http://www.aimspress.com/journal/mine

Mathematics in Engineering, 4(5): 1–25.
DOI:10.3934/mine.2022041
Received: 31 December 2020
Accepted: 06 July 2021
Published: 12 October 2021

Research article

Existence of generalized solutions for Keller-Segel-Navier-Stokes equations with degradation in dimension three†

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† **This contribution is part of the Special Issue:** Advances in the analysis of chemotaxis systems
Guest Editor: Michael Winkler
Link: www.aimspress.com/mine/article/6067/special-articles

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Abstract: We construct generalized solutions for the Keller-Segel system with a degradation source coupled to Navier Stokes equations in three dimensions, in case that the power of degradation is smaller than quadratic. Furthermore, if the logistic type source is purely damping with no growing effect, we prove that solutions converge to zero in some norms and provide upper bounds of convergence rates in time.

Keywords: chemotaxis; generalized solution; Keller-Segel-Navier-Stokes equations; asymptotic behavior

1. Introduction

We consider a mathematical model to decribe the dynamics of biological organism influenced by chemical signal and living in fluid. The original Keller-Segel system was proposed to write the motion of biological individuals sensing gradient of a chemical substance and moving toward its higher concentration (see [9]). Such biological organisms often live in fluid, and thus their behaviors are influenced by motions of viscous fluid flows as well. There are, for example, the bacteria living in fluid such as Bacillus subtilus ([1, 2, 7, 11, 18, 24]) or Escherichia coli ([12, 22]) or phenomena of coral fertilization in sea resulting from the chemotatic behavior of sperm ([4, 6, 10, 24]).

In this note, we study the following Keller-Segel system with degradation coupled to the Navier-Stokes equations in a bounded domain in three dimensions:

$$n_t + u \cdot \nabla n = \Delta n - \nabla\cdot (n\nabla c) + \rho n - \mu n^q, \tag{1.1}$$

$$c_t + u \cdot \nabla c = \Delta c - c + n, \tag{1.2}$$

$$u_t + (u \cdot \nabla) u = \Delta u + \nabla P + n\nabla\phi, \quad \nabla\cdot u = 0 \tag{1.3}$$

in $\Omega \times (0, T)$, where $\Omega \subset \mathbb{R}^3$ is a bounded domain with smooth boundary and $T > 0$. Here n, c, u, and P are the population density of the chemotactic organisms, the concentration of signal substances, the fluid velocity, and the associated pressure, respectively. No flux condition is assigned for n and c at the boundary, and u has no slip boundary condition there, namely

$$\frac{\partial n}{\partial \nu} = \frac{\partial c}{\partial \nu} = 0, \quad u = 0 \quad \text{on } \partial\Omega. \tag{1.4}$$

We assume that initial data (n_0, c_0, u_0) satisfies

$$\begin{cases} 0 \leqslant n_0 \in C^0(\overline{\Omega}) \text{ with } n_0 \not\equiv 0, \\ 0 \leqslant c_0 \in W^{1,\infty}(\Omega), \\ u_0 \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega) \text{ with } \nabla\cdot u_0 = 0. \end{cases} \tag{1.5}$$

In case that the Eq (1.1) has the logistic degradation, i.e., $q = 2$, Tao and Winkler [16] proved global existence and large time behavior of classical solutions to the system (1.1)–(1.3) in two dimensions. Such result was extended to the case of three dimensions, provided that the fluid equation is given by the Stokes system, instead of the Navier-Stokes equations, and μ is sufficiently large (see [15]).

For the chemotaxis-Navier-Stokes system (1.1)–(1.3) with $q = 2$, the existence of generalized solutions was proved by Winkler [22].

To the best of our knowledge, if $q < 2$, it is not known whether or not classical solutions exist globally in time for general data and parameters. Instead of classical solutions, recently it was shown in [8] that generalized solutions to the chemotaxis-Stokes system exists globally in time for $q \in (2 - \frac{1}{d}, 2)$, where d is dimensions two or three, i.e., $d = 2, 3$. (the notion of generalized solutions is reminded in Definition 2). In the absence of fluid, i.e., $u = 0$, one can refer to [19, 20, 23] for generalized solutions.

The main objective of this note is to establish the existence of generalized solutions globally in time, in case that the degradation power q is less than 2, and the Navier-Stokes equations are coupled for the fluid equations in three dimensions.

To begin with, we recall the notion of generalized solution of (1.1)–(1.3). Firstly, we remind the $\gamma-$entrophy super(or sub) solution of the Eq (1.1).

Definition 1. Let $\gamma \in (0, 1)$. Assume that a pair of functions (n, c) and a vector field u satisfy the following:

$$\nabla n \text{ and } \nabla c \text{ are measurable in } \Omega \times (0, \infty),$$
$$n^\gamma, n^{\gamma-2} |\nabla n|^2, n^{\gamma-1}\nabla n \cdot \nabla c, n^{q+\gamma-1} \in L^1_{loc}(\overline{\Omega} \times [0, \infty)),$$
$$n^\gamma \nabla c, n^\gamma u \in L^1_{loc}(\overline{\Omega} \times [0, \infty); \mathbb{R}^3),$$
$$\nabla\cdot u = 0 \text{ in } \mathcal{D}'(\Omega \times (0, \infty)).$$

Then such (n, c, u) is called a weak γ-entropy super-solution(resp., sub-) of the first equation in (1.1)–(1.3) if

$$-\int_0^\infty \int_\Omega n^\gamma \varphi_t - \int_\Omega n_0^\gamma \varphi(\cdot, 0) \underset{(\leqslant)}{\geqslant} \gamma(1-\gamma) \int_0^\infty \int_\Omega n^{\gamma-2} |\nabla n|^2 \varphi + \int_0^\infty \int_\Omega n^\gamma \Delta\varphi$$

$$+ (1-\gamma)\int_0^\infty\int_\Omega n^\gamma \Delta c\,\varphi + \int_0^\infty\int_\Omega n^\gamma \nabla c\cdot\nabla\varphi$$
$$+ \rho\gamma\int_0^\infty\int_\Omega n^\gamma\varphi - \mu\gamma\int_0^\infty\int_\Omega n^{q+\gamma-1}\varphi + \int_0^\infty\int_\Omega n^\gamma u\cdot\nabla\varphi,$$

for all nonnegative $\varphi \in C_0^\infty(\overline{\Omega}\times[0,\infty))$.

Next, we define the notion of the generalized solutions of (1.1)–(1.3).

Definition 2. A triple of two functions and a vector field

$$n \in L^1_{loc}(\overline{\Omega}\times[0,\infty)), c \in L^1_{loc}([0,\infty); W^{1,1}(\Omega)), u \in L^1_{loc}([0,\infty); W_0^{1,1}(\Omega,\mathbb{R}^3))$$

satisfying

$$cu \in L^1_{loc}(\overline{\Omega}\times[0,\infty)), \quad u\otimes u \in L^1_{loc}(\overline{\Omega}\times[0,\infty); \mathbb{R}^3\times\mathbb{R}^3)$$

is called a generalized solution of (1.1)–(1.3), if

$$-\int_0^\infty\int_\Omega c\varphi_t - \int_\Omega c_0\varphi(\cdot,0) = -\int_0^\infty\int_\Omega \nabla c\cdot\nabla\varphi - \int_0^\infty\int_\Omega c\varphi + \int_0^\infty\int_\Omega n\varphi + \int_0^\infty\int_\Omega cu\cdot\nabla\varphi \tag{1.6}$$

for all $\varphi \in C_0^\infty(\overline{\Omega}\times[0,\infty))$ and, if $\nabla\cdot u = 0$ in $\mathcal{D}'(\Omega\times(0,\infty))$ and

$$-\int_0^\infty\int_\Omega u\varphi_t - \int_\Omega u_0\varphi(\cdot,0) = -\int_0^\infty\int_\Omega \nabla u\cdot\nabla\varphi + \int_0^\infty\int_\Omega (u\otimes u)\cdot\nabla\varphi + \int_0^\infty\int_\Omega n\nabla\phi\cdot\varphi \tag{1.7}$$

for all $\varphi \in C_0^\infty(\Omega\times[0,\infty);\mathbb{R}^3)$ with $\nabla\cdot\varphi \equiv 0$, and if there exist $\gamma_1,\gamma_2 \in (0,1)$ such that (n,c,u) is a weak γ_1-entropy super-solution and a weak γ_2-entropy sub-solution of the first equations in (1.1)–(1.3).

For logistic coefficients ρ,μ and the potential function ϕ, we assume that

$$\rho\in\mathbb{R}, \quad \mu>0 \quad \text{and} \quad \phi\in C^1(\Omega). \tag{1.8}$$

We are now ready to state our main result.

Theorem 1.1. *Let $q \in \left(\frac{20}{11},2\right)$. Then the Eqs* (1.1)–(1.5) *with* (1.8) *admit at least one generalized solution in the sense of Definition 2.*

Remark 1. The result Theorem 1.1 is an improvemnt of that of [22], which showed the existence of the generalized solution in case that $q = 2$. Furthermore, it is also an extension to the result of [8], since the Navier-Stokes equations are considered instead the Stokes system. In such case, the range of q is, however, restrictive, compared to the case that $q \in \left(\frac{5}{3},2\right)$ in [8]. This is mainly due to the fact that the control of u is more difficult for the Navier-Stokes equations, which causes lower regularity of $u\cdot\nabla c$ and, in turn, ∇c (see Lemma 3.6 for the details). Therefore, passing to the limit for regularized solutions, convergence to $n\nabla c$ is well understood only for $q \in \left(\frac{20}{11},2\right)$.

Next, in case that $\rho \leqslant 0$, we can show that generalized solutions converge to zero in an appropriate sense, passing time to the limit. More precisely, we obtain the following:

Theorem 1.2. *Let (n, c, u) be the generalized solution established in Theorem 1.1. If $\rho = 0$, then (n, c, u) vanishes in $L^1(\Omega) \times L^l(\Omega) \times L^2(\Omega)$ as time tends to infinity. Furthermore, (n, c, u) satisfies*

$$\int_\Omega n(\cdot, t)\,\mathrm{d}x \leqslant C(1+t)^{-\frac{1}{q-1}}, \qquad \int_\Omega |u(\cdot, t)|^2\,\mathrm{d}x \leqslant C(1+t)^{-\frac{3q(4-q)-10}{3(q-1)^2}}$$

$$\textit{and} \qquad \int_\Omega (c(\cdot, t))^l\,\mathrm{d}x \leqslant \begin{cases} C(1+t)^{-\frac{2lq+q-3l}{3l(q-1)^2}}, & \textit{if} \quad 1 \leqslant l \leqslant 3q-2, \\ C(1+t)^{-\frac{3q-(5-2q)l}{l(3q-5)(q-1)}}, & \textit{if} \quad 3q-2 < l \leqslant \frac{3q}{5-2q}. \end{cases}$$

Morerover, if $\rho < 0$, then (n, c, u) satisfies

$$\int_\Omega n(\cdot, t)\,\mathrm{d}x \leqslant Ce^{\rho t}, \qquad \int_\Omega |u(\cdot, t)|^2\,\mathrm{d}x \leqslant Ce^{-\delta_* t}$$

$$\textit{and} \qquad \int_\Omega (c(\cdot, t))^l\,\mathrm{d}x \leqslant Ce^{-\frac{3q-(5-2q)l}{5(q-1)l}\rho_* t}, \qquad \textit{if} \quad 1 \leqslant l \leqslant \frac{3q}{5-2q}.$$

where $\rho_* = \min\{-\rho, 1\}$, $\delta_* = \frac{1}{2}\min\left\{\frac{C_p}{2}, -\rho\frac{5q-6}{3(q-1)}\right\}$ and C_p is the Poincaré constant for Ω.

Remark 2. The result of Theorem 1.2 can be extended to the case $q = 2$ and $\rho = 0$. In such case, in particular, estimates of c read as follows:

$$\int_\Omega (c(\cdot, t))^l\,\mathrm{d}x \leqslant \begin{cases} C(1+t)^{-\frac{l+2}{3l}}, & \text{if} \quad 1 \leqslant l \leqslant 4, \\ C(1+t)^{-\frac{6-l}{l}}, & \text{if} \quad 4 < l \leqslant 6. \end{cases}$$

This estimate of decay for c is slightly better, compared to those of [22, Section 8]. On the other hand, in case that $q = 2$ and $\rho > 0$, it was also shown in [22] that if $\mu > \chi\sqrt{\rho}/4$, then

$$\limsup_{t\to\infty} \left\|n(\cdot, t) - \frac{\rho}{\mu}\right\|_1 + \left\|c(\cdot, t) - \frac{\rho}{\mu}\right\|_p + \|u(\cdot, t)\|_2 = 0, \qquad 1 \leqslant p < 6.$$

This convergence is based on stabilization of a certain energy functional (see [22, Section 8]). Although similar results are expected, such a method doesn't seem to be valid unless $q = 2$. Therefore, we leave the asymptotic behaviors as an open question in case that $\rho > 0$ and $q < 2$.

This paper is organized as follows: In Section 2, we introduce an approximated system and recall some useful lemma for our purpose. Section 3 is devoted to obtaining estimates, independent of a regularizing parameter, of the approximated system. We then discuss the convergence of approximated solutions to a generalized solution in Section 4. Finally, in Section 5, asymptotic estimates are provided.

Throughout this paper, we shall abbreviate $\|f\|_{L^p(\Omega)}$ as $\|f\|_p$ for simplicity. Further, we denote by $C > 0$ generic constants which may be different from line to line.

2. Preliminaries

In the following proposition we define an appropriate approximated system to (1.1)–(1.3), for which global classical solutions can be verified. The approximated system is given by

$$\begin{cases}\partial_t n_\epsilon + u_\epsilon \cdot \nabla n_\epsilon = \Delta n_\epsilon - \nabla\cdot(n_\epsilon \nabla c_\epsilon) + \rho n_\epsilon - \mu n_\epsilon^q - \epsilon n_\epsilon^\kappa,\\ \partial_t c_\epsilon + u_\epsilon \cdot \nabla c_\epsilon = \Delta c_\epsilon - c_\epsilon + n_\epsilon,\\ \partial_t u_\epsilon + (Y_\epsilon u_\epsilon \cdot \nabla) u_\epsilon = \Delta u_\epsilon + \nabla P_\epsilon + n_\epsilon \nabla\phi,\\ \nabla\cdot u_\epsilon = 0,\\ \frac{\partial n_\epsilon}{\partial \nu} = \frac{\partial c_\epsilon}{\partial \nu} = u_\epsilon = 0,\\ n_\epsilon(x,0) = n_0, \quad c_\epsilon(x,0) = n_0, \quad u_\epsilon(x,0) = u_0.\end{cases} \tag{2.1}$$

Here $\epsilon \in (0,1)$, $\kappa > 2$ and Y_ϵ is the Yosida approximation defined by

$$Y_\epsilon f := (I + \epsilon A)^{-1} f, \quad f \in L^2_\sigma(\Omega),$$

where A is the realization of the stokes operator in $D(A) = W^{2,2}(\Omega) \cap W^{1,2}_{0,\sigma}(\Omega) \subset L^2_\sigma(\Omega)$.

Following method of proofs developped in [8] and [22], one can prove the existence of classical solution of the approximated system (2.1). Since its verification is similar to thoes of [8] and [22], we skip its proof.

Proposition 1. *For each $\epsilon \in (0,1)$, there exist functions*

$$\begin{cases} n_\epsilon \in C^0(\overline{\Omega} \times [0,\infty) \cap C^{2,1}(\overline{\Omega} \times (0\,\infty)),\\ c_\epsilon \in C^0(\overline{\Omega} \times [0,\infty) \cap C^{2,1}(\overline{\Omega} \times (0\,\infty)),\\ u_\epsilon \in C^0(\overline{\Omega} \times [0,\infty) \cap C^{2,1}(\overline{\Omega} \times (0\,\infty)),\\ P_\epsilon \in C^{1,0}(\overline{\Omega} \times (0,\infty)) \end{cases}$$

such that $(n_\epsilon, c_\epsilon, u_\epsilon, P_\epsilon)$ solves (2.1) classically in $\overline{\Omega} \times (0,\infty)$.

We recall an effective inequality in Sobolev spaces called the Gagliardo-Nirenberg interpolation inequality. Here we only consider a version of bounded Lipschitz domain Ω in $\mathbb{R}^3$. The proof can be found in [3, Theorem 1.5.2] and [13].

Lemma 2.1. *Let $1 \leqslant p, r \leqslant \infty$ and $0 \leqslant n < m \in \mathbb{N}$. Then there exist constants C_1 and $C_2 > 0$ such that*

$$\|D^n f\|_q \leqslant C \|D^m f\|_p^\theta \|f\|_r^{1-\theta} + C_2 \|f\|_s, \quad f \in \mathcal{D}'(\Omega) \tag{2.2}$$

where $\frac{1}{q} - \frac{n}{3} = \left(\frac{1}{p} - \frac{m}{3}\right)\theta + \frac{1}{r}(1-\theta)$, $\theta \in [\frac{n}{m}, 1]$, and $s > 0$ is arbitrary.

The following two Lemmas named maximal estimates are crutial to obtain a regularity of approximated solutions (see [5, 8, 14]).

Lemma 2.2. *Let $T > 0$, $v_0 \in W^{1,p}(\Omega)$ and $h \in L^p(0,T; L^p(\Omega; \mathbb{R}^3))$ for $1 < p < \infty$. Then there exists a unique solution $v \in L^p(0,T; W^{1,p}(\Omega))$ solving*

$$\begin{cases} v_t - \Delta v = \nabla\cdot h, & (x,t) \in \Omega \times (0,T),\\ v(x,0) = v_0(x), & x \in \Omega,\\ \frac{\partial v}{\partial \nu} = 0, & (x,t) \in \partial\Omega \times (0,T). \end{cases}$$

Furthermore, v attains the following estimate.

$$\int_0^T \|v(s)\|_p^p \, \mathrm{d}s + \int_0^T \|\nabla v(s)\|_p^p \, \mathrm{d}s \leqslant C_T \left(\int_0^T \|h(s)\|_p^p \, \mathrm{d}s + \|v_0\|_{W^{1,p}(\Omega)}^p \right). \tag{2.3}$$

Lemma 2.3. *Let $T > 0$ and $p \in (1, 2]$. Then for every $v_0 \in W^{1,\infty}(\Omega)$ and $h \in L^p(\Omega \times (0, T))$, the following heat equation with Neumann boundary condition*

$$\begin{cases} v_t - \Delta v = h, & (x,t) \in \Omega \times (0,T), \\ v(x,0) = v_0(x), & x \in \Omega, \\ \frac{\partial v}{\partial \nu} = 0, & (x,t) \in \partial\Omega \times (0,T) \end{cases} \tag{2.4}$$

has a unique solution $v \in W^{1,p}((0,T); L^p(\Omega)) \cap L^p((0,T); W^{2,p}(\Omega))$ satisfying

$$\|v_t\|_{L^p(\Omega\times(0,T))} + \|v\|_{L^p(0,T;W^{2,p}(\Omega))} \leqslant C_T \left(\|h\|_{L^p(\Omega\times(0,T))} + 1 \right) \tag{2.5}$$

with some $C_T > 0$.

Proof. Set $X = L^p(\Omega)$ and $X_1 = W_\nu^{2,p}(\Omega) := \{f \in W^{2,p}(\Omega) : \frac{\partial f}{\partial \nu} = 0 \text{ on } \partial\Omega\}$. From [14] and [19, Proposition 2] we have

$$\|v_t\|_{L^p(\Omega\times(0,T))} + \|v\|_{L^p(0,T;W^{2,p}(\Omega))} \leqslant C_T \left(\|v_0\|_{1-\frac{1}{p},p} + \|h\|_{L^p(\Omega\times(0,T))} \right),$$

where $\|\cdot\|_{1-\frac{1}{p},p}$ stands for the norm in the real interpolation space $(X, X_1)_{1-\frac{1}{p},p}$. Now (2.5) is achieved from the embedding [21, Lemma 2.1.(ii)]

$$W^{1,\infty}(\Omega) \hookrightarrow W^{1,p}(\Omega) \hookrightarrow W^{2(1-\frac{1}{p}),p}(\Omega) \cong (X, X_1)_{1-\frac{1}{p},p}\,,$$

for any $p \in (1, 2]$. □

Remark 3. For the purpose of our analysis, we consider only the case $p \in (1, 2]$ in Lemma 2.3. One can refer to [21] for more general cases, in particular $p \geqslant 3$, where the interpolation space $(X, X_1)_{1-\frac{1}{p},p}$ is not equaivalent to $W^{2(1-\frac{1}{p}),p}(\Omega)$.

Next, we present a compactness theorem called Aubin-Lions Lemma [17, Theorem 2.1] that will be used to give convergence results for the approximated solution $(n_\epsilon, c_\epsilon, u_\epsilon)$.

Lemma 2.4. *Let $T > 0$, $1 \leqslant \alpha_0, \alpha_1 < \infty$ and X_0, X, X_1 be Banach spaces with $X_0 \subset X \subset X_1$. Suppose further that the embedding $X_0 \hookrightarrow X$ is compact and the embedding $X \hookrightarrow X_1$ is continuous. Let*

$$W = \{v \in L^{\alpha_0}(0, T; X_0) \mid \partial_t v \in L^{\alpha_1}(0, T; X_1)\}\,.$$

Then the embedding $W \hookrightarrow L^{\alpha_0}(0, T; X)$ is compact.

3. Regularized solutions

The following basic properties of these solutions are well-known.

Lemma 3.1. *Let $T > 0$. For each $\epsilon \in (0,1)$, the solution of* (2.1) *fulfills*

$$\int_\Omega n_\epsilon(x,t)\,\mathrm{d}x \leqslant m \quad \textit{for all} \;\; t < T \tag{3.1}$$

and

$$\mu \int_0^T \int_\Omega n_\epsilon^q(x,s)\,\mathrm{d}x\,\mathrm{d}s + \epsilon \int_0^T \int_\Omega n_\epsilon^\kappa(x,s)\,\mathrm{d}x\,\mathrm{d}s \leqslant (\rho_+ T + 1)m, \tag{3.2}$$

where $m = \max\left\{\int_\Omega n_0, \left(\frac{\rho_+|\Omega|}{\mu}\right)^{\frac{1}{q-1}}\right\}$ *and* $\rho_+ = \max\{\rho, 0\}$.

Proof. Integrating the first equation in (2.1) over Ω, employing the divergence theorem, and using the Hölder inequality yield that, for all $t > 0$,

$$\frac{d}{dt}\int_\Omega n_\epsilon = \rho \int_\Omega n_\epsilon - \mu \int_\Omega n_\epsilon^q - \epsilon \int_\Omega n_\epsilon^\kappa \leqslant \rho_+ \int_\Omega n_\epsilon - \frac{\mu}{|\Omega|}\left(\int_\Omega n_\epsilon\right)^q. \tag{3.3}$$

An ODE comparison implies (3.1). Integrating (3.3) with respect to time and then using (3.1), we have

$$\mu \int_0^T \int_\Omega n_\epsilon^q + \epsilon \int_0^T \int_\Omega n_\epsilon^\kappa \leqslant \rho_+ \int_0^T \int_\Omega n_\epsilon + \int_\Omega n_0(x)\,\mathrm{d}x - \int_\Omega n_\epsilon(x,T)\,\mathrm{d}x \leqslant (\rho_+ T + 1)m,$$

which implies (3.2). □

The following estimate is easily obtained by (3.1).

Lemma 3.2. *For each $\epsilon \in (0,1)$, we have*

$$\int_\Omega c_\epsilon(x,t)\,\mathrm{d}x \leqslant \max\left\{\int_\Omega c_0, m\right\} \quad \textit{for all}\; t > 0. \tag{3.4}$$

Proof. Integrating the equation for c_ϵ in (2.1) and using (3.1), we have

$$\frac{d}{dt}\int_\Omega c_\epsilon + \int_\Omega c_\epsilon = \int_\Omega n_\epsilon \leqslant m \quad \text{for all } t < T,$$

which yields (3.4) by the ODE comparison. □

We recall a useful result shown in [22, Lemma 3.4].

Lemma 3.3. *Let $T \in (0,\infty]$, $\tau \in (0,T)$, $a > 0$ and $b > 0$. Suppose that a nonnegative function $h \in L^1_{loc}(\mathbb{R})$ be such that*

$$\int_t^{t+\tau} h(s)\,\mathrm{d}s \leqslant b\tau \quad \text{for all } t \in [0, T-\tau).$$

If a nonnegative function $y \in C^0[0,T) \cap C^1(0,T)$ satisfies

$$y'(t) + ay(t) \leqslant h(t),$$

then

$$y(t) \leqslant y(0) + \frac{b\tau}{1-e^{-a\tau}} \quad \text{for all } t > 0.$$

The following lemma is a variant of the result with $q = 2$ in [22, Lemma 3.6].

Lemma 3.4. *Let $T > 0$ and $q \in (\frac{5}{3}, 2)$. Then there exists $C > 0$ such that for any $\epsilon \in (0,1)$ we obtain*

$$\int_\Omega |c_\epsilon(x,t)|^r \,\mathrm{d}x \leqslant C \quad \text{for all } t > 0. \tag{3.5}$$

Moreover,

$$\int_0^T \Big(\int_\Omega |c_\epsilon(x,s)|^{3r} \Big)^{\frac{1}{3}} \,\mathrm{d}x\,\mathrm{d}s \leqslant C(T+1), \tag{3.6}$$

where $r = \frac{3q}{5-2q}$.

Proof. Multiplying the equation for c_ϵ in (2.1) by c_ϵ^{r-1} and integrating over Ω, we have for all $t > 0$,

$$\frac{d}{dt}\frac{1}{r}\int_\Omega c_\epsilon^r + \frac{4(r-1)}{r^2}\int_\Omega \left|\nabla c_\epsilon^{\frac{r}{2}}\right|^2 + \int_\Omega c_\epsilon^r = \int_\Omega n_\epsilon c_\epsilon^{r-1} \leqslant \|n_\epsilon\|_q \left\|c_\epsilon^{r-1}\right\|_{\frac{q}{q-1}}, \tag{3.7}$$

where the Hölder inequality is used. Using the Gagliardo-Nirenberg inequality and (3.4), we note that

$$\begin{aligned}\left\|c_\epsilon^{r-1}\right\|_{\frac{q}{q-1}} = \left\|c_\epsilon^{\frac{r}{2}}\right\|_{\frac{2(r-1)}{r}\frac{q}{q-1}}^{\frac{2(r-1)}{r}} &\leqslant C\left(\left\|\nabla c_\epsilon^{\frac{r}{2}}\right\|_2^{\frac{2(r-1)}{r}\theta}\left\|c_\epsilon^{\frac{r}{2}}\right\|_2^{\frac{2(r-1)}{r}(1-\theta)} + \left\|c_\epsilon^{\frac{r}{2}}\right\|_{\frac{2}{r}}^{\frac{2(r-1)}{r}}\right)\\ &\leqslant C\left\|\nabla c_\epsilon^{\frac{r}{2}}\right\|_2^{\frac{2(r-1)}{r}\theta}\left\|c_\epsilon^{\frac{r}{2}}\right\|_2^{\frac{2(r-1)}{r}(1-\theta)} + C \quad \text{for all } t > 0,\end{aligned}$$

where $\theta = \frac{3}{2}(1 - \frac{r}{r-1}\frac{g-1}{q}) \in (0,1)$ since $r = \frac{3q}{5-2q}$. Employing Young's inequality, we have

$$\begin{aligned}\|n_\epsilon\|_q \left\|c_\epsilon^{r-1}\right\|_{\frac{q}{q-1}} &\leqslant \frac{2(r-1)}{r^2}\left\|\nabla c_\epsilon^{\frac{r}{2}}\right\|_2^2 + C\|n_\epsilon\|_q^q \left\|c_\epsilon^{\frac{r}{2}}\right\|_2^{\frac{4(q-1)}{3}} + \|n_\epsilon\|_q^q + C\\ &\leqslant \frac{2(r-1)}{r^2}\left\|\nabla c_\epsilon^{\frac{r}{2}}\right\|_2^2 + C\|n_\epsilon\|_q^q\left(\left\|c_\epsilon^{\frac{r}{2}}\right\|_2^2 + 1\right) + \|n_\epsilon\|_q^q + C.\end{aligned} \tag{3.8}$$

Combining (3.7) with (3.8) implies that there exist $C_5 > 0$ such that

$$\frac{d}{dt}\frac{1}{r}\int_\Omega c_\epsilon^r + \frac{2(r-1)}{r^2}\int_\Omega \left|\nabla c_\epsilon^{\frac{r}{2}}\right|^2 + \int_\Omega c_\epsilon^r + 1 \leqslant C\left(\|n_\epsilon\|_q^q + 1\right)\left(\|c_\epsilon\|_r^r + 1\right). \tag{3.9}$$

Let $y(t) := \|c_\epsilon(t)\|_r^r + 1$ and $h(t) := \|n_\epsilon(t)\|_q^q + 1$, which is in L^1 locally in time. Then, dividing (3.9) by $y(t)$ yields that

$$\frac{d}{dt}\ln y + \frac{2(r-1)}{r}\frac{1}{y}\left\|\nabla c_\epsilon^{\frac{r}{2}}\right\|_2^2 + 1 \leqslant Ch. \tag{3.10}$$

We use again the Gagliardo-Nirenberg inequality to obtain that for all $t > 0$

$$y(t) \leqslant C\left\|\nabla c_\epsilon^{\frac{r}{2}}\right\|_2^{\frac{6(r-1)}{3r-1}}\left\|c_\epsilon^{\frac{r}{2}}\right\|_{\frac{2}{r}}^{\frac{4}{3r-1}} + C\left\|c_\epsilon^{\frac{r}{2}}\right\|_{\frac{2}{r}}^2 + 1 \leqslant C\left(\left\|\nabla c_\epsilon^{\frac{r}{2}}\right\|_2^{\frac{6(r-1)}{3r-1}} + 1\right),$$

which leads that $\left\|\nabla c_\epsilon^{\frac{r}{2}}\right\|_2^2 \geqslant \left(\frac{1}{C}y(t) - 1\right)^{\frac{3r-1}{3(r-1)}} \geqslant Cy^{\frac{3r-1}{3(r-1)}} - 1$. Hence, it follows that

$$\frac{1}{y}\left\|\nabla c_\epsilon^{\frac{r}{2}}\right\|_2^2 \geqslant Cy^{\frac{2}{3(r-1)}} - \frac{1}{y} \geqslant C\ln y - 1 \quad \text{for all } t > 0, \tag{3.11}$$

where we use the trivial inequality $\ln y \lesssim y^k$ for $k > 0$. Putting the above inequality (3.11) into (3.10), we have

$$\frac{d}{dt}\ln y + C\ln y \leqslant h.$$

By Lemma 3.3, we can conclude that there exists $C > 0$ satisfying $y(t) \leqslant C$ for all $t > 0$ which proves (3.5) as required. Integrating (3.10) with respect to time and exploiting the boundedness of $y(t)$, guaranteed by (3.5), yield that

$$\int_0^T \left\|\nabla c_\epsilon^{\frac{r}{2}}\right\|_2^2 \leqslant C(1+T)$$

for some $C > 0$. Using (2.2) and (3.4), we finally have (3.6). □

We adopt well-known energy estimate for the Navier-Stokes system to gain a bound for u_ϵ in energy class.

Lemma 3.5. *Let $T > 0$ and $q \in \left(\frac{5}{3}, 2\right)$. Then there exists $C > 0$ such that for each $\epsilon \in (0,1)$, we have*

$$\int_\Omega |u_\epsilon(x,t)|^2 \,\mathrm{d}x \leqslant C \quad \textit{for all } t > 0 \tag{3.12}$$

and

$$\int_0^T\int_\Omega |\nabla u_\epsilon(x,s)|^2 \,\mathrm{d}x\,\mathrm{d}s \leqslant C(1+T). \tag{3.13}$$

Proof. We test the fluid equation in (2.1) by u_ϵ to find the following L^2 estimate

$$\frac{d}{dt}\int_\Omega u_\epsilon^2 + \int_\Omega |\nabla u_\epsilon|^2 = \int_\Omega n_\epsilon u_\epsilon \nabla\phi \tag{3.14}$$

We can estimate the right hand side of (3.14) using the Hölder inequality, the Sobolev embedding $W_{0,\sigma}^{1,2} \hookrightarrow L^6$, and the interpolation inequality for n_ϵ that

$$\int_\Omega n_\epsilon u_\epsilon \nabla\phi \leqslant C\|n_\epsilon\|_{\frac{6}{5}}\|u_\epsilon\|_6$$

$$\leqslant C\,\|n_\epsilon\|_{\frac{6}{5}}^2 + \frac{1}{2}\,\|\nabla u_\epsilon\|_2^2$$
$$\leqslant C\,\|n_\epsilon\|_q^{\frac{q}{3(q-1)}}\,\|n_\epsilon\|_1^{\frac{5q-6}{3(q-1)}} + \frac{1}{2}\,\|\nabla u_\epsilon\|_2^2$$
$$\leqslant C\left(\|n_\epsilon\|_q^q + 1\right) + \frac{1}{2}\,\|\nabla u_\epsilon\|_2^2 \quad \text{for all } t > 0, \tag{3.15}$$

where we used that $\frac{q}{3(q-1)} \leqslant q$.
Thus, with the aid of (3.15) and the Poincaré inequality, we have for some C

$$\frac{d}{dt}\int_\Omega u_\epsilon^2 + C\int_\Omega u_\epsilon^2 \leqslant C\,\|n_\epsilon\|_q^q + 1.$$

(3.12) is proved if we use (3.2) and Lemma 3.3, and then (3.13) can be calculated by integrating (3.14) with respect to time and using (3.15). □

A direct consequence of Lemma 3.5 is the following.

Corollary 1. *Let $T > 0$ and $\frac{3}{\alpha} + \frac{2}{\beta} = \frac{3}{2}$, $2 \leqslant \alpha \leqslant 6$. Then*

$$\int_0^T \Big(\int_\Omega |u_\epsilon(x,s)|^\alpha\Big)^{\frac{\beta}{\alpha}}\,\mathrm{d}x\,\mathrm{d}s \leqslant C(1+T), \tag{3.16}$$

in particular, if $\alpha = \beta = \frac{10}{3}$, then

$$\int_0^T \int_\Omega |u_\epsilon(x,s)|^{\frac{10}{3}}\,\mathrm{d}x\,\mathrm{d}s \leqslant C(1+T). \tag{3.17}$$

Proof. In view of Lemma 3.5, (3.16), in particular (3.17), is derived from the Gagliardo-Nirenberg inequality (2.2). □

Since u_ϵ only belong to energy class, we have lower regularity of ∇c_ϵ, due to difficulties of controlling convective term $u\cdot\nabla c$, than the case that the Stokes sysem is coupled. Nevertheless, using the divergence free condtion, we obtain a certain integrability of ∇c_ϵ by the following decompsition, which makes computations easier. More precisely, let w_ϵ be a solution satisfying

$$\begin{cases}\partial_t w_\epsilon - \Delta w_\epsilon = -c_\epsilon + n_\epsilon, & (x,t) \in \Omega\times[0,t),\\ w_\epsilon(x,0) = c_0, & x\in\Omega.\end{cases}$$

Now we set $\tilde{w}_\epsilon := c_\epsilon - w_\epsilon$. Then, due to the divergence free condition for u_ϵ, it follows that $\tilde{w}_\epsilon$ solves

$$\begin{cases}\partial_t \tilde{w}_\epsilon - \Delta \tilde{w}_\epsilon = -\nabla\cdot(u_\epsilon c_\epsilon), & (x,t) \in \Omega\times[0,t),\\ \tilde{w}_\epsilon(x,0) = 0, & x\in\Omega.\end{cases}$$

In next lemma, estimating each solutions of the decompsition, we show that $\nabla c_\epsilon \in L^{10q/(10-q)}(\Omega\times(0,T))$.

Lemma 3.6. *Let $T > 0$ and $q \in \left(\frac{5}{3}, 2\right)$. Then given $\epsilon \in (0,1)$, there exists $C = C(T) > 0$ such that*

$$\int_0^T \int_\Omega |\nabla c_\epsilon(x,s)|^m \, \mathrm{d}x\,\mathrm{d}s \leqslant C, \tag{3.18}$$

where $m = \frac{10q}{10-q}$.

Proof. We first observe reularity of w_ϵ. On account of (2.5), we can find a constant $C = C(T) > 0$ satisfying

$$\begin{aligned}\int_0^T \|\Delta w_\epsilon\|_q^q &\leqslant C \int_0^T \left(\|n_\epsilon\|_q^q + \|c_\epsilon\|_q^q + 1\right) \\ &\leqslant C\left(\left(\sup_{t>0} \|c_\epsilon\|_r\right)^q + \int_0^T \|n_\epsilon\|_q^q + 1\right).\end{aligned} \tag{3.19}$$

Then the Gagliardo-Nirenberg interpolation inequality (2.2) and (3.5) yield that

$$\begin{aligned}\int_0^T \|\nabla w_\epsilon\|_{\frac{5q}{5-q}}^{\frac{5q}{5-q}} &\leqslant C \int_0^T \left(\|\Delta w_\epsilon\|_q^{\frac{5q}{5-q}(1-\frac{q}{5})} \|w_\epsilon\|_{\frac{3q}{5-2q}}^{\frac{5q}{5-q}\cdot\frac{q}{5}} + \|w_\epsilon\|_{\frac{3q}{5-2q}}^{\frac{5q}{5-q}}\right) \\ &\leqslant C\left(\int_0^T \|\Delta w_\epsilon\|_q^q + 1\right).\end{aligned} \tag{3.20}$$

Thus, from (3.19) and (3.20) we see that for some $C = C(T) > 0$

$$\int_0^T \|\nabla w_\epsilon\|_{\frac{5q}{5-q}}^{\frac{5q}{5-q}} \leqslant C\left(\int_0^T \|n_\epsilon\|_q^q + \left(\sup_{t>0}\|c_\epsilon\|_r\right)^q + 1\right).$$

The last term is finite because of (3.2), (3.5) and the fact that $q \leqslant r = \frac{3q}{5-2q}$. Next, let α and β be in Lemma 3.5 with $\alpha = \frac{90q}{11q+40}$ and $\beta = \frac{30q}{17q-20}$. It can be easily checked that $2 < \alpha < 6$ and $2 < \beta$ because $q \in \left(\frac{5}{3}, 2\right)$. Then we can see via the maximal estimate (2.3) and the Hölder inequality that

$$\int_0^T \|\nabla \tilde{w}\|_m^m \leqslant C_T \int_0^T \|u_\epsilon c_\epsilon\|_m^m \leqslant C\left(\int_0^T \|u_\epsilon\|_\alpha^\beta\right)^{\frac{m}{\beta}} \left(\int_0^T \|c_\epsilon\|_{3r}^r\right)^{\frac{m}{r}} \tag{3.21}$$

which is valid since $\frac{1}{m} = \frac{1}{\alpha} + \frac{1}{3r} = \frac{1}{\beta} + \frac{1}{r}$, where $r = \frac{3q}{5-2q}$. The last term in (3.21) is finite due to (3.16) and (3.6). Hence, we have

$$\int_0^T \|\nabla c_\epsilon\|_m^m \leqslant \int_0^T \|\nabla w_\epsilon\|_m^m + \int_0^T \|\nabla \tilde{w}_\epsilon\|_m^m,$$

which is finite since $m < \frac{5q}{5-q}$ and (3.21). Then (3.18) is proved. □

Taking advantage of Lemma 3.6, we can obtain the maximal estimate for c_ϵ.

Lemma 3.7. *Let $T > 0$ and $q \in \left(\frac{5}{3}, 2\right)$. Then there exists $C = C(T) > 0$ such that for any $\epsilon > 0$,*

$$\int_0^T \|\partial_t c_\epsilon\|_{\frac{5q}{5+q}}^{\frac{5q}{5+q}} + \int_0^T \|\Delta c_\epsilon\|_{\frac{5q}{5+q}}^{\frac{5q}{5+q}} \leqslant C. \tag{3.22}$$

Proof. Applying (2.5), we obtain

$$\begin{aligned}
\int_0^T \|\partial_t c_\epsilon\|_{\frac{5q}{5+q}}^{\frac{5q}{5+q}} + \int_0^T \|\Delta c_\epsilon\|_{\frac{5q}{5+q}}^{\frac{5q}{5+q}} &\leqslant C\left(\int_0^T \|c_\epsilon\|_{\frac{5q}{5+q}}^{\frac{5q}{5+q}} + \int_0^T \|n_\epsilon\|_{\frac{5q}{5+q}}^{\frac{5q}{5+q}} + \int_0^T \|u_\epsilon \nabla c_\epsilon\|_{\frac{5q}{5+q}}^{\frac{5q}{5+q}} + 1\right)\\
&\leqslant C\left(\left(\sup_{t>0} \|c_\epsilon\|_r\right)^{\frac{5q}{5+q}} + \int_0^T \|n_\epsilon\|_q^q + \int_0^T \|u_\epsilon\|_{\frac{10}{3}}^{\frac{5q}{5+q}} \|\nabla c_\epsilon\|_m^{\frac{5q}{5+q}} + 1\right)\\
&\leqslant C\left(\left(\sup_{t>0} \|c_\epsilon\|_r\right)^{\frac{5q}{5+q}} + \int_0^T \|n_\epsilon\|_q^q + \int_0^T \|u_\epsilon\|_{\frac{10}{3}}^{\frac{10}{3}} + \int_0^T \|\nabla c_\epsilon\|_m^m + 1\right) < C,
\end{aligned}$$

due to (3.2), (3.5), (3.17) and (3.18). This proves (3.22). □

The following two lemmas are crucial to achieving the convergence property for n_ϵ.

Lemma 3.8. *Let $T > 0$ and $q \in \left(\frac{5}{3}, 2\right)$. Then for any $\gamma \in (0, 1)$ with $\gamma \leqslant \frac{4q-5}{5}$, there exists $C = C(T) > 0$ satisfying*

$$\int_0^T \int_\Omega \left|\nabla (n_\epsilon + 1)^{\frac{\gamma}{2}}(x, s)\right|^2 \mathrm{d}x\, \mathrm{d}s \leqslant C. \tag{3.23}$$

Proof. Testing the first equation in (2.1) by $\gamma n_\epsilon^{\gamma-1}$ and using integration by parts, we obtain

$$\begin{aligned}
\frac{4(1-\gamma)}{\gamma} \int_0^T \int_\Omega \left|\nabla n_\epsilon^{\frac{\gamma}{2}}\right|^2 &= \int_\Omega n_\epsilon^\gamma(\cdot, T) - \int_\Omega n_0^\gamma - (1-\gamma) \int_0^T \int_\Omega n_\epsilon^\gamma \Delta c_\epsilon\\
&\quad - \rho\gamma \int_0^T \int_\Omega n_\epsilon^\gamma + \mu\gamma \int_0^T \int_\Omega n_\epsilon^{\gamma+q-1} + \epsilon\gamma \int_0^T \int_\Omega n_\epsilon^{\kappa+\gamma-1}.
\end{aligned} \tag{3.24}$$

Using Young's inequality and (3.2), we have

$$\int_\Omega n_\epsilon^\gamma(\cdot, T) - \int_\Omega n_0^\gamma \leqslant C\left(\int_\Omega n_\epsilon + 1\right) < C,$$

and

$$-\rho\gamma \int_0^T \int_\Omega n_\epsilon^\gamma + \mu\gamma \int_0^T \int_\Omega n_\epsilon^{\gamma+q-1} + \epsilon\gamma \int_0^T \int_\Omega n_\epsilon^{\kappa+\gamma-1}$$

$$\leqslant C\left(\mu\int_0^T\int_\Omega n_\epsilon^q+\epsilon\int_0^T\int_\Omega n_\epsilon^\kappa+1\right)<C. \tag{3.25}$$

Since $0<\gamma\leqslant\frac{4q-5}{5}$, we see that $\frac{5+q}{5q}+\frac{\gamma}{q}\leqslant 1$. This leads

$$(1-\gamma)\int_0^T\int_\Omega n_\epsilon^\gamma\Delta c_\epsilon\leqslant\int_0^T\|n_\epsilon\|_q^\gamma\,\|\Delta c_\epsilon\|_{\frac{5q}{5+q}}$$

$$\leqslant C\left(\int_0^T\|n_\epsilon\|_q^q+\int_0^T\|\Delta c_\epsilon\|_{\frac{5q}{5+q}}^{\frac{5q}{5+q}}+1\right)<C. \tag{3.26}$$

Collecting (3.24), (3.25) and (3.26), we obtain

$$\int_0^T\int_\Omega n_\epsilon^{\gamma-2}\,|\nabla n_\epsilon|^2=\frac{4}{\gamma^2}\int_0^T\int_\Omega\left|\nabla n_\epsilon^{\frac{\gamma}{2}}\right|^2\leqslant C. \tag{3.27}$$

Since $\gamma-2<0$, we get $(n_\epsilon+1)^{\gamma-2}\leqslant n_\epsilon^{\gamma-2}$, hence (3.23).

□

In the following lemma, we mean by $(W_0^{k,2})^*$ the dual space of $W_0^{k,2}$.

Lemma 3.9. *Let $T>0$ and $q\in\left(\frac{5}{3},2\right)$. Then for any $\gamma\in(0,1)$ with $\gamma\leqslant\frac{4q-5}{5}$, there exists $k\in\mathbb{N}$ and $C=C(T)>0$, independent of ϵ, satisfying*

$$\left\|\partial_t(1+n_\epsilon)^{\frac{\gamma}{2}}\right\|_{L^1(0,T;(W_0^{k,2}(\Omega))^*)}\leqslant C.$$

Proof. Fix $k\in\mathbb{N}$ to be choosen later and let $\varphi\in W_0^{k,2}(\Omega)$ be a test function. We observe that

$$\frac{2}{\gamma}\int_\Omega\partial_t(n_\epsilon+1)^{\frac{\gamma}{2}}\varphi=\int_\Omega(1+n_\epsilon)^{\frac{\gamma}{2}-1}\partial_t n_\epsilon\varphi$$

$$=\int_\Omega(1+n_\epsilon)^{\frac{\gamma}{2}-1}\left(\Delta n_\epsilon-u_\epsilon\cdot\nabla n_\epsilon-\nabla\cdot(n_\epsilon\nabla c_\epsilon)+\rho n_\epsilon-\mu n_\epsilon^q-\epsilon n_\epsilon^\kappa\right)\varphi=:\sum_{i=1}^6 J_i.$$

First, employing integration by parts and Hölder inequality, we can estimate J_1 as follows:

$$|J_1|\leqslant C\int_\Omega(1+n_\epsilon)^{\frac{\gamma}{2}-2}\,|\nabla n_\epsilon|^2\,|\varphi|+C\int_\Omega(1+n_\epsilon)^{\frac{\gamma}{2}-1}\,|\nabla n_\epsilon|\,|\nabla\varphi|$$

$$\leqslant C\,\|\varphi\|_\infty\left\|\nabla n_\epsilon^{\frac{\gamma}{2}}\right\|_2^2+C\,\|\nabla\varphi\|_2\left(1+\left\|\nabla n_\epsilon^{\frac{\gamma}{2}}\right\|_2^2\right), \tag{3.28}$$

where we used the fact that $(1+n_\epsilon)^{\frac{\gamma}{2}-2}\leqslant(1+n_\epsilon)^{\gamma-2}\leqslant n_\epsilon^{\gamma-2}$. Similarly, the second and third terms are controlled as follows:

$$|J_2|\leqslant C\int_\Omega(1+n_\epsilon)^{\frac{\gamma}{2}-2}n_\epsilon^{2-\frac{\gamma}{2}}\left|\nabla n_\epsilon^{\frac{\gamma}{2}}\right|\,|u_\epsilon|\,|\varphi|+C\int_\Omega(1+n_\epsilon)^{\frac{\gamma}{2}-1}\,|n_\epsilon|\,|u_\epsilon|\,|\nabla\varphi|$$

$$
\begin{aligned}
&\leqslant C\left\|\nabla n_\epsilon^{\frac{\gamma}{2}}\right\|_2 \|u_\epsilon\|_{\frac{10}{3}} \|\varphi\|_5 + C\|1+n_\epsilon\|_q^{\frac{\gamma}{2}} \|u_\epsilon\|_{\frac{10}{3}} \|\nabla\varphi\|_{\frac{10q}{7q-5\gamma}} \\
&\leqslant C\left(\left\|\nabla n_\epsilon^{\frac{\gamma}{2}}\right\|_2^{\frac{10}{7}} + C\|u_\epsilon\|_{\frac{10}{3}}^{\frac{10}{3}}\right)\|\varphi\|_5 + C\left(\|1+n_\epsilon\|_q^{\frac{5\gamma}{7}} + \|u_\epsilon\|_{\frac{10}{3}}^{\frac{10}{3}}\right)\|\nabla\varphi\|_{\frac{10q}{3q+5}} \\
&\leqslant C\left(\left\|\nabla n_\epsilon^{\frac{\gamma}{2}}\right\|_2^{2} + C\|u_\epsilon\|_{\frac{10}{3}}^{\frac{10}{3}} + 1\right)\|\varphi\|_5 + C\left(\|n_\epsilon\|_q^{q} + \|u_\epsilon\|_{\frac{10}{3}}^{\frac{10}{3}} + 1\right)\|\nabla\varphi\|_{\frac{10q}{3q+5}} \qquad (3.29)
\end{aligned}
$$

because $\gamma < 1 < \frac{7q}{5}$ and $\frac{10q}{7q-5\gamma} \leqslant \frac{10q}{3q+5}$.

$$
\begin{aligned}
|J_3| &\leqslant C\int_\Omega (1+n_\epsilon)^{\frac{\gamma}{2}-2} n_\epsilon^{2-\frac{\gamma}{2}} \left|\nabla n_\epsilon^{\frac{\gamma}{2}}\right| |\nabla c_\epsilon| |\varphi| + C\int_\Omega (1+n_\epsilon)^{\frac{\gamma}{2}-1} |n_\epsilon| |\nabla c_\epsilon| |\nabla\varphi| \\
&\leqslant C\left\|\nabla n_\epsilon^{\frac{\gamma}{2}}\right\|_2 \|\nabla c_\epsilon\|_q \|\varphi\|_{\frac{2q}{2-q}} + C\|1+n_\epsilon\|_q^{\frac{\gamma}{2}} \|\nabla c_\epsilon\|_q \|\nabla\varphi\|_{\frac{2q}{2q-2-\gamma}} \\
&\leqslant C\left(\left\|\nabla n_\epsilon^{\frac{\gamma}{2}}\right\|_2^2 + \|\nabla c_\epsilon\|_m^m + 1\right)\|\varphi\|_{\frac{2q}{2-q}} \\
&\quad + C\left(\|n_\epsilon\|_q^q + \|\nabla c_\epsilon\|_m^m + 1\right)\|\nabla\varphi\|_{\frac{2q}{2q-2-\gamma}}, \qquad (3.30)
\end{aligned}
$$

where we used the fact that $q < m$ and $\gamma \leqslant \frac{4q-5}{5} < 2q-2$. Estimates for J_4, J_5 and J_6 can be easily obtained by the following calculation

$$
|J_4| \leqslant \int_\Omega (1+n_\epsilon)^{\frac{\gamma}{2}} |\varphi| \leqslant C\left(\|n_\epsilon\|_q^q + 1\right)\|\varphi\|_\infty, \qquad (3.31)
$$

$$
|J_5| \leqslant \int_\Omega (1+n_\epsilon)^{\frac{\gamma}{2}+q-1} |\varphi| \leqslant C\left(\|n_\epsilon\|_q^q + 1\right)\|\varphi\|_\infty, \qquad (3.32)
$$

$$
|J_6| \leqslant \epsilon\int_\Omega (1+n_\epsilon)^{\frac{\gamma}{2}+\kappa-1} |\varphi| \leqslant C\left(\epsilon\|n_\epsilon\|_\kappa^\kappa + 1\right)\|\varphi\|_\infty. \qquad (3.33)
$$

Collecting all of estimates (3.28)-(3.33) and applying the Sobolev embedding theorem, we have

$$
\begin{aligned}
\left|\int_\Omega \partial_t(1+n_\epsilon)^{\frac{\gamma}{2}}\varphi\right| &\leqslant C\left(\left\|\nabla n_\epsilon^{\frac{\gamma}{2}}\right\|_2^2 + \|u_\epsilon\|_{\frac{10}{3}}^{\frac{10}{3}} + \|\nabla c_\epsilon\|_m^m + \|n_\epsilon\|_q^q + \epsilon\|n_\epsilon\|_\kappa^\kappa + 1\right) \\
&\qquad \times \|\varphi\|_{W_0^{1,\infty}(\Omega)}. \qquad (3.34)
\end{aligned}
$$

Choose k sufficiently large that $k > \frac{5}{2}$. Then $W_0^{k,2}(\Omega)$ is embedded into $W^{1,\infty}(\Omega)$ by Sobolev embedding. Finally, integration of (3.34) over $(0,T)$ leads, with the help of (3.1), (3.2), (3.18), (3.16) and (3.23), that

$$
\left\|\partial_t(1+n_\epsilon)^{\frac{\gamma}{2}}\right\|_{L^1(0,T;(W_0^{k,2}(\Omega))^*)} \leqslant C,
$$

as desired. □

The estimate for the time derivative of u_ϵ is obtained by the simple calculation.

Lemma 3.10. *Let $T > 0$. Then there exists $C > 0$ such that for any $\epsilon > 0$,*

$$
\|\partial_t u_\epsilon\|_{L^1(0,T;(W_{0,\sigma}^{1,5}(\Omega))^*)} \leqslant C(1+T). \qquad (3.35)
$$

Proof. Given $\varphi \in C_0^\infty(\Omega \times [0,\infty); \mathbb{R}^3)$ with $\nabla\cdot\varphi = 0$, we compute

$$\begin{aligned}
\left|\int_\Omega \partial_t u_\epsilon \varphi\right| &= \left|-\int_\Omega \nabla u_\epsilon \cdot \nabla\varphi - \int_\Omega (Y_\epsilon u_\epsilon \otimes u_\epsilon)\nabla\varphi + \int_\Omega n_\epsilon \nabla\phi\varphi\right| \\
&\leqslant \|\nabla u_\epsilon\|_2 \|\nabla\varphi\|_2 + \|Y_\epsilon u_\epsilon \otimes u_\epsilon\|_{\frac{5}{4}} \|\nabla\varphi\|_5 + \|n_\epsilon\|_q \|\varphi\|_{\frac{q}{q-1}} \|\nabla\phi\|_\infty \\
&\leqslant \left(\|\nabla u_\epsilon\|_2^2 + 1\right)\|\nabla\varphi\|_2 + C\left(\|Y_\epsilon u_\epsilon\|_2^2 + \|u_\epsilon\|_{\frac{10}{3}}^{\frac{10}{3}} + 1\right)\|\nabla\varphi\|_5 \\
&\quad + C\left(\|n_\epsilon\|_q^q + 1\right)\|\varphi\|_\infty \\
&\leqslant C\left(\|\nabla u_\epsilon\|_2^2 + \|u_\epsilon\|_{\frac{10}{3}}^{\frac{10}{3}} + \|n_\epsilon\|_q^q + 1\right)\|\varphi\|_{W_0^{1,5}(\Omega)}.
\end{aligned} \tag{3.36}$$

Here we used the well-known inequality $\|Y_\epsilon u_\epsilon\|_2^2 \leqslant C\|u_\epsilon\|_2^2$. Thus, integrating (3.36) over $(0,T)$ yields (3.35). □

4. Convergence

We are now ready to prove the convergence property for $(n_\epsilon, c_\epsilon, u_\epsilon)$.

Lemma 4.1. *Let $q \in (\frac{5}{3}, 2)$, $\gamma \in (0,1)$ with $\gamma \leqslant \frac{4q-5}{5}$ and $p \in (1,q)$. A number m is given in Lemma 3.6. Then the classical solution $(n_\epsilon, c_\epsilon, u_\epsilon)$ of* (2.1) *satisfies the following convergence property.*

$$n_\epsilon \to n \quad \text{a.e. in } \Omega\times(0,\infty), \tag{4.1}$$

$$n_\epsilon \rightharpoonup n \quad \text{in } L^q_{loc}(\overline{\Omega}\times[0,\infty)), \tag{4.2}$$

$$n_\epsilon \to n \quad \text{in } L^p_{loc}(\overline{\Omega}\times[0,\infty)), \tag{4.3}$$

$$n_\epsilon^{\frac{\gamma}{2}} \rightharpoonup n^{\frac{\gamma}{2}} \quad \text{in } L^2_{loc}([0,\infty); W^{1,2}(\Omega)), \tag{4.4}$$

$$c_\epsilon \to c \quad \text{a.e. in } \Omega\times(0,\infty), \tag{4.5}$$

$$c_\epsilon \rightharpoonup c \quad \text{in } L^m_{loc}([0,\infty); W^{1,m}(\Omega)), \tag{4.6}$$

$$\Delta c_\epsilon \rightharpoonup \Delta c \quad \text{in } L^{\frac{5q}{5+q}}_{loc}(\overline{\Omega}\times[0,\infty)), \tag{4.7}$$

$$u_\epsilon \to u \quad \text{a.e. in } \Omega\times(0,\infty), \tag{4.8}$$

$$u_\epsilon \to u \quad \text{in } L^2_{loc}(\overline{\Omega}\times[0,\infty)), \tag{4.9}$$

$$u_\epsilon \rightharpoonup u \quad \text{in } L^{\frac{10}{3}}_{loc}(\overline{\Omega}\times[0,\infty)), \tag{4.10}$$

$$\nabla u_\epsilon \rightharpoonup \nabla u \quad \text{in } L^2_{loc}(\overline{\Omega}\times[0,\infty)). \tag{4.11}$$

Proof. For convenience, we denote a subsequence $(\epsilon_j)_{j\in\mathbb{N}}$ of ϵ by ϵ itself. First, Lemma 2.4 gives the pointwise convergence of c_ϵ in (4.5):

$$c_\epsilon \to c \quad \text{a.e. in } \Omega\times(0,\infty).$$

Indeed, using Lemma 2.4, bounds for c_ϵ in $L^m_{loc}([0,\infty); W^{1,m}(\Omega))$ and $\partial_t c_\epsilon$ in $L^{\frac{5q}{5+q}}_{loc}(\overline{\Omega}\times[0,\infty))$, asserted in Lemma 3.6 and Lemma 3.7, yield the strong convergence of c_ϵ in $L^m_{loc}(\overline{\Omega}\times[0,\infty))$ which in particular implies (4.5). Similarly, by Lemma 3.8 and 3.9, we see that $(1+n_\epsilon)^{\frac{\gamma}{2}}_{\epsilon\in(0,1)}$ is relatively compact in $L^2_{loc}(\overline{\Omega}\times[0,\infty))$ with respect to the strong topology by Lemma 2.4. we can thus see that

$$n_\epsilon \to n \quad \text{a.e. in } \Omega\times(0,\infty)\,,$$

which proves (4.1), as well as (4.4) holds. Likewise, exploiting boundedness of u_ϵ and of its time derivative, as proved in Lemma 3.5 and Lemma 3.10, and using Lemma 2.4 again, we have (4.8) and (4.9). The convergence properites (4.2), (4.6), (4.7), (4.10) and (4.11) is a direct consequence of (3.2), (3.18), (3.22), (3.17) and (3.13), respectively. In order to prove (4.3), we use (3.2) again, which implies that $\int_0^T \left\|n_\epsilon^p\right\|_{\frac{q}{p}} \leqslant C$ for all $t>0$. Hence we have

$$n_\epsilon^p \rightharpoonup n^p \quad \text{in } L^{\frac{q}{p}}_{loc}(\overline{\Omega}\times[0,\infty))$$

as $\epsilon \searrow 0$. By this weak convergence we have

$$\int_0^T\int_\Omega n_\epsilon^p \to \int_0^T\int_\Omega n^p \quad \text{for all } t>0,$$

which asserts that $n_\epsilon \to n$ in $L^p_{loc}(\overline{\Omega}\times[0,\infty))$ due to uniform convexity of L^p-space for $p>1$. This proves (4.3). □

We shall prove the limit (n,c,u) in Lemma 4.1 is a solution of our main system (1.1)–(1.3) in the sense of Definition 2. We first focus on c and u which satisfy (1.1) and (1.2) in the standard weak sence. In addition, we show that n is a weak sub-solution in the sense of Definition 1.

Lemma 4.2. *Let (n,c,u) be the limit function and vector field in Lemma 4.1. Then (1.6) and (1.7) hold.*

Proof. We multiply the second equation in (2.1) by the test function $\varphi\in C_0^\infty(\overline{\Omega}\times[0,\infty))$ to get, for all $\epsilon\in(0,1)$,

$$-\int_0^\infty\int_\Omega c_\epsilon\varphi_t - \int_\Omega c_0\varphi(\cdot,0) = -\int_0^\infty\int_\Omega \nabla c_\epsilon\cdot\nabla\varphi - \int_0^\infty\int_\Omega c_\epsilon\varphi + \int_0^\infty\int_\Omega n_\epsilon\varphi + \int_0^\infty\int_\Omega c_\epsilon u_\epsilon\cdot\nabla\varphi.$$

Applying (4.6) and (4.2), we easily obtain

$$\int_0^\infty\int_\Omega c_\epsilon\varphi_t \to \int_0^\infty\int_\Omega c\varphi_t, \qquad \int_0^\infty\int_\Omega c_\epsilon\varphi \to \int_0^\infty\int_\Omega c\varphi, \tag{4.12}$$

$$\int_0^\infty\int_\Omega \nabla c_\epsilon\cdot\nabla\varphi \to \int_0^\infty\int_\Omega \nabla c\cdot\nabla\varphi, \qquad \int_0^\infty\int_\Omega n_\epsilon\varphi \to \int_0^\infty\int_\Omega n\varphi \tag{4.13}$$

as $\epsilon = \epsilon_j \searrow 0$. On the other hand, combining (4.3) and (4.10) infers that $c_\epsilon u_\epsilon \rightharpoonup cu$ in L^s_{loc} for $s := \frac{10+3p}{10p} \geqslant 1$, which proves

$$\int_0^\infty \int_\Omega c_\epsilon u_\epsilon \cdot \nabla\varphi \to \int_0^\infty \int_\Omega cu \cdot \nabla\varphi \tag{4.14}$$

as $\epsilon \searrow 0$. Next we multiply the third equation in (2.1) by $\varphi \in C_0^\infty(\Omega \times [0,\infty); \mathbb{R}^3)$ with $\nabla\cdot\varphi = 0$ that gives

$$-\int_0^\infty \int_\Omega u_\epsilon \cdot \varphi_t - \int_\Omega u_0 \cdot \varphi(\cdot,0) = -\int_0^\infty \int_\Omega \nabla u_\epsilon \cdot \nabla\varphi + \int_0^\infty \int_\Omega (Y_\epsilon u_\epsilon \otimes u_\epsilon) \cdot \nabla\varphi + \int_0^\infty \int_\Omega n_\epsilon \nabla\phi \cdot \varphi$$

for all $\epsilon \in (0,1)$. Similar to the above, (4.10), (4.11), (4.2) and the condition on $\nabla\phi$, as assumed in (1.8), imply that

$$\int_0^\infty \int_\Omega u_\epsilon \cdot \varphi_t \to \int_0^\infty \int_\Omega u \cdot \varphi_t, \qquad \int_0^\infty \int_\Omega \nabla u_\epsilon \cdot \nabla\varphi \to \int_0^\infty \int_\Omega \nabla u \cdot \nabla\varphi, \tag{4.15}$$

$$\int_0^\infty \int_\Omega n_\epsilon \nabla\phi \cdot \varphi \to \int_0^\infty \int_\Omega n \nabla\phi \cdot \varphi \tag{4.16}$$

as $\epsilon \searrow 0$. Since it is well known that $Y_\epsilon u_\epsilon \to u$ in $L^2_{loc}(\Omega \times (0,\infty))$, with the aid of (4.9), we obtain $Y_\epsilon u_\epsilon \otimes u_\epsilon \to u \otimes u$ in $L^1_{loc}(\Omega \times (0,\infty))$. This proves

$$\int_0^\infty \int_\Omega (Y_\epsilon u_\epsilon \otimes u_\epsilon) \cdot \nabla\varphi \to \int_0^\infty \int_\Omega (u \otimes u) \cdot \nabla\varphi \tag{4.17}$$

as $\epsilon \searrow 0$. We collect (4.12)–(4.17) to conclude the proof. □

So far, we used that $q > \frac{5}{3}$. In the next Lemma, however, it is necessary to assume that $q > \frac{20}{11}$, which is crucial to show convergence of $n_\epsilon \nabla c_\epsilon$ (see the estimate (4.21) below).

Lemma 4.3. *Let $q \in \left(\frac{20}{11}, 2\right)$ and (n,c,u) be the limit function and vector field in Lemma 4.1. Then n is a $\gamma-$entropy sub-solution of* (1.1)–(1.3) *with $\gamma = 1$, that is, n satisfies the following integral inequality*

$$-\int_0^\infty \int_\Omega n\varphi_t - \int_\Omega n_0\varphi(\cdot,0) \leqslant \int_0^\infty \int_\Omega n\Delta\varphi + \int_0^\infty \int_\Omega n\nabla c \cdot \nabla\varphi + \rho \int_0^\infty \int_\Omega n\varphi - \mu \int_0^\infty \int_\Omega n^q\varphi + \int_0^\infty \int_\Omega nu \cdot \nabla\varphi$$

for all nonnegative $\varphi \in C_0^\infty(\overline{\Omega} \times [0,\infty))$.

Proof. We multiply the first equation in (2.1) by a nonnegative test function $\varphi \in C_0^\infty(\overline{\Omega} \times [0,\infty))$ and integrate over $\Omega \times (0,\infty)$. By suitable integration by parts,

$$-\int_0^\infty \int_\Omega n_\epsilon \varphi_t - \int_\Omega n_0 \varphi(\cdot,0) = \int_0^\infty \int_\Omega n_\epsilon \Delta\varphi + \int_0^\infty \int_\Omega n_\epsilon \nabla c_\epsilon \cdot \nabla\varphi + \rho \int_0^\infty \int_\Omega n_\epsilon \varphi$$
$$- \mu \int_0^\infty \int_\Omega n_\epsilon^q \varphi - \epsilon \int_0^\infty \int_\Omega n_\epsilon^\kappa \varphi + \int_0^\infty \int_\Omega n_\epsilon u_\epsilon \cdot \nabla\varphi$$

for all $\epsilon \in (0,1)$. Using (4.2), we see that

$$\int_0^\infty \int_\Omega n_\epsilon \varphi_t \to \int_0^\infty \int_\Omega n\varphi_t, \qquad \int_0^\infty \int_\Omega n_\epsilon \Delta\varphi \to \int_0^\infty \int_\Omega n\Delta\varphi, \tag{4.18}$$

$$\text{and} \quad \rho \int_0^\infty \int_\Omega n_\epsilon \varphi \to \rho \int_0^\infty \int_\Omega n\varphi \tag{4.19}$$

as $\epsilon \searrow 0$. Funthermore, applying strong convergence of $(n_\epsilon)_{\epsilon\in(0,1)}$, $(u_\epsilon)_{\epsilon\in(0,1)}$ as asserted in Lemma 4.1, we have

$$\int_0^\infty \int_\Omega n_\epsilon u_\epsilon \cdot \nabla\varphi \to \int_0^\infty \int_\Omega nu \cdot \nabla\varphi \tag{4.20}$$

as $\epsilon \searrow 0$. Since $q \in (\frac{20}{11}, 2)$, we can take $p < q$ close to q satisfying $\frac{1}{p} + \frac{1}{m} < 1$. Then, by (4.3) and (4.6) we see that

$$\int_0^\infty \int_\Omega n_\epsilon \nabla c_\epsilon \cdot \nabla\varphi \to \int_0^\infty \int_\Omega n\nabla c \cdot \nabla\varphi \tag{4.21}$$

as $\epsilon \searrow 0$. Besides, the nonnegativity of n_ϵ and φ leads that

$$- \epsilon \int_0^\infty \int_\Omega n_\epsilon^\kappa \varphi \leqslant 0 \tag{4.22}$$

for all $\epsilon \in (0,1)$. Lastly, we observe that by Fatou's lemma

$$\mu \int_0^\infty \int_\Omega n^q \varphi \leqslant \liminf_{\epsilon\searrow 0} \left\{ \mu \int_0^\infty \int_\Omega n_\epsilon^q \varphi \right\}. \tag{4.23}$$

Hence, combining (4.18)–(4.23), we conclude that n is a $\gamma-$entropy sub-solution with $\gamma = 1$. □

Now we shall prove that (n,c,u) as in Lemma 4.1 is a $\gamma-$entropy super-solution.

Lemma 4.4. *Let $q \in \left(\frac{5}{3}, 2\right)$ and (n,c,u) be the limit functions and vector field in Lemma 4.1. Then for any fixed $\gamma \in \left(0, \frac{4q-5}{5}\right)$, n is a $\gamma-$entropy supersolution of* (1.1)–(1.3).

Proof. Let $0 \leqslant \varphi \in C_0^\infty(\overline{\Omega} \times [0,\infty))$ be arbitralily. Testing the first equation in (2.1) by $\gamma n_\epsilon^{\gamma-1}\varphi$ and integrating by parts, we have

$$\begin{aligned}-\int_0^\infty\int_\Omega n_\epsilon^\gamma\varphi_t - \int_\Omega n_0^\gamma\varphi(\cdot,0) &= \gamma(1-\gamma)\int_0^\infty\int_\Omega n_\epsilon^{\gamma-2}|\nabla n_\epsilon|^2\varphi + \int_0^\infty\int_\Omega n_\epsilon^\gamma\Delta\varphi\\ &+ (1-\gamma)\int_0^\infty\int_\Omega n_\epsilon^\gamma\Delta c_\epsilon\varphi + \int_0^\infty\int_\Omega n_\epsilon^\gamma\nabla c_\epsilon\cdot\nabla\varphi\\ &+ \rho\gamma\int_0^\infty\int_\Omega n_\epsilon^\gamma\varphi - \mu\gamma\int_0^\infty\int_\Omega n_\epsilon^{q+\gamma-1}\varphi - \epsilon\gamma\int_0^\infty\int_\Omega n_\epsilon^{\kappa+\gamma-1}\varphi + \int_0^\infty\int_\Omega n_\epsilon^\gamma u_\epsilon\cdot\nabla\varphi\end{aligned}$$

for all $\epsilon \in (0,1)$. Since $\gamma \in (0,1)$, we obtaing the strong convergence $n_\epsilon^\gamma \to n^\gamma$ in $L^p_{loc}(\Omega \times (0,\infty))$ for $p \in (1,q)$ due to (4.3) which follows

$$\int_0^\infty\int_\Omega n_\epsilon^\gamma\varphi_t \to \int_0^\infty\int_\Omega n^\gamma\varphi_t,\quad \int_0^\infty\int_\Omega n_\epsilon^\gamma\Delta\varphi \to \int_0^\infty\int_\Omega n^\gamma\Delta\varphi,\quad \rho\int_0^\infty\int_\Omega n_\epsilon^\gamma\varphi \to \rho\int_0^\infty\int_\Omega n^\gamma\varphi \tag{4.24}$$

as $\epsilon \searrow 0$. Furthermore, referring to (4.20) and (4.21) we have

$$\int_0^\infty\int_\Omega n_\epsilon^\gamma\nabla c_\epsilon\cdot\nabla\varphi \to \int_0^\infty\int_\Omega n^\gamma\nabla c\cdot\nabla\varphi \quad\text{and}\quad \int_0^\infty\int_\Omega n_\epsilon^\gamma u_\epsilon\cdot\nabla\varphi \to \int_0^\infty\int_\Omega n^\gamma u\cdot\nabla\varphi \tag{4.25}$$

as $\epsilon \searrow 0$. As $n_\epsilon^{q+\gamma-1}$ is bounded in $L^k_{loc}(\Omega \times (0,\infty))$ for $k = \frac{q}{q+\gamma-1} > 1$, uniformly in ϵ, the weak convergence $n_\epsilon^{q+\gamma-1} \rightharpoonup n^{q+\gamma-1}$ in $L^k_{loc}(\Omega \times (0,\infty))$ holds. Thus, we have

$$\int_0^\infty\int_\Omega n_\epsilon^{q+\gamma-1}\varphi \to \int_0^\infty\int_\Omega n^{q+\gamma-1}\varphi \tag{4.26}$$

as $\epsilon \searrow 0$. Since $\frac{5+q}{5q} + \frac{\gamma}{q} < 1$, it follows from (4.3) and (4.7) that

$$\int_0^\infty\int_\Omega n_\epsilon^\gamma\Delta c_\epsilon\varphi \to \int_0^\infty\int_\Omega n^\gamma\Delta c\,\varphi \tag{4.27}$$

as $\epsilon \searrow 0$. For the regularizing term, we note that from Hölder inequality and (3.2)

$$\left|-\gamma\epsilon\int_0^\infty\int_\Omega n_\epsilon^{\kappa+\gamma-1}\varphi\right| \leqslant C_1\gamma\epsilon^{\frac{1-\gamma}{\kappa}}\|\varphi\|_\infty\left(\epsilon\int_0^\infty\int_\Omega n_\epsilon{}^\kappa\right)^{\frac{\kappa+\gamma-1}{\kappa}} \leqslant C_2\epsilon^{\frac{1-\gamma}{\kappa}}$$

for all $\epsilon \in (0,1)$. Hence, we have

$$-\gamma\epsilon\int_0^\infty\int_\Omega n_\epsilon^{\kappa+\gamma-1}\varphi \to 0 \tag{4.28}$$

as $\epsilon \searrow 0$. Finally, from (4.4) and the lower semicontinuity of the seminorm $\|\cdot\|$ defined by $\|f\| := (\int_0^\infty \int_\Omega f^2\varphi)^{\frac{1}{2}}$ with respect to weak convergence, we obtain

$$\gamma(1-\gamma)\int_0^\infty \int_\Omega n^{\gamma-2}\,|\nabla n|^2\,\varphi \leqslant \gamma(1-\gamma)\liminf_{\epsilon\searrow 0}\int_0^\infty \int_\Omega n_\epsilon^{\gamma-2}\,|\nabla n_\epsilon|^2\,\varphi. \tag{4.29}$$

Therefore, collecting (4.24)–(4.29) proves that n is a $\gamma-$entropy super-solution of (1.1)–(1.3). □

Proof of Theorem 1.1. This is the combination of Lemma 4.2, Lemma 4.3 and Lemma 4.4. □

5. Asymptotic behavior

The following Lemma is elementary, but for clarity, we give its detail.

Lemma 5.1. *Let $a > 1$ and $f \in L^1([0,\infty))$. Suppose there is $t_0 > 0$ such that $f(t) \leqslant Nt^{-a}$ for sufficiently large $t \geqslant t_0$. Assume further that a non-negative measurable function $y(t)$ satisfies*

$$y'(t) + y(t) \leqslant f(t).$$

Then, $y(t) \leqslant Ct^{-a}$ for sufficiently large t.

Proof. Firstly we note that $y(t)$ is bounded uniformly in time. Then, using the integrating factor, we have for $t \geqslant t_0$

$$e^{2t}y(2t) - e^t y(t) \leqslant \int_t^{2t} e^\tau f(\tau)\,\mathrm{d}\tau,$$

which yields, using integration by parts,

$$\begin{aligned} y(2t) &\leqslant e^{-t}y(t) + Ne^{-2t}\int_t^{2t} e^\tau \tau^{-a}\,\mathrm{d}\tau \\ &\leqslant Ce^{-t} + Ne^{-2t}\left[e^{2t}\,(2t)^{-\alpha} - e^t t^{-\alpha} + \alpha\int_t^{2t} e^\tau \tau^{-\alpha-1}\,\mathrm{d}\tau\right] \\ &\leqslant C(2t)^{-\alpha}. \end{aligned}$$

□

Proof of Theorem 1.2. • (**The case** $\rho = 0$) Noting that $\rho = 0$, we integrate the equation for n_ϵ in (2.1) over Ω to get

$$\frac{d}{dt}\int_\Omega n_\epsilon(\cdot,t)\,\mathrm{d}x \leqslant -\frac{\mu}{|\Omega|^{q-1}}\left(\int_\Omega n_\epsilon(\cdot,t)\,\mathrm{d}x\right)^q.$$

A standard argument of ODE implies that

$$\int_\Omega n_\epsilon(\cdot,t)\,\mathrm{d}x \leqslant C(1+t)^{-\frac{1}{q-1}} \quad \text{for all } t>0.$$

Next, integrating the equation of c_ϵ, it follows that for all $t>0$,

$$\frac{d}{dt}\int_\Omega c_\epsilon(\cdot,t)\,\mathrm{d}x + \int_\Omega c_\epsilon(\cdot,t)\,\mathrm{d}x = \int_\Omega n_\epsilon(\cdot,t)\,\mathrm{d}x.$$

Let $g(t) = \int_\Omega n_\epsilon(\cdot,t)\,\mathrm{d}x$. Then, since $\frac{1}{q-1} > 1$, we observe that $g \in L^1([0,\infty))$, and thus, via Lemma 5.1, it follows that

$$\int_\Omega c_\epsilon(\cdot,t)\,\mathrm{d}x \leqslant C(1+t)^{-\frac{1}{q-1}} \quad \text{for all } t>0.$$

On the other hand, putting $m = 3q-2$ and testing the equation for c_ϵ in (2.1) by c^{m-1}, we get

$$\begin{aligned}\frac{1}{m}\frac{d}{dt}\int_\Omega c_\epsilon^m(\cdot,t)\,\mathrm{d}x + \int_\Omega \left|\nabla c_\epsilon^{\frac{m}{2}}\right|^2 \mathrm{d}x + \int_\Omega c_\epsilon^m\,\mathrm{d}x &= \int_\Omega n_\epsilon c_\epsilon^{m-1}\,\mathrm{d}x \\ \leqslant \|n_\epsilon\|_{\frac{3m}{2m+1}} \left\|c_\epsilon^{m-1}\right\|_{\frac{3m}{m-1}} &= \|n_\epsilon\|_{\frac{3m}{2m+1}} \left\|c_\epsilon^{\frac{m}{2}}\right\|_6^{\frac{2(m-1)}{m}} \\ \leqslant C\,\|n_\epsilon\|_{\frac{3m}{2m+1}} \left(\left\|\nabla c_\epsilon^{\frac{m}{2}}\right\|_2^{\frac{2(m-1)}{m}} + 1\right) &\leqslant C\,\|n_\epsilon\|_{\frac{3m}{2m+1}}^m + \frac{1}{2}\left\|\nabla c_\epsilon^{\frac{m}{2}}\right\|_2^2.\end{aligned}$$

Since $m = 3q-2$, we observe that

$$\|n_\epsilon\|_{\frac{3m}{2m+1}}^m = \|n_\epsilon\|_{\frac{3q-2}{2q-1}}^{3q-2} \leqslant \|n_\epsilon\|_1^{2(q-1)}\|n_\epsilon\|_q^q \leqslant C(1+t)^{-2}\,\|n_\epsilon(t)\|_q^q.$$

Let $h(t) = (1+t)^{-2}\,\|n_\epsilon(t)\|_q^q$. Then, it is direct that $h \in L^1((0,\infty))$. Setting $Z(t) = \int_\Omega c_\epsilon^m(\cdot,t)\,\mathrm{d}x$, we have $Z'(t) + Z(t) \leqslant h(t)$, which yields

$$e^{2t}Z(2t) - e^tZ(t) = \int_t^{2t} e^\tau h(\tau)d\tau,$$

which implies that

$$Z(2t) \leqslant e^{-t}Z(t) + C(1+t)^{-2}\int_t^{2t}\|n_\epsilon(\tau)\|_q^q\,d\tau \leqslant C(1+t)^{-2}.$$

Noting that $Z(t) \leqslant C$ for all $t>0$, we have

$$\|c_\epsilon(t)\|_{3q-2} \leqslant C(1+t)^{-\frac{2}{3q-2}}.$$

Hence, interpolation gives

$$\|c_\epsilon(t)\|_l \leqslant \|c_\epsilon(t)\|_1^{1-\theta}\,\|c_\epsilon(t)\|_{3q-2}^{\theta} \leqslant C(1+t)^{-\frac{2lq+q-3l}{3l(q-1)^2}},$$

where $1 \leqslant l \leqslant 3q-2$ and $\theta = \frac{(l-1)(3q-2)}{3l(q-1)}$. On the other hand, in case that $3q-2 \leqslant l \leqslant \frac{3q}{5-2q}$, interpolation gives

$$\|c_\epsilon(t)\|_k \leqslant \|c_\epsilon(t)\|_{3q-2}^{\theta_1}\,\|c_\epsilon(t)\|_{\frac{3q}{5-2q}}^{1-\theta_1} \leqslant C(1+t)^{-\frac{3q-(5-2q)k}{k(3q-5)(q-1)}},$$

where $\theta_1 = \frac{(3q-(5-2q)k)(3q-2)}{2k(3q-5)(q-1)}$. Finally, recalling (3.14) and (3.15), we have

$$\begin{aligned}\frac{d}{dt}\int_\Omega |u_\epsilon(\cdot,t)|^2\,\mathrm{d}x + \frac{1}{2}\int_\Omega |\nabla u_\epsilon(\cdot,t)|^2\,\mathrm{d}x &\leqslant C\left(\int_\Omega |n_\epsilon(\cdot,t)|^{\frac{6}{5}}\right)^{\frac{5}{3}}\\ &\leqslant C\,\|n_\epsilon(t)\|_1^{\frac{5q-6}{3(q-1)}}\,\|n_\epsilon(t)\|_q^{\frac{q}{3(q-1)}},\end{aligned}$$

where we used

$$\|n_\epsilon\|_{\frac{6}{5}} \leqslant \|n_\epsilon\|_1^{\theta}\,\|n_\epsilon\|_q^{1-\theta}, \qquad \theta = \frac{5q-6}{6(q-1)}.$$

We set $h(t) = \|n_\epsilon(t)\|_1^{\frac{5q-6}{3(q-1)}}\,\|n_\epsilon(t)\|_q^{\frac{q}{3(q-1)}} \leqslant C(1+t)^{-\frac{5q-6}{3(q-1)^2}}\,\|n_\epsilon(t)\|_q^{\frac{q}{3(q-1)}}$. We note that $h \in L^1((0,\infty))$, since $n_\epsilon \in L^q(\Omega\times(0,\infty))$ and

$$\int_0^\infty h(t)\,\mathrm{d}t \leqslant \left(\int_0^\infty (1+t)^{-\frac{5q-6}{(3q-4)(q-1)}}\,\mathrm{d}t\right)^{\frac{3q-4}{3(q-1)}}\left(\int_0^\infty \|n_\epsilon(t)\|_q^q\,\mathrm{d}t\right)^{\frac{1}{3(q-1)}} < C.$$

Using the Poincaré inequality, it follows that

$$\frac{d}{dt}\int_\Omega |u_\epsilon(\cdot,t)|^2\,\mathrm{d}x + \frac{C_p}{2}\int_\Omega |u_\epsilon(\cdot,t)|^2\,\mathrm{d}x \leqslant h(t). \tag{5.1}$$

Since h is in L^1, we have $\|u_\epsilon(\cdot,t)\|_2 \leqslant C$ for all t. In addition, we obtain, for sufficiently large t,

$$\|u_\epsilon(t)\|_2 \leqslant C(1+t)^{-\frac{-3q^2+12q-10}{3(q-1)^2}}.$$

Indeed, setting $z(t) := \|u_\epsilon(t)\|_2^2$, it leads that

$$\begin{aligned}z(2t) &\leqslant e^{-t}z(t) + e^{-2t}\int_t^{2t} e^{\tau}h(\tau)d\tau \leqslant e^{-t}z(t) + \int_t^{2t} h(\tau)d\tau\\ &\leqslant Ce^{-t} + C\left(\int_t^{2t}(1+t)^{-\frac{5q-6}{(3q-4)(q-1)}}\right)^{\frac{3q-4}{3(q-1)}}\\ &\leqslant Ce^{-t} + C(1+t)^{\frac{3q^2-12q+10}{3(q-1)^2}} \leqslant C(1+t)^{-\frac{-3q^2+12q-10}{3(q-1)^2}}.\end{aligned}$$

• (**The case** $\rho < 0$) Firstly, we integrate the equation for n_ϵ over Ω to get

$$\frac{d}{dt}\int_\Omega n_\epsilon - \rho\int_\Omega n_\epsilon \leqslant -\mu\int_\Omega n_\epsilon^\kappa \leqslant 0,$$

which directly yields

$$\int_\Omega n_\epsilon(\cdot,t)\,\mathrm{d}x \leqslant me^{\rho t} \quad \text{for all } t>0, \tag{5.2}$$

where m is as in Lemma 3.1. Next, again integrating the equation for c_ϵ over Ω and letting $z(t) := \int_\Omega c_\epsilon(\cdot,t)\,\mathrm{d}x$, it follows that

$$z'(t) + z(t) \leqslant me^{\rho t},$$

which leads that for all $t > 0$,

$$z(t) \leqslant e^{-t}z_0 + me^{-t}\int_0^t e^{(1+\rho)\tau}\,\mathrm{d}\tau \leqslant C\left(e^{-t} + \frac{1}{1+\rho}\left(e^{\rho t} - e^{-t}\right)\right),$$

where $C = \max\left\{m, \int_\Omega c_0\right\}$. Thus, we have

$$\int_\Omega c_\epsilon(\cdot,t)\,\mathrm{d}x \leqslant Ce^{-\rho_* t} \quad \text{for all } t>0, \tag{5.3}$$

where $\rho_* = \min\{-\rho, 1\} > 0$. Using the interpolation inequality, (3.5) and (5.3), we obtain for $1 \leqslant l \leqslant \frac{3q}{5-2q}$,

$$\|c_\epsilon(t)\|_l \leqslant \|c_\epsilon(t)\|_1^{\frac{3q-(5-2q)l}{5(q-1)l}}\,\|c_\epsilon(t)\|_{\frac{3q}{5-2q}}^{\frac{3q(l-1)}{5(q-1)l}} \leqslant Ce^{-\frac{3q-(5-2q)l}{5(q-1)l}\rho_* t} \quad \text{for all } t>0.$$

Lastly, we recall the inequality (5.1):

$$\frac{d}{dt}\int_\Omega |u_\epsilon(\cdot,t)|^2\,\mathrm{d}x + C_*\int_\Omega |u_\epsilon(\cdot,t)|^2\,\mathrm{d}x \leqslant h(t).$$

Here $h(t) = \|n_\epsilon\|_1^{\frac{5q-6}{3(q-1)}}\,\|n_\epsilon\|_q^{\frac{q}{3(q-1)}} \leqslant C_3e^{-\delta t}\,\|n_\epsilon(t)\|_q^{\frac{q}{3(q-1)}}$ with $\delta = -\frac{5q-6}{3(q-1)}\rho > 0$ and $C_* = \frac{C_p}{2} > 0$, where C_p is the constant appeared in the Poincaré inequality. Letting $z(t) := \|u_\epsilon(t)\|_2^2$, we have

$$\begin{aligned}
z(t) \leqslant & e^{-C_*t}z(0) + e^{-C_*t}\int_0^t e^{C_*\tau}h(\tau)d\tau \\
\leqslant & e^{-C_*t}z(0) + C_3e^{-C_*t}\int_0^t e^{(C_*-\delta)\tau}\,\|n_\epsilon(\tau)\|_q^{\frac{q}{3(q-1)}}\,d\tau \\
\leqslant & e^{-C_*t}z(0) + C_3e^{-C_*t}e^{(C_*-\delta)_+t}\,t^{\frac{3q-4}{3(q-1)}}\left(\int_0^t \|n_\epsilon(\tau)\|_q^q\,d\tau\right)^{\frac{1}{3(q-1)}}
\end{aligned}$$

$$\leqslant C_4\left(e^{-C_*t}+e^{-\min\{C_*,\delta\}\frac{t}{2}}\right)$$
$$\leqslant C_5 e^{-\delta_* t},$$

where $\delta_* = \frac{1}{2}\min\{C_*, \delta\}$. In both cases $\rho = 0$ and $\rho < 0$, we finally get the estimates for (n, c, u) in Theorem 1.2 by passing ϵ to the limit via the Fatou's Lemma which is guaranteed by (4.1), (4.5) and (4.8). □

Acknowledgements

K. Kang is partially supported by NRF-2019R1A2C1084685 and NRF-2015R1A5A1009350. D. Kim is supported by NRF-2019R1A2C1084685.

Conflict of interest

The authors declare no conflict of interest.

References

1. M. Chae, K. Kang, J. Lee, Existence of smooth solutions to coupled chemotaxis-fluid equations, *DCDS*, **33** (2013), 2271–2297.
2. M. Chae, K. Kang, J. Lee, Global existence and temporal decay in keller-segel models coupled to fluid equations, *Commun. Part. Diff. Eq.*, **39** (2014), 1205–1235.
3. P. Cherrier, A. Milani, *Linear and quasi-linear evolution equations in Hilbert spaces*, Providence, RI: American Mathematical Society, 2012.
4. J. C. Coll, B. F. Bowden, G. V. Meehan, G. M. Konig, A. R. Carroll, D. M. Tapiolas, et al., Chemical aspects of mass spawning in corals. I. sperm-attractant molecules in the eggs of the scleractinian coral montipora digitata, *Marine Biology*, **118**, (1994), 177–182.
5. R. Denk, M. Hieber, J. Prüss, Optimal L^p-L^q-estimates for parabolic boundary value problems with inhomogeneous data, *Math. Z.*, **257** (2007), 193–224.
6. E. Espejo, T. Suzuki, Reaction terms avoiding aggregation in slow fluids, *Nonlinear Anal. Real*, **21** (2015), 110–126.
7. S. Ishida, Global existence and boundedness for chemotaxis-navier-stokes systems with position-dependent sensitivity in 2d bounded domains, *DCDS*, **35** (2015), 3463–3482.
8. K. Kang, K. Kim, C. Yoon, Existence of weak and regular solutions for keller-segel system with degradation coupled to fluid equations, *J. Math. Anal. Appl.*, **485** (2020), 123750.
9. E. F. Keller, L. A. Segel, Model for chemotaxis, *J. Theor. Biol.*, **30** (1971), 225–234.
10. A. Kiselev, L. Ryzhik, Biomixing by chemotaxis and enhancement of biological reactions, *Commun. Part. Diff. Eq.*, **37** (2012), 298–318.
11. J. Lankeit, Long-term behaviour in a chemotaxis-fluid system with logistic source, *Math. Mod. Meth. Appl. S*, **26** (2016), 2071–2109.

12. N. Mittal, E. O. Budrene, M. P. Brenner, A. Van Oudenaarden, Motility of escherichia coli cells in clusters formed by chemotactic aggregation, *PNAS*, **100** (2003), 13259–13263.

13. L. Nirenberg, An extended interpolation inequality, *Ann. Scuola Norm. Sup. Pisa Cl. Sci.*, **20** (1966), 733–737.

14. J. Prüss, R. Schnaubelt, Solvability and maximal regularity of parabolic evolution equations with coefficients continuous in time, *J. Math. Anal. Appl.*, **256** (2001), 405–430.

15. Y. Tao, M. Winkler, Boundedness and decay enforced by quadratic degradation in a three-dimensional chemotaxis-fluid system, *Z. Angew. Math. Phys.*, **66** (2015), 2555–2573.

16. Y. Tao, M. Winkler, Blow-up prevention by quadratic degradation in a two-dimensional keller-segel-navier-stokes system, *Z. Angew. Math. Phys.*, **67** (2016), 138.

17. R. Temam, *Navier-Stokes equations. Theory and numerical analysis*, Amsterdam-New York-Oxford: North-Holland Publishing Co., 1977.

18. I. Tuval, L. Cisneros, C. Dombrowski, C. W. Wolgemuth, J. O. Kessler, R. E. Goldstein, Bacterial swimming and oxygen transport near contact lines, *PNAS*, **102** (2005), 2277–2282.

19. G. Viglialoro, Very weak global solutions to a parabolic-parabolic chemotaxis-system with logistic source, *J. Math. Anal. Appl.*, **439** (2016), 197–212.

20. G. Viglialoro, Boundedness properties of very weak solutions to a fully parabolic chemotaxis-system with logistic source, *Nonlinear Anal. Real*, **34** (2017), 520–535.

21. W. Wang, A quasilinear fully parabolic chemotaxis system with indirect signal production and logistic source, *J. Math. Anal. Appl.*, **477** (2019), 488–522.

22. M. Winkler, A three-dimensional keller-segel-navier-stokes system with logistic source: global weak solutions and asymptotic stabilization, *J. Funct. Anal.*, **276** (2019), 1339–1401.

23. M. Winkler, Chemotaxis with logistic source: very weak global solutions and their boundedness properties, *J. Math. Anal. Appl.*, **348** (2008), 708–729.

24. M. Winkler, Stabilization in a two-dimensional chemotaxis-navier-stokes system, *Arch. Ration. Mech. Anal.*, **211** (2014), 455–487.

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