^p(\Omega_0)$, $u \geqslant 0$*, its Schwarz rearrangement is the unique radially symmetric function* $u^* \in L^p(\Omega_0^*)$ *having the same distribution function as u.* $u^\#$ *stands for the one-dimensional function such that* $u^* = u^\#(c_d|\cdot|^d)$ *where* $c_d :=$ $\mathrm{Vol}(\mathbb{B}(0; 1))$.

As a consequence of the equimeasurability of the function and of its rearrangement* there holds:

$$\forall p \in [1; +\infty)\,, \forall u \in L^p(\Omega_0)\,, u \geqslant 0\,, \int_\Omega u^p = \int_{\Omega_0^*} (u^*)^p\,. \tag{1.13}$$

Two results are particularly important in the study of the Schwarz rearrangement:

- Hardy-Littlewood inequality: for any two non-negative functions $f, g \in L^2(\Omega)$,

$$\int_{\Omega_0} fg \leqslant \int_{\Omega_0^*} f^* g^*. \tag{1.14}$$

- PólyaSzegö inequality: for any $p \in [1; +\infty)$, for any $u \in W_0^{1,p}(\Omega)\,, u \geqslant 0$,

$$u^* \in W_0^{1,p}(\Omega_0) \text{ and } \int_{\Omega_0^*} |\nabla u^*|^p \leqslant \int_{\Omega_0} |\nabla u|^p. \tag{1.15}$$

Finally, we will rely on an ordering of the set of functions.

Definition 4. *Let* $R_{\Omega_0} > 0$ *be the radius of the ball* Ω_0^**. For any two non-negative functions* $f, g \in L^1(\Omega_0)$ *we write*

$$f \prec g$$

if

$$\forall r \in [0; R_{\Omega_0}]\,, \int_{\mathbb{B}(0;r)} f^* \leqslant \int_{\mathbb{B}(0;r)} g^*. \tag{1.16}$$

This ordering [17] provides the natural framework for comparison theorems in elliptic and parabolic equations [3, 4, 28, 29, 36–38]. The following property is proved in [4, Proposition 2]: for any non-decreasing convex function F such that $F(0) = 0$, for any two non-negative functions $f, g \in L^1(\Omega)$,

$$f \prec g \to F(f) \prec F(g). \tag{1.17}$$

We now pass to the definition of rearrangement sets:

*Two functions are called equimeasurable if they have the same distribution functions.

Definition 5 (Rearrangement sets)**.** *For any non-negative function $f \in L^1(\Omega_0)$ we define*

$$C_{\Omega_0}(f) := \{\varphi \in L^1(\Omega_0)\,, \varphi^* = f^*\} \tag{1.18}$$

and

$$\mathscr{K}_{\Omega_0}(f) := \left\{\varphi \in L^1(\Omega_0)\,, \varphi \geqslant 0 \text{ a.e.}, \varphi \prec f\,, \int_{\Omega_0} \varphi = \int_{\Omega_0} f\right\}. \tag{1.19}$$

The following result can be found in [8,27,34]: for any non-negative $f \in L^1(\Omega)$, $\mathscr{K}_\Omega(f)$ is a weakly compact, convex set; its extreme points are the elements of $C_\Omega(f)$.

The Alvino-Trombetti rearrangement: definition and property The Alvino-Trombetti rearrangement is very useful when handling two-phase problems, and was introduced in [6, 7] to establish some comparison principles for some elliptic equations with a diffusion matrix. The goal is the following: let $u \in W^{1,2}_0(\Omega) \cap L^\infty(\Omega)$ be a non-negative function and let $a \in \mathcal{A}(\Omega)$. We want to prove that there exists $\tilde{a}$ that is radially symmetric, such that $\tilde{a}\,\mathrm{Id}$ is uniformly elliptic and such that

$$\int_\Omega a|\nabla u|^2 \geqslant \int_{\Omega^*} \tilde{a}\,|\nabla u^*|^2\,. \tag{1.20}$$

One defines $\tilde{a}$ as the unique radially symmetric function such that

$$\text{For a.e. } t \in (0; \|u\|_{L^\infty})\,, \int_{\{u^* \geqslant t\}} \frac{1}{\tilde{a}} = \int_{\{u \geqslant t\}} \frac{1}{a}. \tag{1.21}$$

It can be checked [15] that

$$\tilde{a}^{-1} \in \mathscr{K}_{\Omega^*}\left((a^*)^{-1}\right).$$

Remark 6. *In particular, if all the level-sets of u have Lebesgue measure zero and the gradient of u does not vanish on these level-sets, this definition rewrites as*

$$\textit{For a.e. } t \in (0; \|u\|_{L^\infty})\,, \int_{\{u^* = t\}} \frac{1}{\tilde{a}|\nabla u^*|} = \int_{\{u=t\}} \frac{1}{a|\nabla u|}. \tag{1.22}$$

This fact follows from the co-area formula, which states in particular that

$$\int_{\{u \geqslant c\}} \frac{1}{a} = \int_c^\infty \int_{\{u=t\}} \frac{1}{a|\nabla u|}.$$

[7, Lemma 1.2] or [15, Proposition 4.9] assert that: for any $u \geqslant 0\,, u \in W^{1,2}_0(\Omega)$, for any $a \in \mathcal{A}(\Omega)$, $\tilde{a}$ being defined by (1.21), there holds

$$\int_\Omega a|\nabla u|^2 \geqslant \int_{\Omega^*} \tilde{a}|\nabla u^*|^2. \tag{1.23}$$

1.3. Main results

1.3.1. The elliptic case: Talenti inequalities and bang-bang property

Talenti inequalities for the relaxed problem Let us startby recalling an application of the Alvino-Trombetti rearrangement to Talenti-like inequalities. Talenti inequalities originated in the seminal [36] and have, since then, been widely studied [2–4, 8, 9, 28, 29, 37, 38]. Roughly speaking, they amount to comparing, using the relation $\prec$, the solution u of an elliptic problem with the solution u' of a "symmetrised" elliptic equation. This first result [6] is the stepping stone to our main theorem and holds for the relaxed version of the problem:

Theorem A ([6], Comparison results, optimisation w.r.t. a and f)**.** *Let Ω be a centred ball. For any $a \in \mathcal{A}(\Omega)$ and any $f \in \mathcal{F}(\Omega)$, $\tilde{a}$ being defined by* (1.21)*, there holds*

$$u^*_{\mathrm{ell},a,f} \leqslant u_{\mathrm{ell},\tilde{a},f^*}. \tag{1.24}$$

As a consequence, for any increasing function j,

$$\int_\Omega j(u_{\mathrm{ell},a,f}) \leqslant \int_\Omega j(u_{\mathrm{ell},\tilde{a},f^*}).$$

Thus, it seems quite interesting to investigate whether the optimisation problem ($\mathbf{P}_{\mathrm{ell},j}$) has a radial solution. This would seem natural given the equation above. However, the Alvino-Trombetti rearrangement only provides us with a rearranged coefficient $\tilde{a}$ such that the inverse $(\tilde{a})^{-1} \in \mathscr{K}_{\Omega^*}(a^{-1})$. This last set is however different from $\mathcal{A}(\Omega)$. The same problem arises when considering bang-bang functions a. We nonetheless obtain existence properties for the unrelaxed problem.

Theorem I (Existence and bang-bang property in radial geometry for convex integrand, optimisation w.r.t. a and f)**.** *Assume j is a convex C^2 function. Let $R > 0$. Let $\Omega = \mathbb{B}(0; R)$. The optimisation problem*

$$\sup_{a \in \mathfrak{M}(\Omega), f \in \mathcal{F}(\Omega)} \int_\Omega j(u_{\mathrm{ell},a,f})$$

has a solution $(\overline{a}, \overline{f}) \in \mathfrak{M}(\Omega) \times \mathcal{F}(\Omega)$.

The proof of this theorem is inspired by the proof of existence of optimal profiles for eigenvalue problems in [15].

Finally, the last result for elliptic problems deals with a bang-bang property when optimising only with respect to a: is it true that, if we just assume that j is increasing, if a solution $\overline{a}$ of ($\mathbf{P}_{\mathrm{ell},j,f}$) exists, then it is bang-bang? We can only partially answer this question, in the next theorem. It is the main result of our paper.

Theorem II (Weak bang-bang property for increasing cost functions, optimisation w.r.t radially symmetric a)**.** *Assume j is an increasing function such that $j' > 0$ on $\mathbf{R}_+^*$. Let $R > 0$. Let $\Omega = \mathbb{B}(0; R)$ and $f \in \mathcal{F}(\Omega)$. Then, if the optimisation problem*

$$\sup_{a \in \mathcal{A}(\Omega), a \text{ radially symmetric}} \int_\Omega j(u_{\mathrm{ell},a,f}) \tag{$\mathbf{P}_{\mathrm{ell},j,a}$}$$

has a solution $\overline{a}$, there holds

$$\boldsymbol{\overline{a} \in \mathfrak{M}(\Omega)} \text{ *or, in other words,* } \boldsymbol{\overline{a} = \alpha + (\beta - \alpha)\mathbf{1}_\omega} \text{ *for some measurable* } \boldsymbol{\omega \subset \Omega}.$$

The proof of this theorem is based on the development of an oscillatory method recently introduced in [25].

1.3.2. The parabolic case: time-dependent optimal design problems & application to parabolic eigenvalue optimisation problems

In this second part, we state our main result devoted to the parabolic optimisation problem ($\mathbf{P}_{\text{parab},j_1,j_2}$). The proof of the parabolic isoperimetric inequality is done by adapting the proofs of Theorem A and of [29, Theorem 2.1]. For the sake of clarity, for a function u of two variables $u = u(t,x)$, the notation $u^*(t,\cdot)$ stands for the Schwarz rearrangement of $u(t,\cdot)$ with respect to the space variable x.

Theorem III (Comparison results, optimisation w.r.t. a and f)**.** *Let $\Omega = \mathbb{B}(0;R)$. Let $a \in \mathcal{M}(\Omega;T)$ and $f \in \mathcal{F}(\Omega;T)$. Then there exists a radially symmetric function $\tilde{a}$ defined on $(0;T)\times\Omega$ such that $\alpha \leqslant \tilde{a} \leqslant \beta$ almost everywhere and such that, for almost every $t \in (0;T)$ and every $r \in (\Omega;R)$ there holds*

$$\int_{\mathbb{B}(0;r)} u^*_{\text{parab},a,f} \leqslant \int_{\mathbb{B}(0;r)} u_{\text{parab},\tilde{a},f^*}.$$

In particular, if j_1 and j_2 are convex increasing functions there holds

$$\mathcal{J}_{\text{parab}}(a,f) \leqslant \mathcal{J}_{\text{parab}}(\tilde{a},f^*).$$

Let us now offer some comments about this result, and about the method of proof.

Remark 7 (Comments on Theorem III)**.** *1) The first thing that has to be noted is that, exactly as in the elliptic case, although the new weight $\overline{a}$ satisfies the correct upper and lower bounds $\alpha \leqslant \overline{\beta}$, there is a priori no guarantee that $\overline{a} \in \mathcal{M}(\Omega)$. Some other arguments would then be needed in order to conclude as to the integral constraint. It is not clear at this stage how one may go about this question.*

2) The second remark has to do with the method of proof that is employed. The two main available approaches in the context of parabolic equations are, on the one hand, dealing with the parabolic problem directly, as is done in [29] and as we do here, and on the other hand by time-discretisation of the evolution problem, as in [3]. We believe the second of these approaches may prove more delicate. To see why, let us recall the main steps of the proof of [3]: the authors, which in particular try to prove a comparison result for rearrangement of the source f in the parabolic equation

$$\frac{\partial u}{\partial t} - \Delta u = f,$$

approximate this equation by the discretisation with time step $N \in \mathbf{N}^*$

$$N(u_{k+1,N} - u_{k,N}) - \Delta u_{k+1,N} = f_{k,N}$$

where $f_{k,N} = N\int_{k/N}^{(k+1)/N} f$. On each of these discretised problem, they use an elliptic Talenti inequality, yielding a comparison with the solution of the same system with $f^*_{k,N}$ as a right-hand side. This Schwarz symmetrisation operation is independent of the time-step (in the sense that the definition of f^* does not depend on the time step N). In our case however, since the

Alvino-Trombetti rearrangement depends on the function $u_{\mathrm{parab},a,f}$ evaluated at the time t, this would translate, at the discretised level, as a rearrangement that would depend on both indexes k and N. This may lead to potential difficulties in passing to the limit.

Let us underline that this type of parabolic comparison results can be very useful when dealing with parabolic eigenvalue optimisation problems, as is done for instance in [32, Theorem 3.9].

1.4. Bibliographical references

In this paper, we offer contributions that may be viewed from several point of views, each of which stemming from very rich domains in mathematical analysis.

Two-phase spectral optimisation problems Two-phase optimisation problems have a rich history, and are deeply linked to homogenisation phenomenas. We refer, for instance, to [1, 31] for a presentation of this rich theory, and we underline that one of the striking features of these problems is that there is often a lack of existence results. These results are typically obtained by proving that should an optimiser exist, then an overdetermined problem that can only solved in radial geometries should have a solution. This is done by using Serrin type theorems [35], and this phenomenon occurs in dimension $d \geqslant 2$. A typical and famous example of such problems is the optimisation of the first Dirichlet eigenvalue of the operator $-\nabla \cdot (a\nabla)$ under the constraint that $a \in \mathcal{A}(\Omega)$. To the best of our knowledge, the proof of non-existence of an optimal $a^* \in \mathcal{A}(\Omega)$ when Ω is not a ball was only recently completed in a series of papers by Casado-Diaz [10–12]. However, these negative results in the case of non-radial geometries do not allow to conclude as for the existence and/or characterisation of optimisers in radially symmetric domains. In this case, the same spectral optimisation problem being under consideration, the first proof of existence can be found in [15], using the Alvino-Trombetti rearrangement. We borrow from their ideas in the proof of Theorem I (and we highlight the fact that we do not consider here energetic problems). To underline the complexity of this spectral optimisation problem, let us also mention [22], in which it is shown that, in the ball, the qualitative features of the optimiser a^* strongly depend on the volume constraint. We also refer to [13, 26] for the study of the spectral optimisation of operators with respect to a weight $a \in \mathcal{A}(\Omega)$ that appears both in the principal symbol $-\nabla \cdot (a\nabla)$ and as a potential.

Elliptic and parabolic Talenti inequalities Talenti inequalities, which originate in the seminal [36] have been the subject of an intense research activity. For parabolic equations, the study of such inequalities started, as far as we are aware, in the works of Bandle [9], Vazquez [38] and were later deeply analysed by Alvino, Trombetti and Lions [2, 3] on the one hand, and by Mossino and Rakotoson on the other [29]. We would like to mention that we have recently obtained a quantitative parabolic isoperimetric inequality for the source term in [24]. Alvino, Nitsch and Trombetti have recently proved an elliptic Talenti inequality under Robin boundary conditions, using a very fine analysis of the Robin problem [5]. This Robin Talenti inequality was then used in, for instance, [21, 33].

2. Proof of Theorem I

Proof of Theorem I. For the first part of the theorem, we consider the case $\Omega = \mathbb{B}(0; R)$ where $R > 0$ is a fixed constant. We work with functions $a \in \mathfrak{M}(\Omega)$. In other words, there exists $\omega \subset \Omega$ measurable such that

$$a = \alpha + (\beta - \alpha)1_{\omega},$$

and we aim at solving

$$\sup_{a \in \mathfrak{M}(\Omega), f \in \mathcal{F}(\Omega)} \mathcal{J}(f, a) = \int_{\Omega} j(u_{\mathrm{ell},a,f}),$$

under the assumption that j is convex on $\mathbf{R}_+$.

Let us first note that for any $a \in \mathfrak{M}(\Omega)$ we have $a^* = \alpha + (\beta - \alpha)1_{\mathbb{B}^*}$ where $\mathbb{B}^* = \mathbb{B}(0; r^*)$ satisfies $\beta \mathrm{Vol}(\mathbb{B}^*) + \mathrm{Vol}(\Omega \backslash \mathbb{B}^*) = V_0$. As a consequence, for any $a_1, a_2 \in \mathfrak{M}(\Omega)$,

$$\mathscr{K}((a_1)^{-1}) = \mathscr{K}((a_2)^{-1})$$

where $\mathscr{K}(\cdot)$ is the rearrangement class defined in definition 5. For the sake of notational convenience, we define

$$\mathfrak{K} := \mathscr{K}(a^{-1}) \text{ where } a \text{ is any element of } \mathfrak{M}(\Omega).$$

By Theorem A, for any $a \in \mathcal{A}(\Omega)$ there exists a radially symmetric $\tilde{a}$ such that

$$\tilde{a}^{-1} \in \mathfrak{K}\,, J(\tilde{a}, \overline{f}) \geqslant J(a, f)$$

where $\overline{f}$ is simply the Schwarz rearrangement of f. By convexity of the functional with respect to f, $\overline{f}$ is a bang-bang function. We henceforth consider it fixed and focus on optimisation with respect to a.

The problem with this reformulation is that there is a priori no guarantee that $\tilde{a} \in \mathfrak{M}(\Omega)$, and it is in general false. To overcome this difficulty, we now focus on a slightly simplified version of our problem:

$$\sup_{a \text{ radially symmetric s.t. } \tilde{a} \in \mathfrak{K}} \int_{\Omega} j(u_a) \text{ where } u_a \text{ solves } \begin{cases} -\nabla \cdot (a \nabla u_a) = \overline{f} & \text{in } \Omega\,, \\ u_a = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

Another refomulation, a priori encompassing a larger class, is

$$\sup_{\tilde{\mu} \text{ radially symmetric s.t. } \mu \in \mathfrak{K}} \left(\mathcal{H}(\mu) := \int_{\Omega} j(v_{\mu}) \right) \text{ where } v_{\mu} \text{ solves } \begin{cases} -\nabla \cdot \left(\frac{1}{\mu} \nabla v_{\mu} \right) = \overline{f} & \text{in } \Omega\,, \\ v_{\mu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.2}$$

We now proceed in several steps, following the ideas of [15]:

1) Existence of solutions to (2.2): we first prove, in lemma 8, that there exists a solution $\overline{\mu}$ to (2.2). This is done via the direct method in the calculus of variations.
2) The bang-bang property for $\overline{\mu}$: we then prove, in lemma 9, that any solution $\overline{\mu}$ of (2.2) is a bang-bang function. In other words, there exists a measurable subset $\omega \subset \Omega$ such that

$$\overline{\mu} = \frac{1}{\alpha + (\beta - \alpha)1_{\omega}}.$$

This is done via a convexity argument. As a consequence, $\frac{1}{\overline{\mu}} \in \mathfrak{M}(\Omega)$, hence concluding the proof.

Existence of solutions to (2.2) The main result of this paragraph is the following lemma:

Lemma 8. *There exists a solution* $\overline{\mu}$ *of the variational problem* (2.2).

Proof of Lemma 8. We first note the following fact: if a sequence of radially symmetric functions $\{\mu_k\} \in \mathfrak{R}^{\mathbf{N}}$ weakly converges in $L^\infty - *$ to μ_∞ (which is an element of $\mathfrak{R}$ by the closedness of $\mathfrak{R}$ for the weak convergence, see [27]) then, up to a subsequence,

$$\mathcal{H}(\mu_k) \underset{k\to\infty}{\to} \mathcal{H}(\mu_\infty).$$

Obtaining this result boils down to proving that, for any sequence of radially symmetric functions $\{\mu_k\} \in \mathfrak{R}^{\mathbf{N}}$ weakly converges in $L^\infty - *$ to μ_∞ there holds

$$\left(\mu_k \underset{k\to\infty}{\rightharpoonup} \mu_\infty\right) \Rightarrow \left(v_{\mu_k} \underset{k\to\infty}{\to} v_{\mu_\infty} \text{ a.e. up to a subsequence}\right).$$

Indeed, it then simply suffices to use the dominated convergence theorem to obtain the required result. Let us then prove that for any sequence of radially symmetric functions $\{\mu_k\} \in \mathfrak{R}^{\mathbf{N}}$ weakly converging in $L^\infty - *$ to μ_∞,

$$v_{\mu_k} \underset{k\to\infty}{\to} v_{\mu_\infty} \text{ in } L^2(\Omega). \tag{2.3}$$

However, since we are working with radially symmetric functions, this follows from explicit integration in radial coordinates of

$$\begin{cases} -\nabla \cdot \left(\frac{1}{\mu}\nabla v_{\mu_k}\right) = \overline{f} & \text{in } \Omega\,, \\ v_{\mu_k} = 0 & \text{on } \partial\Omega \end{cases}$$

which gives, for any $k \in \mathbf{N}$ (and with a slight abuse of notation),

$$r^{d-1}v'_{\mu_k}(r) = -\mu_k(r)\int_0^r s^{d-1}f(s)ds.$$

Thus, from the Rellich-Kondrachov embedding

$$v_{\mu_k} \underset{k\to\infty}{\to} v_{\mu_\infty} \begin{cases} \text{weakly in } W_0^{1,2}(\Omega)\,, \\ \text{strongly in } L^2(\Omega). \end{cases}$$

It suffices to extract a subsequence that is converging almost everywhere.

We turn back to the proof of the lemma: let $\{\mu_k\}_{k\in\mathbf{N}}$ be a maximising sequence for (2.2). Since the set $\mathfrak{R}$ is weakly compact, and since for any $k \in \mathbf{N}$ μ_k is radially symmetric, there exists a radially symmetric $\mu_\infty \in \mathfrak{R}$ such that, up to a subsequence,

$$\mu_k \underset{k\to\infty}{\to} \mu_\infty \text{ weakly in } L^\infty - *.$$

Hence, up to a subsequence,

$$\mathcal{H}(\mu_k) \underset{k\to\infty}{\to} \mathcal{H}(\mu_\infty)$$

so that μ_∞ is a solution of (2.2). □

The bang-bang property for $\overline{\mu}$ We now present the key point of the proof of Theorem I, the bang-bang property.

Lemma 9. *Any solution $\overline{\mu}$ of* (2.2) *is of bang-bang type: there exists $\omega \subset \Omega$ such that*

$$\mu = \frac{1}{\alpha + (\beta - \alpha)1_{\omega}}.$$

Proof of lemma 9. We argue by contradiction and assume that there exists a solution $\overline{\mu}$ of (2.2) that is not of bang-bang type. We will reach a conclusion using a second order information on the functional $\mathcal{H}$, namely, by using the first and second order Gâteaux-derivative of the functional $\mathcal{H}$. Let us first observe that it is standard [18] to see that the map $\mathfrak{K} \ni \mu \mapsto v_\mu$ is Gâteaux-differentiable. Furthermore, for a given $\mu \in \mathfrak{K}$ and an admissible perturbation h at μ (i.e., such that $\mu + th \in \mathfrak{K}$ for $t > 0$ small enough) the first order Gâteau-derivative of v_μ in the direction h is the unique solution $\dot{v}_\mu(h)$ of

$$\begin{cases} -\nabla \cdot \left(\frac{1}{\mu}\nabla \dot{v}_\mu(h)\right) + \nabla \cdot \left(\frac{h}{\mu^2}\nabla v_\mu\right) = 0 & \text{in } \Omega\,, \\ \dot{v}_\mu(h) = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.4}$$

and the first Gâteaux-derivative of $\mathcal{H}$ at μ in the direction h is given by

$$\dot{\mathcal{H}}(\mu)[h] = \int_\Omega j'(v_\mu)\dot{v}_\mu(h). \tag{2.5}$$

This leads to introducing the adjoint state p_μ as the unique solution of

$$\begin{cases} -\nabla \cdot \left(\frac{1}{\mu}\nabla p_\mu\right) = j'(v_\mu) & \text{in } \Omega\,, \\ p_\mu = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.6}$$

Remark 10. *It should be noted that by explicit integration of the equation on v_μ in radial coordinate, $v_\mu \in L^\infty(\Omega)$; as a consequence, $j'(v_\mu)$ is an L^∞ function, so that p_μ is well-defined.*

Multiplying, on the one hand (2.4) by p_μ, on the other hand (2.6) by v_μ, and integrating by parts gives

$$\dot{\mathcal{H}}(\mu)[h] = \int_\Omega \frac{1}{\mu}\langle \nabla p_\mu, \nabla \dot{v}_\mu(h)\rangle = -\int_\Omega \frac{h}{\mu^2}\langle \nabla v_\mu\,, \nabla p_\mu\rangle = -\int_\Omega h\left\langle \frac{\nabla v_\mu}{\mu}, \frac{\nabla p_\mu}{\mu}\right\rangle. \tag{2.7}$$

We now compute the second order derivative of the criterion in a similar manner: the second order Gâteaux derivative of v_μ at μ in the direction h is zero. In other words, denoting by $\ddot{v}_\mu$ this second order derivative, we have

$$\ddot{v}_\mu = 0.$$

Indeed, this follows from the explicit computation of v_μ as

$$r^{d-1}v'_{\mu_k}(r) = -\mu_k(r)\int_0^r s^{d-1}f(s)ds.$$

Thus, it appears that $\mu \mapsto v_\mu$ is linear. As a consequence we have that the second order Gâteaux derivative of $\mathcal{H}$ at μ in the direction h, henceforth abbreviated as $\ddot{\mathcal{H}}(\mu)$, is given by

$$\ddot{\mathcal{H}}(\mu) = \int_\Omega j''(v_\mu)\left(\dot{v}_\mu\right)^2 + \int_\Omega j'(v_\mu)\ddot{v}_\mu = \int_\Omega j''(v_\mu)\left(\dot{v}_\mu\right)^2. \tag{2.8}$$

Hence, if j is convex, so is $\mathcal{H}$. Thus any solution $\overline{\mu}$ of (2.2) is an extreme point of $\mathfrak{K}$. In other words

$$\overline{\mu} \in C(a^{-1}) \text{ where } a \text{ is any element of } \mathfrak{M}(\Omega).$$

It follows that $\frac{1}{\overline{\mu}} \in \mathfrak{M}(\Omega)$. □

Conclusion of the proof As noted at the beginning of the proof, for any $a \in \mathfrak{M}(\Omega)$ there exists $\tilde{a}$ such that $\tilde{a}^{-1} \in \mathfrak{K}$ and such that $\mathcal{J}(\tilde{a}, \overline{f}) \geqslant \mathcal{J}(a, f)$. Since

$$\mathcal{H}\left(\frac{1}{\tilde{a}}\right) = \mathcal{J}(\tilde{a}, \overline{f})$$

it follows that

$$\mathcal{J}(\tilde{a}, \overline{f}) \leqslant \mathcal{H}(\overline{\mu}) \text{ where } \overline{\mu} \text{ is the solution of (2.2) given by lemma 8.}$$

From proposition 9 $\overline{\mu}$ is a bang-bang function. As a consequence, $\frac{1}{\overline{\mu}} := \overline{a}$ is an element of $\mathfrak{M}(\Omega)$. Thus

$$\mathcal{J}(a, f) \leqslant \mathcal{J}(\tilde{a}, \overline{f}) \leqslant \mathcal{H}(\overline{\mu}) = \mathcal{J}(\overline{a}, \overline{f}).$$

Thus $\overline{a}$ is a solution of the initial optimisation problem.

The proof of the theorem is now complete. □

3. Proof of Theorem II

Proof of Theorem II. Throughout this proof we assume that we are given a radially symmetric solution $\overline{a}$ of the optimisation problem

$$\sup_{a \in \mathcal{A}(\Omega), a \text{ radially symmetric}} \int_\Omega j(u_{\text{ell},a,f})$$

and we want to prove that $\overline{a} \in \mathfrak{M}(\Omega)$. To reach the desired conclusion we argue by contradiction and we assume that $\overline{a} \notin \mathfrak{M}(\Omega)$. We emphasise once again that this proof does not require rearranging the source term f. Since f is assumed to be fixed, we write $\mathcal{J}(a)$ for $\mathcal{J}(a, f) = \int_\Omega j(u_{\text{ell},a,f})$ and u_a for $u_{\text{ell},a,f}$.

Let us single out the following result, that follows from direct integration in radial coordinates of (1.10):

Lemma 11. *For any radially symmetric $a \in \mathcal{A}(\Omega)$ and $f \in \mathcal{F}(\Omega)$, $u_a \in W^{1,\infty}(\Omega)$, u_a is radial and we furthermore have, with a slight abuse of notation, for a.e. $r \in (0; R)$,*

$$u_a'(r) = -\frac{1}{a(r)r^{d-1}} \int_0^r \xi^{d-1} f(\xi) d\xi. \tag{3.1}$$

In particular, u_a' is a non-positive function and, for any $\varepsilon > 0$, $\sup_{[\varepsilon;R]} u_a' < 0$. It is strictly decreasing if $f > 0$ in a neighbourhood of 0.

We now compute the Gateaux derivatives of both the maps $a \mapsto u_{\mathrm{ell},a,f}$ and of $a \mapsto \mathcal{J}(a)$ (we note that the fact that both maps are Gateaux differentiable follow from standard arguments). We note that, to compute them, it is not necessary to assume that the coefficients a and f are radiallly symmetric.

The first-order Gateaux derivative of $u_{\mathrm{ell},a,f}$ at a in an admissible direction h (i.e., such that $a + th \in \mathcal{A}(\Omega)$ for $t > 0$ small enough), denoted by $\dot{u}_a$, is the unique solution to

$$\begin{cases} -\nabla \cdot (a \nabla \dot{u}_a) = \nabla \cdot (h \nabla u_a) & \text{in } \Omega\,, \\ \dot{u}_a = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.2}$$

The Gateaux derivative of $\mathcal{J}$ at a in the direction h is given by

$$\dot{\mathcal{J}}(a)[h] = \int_\Omega j'(u_a)\dot{u}_a. \tag{3.3}$$

This leads to introducing the adjoint state p_a as the unique solution to

$$\begin{cases} -\nabla \cdot (a \nabla p_a) = j'(u_a) & \text{in } \Omega\,, \\ p_a = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.4}$$

Multiplying (3.4) by $\dot{u}_a$ and (3.2) by p_a and integrating by parts gives

$$\dot{\mathcal{J}}(a)[h] = \int_\Omega j'(u_a)\dot{u}_a = \int_\Omega a\langle \nabla p_a\,, \nabla \dot{u}_a\rangle = -\int_\Omega h\langle \nabla u_a, \nabla p_a\rangle. \tag{3.5}$$

In the same way, the second order Gateaux derivative of u_a at a in the direction h, denoted by $\ddot{u}_a$, is the unique solution to

$$\begin{cases} -\nabla \cdot (a \nabla \ddot{u}_a) = 2\nabla \cdot (h \nabla \dot{u}_a) & \text{in } \Omega\,, \\ \ddot{u}_a = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.6}$$

and the second order Gateaux derivative of $\mathcal{J}$ at a in the direction h is given by

$$\ddot{\mathcal{J}}(a)[h,h] = \int_\Omega j''(u_a)(\dot{u}_a)^2 + \int_\Omega j'(u_a)\ddot{u}_a. \tag{3.7}$$

However, multiplying (3.6) by p_a, integrating by parts and using the weak formulation of (3.4) yields

$$\int_\Omega j'(u_a)\ddot{u}_a = \int_\Omega a\langle \nabla p_a\,, \nabla \ddot{u}_a\rangle = -2\int_\Omega h\langle \nabla \dot{u}_a\,, \nabla p_a\rangle. \tag{3.8}$$

Plugging (3.8) in (3.7) gives

$$\ddot{\mathcal{J}}(a)[h,h] = \int_\Omega j''(u_a)(\dot{u}_a)^2 - 2\int_\Omega h\langle \nabla \dot{u}_a\,, \nabla p_a\rangle. \tag{3.9}$$

We now use the radial symmetry assumption: since h, a and f are radially symmetric, and since $u_a'(0) = \dot{u}_a'(0) = 0$, (3.2) implies, in radial coordinates, as

$$-a\dot{u}_a' = hu_a'. \tag{3.10}$$

Furthermore, we have the following lemma:

Lemma 12. *If $j' > 0$ on $\mathbf{R}_+^*$ then p_a is a radially symmetric decreasing function:*

$$p_a' < 0 \text{ in } (0;R).$$

Proof of Lemma 12. The fact that p_a is decreasing simply follows from, first, the strong maximum principle which implies that

$$u_a > 0 \text{ in } [0;R)$$

and, second, from explicit integration of the equation on p_a in radial coordinates, which gives

$$p_a'(r) = -\frac{1}{a(r)r^{d-1}}\int_0^r j'(u_a)r^{d-1}dr < 0 \text{ for } r > 0.$$

□

The radiality of p_a implies

$$\langle \nabla \dot{u}_a\,, \nabla p_a\rangle = \dot{u}_a' p_a'.$$

As a consequence of (3.10), we have, for a constant $M_d > 0$

$$\begin{aligned}
-\int_\Omega h\langle \nabla \dot{u}_a\,, \nabla \dot{p}_a\rangle &= -M_d\int_0^R r^{d-1}h(r)\dot{u}_a'(r)p_a'(r)dr\\
&= M_d\int_0^R a(r)\left(\dot{u}_a'\right)^2\frac{p_a'}{u_a'}(r)dr\\
&= M_d\int_0^R a(r)\left(\dot{u}_a'\right)^2\frac{ap_a'}{au_a'}(r)dr.
\end{aligned}$$

Let us first define

$$\varphi : r \in (0;R] \mapsto \frac{ap_a'}{au_a'}(r).$$

We observe that $\varphi > 0$ in $(0;R]$ and that, as $r \to 0$,

$$\liminf_{r\to 0}\varphi(r) \geqslant \liminf_{r\to 0}\frac{j'(u_a(0))}{f(r)} > 0. \tag{3.11}$$

If $f > 0$ in a neighbourhood of 0 then we can extend φ by $\frac{j'(u_a(0))}{f(0)} > 0$ in 0. If on the other hand $f = 0$ in a neighbourhood of 0, then $\varphi \to +\infty$ when $r \to 0$. In any case, there exists a constant $A > 0$ such that

$$\varphi \geqslant \frac{A_0}{2} > 0 \text{ in } [0;R].$$

We define the function

$$\Phi : \Omega \ni x \mapsto \varphi\left(|x|\right),$$

and we thus have

$$-\int_\Omega h\langle \nabla \dot{u}_a\,, \nabla \dot{p}_a\rangle = -2M_d\int_0^R r^{d-1}h(r)\dot{u}_a'(r)p_a'(r)dr$$

$$
\begin{aligned}
&= 2M_d \int_0^R a(r)\,(\dot{u}_a')^2\,\frac{p_a'}{u_a'}(r)dr \\
&= 2\int_\Omega \Phi a|\nabla \dot{u}_a|^2 \\
&\geqslant A_0 \int_\Omega a|\nabla \dot{u}_a|^2.
\end{aligned}
$$

Finally, as $j \in C^2(\mathbb{R}_+)$ and $u_a \in L^\infty(\Omega)$ there exists a constant $B > 0$ such that

$$
j''(u_a) \geqslant -B \text{ in } \Omega. \tag{3.12}
$$

We end up with the following estimate on $\ddot{\mathcal{J}}(a)[h,h]$:

$$
\ddot{\mathcal{J}}(a)[h,h] = \int_\Omega j''(u_a)(\dot{u}_a)^2 - 2\int_\Omega h\langle \nabla \dot{u}_a\,,\nabla p_a\rangle \tag{3.13}
$$

$$
\geqslant A_0 \int_\Omega a|\nabla \dot{u}_a|^2 - B\int_\Omega \dot{u}_a^2. \tag{3.14}
$$

Remark 13. *It should be noted that, at this level, we recover the convexity of the functional $\mathcal{J}$ if we assume that $j'' \geqslant 0$. Indeed, in that case we can take $B = 0$.*

Let us now turn back to the core of the proof: we have a maximiser $\overline{a} \notin \mathfrak{M}(\Omega)$. Let

$$
\tilde{\omega} := \{\alpha < a < \beta\}.
$$

Since $\overline{a} \notin \mathfrak{M}(\Omega)$,

$$
\mathrm{Vol}(\tilde{\omega}) > 0.
$$

Furthermore, for any $h \in L^\infty(\tilde{\omega})$, extended by 0 outside of $\tilde{\omega}$ and such that $\int_{\tilde{\omega}} h = 0$, we have (since both h and $-h$ are admissible perturbations at $\overline{a}$)

$$
\dot{\mathcal{J}}(a)[h] = 0. \tag{3.15}
$$

To reach a contradiction, it suffices to build $h \in L^\infty(\tilde{\omega})\backslash\{0\}$ such that

$$
\boldsymbol{\int_\Omega h\mathbf{1}_{\tilde{\omega}} = 0 \text{ and } \ddot{\mathcal{J}}(a)[h\mathbf{1}_{\tilde{\omega}}, h\mathbf{1}_{\tilde{\omega}}] > 0.} \tag{3.16}
$$

Actually, by approximation it suffices to build $h \in L^2(\tilde{\omega})$ such that (3.16) holds. For the sake of notational simplicity, for any $h \in L^2(\tilde{\omega})$, we identify h with $h1_{\tilde{\omega}} \in L^2(\Omega)$. To obtain the existence of such a perturbation we single out the estimate

$$
\ddot{\mathcal{J}}(a)[h,h] \geqslant A_0 \int_\Omega a|\nabla \dot{u}_a|^2 - B\int_\Omega \dot{u}_a^2. \tag{3.17}
$$

We introduce the sequence $\{\sigma_k\,,\psi_k\}_{k\in\mathbf{N}^*}$ of eigenvalues of the operator $-\nabla\cdot(a\nabla)$. We pick a non-decreasing of the eigenvalue sequence:

$$
0 < \sigma_0 \leqslant \sigma_1 \leqslant \ldots \leqslant \sigma_k \underset{k\to\infty}{\to} +\infty.
$$

The eigenequations are given by

$$\forall k \in \mathbb{N}^*\,, \begin{cases} -\nabla\cdot(a\nabla\psi_k) = \sigma_k\psi_k & \text{in } \Omega\,, \\ \psi_k = 0 & \text{on } \partial\Omega\,, \\ \int_\Omega \psi_k^2 = 1. \end{cases} \tag{3.18}$$

For any admissible perturbation h at a, we decompose $\dot{u}_a$ in this basis as

$$\dot{u}_a = \sum_{k=1}^{\infty} \alpha_k(h)\psi_k, \tag{3.19}$$

where the coefficients $\{\alpha_k(h)\}_{k\in\mathbb{N}^*}$ are determined by equation (3.2). If we assume that, for an integer K large enough, we have

$$\forall k \leqslant K\,, \alpha_k(h) = 0\,, \sum_{k=K}^{\infty} \alpha_k(h)^2 > 0 \tag{3.20}$$

then we obtain, by expanding the right hand-side of (3.17),

$$\boldsymbol{\ddot{\mathcal{J}}(a)[h\mathbb{1}_{\tilde{\omega}}, h\mathbb{1}_{\tilde{\omega}}] \geqslant A\sum_{k=K}^{\infty}\sigma_k\alpha_k(h)^2 - B\sum_{k=K}^{\infty}\alpha_k(h)^2 \geqslant (A\sigma_K - B)\sum_{k=K}^{\infty}\frac{\alpha_k(h)^2}{\sigma_k} > 0.} \tag{3.21}$$

As a consequence it remains to construct a perturbation h such that

$$\nabla\cdot(h\nabla u_a) = \sum_{k\geqslant K}\eta_k\psi_k. \tag{3.22}$$

We need however to be careful, since $\nabla\cdot(h\nabla u_a)$ merely lies in $W^{-1,2}(\Omega)$. To overcome this difficulty, we define, for any $h \in L^2(\tilde{\omega})$, the coefficient

$$\eta_k(h) := \int_\Omega h\langle\nabla u_a\,, \nabla\psi_k\rangle. \tag{3.23}$$

Since $\nabla\cdot(h\nabla u_a) \in W^{-1,2}(\Omega)$, each of this quantities is well-defined. Furthermore, setting, for any $k \in \mathbb{N}$,

$$\alpha_k(h) := \int_\Omega \psi_k\dot{u}_a \tag{3.24}$$

we have

$$\alpha_k(h) := \int_\Omega \psi_k\dot{u}_a = \frac{1}{\sigma_k}\int_\Omega a\langle\nabla\psi_k\,, \nabla\dot{u}_a\rangle = \frac{\eta_k(h)}{\sigma_k}. \tag{3.25}$$

Since $\dot{u}_a \in W_0^{1,2}(\Omega)$ we have, in $W_0^{1,2}(\Omega)$, the decomposition

$$\dot{u}_a = \sum_{k=1}^{\infty}\frac{\eta_k(h)}{\sigma_k}\psi_k. \tag{3.26}$$

As a consequence, to ensure a decomposition of the form (3.19) it suffices to find, for K large enough, an $h \in L^2(\tilde{\omega})$ such that

- h is radially symmetric,
- $\|h\|_{L^2(\tilde\omega)} = 1$,
- For any $k = 1, \dots, K-1$, $\eta_k(h) = 0$,
- There holds $\int_\Omega h 1_{\tilde\omega} = \int_{\tilde\omega} h = 0$.

We define $L^2_{\rm rad}(\tilde\omega)$ as the space of radially symmetric functions in $L^2(\tilde\omega)$. Let us first note that for any $k \in \{1, \dots, K\}$ the linear maps $\eta_k := h \mapsto \eta_k(h)$ are continuous on $L^2_{\rm rad}(\tilde\omega)$. This continuity property is a consequence of the radial symmetry assumption on the coefficients, which from Lemma 11 implies that $\nabla u_a \in L^\infty(\Omega)$. Indeed, we can then simply write

$$\begin{aligned}
|\eta_k(h)| &\leqslant \|\nabla u_a\|_{L^\infty(\Omega)} \int_\Omega |h| \cdot |\nabla \psi_k| \\
&\leqslant \|\nabla u_a\|_{L^\infty(\Omega)} \|\nabla \psi_k\|_{L^2(\Omega)} \|h\|_{L^2(\Omega)} \\
&= \|\nabla u_a\|_{L^\infty(\Omega)} \|\nabla \psi_k\|_{L^2(\Omega)} \|h\|_{L^2(\tilde\omega)},
\end{aligned}$$

whence the continuity.

Defining $R_0 \in L^2_{\rm rad}(\tilde\omega)'$ as

$$R_0(h) := \int_{\tilde\omega} h,$$

which is obviously continuous on $L^2(\tilde\omega)$ we are hence looking for h_k such that

$$\|h_K\|_{L^2(\tilde\omega)} , h_K \in \ker(R_0) \cap \left(\bigcap_{k=1}^{K-1} \ker(\eta_k) \right). \tag{3.27}$$

However, $L^2_{\rm rad}(\tilde\omega)$ is an infinite dimensional Hilbert space, $\ker(R_0)$ and $\bigcap_{k=1}^{K-1} \ker(\eta_k)$ are closed subspaces of finite co-dimension, hence $\ker(R_0) \cap \left(\bigcap_{k=1}^{K-1} \ker(\eta_k) \right)$ has finite co-dimension. In particular it is non empty, so there exists h_K such that (3.27) holds. The conclusion follows.

□

4. Proof of Theorem III

Proof of Theorem III. For the proof of the parabolic Talenti inequalities we follow the main ideas of [29, Theorem 2.1] and of [7]. Since the proof is very similar we mostly present the main steps. To alleviate notations, we simply write $u_{a,f}$ for $u_{{\rm parab},a,f}$. For a fixed $a \in \mathcal{M}(\Omega; T)$, we define, for almost every $t \in (0; T)$, $\tilde a(t, \cdot)$ as the Alvino-Trombetti rearrangement of $a(t, \cdot)$ with respect to $u_{a,f}(t, \cdot)$. In other words, for almost every $t \in (0; T)$ and almost every $s \in (0; \|u_{a,f}(t, \cdot)\|_{L^\infty})$,

$$\int_{\{u^*_{a,f}(t,\cdot) \geqslant s\}} \frac{1}{\tilde a(t, \cdot)} = \int_{\{u_{a,f}(t,\cdot) \geqslant s\}} \frac{1}{a(t, \cdot)}.$$

From [7, Proof of Lemma 1.2] we have, with $S_d = d{\rm Vol}(\mathbb{B}(0, 1))^{\frac{1}{d}}$,

$$S_d^2 \mu_{u_{a,f}}(t, s)^{2-\frac{2}{d}} \leqslant \left(-\frac{d}{ds} \int_{\{u_{a,f}(t,\cdot) > s\}} \frac{1}{a} \right) \left(-\frac{d}{ds} \int_{\{u_{a,f}(t,\cdot) > s\}} a |\nabla u_{a,f}|^2 \right).$$

Let $\overline{a}$ is the one-dimensional counterpart of $\tilde{a}$ (i.e., $\tilde{a} = \overline{a}(c_d|\cdot|^d)$). Then, as in [7], there holds, almost everywhere,

$$-\frac{d}{ds}\int_{\{u_{a,f}(t,\cdot)>s\}} \frac{1}{a} = -\frac{\frac{\partial \mu_{u_{a,f}}}{\partial s}}{\overline{a}}(t,s).$$

On the other hand the same arguments as in [3, Proof of Theorem 1] (see also [36]) show that, almost everywhere

$$-\frac{d}{ds}\int_{\{u_{a,f}(t,\cdot)>s\}} a|\nabla u_{a,f}|^2 = \int_{\{u_{a,f}(t,\cdot)>s\}} \left(f - \frac{\partial u_{a,f}}{\partial t}\right).$$

We can hence conclude that

$$S_d^2 \overline{a}\mu_{u_{a,f}}(t)^{2-\frac{2}{d}} \leqslant -\frac{\partial \mu_{u_{a,f}}}{\partial s}(t,s)\int_{\{u_{a,f}>t\}} \left(f - \frac{\partial u_{a,f}}{\partial t}\right). \tag{4.1}$$

We now rewrite

$$\int_{\{u_{a,f}(t,\cdot)>s\}} \frac{\partial u_{a,f}}{\partial t} = \int_0^{\mu_{u_{a,f}}(t,s)} \frac{\partial u^\#_{a,f}}{\partial t}.$$

Introducing as in [29] the function k defined as

$$k(t,\xi) := \int_0^\xi u^\#(t,\cdot)$$

we hence obtain

$$\int_{\{u_{a,f}(t,\cdot)>s\}} \frac{\partial u_{a,f}}{\partial t} = \frac{\partial k}{\partial t}(t,\mu(t,s)). \tag{4.2}$$

By the Hardy-Littlewood inequality we have

$$\int_{\{u_{a,f}>t\}} f \leqslant \int_0^{\mu_{u_{a,f}}(t,s)} f^\#. \tag{4.3}$$

Combining these estimates we are left with

$$1 \leqslant -S_d^{-2}\overline{a}^{-1}\mu_{u_{a,f}}(t,s)^{-2+\frac{2}{d}}\frac{\partial \mu_{u_{a,f}}}{\partial s}(t,s)\left(\int_0^{\mu_{u_{a,f}}(t,s)} f^\# - \frac{\partial k}{\partial t}(t,\mu(t,s))\right) \tag{4.4}$$

which, after integration, gives

$$0 \leqslant -\frac{\partial^2 k}{\partial \xi^2} \leqslant S_d^{-2}\overline{a}^{-1}\xi^{-2+\frac{2}{d}}\left(\int_0^\xi f^\# - \frac{\partial k}{\partial t}(t,\xi)\right). \tag{4.5}$$

We denote by k_* the function obtained by replacing $u_{a,f}$ by $u_{\tilde{a},f^*}$ in the definition of k. Since all the previous inequalities become equalities in this case it follows that the function $K := k - k^*$ satisfies

$$\begin{cases} \frac{\partial k}{\partial t} - S_d^2\overline{a}\xi^{2-\frac{2}{d}}\frac{\partial^2 K}{\partial^2 \xi^2} \leqslant 0 & \text{in } (0;\mathrm{Vol}(\Omega))\times(0;T) \\ K(0,\cdot) = 0\,, \\ K(t,0) = 0 = \frac{\partial K}{\partial \xi}(t,\mathrm{Vol}(\Omega)). \end{cases} \tag{4.6}$$

From the maximum principle, we have $K \leqslant 0$, so that the conclusion follows. If j_1 and j_2 are convex non-decreasing functions, the second conclusion of the theorem follows from [4, Proposition 2].

□

5. Conclusions and open problems

In this paper, we have undertaken the study of certain non-energetic two-phase optimisation problems. Of course, our results are partial, and we now present some open problems that we think are worth investigating.

Open problem I: rearrangements for the time-independent case The first crucial question has to do with the parabolic problem. Indeed, since the Alvino-Trombetti rearrangement we use is defined differently for every time t, the question of time-independent a remains completely open, and we believe it may be fruitful to investigate in the future.

Open problem II: possible relaxations of the problem, bang-bang property for the parabolic optimisation problem The second problem has to do with the conclusion of Theorem I. A more satisfying conclusion that we could not reach would have been a weak bang-bang property, namely that, a profile $a \in \mathcal{A}(\Omega)$ being given, there exists $\tilde{a} \in \mathfrak{M}(\Omega)$ that improves the criterion. Usually, this type of property is obtained using the convexity or concavity of the functional. However, here, what we proved in Theorem II was that the second-order derivative of the functional is positive on an infinite dimensional subspace of the space of admissible perturbations. It is unclear whether this weaker information may be sufficient.

Open problem III: Robin boundary conditions Finally, let us note that, following the recent progresses in the study of Robin Talenti inequalities [5], it may be very interesting to try and understand which type of rearrangement of the weight a may be suitable to obtain Talenti inequalities for two-phases problems under Robin boundary conditions.

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Conflict of interest

The author declares no conflict of interest.

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