http://www.aimspress.com/journal/mine

Mathematics in Engineering, 4(6): 1–14.
DOI:10.3934/mine.2022048
Received: 01 January 2021
Accepted: 23 July 2021
Published: 10 November 2021

Research article

On the uniqueness of mild solutions for the parabolic-elliptic Keller-Segel system in the critical L^p-space†

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† **This contribution is part of the Special Issue:** Advances in the analysis of chemotaxis systems
Guest Editor: Michael Winkler
Link: www.aimspress.com/mine/article/6067/special-articles

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Abstract: We are concerned with the uniqueness of mild solutions in the critical Lebesgue space $L^{\frac{n}{2}}(\mathbb{R}^n)$ for the parabolic-elliptic Keller-Segel system, $n \geq 4$. For that, we prove the bicontinuity of the bilinear term of the mild formulation in the critical weak-$L^{\frac{n}{2}}$ space, without using Kato time-weighted norms, time-spatial mixed Lebesgue norms (i.e., $L^q((0,T);L^p)$-norms with $q \neq \infty$), and any other auxiliary norms. Our proofs are based on Yamazaki's estimate, duality and Hölder's inequality, as well as an adapted Meyer-type argument. Since they are different from those of Kozono, Sugiyama and Yahagi [J. Diff. Eq. 253 (2012)] and it is not clear whether mild solutions are weak solutions in the critical $C([0,T);L^{\frac{n}{2}})$, our results complement theirs in a twofold way. Moreover, the bilinear estimate together heat semigroup estimates yield a well-posedness result whose dependence with respect to the decay rate γ of the chemoattractant is also analyzed.

Keywords: Keller-Segel system; uniqueness; critical spaces; bilinear estimates; Lorentz spaces

1. Introduction

We are concerned with the parabolic-elliptic Keller-Segel (or Patlak-Keller-Segel) system

$$\begin{cases} \partial_t u = \nabla \cdot (\nabla u - u\nabla v), & \text{in} \quad x \in \mathbb{R}^n,\ t \in (0,T), \\ -\triangle v + \gamma v = \kappa u, & \text{in} \quad x \in \mathbb{R}^n,\ t \in (0,T), \\ u|_{t=0} = u_0 \geq 0, & \text{in} \quad x \in \mathbb{R}^n, \end{cases} \tag{1.1}$$

where $0 < T \leq \infty$, $u(x,t) \geq 0$ represents the density of cells and $v(x,t) \geq 0$ is the concentration of the chemoattractant. The parameters $\gamma \geq 0$ and $\kappa > 0$ denote the decay and production rate of the chemoattractant, respectively. The model works for $n \geq 2$ but, as will be seen later, we restrict themselves to $n \geq 4$ due to technical issues.

System (1.1) is a chemotaxis model with a mathematical structure of parabolic-elliptic type. It is used in the study of aggregation of biological species, describing the behavior of organisms (e.g., bacteria) that move towards high concentration of a chemical secreted by themselves.

In view of $(1.1)_2$, we can express $v = \kappa(-\Delta + \gamma I)^{-1}u$ and, according to Duhamel's principle, the Cauchy problem (1.1) can be formally converted to the integral equation

$$u(t) = G(t)u_0 + B(u,u)(t), \tag{1.2}$$

where $G(t) = e^{\Delta t}$ stands for the heat semigroup and the bilinear term B is given by

$$B(u,w)(t) = -\kappa \int_0^t \nabla_x \cdot G(t-s)\left[u\nabla_x(-\Delta + \gamma I)^{-1}w\right](s)ds. \tag{1.3}$$

Functions $u(x,t)$ satisfying (1.2) are called mild solutions for (1.1). Here the mild formulation (1.2) is considered in a suitable dual sense, see Section 3 for details.

For $\gamma = 0$, Eqs. $(1.1)_1$–$(1.1)_2$ has the scaling

$$u(x,t) \to \lambda^2 u(\lambda x, \lambda^2 t), \tag{1.4}$$

which, for the initial data, induces

$$u_0(x) \to \lambda^2 u_0(\lambda x). \tag{1.5}$$

Spaces invariant under the scaling (1.5), namely $\|u_0\|_X \approx \left\|\lambda^2 u_0(\lambda x)\right\|_X$ for all $\lambda > 0$, are called critical spaces for (1.1).

In the dimension $n = 2$, it is well-known that there exists a threshold value for the initial mass $M = \int u_0 dx$ that decides if solutions exist globally ($M < 8\pi/\kappa$) or blow up in a finite time ($M > 8\pi/\kappa$) (see, e.g., [6,7]). Note that the space $L^1(\mathbb{R}^2)$ is critical for (1.1). For $n \geq 3$, one might wonder if some critical space could play a similar role as the L^1-space in $n = 2$ (for example, $L^{\frac{n}{2}}(\mathbb{R}^n)$), however, it is still an open problem to know whether there exists such a suitable space. In connection with that, in dimensions $n \geq 3$, there is a huge literature about existence of mild solutions for (1.1) and its parabolic-parabolic version with smallness conditions on the initial data in critical spaces. Without making a complete list, we mention the results in $L^1 \cap L^{\frac{n}{2}}$ [10], $L^{\frac{n}{2}}$ [22] (weak solutions), Marcinkiewicz $L^{\frac{n}{2},\infty}$ (weak$-L^p$ spaces) [23], $\mathcal{PM}^{n-2}$ [3], Besov $\dot{B}^{\frac{n}{q}-2}_{q,\infty}$ [18], Triebel-Lizorkin $\dot{F}^{-2}_{\infty,2}$ [19], Morrey $\mathcal{M}_{q,n-2q}$ [2], Fourier-Besov $F\dot{B}^{-2}_{1,r}$ [18], Besov-Morrey $\mathcal{N}^{\frac{n-\mu}{q}-2}_{q,\mu,\infty}$ [13], and Fourier-Besov-Morrey spaces $\mathcal{FN}^{n-2-\frac{n-\mu}{q}}_{q,\mu,\infty}$ [9], among others. It is worth noting that most of the above existence results of small mild solutions in critical spaces were inspired by those for Navier-Stokes equations, see, e.g., [1,11,15,20,21,24,25,30], and their references.

On the other hand, the uniqueness in critical spaces X is more subtle and needs some care. For $n \geq 3$, most of the above existence results are proved by constructing a fixed point argument in time-dependent spaces with norms composed of two ou more parts. One is the norm of the persistence space $L^\infty((0,\infty);X)$ and the others are auxiliary norms such as Kato time-weighted type norms, time-spatial mixed Lebesgue norms (i.e., $L^q((0,T);L^p)$-norms or, more generally, $L^q((0,T);Y)$ with $q \neq \infty$) and Chemin-Lerner type norms, which are used to control the bilinear term $B(u,w)$. Also, solutions are continuous at $t > 0$ but only time-weakly continuous at $t = 0^+$, since the heat semigroup $\{e^{t\Delta}\}_{t\geq 0}$ is not strongly continuous at $t = 0^+$. This lack of continuity can be overcome by considering either the maximal subspace $\tilde{X}$ in which $\{e^{t\Delta}\}_{t\geq 0}$ is continuous or the closure of $C_0^\infty(\mathbb{R}^n)$ in X, and then solutions belong to $C([0,T);\tilde{X})$ with large initial data $u_0 \in \tilde{X}$ and small $T > 0$. The estimates involving auxiliary norms in the proof of existence results, in principle, provide only a conditional uniqueness result, that is, uniqueness in a space more restricted than the natural one $C([0,T);\tilde{X})$. For the sake of completeness, in the case $n = 2$ we would like to mention the uniqueness results of weak/mild solutions for (1.1) (and its parabolic-parabolic version) with finite mass, finite second moment and finite entropy (see, e.g., [8, 12, 27] and their references).

A way to obtain unconditional uniqueness in the critical class $C([0,T);\tilde{X})$ (or uniqueness of small solutions in $L^\infty((0,\infty);X)$) is to prove the bilinear estimate

$$\|B(u,w)\|_{L^\infty((0,T);X)} \leq C\,\|u\|_{L^\infty((0,T);X)}\,\|w\|_{L^\infty((0,T);X)}\,, \tag{1.6}$$

where $C > 0$ is a constant. This approach has already been employed in the context of Navier-Stokes equations. For example, see [11, 14, 25, 26, 28, 30] to results in the framework of critical Lebesgue, Marcinkiewicz, Morrey and weak-Morrey spaces.

Next, let us discuss in more detail the works [22] and [23], which are more directly related to our results. In [23], Kozono-Sugiyama proved local well-posedness of mild solutions for (1.1) with small data $u_0 \in L^{\frac{n}{2},\infty}$ and $n \geq 3$, where the existence and uniqueness are obtained in the class

$$u \in BC((0,T);L^{\frac{n}{2},\infty}) \text{ and } t^\beta u \in BC((0,T);L^q) \text{ with } \frac{n}{2} < q < n, \tag{1.7}$$

where u is time-weakly continuous at $t = 0^+$ and $\beta = 1 - \frac{n}{2q}$. Also, $u \in BC([0,T);L^{\frac{n}{2}})$ provided that $u_0 \in L^{\frac{n}{2}} \hookrightarrow L^{\frac{n}{2},\infty}$. They employed a point-fixed argument by using Kato's approach (see [20]) and their bilinear estimates are performed with the help of the auxiliary time-weighted norm $\sup_{t\in(0,T)} t^\alpha \|u(\cdot,t)\|_{L^q}$. The solution u is global ($T = \infty$) if $\|u_0\|_{L^{\frac{n}{2},\infty}}$ is small enough. For the uniqueness, besides assuming (1.7), it is required a smallness condition in the auxiliary norm, that is, the uniqueness is obtained in a space more restricted than $BC((0,T);L^{\frac{n}{2},\infty})$. Moreover, assuming additional regularity on the initial data, they obtained the existence of local (or small global) strong L^p-solutions.

In [22], Kozono-Sugiyama-Yahagi proved the existence of local weak solutions u with respect to a suitable set of test functions. More precisely, for initial data $u_0 \in L^{\frac{n}{2}}(\mathbb{R}^n)$, they obtained local-in-time solutions in

$$BC([0,T);L^{\frac{n}{2}}(\mathbb{R}^n)) \cap L^q((0,T);L^p(\mathbb{R}^n)),$$

where $T > 0$ is small enough, $n \geq 3$, $2/q + n/p = 2$, $2 < q < \infty$, $n/2 < p < n$, and $p \leq n^2/2(n-2)$. These weak solutions can be extended globally ($T = \infty$) if $\|u_0\|_{L^{\frac{n}{2}}}$ is small enough. They constructed solutions via an approximation scheme of strong solutions whose existence was obtained in [23]. The uniqueness was obtained in the class $L^q((0,T);L^p(\mathbb{R}^n))$ with $n \geq 3$, $2/q + n/p = 2$ and $n/2 < p < n$. For $n \geq 4$, the uniqueness in the natural persistence space $C([0,T);L^{\frac{n}{2}}(\mathbb{R}^n))$ was obtained without any further condition, while the uniqueness in $L^\infty((0,T);L^{\frac{n}{2}}(\mathbb{R}^n))$ required small conditions on u and u_0. For that, Kozono-Sugiyama-Yahagi converted the uniqueness problem to the one of obtaining global strong solutions for the associated adjoint equation, where coefficients depend on weak solutions, by using maximal regularity of the heat equation and suitable estimates involving $L^q((0,T);L^p(\mathbb{R}^n))$-norms with $q \neq \infty$.

As pointed out by the authors of [22], it is not clear whether mild solutions satisfy their weak formulation. So, a natural question is to know if we have unconditional uniqueness of mild solutions for (1.1) in $C([0,T);L^{\frac{n}{2}}(\mathbb{R}^n))$, as well as the uniqueness in $L^\infty((0,T);L^{\frac{n}{2}}(\mathbb{R}^n))$ with $\sup_{t\in(0,T)}\|u(\cdot,t)\|_{L^{\frac{n}{2}}}$ small enough. Another question is to know if it is possible to obtain these uniqueness properties (and also existence and continuous dependence on initial data) without using auxiliary norms such as, for example, those in [22] and [23].

In this paper we obtain positive answers for the above questions for $n \geq 4$. First, inspired by [30], we prove estimate (1.6) with $X = L^{\frac{n}{2},\infty}(\mathbb{R}^n)$ by means of careful estimates on the predual space of $L^{\frac{n}{2},\infty}(\mathbb{R}^n)$, that is, the Lorentz space $L^{\frac{n}{n-2},1}(\mathbb{R}^n)$ (see Section 2 for the definition). So, adapting arguments found in [28], we obtain the unconditional uniqueness of mild solutions in the class $C([0,T);L^{\frac{n}{2}}(\mathbb{R}^n))$ with $0 < T \leq \infty$ and initial data $u_0 \in L^{\frac{n}{2}}(\mathbb{R}^n)$ (see Theorem 3.1), because $L^{\frac{n}{2}}(\mathbb{R}^n)$ is contained in $\tilde{X}$ (maximal subspace of $L^{\frac{n}{2},\infty}(\mathbb{R}^n)$ where $\{e^{t\Delta}\}_{t\geq 0}$ is continuous). Moreover, using heat semigroup estimates and (1.6), we have the well-posedness of small solutions in $L^\infty((0,T);L^{\frac{n}{2},\infty}(\mathbb{R}^n))$ with $0 < T \leq \infty$. Since we have the continuous inclusion $L^{\frac{n}{2}} \hookrightarrow L^{\frac{n}{2},\infty}$, the uniqueness of mild solutions in $L^\infty((0,T);L^{\frac{n}{2}}(\mathbb{R}^n))$ holds true provided that $\sup_{t\in(0,T)}\|u(\cdot,t)\|_{L^{\frac{n}{2}}}$ is small enough. Furthermore, we analyze the dependence of the well-posedness result with respect to the decay rate γ of the chemoattractant (see Remark 3.2).

Finally, we observe that our results work well by considering non-negative u_0 and u in (1.1) as well as without any sign restrictions on them. However, we have considered the former for physical reasons associated to the model.

This paper is organized as follows. In Section 2, we give some preliminaries about Lorentz spaces and, in particular, weak-L^p spaces. Section 3 is dedicated to the statements of our results and some further remarks. The proofs of results are performed in Section 4.

2. Preliminaries

This section is devoted to recalling some basic definitions and properties about Lorentz spaces.

Let $|\cdot|$ stands for the Lebesgue measure and let $\lambda_f(s) = |\{x \in \mathbb{R}^n : |f(x)| > s\}|$ be the distribution function of a measurable function $f : \mathbb{R}^n \to \mathbb{R}$. The rearrangement of f is defined as

$$f^*(t) = \inf\{s > 0 : \lambda_f(s) \leq t\}, \text{ for } t > 0. \tag{2.1}$$

The Lorentz space is the complete quasi-normed space

$$L^{p,d}(\mathbb{R}^n) = \left\{f : \mathbb{R}^n \to \mathbb{R} \text{ measurable};\ \|f\|^*_{p,d} < \infty\right\}, \tag{2.2}$$

where the quasi-norm $\|f\|^*_{p,d}$ is given by

$$\|f\|^*_{p,d} = \begin{cases} \left[\int_0^\infty \left(t^{\frac{1}{p}}[f^*(t)]\right)^d \frac{dt}{t}\right]^{\frac{1}{d}}, & 0 < p < \infty, 1 \leq d < \infty \\ \sup\limits_{t>0} t^{\frac{1}{p}}[f^*(t)], & 0 < p \leq \infty,\ d = \infty. \end{cases} \tag{2.3}$$

Taking $d = p$ and $d = \infty$ in (2.2), we obtain the Lebesgue space $L^{p,p}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ and the Marcinkiewicz space $L^{p,\infty}$, also called weak-L^p, respectively. The case $p = \infty$ and $1 \leq d < \infty$ was removed from (2.3) because $L^{\infty,d}$ is the trivial space.

In general, the quantity (2.3) is not a norm on $L^{p,d}$. Consider the double-rearrangement

$$f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)ds,$$

and define the quantity $\|\cdot\|_{p,d}$ exactly as in (2.3) but replacing f^* with f^{**}. For $1 < p \leq \infty$, the quantities $\|\cdot\|_{p,d}$ and $\|\cdot\|_{p,d}^*$ are topologically equivalent on $L^{p,d}$, since we have

$$\|\cdot\|_{L^{p,d}}^* \leq \|\cdot\|_{L^{p,d}} \leq \frac{p}{p-1}\|\cdot\|_{L^{p,d}}^*.$$

The pair $(L^{p,r}, \|\cdot\|_{p,d})$ is a Banach space and, unless mentioned otherwise, we consider it when $1 < p \leq \infty$. For $0 < p \leq 1$, $L^{p,d}$ is endowed with $\|\cdot\|_{p,d}^*$. In the case $p = d = 1$ we have $\|\cdot\|_{1,1}^*$ coincides with the standard L^1-norm and $L^{1,1} = L^1$.

Lorentz spaces have the scaling property

$$\|f(\lambda x)\|_{p,d} = \lambda^{-\frac{n}{p}}\|f(x)\|_{p,d}. \tag{2.4}$$

For $1 \leq d_1 \leq p \leq d_2 \leq \infty$ and $1 \leq p \leq \infty$, we have the continuous strict inclusions

$$L^{p,1} \subset L^{p,d_1} \subset L^p \subset L^{p,d_2} \subset L^{p,\infty} \tag{2.5}$$

and then $L^{p,d}$ becomes larger as the second index d goes from 1 to ∞.

Next we recall the Hölder inequality in Lorentz spaces (see [17, 29]). Let $1 < p_1, p_2, p_3 \leq \infty$ and $1 \leq d_1, d_2, d_3 \leq \infty$ be such that $1/p_3 = 1/p_1 + 1/p_2$ and $1/d_1 + 1/d_2 \geq 1/d_3$. Then, there exists a constant $C > 0$ (independent of f and g) such that

$$\|fg\|_{p_3,d_3} \leq C\|f\|_{p_1,d_1}\|g\|_{p_2,d_2}. \tag{2.6}$$

For $1 \leq p, d < \infty$ (with $d = 1$ when $p = 1$), we have that the dual space of $L^{p,d}$ is $L^{p',d'}$(see [16]). In particular, the dual of $L^{p,1}$ is $L^{p',\infty}$ for $1 \leq p < \infty$. The space of compactly-supported smooth functions $C_0^\infty(\mathbb{R}^n)$ is dense in $L^{p,d}(\mathbb{R}^n)$ for $1 \leq p, d < \infty$.

Young inequality works well in the framework of Lorentz spaces. In fact, if $1 < p_1, p_2, p_3 \leq \infty$ and $1 \leq d_1, d_2, d_3 \leq \infty$ with $1/p_3 = 1/p_1 + 1/p_2 - 1$ and $1/d_1 + 1/d_2 \geq 1/d_3$, then (see [29])

$$\|f * g\|_{p_3,d_3} \leq C\|f\|_{p_1,d_1}\|g\|_{p_2,d_2}, \tag{2.7}$$

where $C > 0$ is a universal constant. Moreover, for $p_1 = 1$ and $1 < p = p_2 = p_3 \leq \infty$, we have the inequality (see [4])

$$\|f * g\|_{p,\infty} \leq p^{\frac{p+1}{p}}(p-1)^{-1}\|f\|_{L^1}\|g\|_{p,\infty}. \tag{2.8}$$

Lorentz spaces has nice interpolation properties. For that, recall that the functor $(\cdot,\cdot)_{\theta,r}$ constructed via the $K_{\theta,q}$-method is exact of exponent θ on the categories of quasi-normed and normed spaces. Let

$0 < p_1 < p_2 \leq \infty,\ 0 < \theta < 1,\ \frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}$ and $1 \leq d_1, d_2, d \leq \infty$. Employing $(\cdot,\cdot)_{\theta,r}$ in Lorentz spaces, we obtain (see [5, Chapter 5])

$$\left(L^{p_1,d_1}, L^{p_2,d_2}\right)_{\theta,d} = L^{p,d}. \tag{2.9}$$

For $1 < p \leq \infty$ and $1 \leq d \leq \infty$, by interpolating (2.8), we arrive at the inequality

$$\|f * g\|_{p,d} \leq C\|f\|_{L^1}\|g\|_{p,d}, \tag{2.10}$$

where $C > 0$ is a universal constant.

3. Main results

In this section we state the bilinear estimate (1.6) in weak- $L^{\frac{n}{2}}$ and uniqueness result. Also, we make some comments about global existence and its dependence on the decay rate γ of the chemoattractant, non-negativity and mass conservation.

Before proceeding, we point out that the mild formulation (1.2) and its bilinear term should be meant in a suitable dual sense in the $L^{\frac{n}{2},\infty}$-setting by using its predual space $L^{\frac{n}{n-2},1}$ and the duality pairing $\langle f, g\rangle = \int_{\mathbb{R}^n} fg dx$ (see [30]). More precisely, for $u, w \in L^\infty((0,T); L^{\frac{n}{2},\infty})$ we define $B(u,w)$ by

$$\langle B(u,w), \phi\rangle = -\kappa \int_0^t \left\langle \left(u\nabla_x(-\Delta + \gamma I)^{-1}w\right), \nabla_x G(t-s)\phi \right\rangle ds, \tag{3.1}$$

for all $\phi \in L^{\frac{n}{n-2},1}(\mathbb{R}^n)$ and a.e. $t \in (0,T)$. Note also that, for $u_0 \in L^{\frac{n}{2},\infty}(\mathbb{R}^n)$, the convolution $G(t)u_0$ is well-defined and

$$\langle G(t)u_0, \phi\rangle = \langle u_0, G(t)\phi\rangle\,,\ \text{for all } \phi \in L^{\frac{n}{n-2},1}(\mathbb{R}^n). \tag{3.2}$$

Thus, the formulation (1.2) should be meant as

$$\langle u(\cdot,t), \phi\rangle = \langle u_0, G(t)\phi\rangle - \kappa \int_0^t \left\langle \left[u\nabla_x(-\Delta + \gamma I)^{-1}u\right](s), \nabla_x G(t-s)\phi \right\rangle ds,$$

for all $\phi \in L^{\frac{n}{n-2},1}(\mathbb{R}^n)$ and a.e. $t \in (0,T)$. In other words, taking into account (3.2), u verifies (1.2) with $B(u,u)$ given by (3.1).

Our results read as follows.

Theorem 3.1. *Let $n \geq 4, \gamma \geq 0, \kappa > 0$ and $0 < T \leq \infty$.*

(i) (Bilinear estimate) Let $B(\cdot,\cdot)$ be the bilinear form (1.3). There exists a constant $K > 0$ (independent of γ) such that

$$\|B(u,w)\|_{L^\infty((0,T);L^{\frac{n}{2},\infty})} \leq \kappa g(\gamma) K \|u\|_{L^\infty((0,T);L^{\frac{n}{2},\infty})} \|w\|_{L^\infty((0,T);L^{\frac{n}{2},\infty})}\,, \tag{3.3}$$

for all $u, v \in L^\infty((0,T); L^{\frac{n}{2},\infty}(\mathbb{R}^n))$, where $g(\gamma) = 1$ and $g(\gamma) = \gamma^{-(n-1)}$ if $\gamma = 0$ and $\gamma > 0$, respectively.

(ii) (Uniqueness) Let u and w be mild solutions of (1.1) in the class $C([0,T); L^{\frac{n}{2}}(\mathbb{R}^n))$ with the same initial data u_0. Then $u(\cdot,t) = w(\cdot,t)$ in $L^{\frac{n}{2}}(\mathbb{R}^n)$, for all $t \in [0,T)$.

Remark 3.2. *(Further comments)*

(i) *(The restriction $n \geq 4$) Due to the duality arguments in our proofs, we need to employ Lemma 4.3 with $p = \frac{n}{2}$ and $r = \frac{n}{3} > 1$, and then $n \geq 4$. In turn, we need $r > 1$ for Hölder's inequality (2.6) and the quantity $\|\cdot\|_{r,\infty}$ in $L^{r,\infty}$ to be a norm. An interesting question would be to know whether, with some suitable relaxed integrability conditions, those arguments could be adapted to obtain a similar uniqueness result in lower dimensions.*

(ii) *(Well-posedness) Let $0 < T \leq \infty$, $\gamma \geq 0$, $n \geq 4$ and $u_0 \in L^{\frac{n}{2},\infty}(\mathbb{R}^n)$. Under a small assumption on $\|u_0\|_{\frac{n}{2},\infty}$ and a fixed-point argument, Theorem 3.1 together with heat semigroup estimates (see (4.1)) imply the well-posedness of small mild solutions in the class $L^\infty((0,T); L^{\frac{n}{2},\infty}(\mathbb{R}^n))$. In particular, we obtain the uniqueness of sufficiently small mild solutions in $L^\infty((0,T); L^{\frac{n}{2},\infty}(\mathbb{R}^n))$ or, using the continuous inclusion $L^{\frac{n}{2}}(\mathbb{R}^n) \hookrightarrow L^{\frac{n}{2},\infty}(\mathbb{R}^n)$, in $L^\infty((0,T); L^{\frac{n}{2}}(\mathbb{R}^n))$. Moreover, the solution u belongs to $BC((0,T); L^{\frac{n}{2},\infty}(\mathbb{R}^n))$ with time-weak continuity at $t = 0^+$. For $u_0 \in L^{\frac{n}{2}}(\mathbb{R}^n)$, one obtains $u \in BC([0,T); L^{\frac{n}{2}}(\mathbb{R}^n))$ and the smallness condition on u_0 can be replaced with a smallness one on the existence-time $T > 0$, and $u \in C([0,T); L^{\frac{n}{2}}(\mathbb{R}^n))$ can be large. For $\gamma = 0$ and $T = \infty$, the obtained solution is self-similar provided that u_0 is homogeneous of degree -2.*

(iii) *(Non-negativity and mass conservation) Due to the fixed-point argument, the solution u in item (ii) of this remark can be obtained as the limit of the Picard sequence $u^{(k+1)} = u^{(1)} + B(u^{(k)}, u^{(k)})$, $k \in \mathbb{N}$, and $u^{(1)} = G(t)u_0$. Let $u_0 \in L^{\frac{n}{2},\infty}(\mathbb{R}^n)$ be non-negative. Using the parabolic regularization of the heat semigroup (see, e.g., (4.1)), an induction procedure, and the divergence structure of $B(\cdot,\cdot)$, one can show that $u^{(k)}$ is smooth and non-negative, for each k. Since the convergence $u^{(k)} \to u$ in $L^\infty((0,T); L^{\frac{n}{2},\infty}(\mathbb{R}^n))$ preserves non-negativity, it follows that u is non-negative. Furthermore, for $u_0 \in L^1_+(\mathbb{R}^n) \cap L^{\frac{n}{2},\infty}(\mathbb{R}^n)$, using (4.1) and reducing the size of $\|u_0\|_{\frac{n}{2},\infty}$ (if necessary), one can show further integrability properties and polynomial time decay of L^p-norms for $u(\cdot,t)$ and $B(\cdot,t)$, for $t > 0$, and then obtain $u(\cdot,t) \in L^1_+$ and $B(u,u)(\cdot,t) \in L^1$. After, using the divergence form of $B(\cdot,\cdot)$, one can obtain the mass conservation of the solution, that is, $\int_{\mathbb{R}^n} u(\cdot,t)dx = \int_{\mathbb{R}^n} u_0 dx$, for $t > 0$.*

(iv) *(Large decay rate of the chemoattractant) Let $0 < T \leq \infty$. Considering $\gamma > 0$ large enough, we can make $\kappa g(\gamma)K$ small enough and then obtain the well-posedness of mild solutions for (1.1) in $L^\infty((0,T); L^{\frac{n}{2},\infty}(\mathbb{R}^n))$, without smallness conditions on the existence-time T and initial data u_0. More precisely, in order to employ a fixed-point argument, we need $4\|u_0\|_{\frac{n}{2},\infty}(\kappa g(\gamma)K) < 1$ which leads us to*

$$\gamma > \left(4\kappa K \|u_0\|_{\frac{n}{2},\infty}\right)^{\frac{1}{n-1}}.$$

4. Proof of Theorem 3.1

4.1. Proof of item (i) (Bilinear estimate)

We start with a lemma that will be useful to handle the coupling operator in (1.1) in Lorentz spaces.

Lemma 4.1. *Let $n \geq 2$, $1 < p < n$, $1 \leq d \leq \infty$ and $\frac{1}{q} = \frac{1}{p} - \frac{1}{n}$. The operator $L_j = \partial_j(-\Delta + \gamma I)^{-1}$ is continuous from $L^{p,d}(\mathbb{R}^n)$ to $L^{q,d}(\mathbb{R}^n)$, for each $j = 1, 2, ..., n$. Moreover, for $\gamma > 0$, there exists a constant $C > 0$ (independent of f and γ) such that*

$$\left\|L_j f\right\|_{q,d} \leq C\gamma^{-(n-1)} \|f\|_{p,d}.$$

Proof. We can write the multiplier operator L_j as

$$L_j f = \mathcal{K}_{j,\gamma} * f, \text{ where } \widehat{\mathcal{K}}_{j,\gamma}(\xi) = \frac{-i\xi_j}{|\xi|^2 + \gamma}.$$

Taking $\gamma = 0$ and $\gamma = 1$, we have that

$$\mathcal{K}_{j,0} = \left(\frac{-i\xi_j}{|\xi|^2}\right)^{\vee} \in L^{\frac{n}{n-1},\infty} \text{ and } \mathcal{K}_{j,1} = \left(\frac{-i\xi_j}{|\xi|^2+1}\right)^{\vee} \in L^{\frac{n}{n-1},\infty}.$$

For $\gamma = 0$, by using Young's inequality in Lorentz space (2.7) with $p_3 = q$, $p_1 = \frac{n}{n-1}$, and $p_2 = p$, we obtain that

$$\left\|L_j f\right\|_{q,d} = \left\|\mathcal{K}_{j,0} * f\right\|_{q,d} \leq C \left\|\mathcal{K}_{j,0}\right\|_{p_1,\infty} \|f\|_{p,d} = C \|f\|_{p,d}.$$

Next we deal with the case $\gamma > 0$. By a scaling argument, note that $\widehat{\mathcal{K}}_{j,\gamma}(\xi) = \gamma^{-1/2}\widehat{\mathcal{K}}_{j,1}(\gamma^{-1/2}\xi)$ and then

$$\mathcal{K}_{j,\gamma}(x) = \gamma^{-1/2}\gamma^{n/2}\mathcal{K}_{j,1}(\gamma^{1/2}x) = \gamma^{(n-1)/2}\mathcal{K}_{j,1}(\gamma^{1/2}x).$$

Thus, again using (2.7) with the same indexes above, it follows that

$$\begin{aligned}
\left\|L_j f\right\|_{q,d} &= \left\|\mathcal{K}_{j,\gamma} * f\right\|_{q,d} \\
&\leq C \left\|\gamma^{(n-1)/2}\mathcal{K}_{j,1}(\gamma^{1/2}x)\right\|_{p_1,\infty} \|f\|_{p,d} \\
&= C\gamma^{-\frac{n-1}{2}}\gamma^{-\frac{n}{2p_1}} \left\|\mathcal{K}_{j,1}\right\|_{p_1,\infty} \|f\|_{p,d} \\
&= C\gamma^{-(n-1)} \|f\|_{p,d}.
\end{aligned}$$

◇

In the lemma below, we recall some known estimates in Lorentz spaces for the heat semigroup (see [30]).

Lemma 4.2. *(i) Let* $m \in \{0\} \cup \mathbb{N}$, $1 < r \leq p \leq \infty$, *and* $1 \leq d_1, d_2 \leq \infty$. *Then, there exists a constant* $C > 0$ *such that*

$$\|\nabla_x^m G(t)\varphi\|_{p,d_2} \leq Ct^{-\frac{m}{2}-\frac{n}{2}(\frac{1}{r}-\frac{1}{p})}\|\varphi\|_{r,d_1}, \ \text{for all } \varphi \in L^{r,d_1}. \tag{4.1}$$

(ii) (Yamazaki's estimate) Let $1 < r < p < \infty$. *There is a constant* $C > 0$ *such that*

$$\int_0^\infty s^{\frac{n}{2}(\frac{1}{r}-\frac{1}{p})-\frac{1}{2}}\|\nabla_x \cdot G(s)\phi\|_{p,1}\, ds \leq C\|\phi\|_{r,1}, \text{ for all } \phi \in L^{r,1}. \tag{4.2}$$

In the next lemma, by means of a duality argument and (4.2), we provide estimates for the linear operator

$$Q(f)(x) = \int_0^\infty \nabla_x \cdot G(s) f(\cdot, s)\, ds, \tag{4.3}$$

which is linked to "Duhamel structure" of (1.3). Just like (1.3), the operator (4.3) is understood in the sense of duality, as explained in Section 3. Note that the lemma is valid for $n \geq 2$.

Lemma 4.3. *Let $n \geq 2$ and $1 < r < p < \infty$ be such that $\frac{n}{r} - \frac{n}{p} = 1$. There exists a constant $C > 0$ such that*

$$\|Q(f)\|_{p,\infty} \leq C \sup_{t>0} \|f(\cdot, t)\|_{r,\infty}, \tag{4.4}$$

for all $f \in L^\infty((0,\infty), L^{r,\infty})$, where the supremum over $t > 0$ is taken in the essential sense.

Proof. First, denoting the heat kernel by $\varsigma(x,t)$, we have that $G(t)\phi = \varsigma(\cdot,t) * \phi(\cdot)$. A duality argument and Hölder's inequality (2.6) allow us to estimate the $L^{p,\infty}$-norm of $Q(f)$ as

$$\begin{aligned}
\|Q(f)\|_{p,\infty} &\leq C \sup_{\|\phi\|_{L^{p'},1}=1} \left| \int_{\mathbb{R}^n} Q(f)\phi(x)dx \right| \\
&\leq C \sup_{\|\phi\|_{L^{p'},1}=1} \int_{\mathbb{R}^n} \int_0^\infty |((\nabla \cdot g)(x,s) * f(x,s))\phi(x)|\, dsdx \\
&\leq C \sup_{\|\phi\|_{L^{p'},1}=1} \int_0^\infty \int_{\mathbb{R}^n} |(\nabla \cdot g(-x,s) * \phi(x)) f(x,s)|\, dxds \\
&\leq C \sup_{\|\phi\|_{L^{p'},1}=1} \int_0^\infty \|f(\cdot,s)\|_{r,\infty} \|\nabla_x g(x,s) * \phi\|_{r',\infty} ds.
\end{aligned} \tag{4.5}$$

Now note that $1 < p' < r' < \infty$ and that the condition $\frac{n}{r} - \frac{n}{p} = 1$ implies

$$\frac{n}{2}\left(\frac{1}{p'} - \frac{1}{r'}\right) - \frac{1}{2} = \frac{n}{2}\left(\frac{1}{r} - \frac{1}{p}\right) - \frac{1}{2} = 0. \tag{4.6}$$

Using (4.6) and (4.2) in order to estimate the integral in the right-hand side of (4.5), we arrive at

$$\begin{aligned}
\|Q(f)\|_{p,\infty} &\leq C \sup_{\|\phi\|_{L^{p'},1}=1} \int_0^\infty \|f(\cdot,s)\|_{r,\infty}\, s^{\frac{n}{2}(\frac{1}{p'} - \frac{1}{r'}) - \frac{1}{2}} \|g(x,s) * \phi\|_{r',\infty} ds \\
&\leq C \sup_{t>0} \|f(\cdot,t)\|_{r,\infty} \left(C \sup_{\|\phi\|_{L^{p'},1}=1} \|\phi\|_{p',1} \right) \\
&= C \sup_{t>0} \|f(\cdot,t)\|_{r,\infty},
\end{aligned}$$

as required.

◇

Proof of item (i). Let $0 < T \leq \infty$. For each $t \in (0,T)$, consider $f_t(x,s)$ given by

$$\begin{aligned}
&f_t(\cdot,s) = -\kappa(u\nabla(-\Delta + \gamma I)^{-1}w)(\cdot, t-s), \text{ for a.e. } s \in (0,t), \\
&f_t(\cdot,s) = 0, \text{ for } s \in (t,\infty),
\end{aligned}$$

and note that (1.3) can be rewritten as

$$B(u,w)(t) = -\kappa \int_0^t \nabla \cdot G(t-s)(s) \left[u\nabla(-\Delta + \gamma I)^{-1}w \right] ds = Q(f_t).$$

Taking $p = \frac{n}{2}$, $l = n$ and $r = \frac{n}{3}$, observe that $\frac{1}{r} = \frac{1}{p} + \frac{1}{l}$ and $\frac{1}{l} = \frac{1}{p} - \frac{1}{n}$. Then, we can employ Hölder's inequality and Lemma 4.1 to estimate

$$\begin{aligned}\sup_{0<s<T} \|f_t(\cdot, s)\|_{r,\infty} &= \kappa \sup_{0<s<t<T} \|(u\nabla(-\Delta + \gamma I)^{-1}w)(\cdot, t-s)\|_{r,\infty} \\ &\leq C\kappa \sup_{0<s<T} \|u(\cdot, s)\|_{p,\infty} \sup_{0<s<T} \|\nabla(-\Delta + \gamma I)^{-1}w(\cdot, s)\|_{l,\infty} \\ &\leq C\kappa g(\gamma) \sup_{0<s<T} \|u(\cdot, s)\|_{p,\infty} \sup_{0<s<T} \|w(\cdot, s)\|_{p,\infty}, \end{aligned} \tag{4.7}$$

where $g(\gamma) = 1$ and $g(\gamma) = \gamma^{-(n-1)}$ if $\gamma = 0$ and $\gamma > 0$, respectively. It follows that $f_t \in L^\infty((0, \infty), L^{r,\infty}(\mathbb{R}^n))$, for all $t \in (0, T)$.

Now, noting that

$$\frac{n}{r} - \frac{n}{p} = 3 - 2 = 1,$$

using Lemma 4.3 and afterwards (4.7), we arrive at

$$\begin{aligned}\sup_{0<t<T} \|B(u, w)\|_{p,\infty} &= \sup_{0<t<T} \|Q(f_t)\|_{p,\infty} \\ &\leq C\kappa g(\gamma) \sup_{0<t<T} \left(\sup_{0<s<T} \|f_t(\cdot, s)\|_{r,\infty} \right) \\ &\leq \kappa g(\gamma) K \sup_{0<t<T} \|u(\cdot, t)\|_{p,\infty} \sup_{0<t<T} \|w(\cdot, t)\|_{p,\infty}. \end{aligned}$$

◇

4.2. Proof of item (ii) (Uniqueness)

With the bilinear estimate (3.3) in hands , the uniqueness of solutions in $C([0, T); L^{\frac{n}{2},\infty})$ follows by adapting an argument due to Meyer [28] for our mild formulation (see also [11, 25]).

Let u and w be mild solutions of (1.1) in $C([0, T); L^{\frac{n}{2}}(\mathbb{R}^n))$ such that $u(0) = w(0) = u_0 \in L^{\frac{n}{2}}(\mathbb{R}^n)$. We claim that there exists $0 < T_1 < T$ such that $u(\cdot, t) = w(\cdot, t)$ in $L^{\frac{n}{2}}(\mathbb{R}^n)$, for all $t \in [0, T_1)$. Considering $h = u - w$, $h_1 = G(t)u_0 - u$ and $h_2 = G(t)u_0 - w$, we can rewrite the difference of the quadratic terms inside (1.3) as follows

$$\begin{aligned} u\nabla(-\Delta + \gamma I)^{-1}u) - w\nabla(-\Delta + \gamma I)^{-1}w) &= h\nabla(-\Delta + \gamma I)^{-1}u) + w\nabla(-\Delta + \gamma I)^{-1}h) \\ &= h\nabla(-\Delta + \gamma I)^{-1}G(t)u_0) + G(t)u_0\nabla(-\Delta + \gamma I)^{-1}h) \\ &\quad - h\nabla(-\Delta + \gamma I)^{-1}h_1) - h_2\nabla(-\Delta + \gamma I)^{-1}h). \end{aligned}$$

Thus, we can estimate h as

$$\begin{aligned} \|h(\cdot, t)\|_{\frac{n}{2},\infty} &= \left\| \int_0^t \nabla_x \cdot G(t-s) \left[u\nabla(-\Delta + \gamma I)^{-1}u) - w\nabla(-\Delta + \gamma I)^{-1}w) \right] ds \right\|_{\frac{n}{2},\infty} \\ &\leq \left\| \int_0^t \nabla_x \cdot G(t-s) \left[h\nabla(-\Delta + \gamma I)^{-1}h_1) + h_2\nabla(-\Delta + \gamma I)^{-1}h) \right] ds \right\|_{\frac{n}{2},\infty} \end{aligned}$$

$$+\left\|\int_0^t \nabla_x \cdot G(t-s)\left[h\nabla(-\Delta+\gamma I)^{-1}G(t)u_0) + G(t)u_0\nabla(-\Delta+\gamma I)^{-1}h)\right]ds\right\|_{\frac{n}{2},\infty}$$
$$:= J_1(t) + J_2(t). \tag{4.8}$$

Using (3.3), the parcel $J_1(t)$ can be estimated as

$$J_1(t) \le K\kappa g(\gamma) \sup_{0<t<T} \|h\|_{\frac{n}{2},\infty}(\sup_{0<t<T} \|h_1\|_{\frac{n}{2},\infty} + \sup_{0<t<T} \|h_2\|_{\frac{n}{2},\infty}). \tag{4.9}$$

Next we turn to I_2. Take $1 < l < \frac{n}{2} < \beta < n$ and $b > \beta$ satisfying $\frac{1}{l} = \frac{2}{n} + \frac{1}{b}$, $\frac{1}{b} = \frac{1}{\beta} - \frac{1}{n}$ and let $\eta_\beta = 1 - \frac{n}{2\beta}$. Also, note that $\frac{1}{l} = \frac{1}{\beta} + \frac{1}{n}$ and $\frac{1}{n} = \frac{2}{n} - \frac{1}{n}$. Then, using (4.1), Hölder's inequality and afterwards (4.1), we arrive at

$$\begin{aligned}
J_2(t) &\le C\int_0^t (t-s)^{-\frac{1}{2}(\frac{n}{l}-\frac{n}{n/2})-\frac{1}{2}} \|h(\cdot,s)\|_{\frac{n}{2},\infty} \|\nabla(-\Delta+\gamma I)^{-1}G(t)u_0\|_{b,\infty} ds \\
&\quad + C\int_0^t (t-s)^{-\frac{1}{2}(\frac{n}{l}-\frac{n}{n/2})-\frac{1}{2}} \|G(t)u_0\|_{\beta,\infty} \|\nabla(-\Delta+\gamma I)^{-1}h(\cdot,s)\|_{n,\infty} ds \\
&\le C\int_0^t (t-s)^{-\frac{1}{2}(\frac{n}{l}-\frac{n}{n/2})-\frac{1}{2}} \|h(\cdot,s)\|_{\frac{n}{2},\infty} \|G(t)u_0\|_{\beta,\infty} ds \\
&\quad + C\int_0^t (t-s)^{-\frac{1}{2}(\frac{n}{l}-\frac{n}{n/2})-\frac{1}{2}} \|G(t)u_0\|_{\beta,\infty} \|h(\cdot,s)\|_{\frac{n}{2},\infty} ds \\
&\le C \sup_{0<t<T} \|h(\cdot,s)\|_{\frac{n}{2},\infty} (\sup_{0<t<T} t^{\eta_\beta} \|G(t)u_0\|_{\beta,\infty}) \int_0^t (t-s)^{-\frac{n}{2\beta}} s^{-\eta_\beta} ds \\
&\le C \sup_{0<t<T} \|h(\cdot,s)\|_{\frac{n}{2},\infty} (\sup_{0<t<T} t^{\eta_\beta} \|G(t)u_0\|_{\beta,\infty}),
\end{aligned} \tag{4.10}$$

for all $t \in (0,T)$, where we used above that $-\frac{n}{2\beta} - \eta_\beta + 1 = 0$ and

$$\int_0^t (t-s)^{-\frac{n}{2\beta}} s^{-\eta_\beta} ds = t^{-\frac{n}{2\beta}-\eta_\beta+1} \int_0^1 (1-s)^{-\frac{n}{2\beta}} s^{-\eta_\beta} ds = C < \infty.$$

Inserting (4.9)–(4.10) into (4.8) yields

$$\sup_{0<t<T} \|h(\cdot,t)\|_{\frac{n}{2},\infty} \le CM(T) \sup_{0<t<T} \|h(\cdot,t)\|_{\frac{n}{2},\infty}, \tag{4.11}$$

where

$$M(T) = (\sup_{0<t<T} \|h_1(\cdot,t)\|_{\frac{n}{2},\infty} + \sup_{0<t<T} \|h_2(\cdot,t)\|_{\frac{n}{2},\infty} + \sup_{0<t<T} t^{\eta_\beta} \|G(t)u_0\|_{\beta,\infty}). \tag{4.12}$$

Using that $G(t)u_0 \to u_0$ and $u, v \to u_0$ in $L^{\frac{n}{2}}(\mathbb{R}^n)$, as $t \to 0^+$, and $L^{\frac{n}{2}}(\mathbb{R}^n) \hookrightarrow L^{\frac{n}{2},\infty}(\mathbb{R}^n)$, we obtain

$$\limsup_{t\to 0^+} (\|h_1(\cdot,t)\|_{\frac{n}{2},\infty} + \|h_2(\cdot,t)\|_{\frac{n}{2},\infty}) = 0. \tag{4.13}$$

Moreover, since $u_0 \in L^{\frac{n}{2}}(\mathbb{R}^n)$, there exists a sequence $\{u_{0,k}\}_k \subset L^{\frac{n}{2}}(\mathbb{R}^n) \cap L^{\beta,\infty}(\mathbb{R}^n)$ such that $u_{0,k} \to u_0$ in $L^{\frac{n}{2}}(\mathbb{R}^n) \hookrightarrow L^{\frac{n}{2},\infty}(\mathbb{R}^n)$. In fact, it is sufficient to take $u_{0,k} = G(\frac{1}{k})u_0$ and use (4.1). Then, we can estimate

$$\limsup_{t\to 0^+} t^{\eta_\beta} \|G(t)u_0\|_{\beta,\infty} \le \sup_{0<t<\infty} t^{\eta_\beta} \|G(t)(u_0 - u_{0,k})\|_{\beta,\infty} + \limsup_{t\to 0^+} t^{\eta_\beta} \|G(t)u_{0,k}\|_{\beta,\infty}$$

$$\leq C\|u_0 - u_{0,k}\|_{\frac{n}{2},\infty} + C\|u_{0,k}\|_{\beta,\infty} \limsup_{t\to 0^+} t^{\eta_\beta}$$
$$\leq C\|u_0 - u_{0,k}\|_{\frac{n}{2}} \to 0, \text{ as } k \to \infty. \tag{4.14}$$

In view of (4.12), (4.13) and (4.14), we can take $T = T_1 > 0$ small enough such that $CA(T_1) < 1$. Now, estimate (4.11) implies that $h(\cdot, t) = 0$ for all $t \in [0, T_1)$,which gives the desired claim.

Finally, we are going to show that the smallness condition on T_1 can be removed. For that, consider

$$T^* = \sup\{\widetilde{T} : 0 < \widetilde{T} < T,\ u(\cdot, t) = w(\cdot, t) \text{ in } L^{\frac{n}{2}} \text{ for all } t \in [0, \widetilde{T})\}.$$

If $T^* = T$, then $u = w$ in $[0, T)$, as desired. If $T^* < T \leq \infty$, we have that $u(\cdot, t) = w(\cdot, t)$ for all $t \in [0, T^*)$. By time-continuity, we obtain that $u(\cdot, T^*) = w(\cdot, T^*)$ and then, by the first part, there exists a sufficiently small $\rho > 0$ such that $u(\cdot, t) = w(\cdot, t)$ for all $t \in [T^*, T^* + \rho)$. So, $u(\cdot, t) = w(\cdot, t)$ in $[0, T^* + \rho)$, which contradics the maximality of $T^* < \infty$.

◇

Acknowledgments

L. C. F. Ferreira was supported by FAPESP and CNPQ, Brazil.

Conflict of interest

The author declares no conflict of interest.

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