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Research article

On the local theory of prescribed Jacobian equations revisited†

Neil S. Trudinger*

Mathematical Sciences Institute, Australian National University, Canberra, ACT 0200, Australia

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* **Correspondence:** Email: neil.trudinger@anu.edu.au.

Abstract: In this paper we revisit our previous study of the local theory of prescribed Jacobian equations associated with generating functions, which are extensions of cost functions in the theory of optimal transportation. In particular, as foreshadowed in the earlier work, we provide details pertaining to the relaxation of a monotonicity condition in the underlying convexity theory and the consequent classical regularity. Taking advantage of recent work of Kitagawa and Guillen, we also extend our classical regularity theory to the weak form A3w of the critical matrix convexity conditions.

Keywords: prescribed Jacobian equations; generating functions; convexity theory; existence and regularity

1. Introduction

In this paper we revisit our previous study [18] of the local theory of prescribed Jacobian equations associated with generating functions, which are extensions of cost functions in the theory of optimal transportation. In particular we elaborate further our remark there pertaining to relaxing the monotonicity condition on the matrix function A in our convexity theory, thereby enabling the use of duality properties in the ensuing convexity and local regularity theory.

We begin by describing the class of equations under consideration, which we called generated prescribed Jacobian equations and is now typically abbreviated to just generated Jacobian equations (GJEs). Let Ω be a domain in Euclidean n-space, $\mathbb{R}^n$, and Y a C^1 mapping from $\Omega \times \mathbb{R} \times \mathbb{R}^n$ into $\mathbb{R}^n$. The prescribed Jacobian equation (PJE), is a partial differential equation of the form,

$$\det DY(\cdot, u, Du) = \psi(\cdot, u, Du), \tag{1.1}$$

where ψ is a given scalar function on $\Omega\times\mathbb{R}\times\mathbb{R}^n$ and Du denotes the gradient of the function $u : \Omega \to \mathbb{R}$.

Denoting points in $\Omega\times\mathbb{R}\times\mathbb{R}^n$ by (x, z, p), we always assume that the matrix Y_p is invertible, that is $\det Y_p \neq 0$, so that we may write (1.1) as a general Monge-Ampère type equation (MATE),

$$\det[D^2u - A(\cdot, u, Du)] = B(\cdot, u, Du), \tag{1.2}$$

where

$$A = -Y_p^{-1}(Y_x + Y_z \otimes p), \quad B = (\det Y_p)^{-1}\psi. \tag{1.3}$$

A function $u \in C^2(\Omega)$ is degenerate elliptic, (elliptic), for Eq (1.2), whenever

$$D^2u - A(\cdot, u, Du) \geq 0, \quad (> 0), \tag{1.4}$$

in Ω. If u is an elliptic solution of (1.2), then the function $B(\cdot, u, Du)$ is positive. Accordingly we assume throughout that B is at least non-negative in $\Omega\times\mathbb{R}\times\mathbb{R}^n$, that is ψ and $\det Y_p$ have the same sign.

The second boundary value problem for the prescribed Jacobian equation is to prescribe the image,

$$Tu(\Omega) := Y(\cdot, u, Du)(\Omega) = \Omega^*, \tag{1.5}$$

where Ω^* is another given domain in $\mathbb{R}^n$. When ψ is separable, in the sense that

$$|\psi(x, z, p)| = f(x)/f^* \circ Y(x, z, p), \tag{1.6}$$

for positive $f, f^* \in L^1(\Omega)$, $L^1(\Omega^*)$ respectively, then a necessary condition for the existence of an elliptic solution, for which the mapping T is a diffeomorphism, to the second boundary value problem (1.1), (1.5) is the *mass balance* condition,

$$\int_\Omega f = \int_{\Omega^*} f^*. \tag{1.7}$$

We note that Y need only be defined on the one jet of u, $J_1 = J_1[u](\Omega) = (\cdot, u, Du)(\Omega)$ in order to formulate (1.1) and (1.5) and typically we will only have Y defined on an open set $\mathcal{U} \subset \mathbb{R}^n\times\mathbb{R}\times\mathbb{R}^n$ with the resultant Monge-Ampère type Eq (1.2), accompanied by a constraint, $J_1[u] \subset \mathcal{U}$.

We will also change our notation slightly from [18] and let $g \in C^2(\Gamma)$ denote a generating function, where Γ is a domain in $\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}$ whose projections,

$$I(x, y) = \{z \in \mathbb{R} |\ (x, y, z) \in \Gamma\},$$

are open intervals. For convenience, we also denote the following projections,

$$\Gamma_x = \{(y, z) \in \mathbb{R}^n\times\mathbb{R} |\ (x, y, z) \in \Gamma\}, \quad \Gamma_{y,z} = \{x \in \mathbb{R}^n |\ (x, y, z) \in \Gamma\}.$$

Note that these projections may be empty for some values of x, y and z.

Denoting points in $\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}$, by (x, y, z), we assume that $g_z \neq 0$ in Γ, together with the following two conditions which extend the corresponding conditions in the optimal transportation case [14]:

A1: The mapping $(g_x, g)(x, \cdot, \cdot)$ is one-to-one in Γ_x, for each $x \in \mathbb{R}^n$.

A2: $\det E \neq 0$ in Γ, where E is the $n \times n$ matrix given by

$$E = [E_{i,j}] = g_{x,y} - (g_z)^{-1} g_{x,z} \otimes g_y. \tag{1.8}$$

From A1 and A2, the vector field Y, together with the *dual* function Z, are generated by g through the equations,

$$g_x(x, Y, Z) = p, \quad g(x, Y, Z) = u. \tag{1.9}$$

The Jacobian determinant of the mapping $(y, z) \to (g_x, g)(x, y, z)$ is $g_z \det E, \neq 0$ by A2, so that $Y$and Z are accordingly C^1 smooth. Also by differentiating (1.9), with respect to p, we obtain $Y_p = E^{-1}$. Using (1.3) or differentiating (1.9) for $p = Du$, with respect to x, we obtain that the corresponding prescribed Jacobian Eq (1.1) is a Monge-Ampère equation of the form (1.2) with

$$\begin{aligned} A(x, u, p) &= g_{xx}[x, Y(x, u, p), Z(x, u, p)], \\ B(x, u, p) &= \det E(x, Y, Z) \psi(x, u, p) \end{aligned} \tag{1.10}$$

and is well defined in domains Ω for $J_1 = J_1[u](\Omega) \subset \mathcal{U}$, where

$$\mathcal{U} = \{(x, u, p) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \mid u = g(x, y, z), p = g_x(x, y, z), (x, y, z) \in \Gamma\}. \tag{1.11}$$

Following [18] we also have the dual condition to A1:

A1*: The mapping $Q := -g_y/g_z(\cdot, y, z)$ is one-to-one in $\Gamma_{y,z}$, for all $(y, z) \in \mathbb{R}^n \times \mathbb{R}$.

Condition A1* arises through the notion of duality introduced in [18], where the dual generating function g^* is defined by

$$g[x, y, g^*(x, y, u)] = u. \tag{1.12}$$

Clearly g^* is well defined on the dual set,

$$\Gamma^* = \{(x, y, u) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \mid u \in J(x, y)\},$$

where $J(x, y) = g(x, y, \cdot)I(x, y)$, and $g_y^*(x, y, u) = Q(x, y, z)$ for $u = g(x, y, z)$ so that condition A1* may be equivalently expressed as the mapping g_y^* is one-to-one in x, u for all $(x, y, u) \in \Gamma^*$. Furthermore the Jacobian matrix of the mapping $x \to Q(x, y, z)$ is $-E^t/g_z$, where E^t denotes the transpose of E, so its determinant is automatically non-zero when condition A2 holds. From condition A1*, we then infer the existence of a C^1 dual mapping X, defined uniquely by

$$Q\Big(X(y, z, q), y, z\Big) = q \tag{1.13}$$

for all $q \in Q(\cdot, y, z)(\Gamma_{y,z})$. Note also that by setting

$$P(x, y, u) = g_x(x, y, g^*(x, y, u)),$$

we may express condition A1 in the same form as A1*, namely the mapping P is one-to-one in y, for all (x, u) such that $(x, y, u) \in \Gamma^*$.

In the special case of optimal transportation, we have

$$
\begin{aligned}
&g(x,y,z) = -c(x,y) - z, \quad \Gamma = \mathcal{D}\times\mathbb{R} \quad g_z = -1, \quad I = I(x,y) = J(x,y) = \mathbb{R}, \\
&E = -c_{x,y}, \quad g^*(x,y,u) = -c(x,y) - u,
\end{aligned} \tag{1.14}
$$

where $\mathcal{D}$ is a domain in $\mathbb{R}^n \times \mathbb{R}^n$ and $c \in C^2(\mathcal{D})$ is a *cost function*, satisfying conditions A1 and A2 in [14]. The essential difference here is that Y and A are independent of u so that our arguments here and in [18] are primarily concerned with handling such a dependence.

As in [18] we will assume throughout that g has been normalised so that $g_z < 0$ in accordance with (1.14).

Our next conditions extend the conditions A3 and A3w introduced for optimal transportation in [14, 16, 20] and are expressed in terms of the matrix function A in (1.2), which for the purpose of classical regularity is assumed twice differentiable.

A3 (**A3w**)

$$A^{kl}_{ij}\xi_i\xi_j\eta_k\eta_l := (D_{p_kp_l}A_{ij})\xi_i\xi_j\eta_k\eta_l > (\geq)\ 0,$$

for all $(x,u,p) \in \mathcal{U}, \xi, \eta \in \mathbb{R}^n$ such that $\xi.\eta = 0$.

Conditions A3w (A3) express a co-dimension one convexity (strict codimension one convexity) of the matrix function A with respect to the gradient variable p in the set $\mathcal{U}$, which we can generally assume is convex in p for fixed x and u. As in [16], we may write equivalently that A is regular, (strictly regular), in $\mathcal{U}$. It is proved in [18] that conditions A3 and A3w are invariant under duality, through explicit formulae for $D_p^2 A$ in terms of the generating function g and its derivatives up to order four. This result is extended to non smooth A in [13], where A co-dimension one convex (strictly co-dimension one convex) means that the form $A\xi.\xi = A_{ij}\xi_i\xi_j$ is convex, (locally uniformly convex), along line segments in p, orthogonal to ξ for all $\xi \in \mathbb{R}^n$.

In [18] we also introduced conditions expressing the monotonicity of A with respect to u, namely:

A4 (**A4w**)

$$D_u A_{ij}\xi_i\xi_j > (\geq 0),$$

for all $(x,u,p) \in \mathcal{U}, \xi \in \mathbb{R}^n$.

Only the weak monotonicity A4w was used in [18].

In the next section, we revisit the corresponding section in [18] and show that conditions A1, A2, A1* and A3w suffice for the convexity theory developed there, without using the monotonicity condition A4w. This will entail some upgrading of our conditions on domains, relative to Γ, but will facilitate better the use of duality in ensuing regularity arguments. In fact we had already worked out versions of these out at the time of writing [18] but omitted them in order to avoid the messier statements which pertained to ensuring the sets where condition A3w is used are contained in Γ.

In Section 3, as foreshadowed in [18] we revisit our existence and interior regularity theory. We work with a modified version of our gradient control assumption in [18]. For this and throughout this paper, it will be convenient to fix domains U and V in $\mathbb{R}^n$ such that $U \times V \times g^*\{U \times V \times J(U,V)\} \subset \Gamma$, where $J(U,V) = \cap_{U\times V} J$. For domains Ω and Ω^* with $\bar{\Omega} \subset U$ and $\bar{\Omega}^* \subset V$, we will then assume for our existence results:

A5: There exists an open interval $J_0 = (m_0, M_0) \subset J(U, V)$, $-\infty \le m_0 < M_0 \le \infty$ and positive constant $K_0 < (M_0 - m_0)/2d$, $d = \mathrm{diam}(\Omega)$, such that

$$|g_x(x, y, z)| \le K_0$$

for all $x \in \bar\Omega, y \in \bar\Omega^*, g(x, y, z) \in J_0$.

Following [18], we then have the following classical existence theorem, which improves the corresponding result in Corollary 4.7 there. For the notions of domain convexity used here, the reader is referred to [18] or Section 2 of this paper.

Theorem 1.1. *Let Ω and Ω^* be bounded domains in $\mathbb{R}^n$, and let g be a generating function satisfying A1, A2, A1*, A3 and A5. Suppose that $f > 0, \in C^{1,1}(\Omega), f^* > 0, \in C^{1,1}(\Omega^*)$, with $f, 1/f \in L^\infty(\Omega), f^*, 1/f^* \in L^\infty(\Omega^*)$ and that f and f^* satisfy the mass balance condition* (1.7). *Then for any $x_0 \in \Omega$ and u_0 satisfying $m_0 + K_1 < u_0 < M_0 - K_1$, $K_1 = K_0 diam(\Omega)$, there exists a g-convex, elliptic solution $u \in C^3(\Omega)$ of the second boundary value problem* (1.2), (1.5)*, satisfying $u(x_0) = u_0$, provided Ω^* is g^*-convex with respect to $\Omega \times J_1$, where $J_1 = (u_0 - K_1, u_0 + K_1)$, and Ω is g-convex with respect to all $y \in \Omega^*$ and $z \in g^*(\cdot, y, J_1)(\Omega)$.*

Theorem 1.1 is an immediate consequence of the local regularity result Theorem 3.2, which extends Theorem 4.6 in [18]. Taking account of recent developments, notably the strict convexity result in [2], we can now extend Theorem 1.1 to A3w, by extending the regularity argument in [9] for the optimal transportation case and moreover by [15] the solution is unique. We will also treat this extension in Section 3, (see Theorem 3.4 and Corollaries 3.5, 3.6), together with the necessary local Pogorelov estimate for its proof, Lemma 3.3. These results have also been presented by us in recent lectures, at Peking University in 2019 and Okinawa Institute of Science and Technology in early 2020.

Finally in Section 4, we revisit again our convexity theory, providing an extension of Theorem 3.2 to non-smooth densities and a variant of our key convexity property Lemma 2.2, which does not need duality for its proof.

We conclude this introduction by noting that our introduction of the concept of generating function in [17, 18] was to provide a framework for extending the theory of optimal transportation to embrace near field geometric optics, where the associated ray mappings depended also on the position of a reflecting or refracting surface as well as its gradient. Particular motivation came from the point source reflection regularity theory in [7] which for graph targets is modelled by the generating function in equation (4.15) in [18]. Note that it is $-A$ in [18] (4.17) which satisfies A4 for $\tau < 0$ so the local regularity theory in this case is covered here. The reader is also referred to the papers [2, 5] for further examples of generating functions in optics which fit our theory here.

2. Convexity theory

We begin by repeating the definitions in [18]. We consider bounded domains Ω and $\Omega^* \subset \mathbb{R}^n$ and a generating function g, satisfying conditions A1 and A2 on Γ. For $x_0, y_0 \in \mathbb{R}^n$, we also denote

$$I(\Omega, y_0) = \cap_\Omega I(\cdot, y_0), \quad J(x_0, \Omega^*) = \cap_{\Omega^*} J(x_0, \cdot).$$

A function $u \in C^0(\Omega)$ is called *g-convex* in Ω, if for each $x_0 \in \Omega$, there exists $y_0 \in \mathbb{R}^n$ and $z_0 \in$

$I(\Omega, y_0)$ such that

$$u(x_0) = g(x_0, y_0, z_0), \tag{2.1}$$
$$u(x) \geq g(x, y_0, z_0)$$

for all $x \in \Omega$. If $u(x) > g(x, y_0, z_0)$ for all $x \neq x_0$, then u is called *strictly g-convex*. If u is differentiable at x_0, then $y_0 = Tu(x_0) := Y(x_0, u(x_0), Du(x_0))$, while if u is twice differentiable at x_0, then

$$D^2u(x_0) \geq g_{xx}(x_0, y_0, z_0) = A(\cdot, u, Du)(x_0) \tag{2.2}$$

that is, u is degenerate elliptic for Eq (1.4) at x_0. If $u \in C^2(\Omega)$, we call u locally g-convex in Ω if $J_1[u](\Omega) \subset \mathcal{U}$ and (2.2) holds for all $x_0 \in \Omega$. We will also refer to functions of the form $g(\cdot, y_0, z_0)$ as g-affine and as a g-support at x_0 if (2.1) is satisfied. Note also that the g-convexity of a function u in Ω implies its local semi-convexity. When the inequality in (2.2) is strict, that is u is elliptic for equation (1.4), then we call u locally uniformly g-convex. It follows readily that a locally g-convex C^2 function u will also be locally g-convex in the sense that $u \geq g(\cdot, y_0, z_0)$ in some neighbourhood of x_0, while a locally uniformly g-convex function is strictly g-convex in some neighbourhood of x_0.

The domain Ω is g-convex with respect to $y_0 \in \mathbb{R}^n$, $z_0 \in I(\Omega, y_0)$ if the image $Q_0(\Omega) := -g_y/g_z(\cdot, y_0, z_0)(\Omega)$ is convex in $\mathbb{R}^n$.

The domain Ω^* is g^*-convex with respect to $x_0 \in \mathbb{R}^n$, $u_0 \in J(x_0, \Omega^*)$, if the image $P_0(\Omega^*) := P(x_0, \cdot, u_0)(\Omega^*) = g_x[x_0, \cdot, g^*(x_0, \cdot, u_0)](\Omega^*)$ is convex in $\mathbb{R}^n$.

We may also consider a corresponding notion of domain convexity when u is fixed which agrees with that associated with the vector field Y in [16]. Namely, the domain Ω is Y-convex with respect to $y_0 \in \mathbb{R}^n$, $u_0 \in J(\Omega, y_0)$ if the image $Q_0^*(\Omega) := Q^*(\Omega, y_0, u_0) = -g_y/g_z[\cdot, y_0, g^*(\cdot, y_0, u_0)](\Omega)$ is convex in $\mathbb{R}^n$. It follows then that Ω is g-convex with respect to $y_0 \in \mathbb{R}^n$, $z_0 \in I(\Omega, y_0)$ if Ω is Y-convex with respect to y_0 and $u_0 = g(x, y_0, z_0)$ for every $x \in \Omega$. We remind the reader that our definition of g*-convexity is already a special case of the notion of Y^*-convexity in [16] since $P_0(\Omega^*) = \{p \in \mathbb{R}^n \mid Y(x_0, u_0, p) \in \Omega^*\}$.

It will also be convenient to introduce a more general "sub convexity" notion as follows. The domain Ω is sub g-convex with respect to $y_0 \in \mathbb{R}^n$, $z_0 \in I(\Omega, y_0)$ if the convex hull of $Q_0(\Omega) \subset Q(\Gamma)$ and the domain Ω^* is sub g^*-convex with respect to $x_0 \in \mathbb{R}^n$, $u_0 \in J(x_0, \Omega^*)$ if the convex hull of $P_0(\Omega^*) \subset P(\Gamma^*)$. Analogously the domain Ω is sub Y-convex with respect to $y_0 \in \mathbb{R}^n$, $u_0 \in J(\Omega, y_0)$ if the image $Q_0^*(\Omega) \subset Q^*(\Gamma^*)$.

We also define the domain Ω^* to be g^*-convex (sub g^*-convex) with respect to a function $u \in C^0(\Omega)$ if Ω^* is g^*-convex (sub g^*-convex) with respect to each point on the graph of u.

Note that the above definitions also can be applied to general sets, in place of the domains Ω and Ω^*.

Next we define the relevant notions of normal mapping and section.

Let $u \in C^0(\Omega)$ be g-convex in Ω. We define the *g-normal* mapping of u at $x_0 \in \Omega$ to be the set:

$$Tu(x_0) = \{y_0 \in \mathbb{R}^n \mid \Omega \subset \Gamma_{y_0,z_0} \text{ and } \quad u(x) \geq g(x, y_0, z_0) \text{ for all } x \in \Omega\},$$

where $z_0 = g^*(x_0, y_0, u_0), u_0 = u(x_0)$. Clearly Tu agrees with our previous terminology when u is differentiable. In the non differentiable case we at least have the inclusion,

$$Tu(x_0) \subseteq \Sigma_0 := Y(x_0, u(x_0), \partial u(x_0)), \tag{2.3}$$

where ∂u denotes the subdifferential of u, provided the extended one jet, $J_1[u](x_0) = [x_0, u(x_0), \partial u(x_0)] \subset \mathcal{U}$. Moreover from the semi-convexity of u, it follows that $\partial u(x_0)$ is the convex hull of $P_0(Tu(x_0))$ and $\operatorname{dist}\{Tu(x), Tu(x_0)\} \to 0$ as as $x \to x_0$.

Next if $g_0 = g(\cdot, y_0, z_0)$ is a g-affine function on Ω, we define the *g-section S* of a g-convex function u with respect to g_0 by

$$S = S(u, g_0) = S(u, y_0, z_0) = \{x \in \Omega \mid u(x) < g(x, y_0, z_0)\}$$

If g_0 is also a g-affine support to u at x_0, we define the *contact set S_0* by

$$S_0 = S_0(u, g_0) = S_0(u, y_0, z_0) = \{x \in \Omega \mid u(x) = g(x, y_0, z_0)\}.$$

Note that we have defined sections here differently to [18].

We now have the following variant of Lemma 2.1 in [18].

Lemma 2.1. *Assume that g satisfies A1, A2, A1* and A3w and that $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ is locally g-convex in Ω and $u(\Omega) \subset\subset J(\Omega, Tu(\Omega))$ with $Tu(\Omega)$ sub g^*-convex with respect to u. Then if Ω is g-convex with respect to (y, z) for all $y \in Tu(\Omega), z \in g^*(\cdot, y, u)(\Omega)$, it follows that u is g-convex in Ω.*

Remark 2.1. More specifically, we have for any $x_0 \in \Omega, y_0 = Tu(x_0), z_0 = g^*(x_0, y_0, u(x_0))$, the g-affine function $g(\cdot, y_0, z_0)$ is a g-support, provided $g(\cdot, y_0, z_1) \geq u$ in Ω for some $z_1 < z_0, \in I(\Omega, y_0)$ and Ω is g-convex with respect to (y_0, z), for all $z \in (z_1, z_0)$. More generally we can weaken the assumption that $u(\Omega) \subset\subset J(\Omega, Tu(\Omega))$ to just requiring $(\cdot, Tu(\Omega), u)(\bar{\Omega}) \subset \Gamma^*$.

In order to prove Lemma 2.1 and the ensuing results concerning g-normal mappings and sections, we first recall a fundamental inequality from [18], for which we assume the generating function g satisfies conditions A1, A2, A1* and A3w. Let $u \in C^2(\Omega)$ be locally g-convex in Ω and $g_0 = g(\cdot, y_0, z_0)$ be a g-affine function defined on Ω, where the domain Ω is assumed to be g-convex with respect to (y_0, z_0). Defining the height function $h = u - g_0$ and making the coordinate transformation $x \to q = Q(x, y_0, z_0)$, we then have, following the computation in (2.9) and (2.10) in [18], (without using condition A4w), the differential inequality,

$$D_{q_\xi q_\xi} h \geq -K|D_{q_\xi} h| - K_0|h|, \tag{2.4}$$

for any unit vector ξ, at any point $\hat{x} \in \Omega' \subset\subset \Omega$, for which $(\cdot, [g_0, u] \cup [u, g_0], Dg_0)(\hat{x}) \subset \mathcal{U}$ and the set $Tu(\Omega) \cup \{y_0\}$, is sub g^*-convex with respect to $(\cdot, g_0)(\hat{x})$, where K and K_0 are constants depending on g, g_0, Ω, Ω' and $J_1[u]$. If additionally condition A4w holds, the differential inequality (2.4) holds with $K_0 = 0$, for $h(\hat{x}) \geq 0$, as in inequality (2.10) in [18], while if A3w is replaced by the strict condition A3 or u is locally uniformly g-convex at $\hat{x}$, we have strict inequality in (2.4), and our proofs here can be simplified by only using the simpler inequalities

$$D_{q_\xi q_\xi} h > 0 \tag{2.5}$$

whenever $h(\hat{x}) = D_{q_\xi} h(\hat{x}) = 0$.

Lemma 2.1 now follows by a modification of the proof of the corresponding lemma in [18], which we indicate after formulating the extensions to normal mappings and sections. The crucial convexity property for our regularity considerations is the characterisation of the g-normal mapping in terms of the sub differential through equality in the inclusion (2.3). In our earlier versions of this paper going back to 2014, we had various hypotheses for this result depending on whether we raise or lower g-affine functions or use duality in the proofs. The following version, depending on duality, will be convenient for our purposes here.

Lemma 2.2. *Assume g satisfies A1, A2, A1* and A3w and suppose $u \in C^0(\Omega)$ is g-convex in Ω, with Ω sub g-convex with respect to all $y \in \Sigma_0$, $z = z_y$ for some $x_0 \in \Omega$. Then we have $Tu(x_0) = \Sigma_0$.*

We will deduce Lemma 2.2 from a convexity result for g-sections which extends Lemma 2.3 in [18].

Lemma 2.3. *Assume g satisfies A1, A2, A1* and A3w and suppose $u \in C^0(\Omega)$ is g-convex in Ω. Assume also:*

(i) Ω is g-convex with respect to some $y_0 \in \mathbb{R}^n$, $z_0 \in I(\Omega, y_0)$;

(ii) $Tu(\Omega) \cup \{y_0\}$ is sub g^*-convex with respect to $g_0 := g(\cdot, y_0, z_0)$ in Ω.

Then the section $S = S(u, g_0)$ is also g-convex with respect with respect to (y_0, z_0), while if g_0 is a g-support to u, the contact set $S_0 = S_0(u, g_0)$ is g-convex with respect with respect to (y_0, z_0).

We note that Lemmas 2.2 and 2.3 are direct extensions of the corresponding lemmas in [18] under condition A4w but Lemma 2.1 needs stronger domain convexity conditions in its hypotheses. We will consider further variants of our convexity results in Section 4, including a version of Lemma 2.2 which does not need duality in its proof. We also remark here that Lemmas 2.2 and 2.3 are also extended to C^2 generating functions in [13].

Proofs. We indicate here the necessary modifications of the corresponding proofs of Lemmas 2.1 and 2.3 in [18] which follow by using the more general differential inequality (2.4) in conjunction with decreasing or increasing the appropriate g-affine functions. Letting u be locally g-convex in Ω, for a fixed point $x_0 \in \Omega$, $y_0 = Tu(x_0)$, $z_0 = g^*(x_0, y_0, u_0)$, we modify the height function h in Eq (2.8) in [18] by setting for $\delta \geq 0$,

$$h(x) = h_\delta(x) = u(x) - g(x, y_0, z_0 - \delta). \tag{2.6}$$

As in [18], the function $g_0 = g(\cdot, y_0, z_0)$ is a local support near x_0, that is $h_0 \geq 0$ near x_0. If $h_0(x) < 0$ at some point $x \in \Omega$, from the hypotheses of Lemma 2.1, there exists $\delta \geq 0$ such that h_δ attains a zero maximum along the closed g-segment joining x_0 and x, with respect to $y_0, z_0 - \delta$, (which will lie in some subdomain $\Omega' \subset\subset \Omega$) and $h_\delta(x) < 0$. Setting $q_0 = Q(x_0, y_0, z_0 - \delta)$, $q_t = tq + (1-t)q_0$, $x_t = X(q_t, y_0, z_0 - \delta)$, $0 \leq t \leq 1$ and defining $f(t) = h_\delta(x_t)$, it follows that f attains a zero maximum at some point $\hat{t} \in (0, 1)$ for $\delta > 0$ and $\hat{t} = 0$ for $\delta = 0$, so that also $f'(\hat{t}) = 0$ in both cases. From the differential inequality (2.4) and our sub g-convexity hypothesis, we have the corresponding differential inequality for f,

$$f'' \geq -K|f'| - K_0|f| \tag{2.7}$$

holding, at least in some neighbourhood of the set where f vanishes, which is a contradiction. The latter assertion is easily seen as (2.7) clearly implies a uniform bound from above for v'' where $v = \log(\epsilon - f)$, $\epsilon > 0$, and is a one dimensional version of the strong maximum principle. Accordingly g_0 must be a g-support and Lemma 2.1 is proved.

Next we show that Lemma 2.3 also can be proved by a corresponding modification of the proof of Lemma 2.3 in [18]. By modifying Ω we may assume if necessary that u and g_0 extend to a neighbourhood of $\bar{\Omega}$. If S is not g-convex, with respect to y_0, z_0, there must be a g-segment in $\bar{\Omega}$

joining two points in S containing a point $x_1 \in \partial S$. Setting $u_1 = u(x_1), y_1 \in Tu(x_1), z_1 = g^*(x_1, y_1, u_1)$, the inequality

$$g(x, y_1, z_1) < g(x, y_0, z_0), \tag{2.8}$$

holds for all $x \in S$. Now replacing h in (2.6) by

$$h(x) = h_\delta(x) = g(x, y_1, z_1 + \delta) - g(x, y_0, z_0), \tag{2.9}$$

where g is extended so that $g_\delta(x) := g(x, y_1, z_1 + \delta) = -\infty$ where $(x, y_1, z_1 + \delta) \notin \Gamma$, we again obtain a contradiction with the differential inequality (2.4) for some $\delta \geq 0$. Note that in this case we are lowering the support $g(\cdot, y_1, z_1)$ by increasing z_1 and also using condition (ii) in the sense that $(x, y_1, z_1 + \delta) \in \Gamma$ whenever $g_\delta = g_0$.

Finally, the g-convexity of the contact set S_0 follows by replacing S by S_0 in the above proof, (or alternatively using $S_0 = \cap_{\sigma > 0} S_\sigma$, where $S_\sigma = S(u, y_0, z_0 - \sigma)$), and hence Lemma 2.3 is proved.

Remark 2.2. In the case when u is also a g-affine function $g_1 = g(\cdot, y_1, z_1)$, we can improve Lemma 2.3 so that Ω can be replaced by a closed set, with possibly empty interior, and the section S replaced by the closed section $\tilde{S} = \tilde{S}(g_1, g_0) = \{x \in \Omega \mid g_1 \leq g_0\}$, thereby inferring that $\tilde{S}$ is g-convex with respect to g_0. In this case we can take $x_1 \in \Omega - \tilde{S}$ in the proof since $Tg_1 = y_1$.

Now we can prove Lemma 2.2. Note that we can infer a version of Lemma 2.2 from Lemma 2.3 through the g-transform v, given by

$$v(y) = u^*_g(y) = \sup_\Omega g^*(\cdot, y, u) \tag{2.10}$$

as the g^*-convexity of $Tu(x_0)$ is equivalent to that of the contact set $S^*_0[v, x_0, u_0]$ of v. Here we will proceed somewhat differently with the technicalities. We begin by taking two g affine supports to u at x_0, $g_0 = g(\cdot, y_0, z_0), g_1 = g(\cdot, y_1, z_1)$ and fixing a point $x = x_1$ in Ω, where $g_1 \leq g_0$. Letting $y = y_\theta = Y(x_0, u_0, p_\theta)$, for $p_\theta = \theta Dg_0(x_0) + (1 - \theta) Dg_1(x_0)$, $\theta \in [0, 1]$ denote a point on the closed g^*-segment I_0, with respect to (x_0, u_0), joining y_0 and y_1, we now define $u_y = g(x_1, y_0, z_y)$. We then obtain, from Remark 2.2 and the g^*-convexity of I_0, that $g^*(x_1, y_\theta, u_\theta) \leq g^*(x_0, y, u_0) = z_y$ and hence $g(x_1, y, z_y) \leq g(x_1, y_0, z_0)$, for all $y \in I_0$.

Consequently we have the following extension of the Loeper maximum principle in optimal transportation [11]

$$g(\cdot, y, z_y) \leq \max\{g_0, g_1\} \tag{2.11}$$

in Ω for all $y \in I_0$ which, by virtue of the semi-convexity of u implies that $Tu(x_0)$ is g^*- convex with respect to x_0, u_0 and hence completes the proof of Lemma 2.2.

Remark 2.3. From the proof of Lemma 2.2 we see that condition (i) can be weakened to just requiring the pair of points $\{x_0, x\}$ is sub g-convex with respect to all $y \in \Sigma_0$, $z = z_y$, for all $x \in \Omega$, that is the g-segment joining x_0 to any point in Ω is well defined with respect to all $y \in \Sigma_0$, $z = z_y$.

We also remark that the technicalities in using the differential inequalities (2.4) in the general A3w case can also be simplified if we use an approximation of g_0 by a uniformly g-convex function, as in [6], so that we only need the simpler strict inequality $f'' > 0$ from (2.5) whenever $f = f' = 0$, which requires less smoothness of g for its validity when, additionally to $g \in C^2$ and A regular, either u is uniformly g-convex or A is strictly regular. In fact here the differentiability of $A\xi.\xi$ with respect to p in directions orthogonal to ξ would suffice. However in using our differential inequality approach, (which for the optimal transportation case goes back to [8], with simplified versions in [19, 22]), we would still need some smoothness of the generating function g beyond C^2 smoothness. A substantially different geometric approach to our convexity theory, which has its optimal transportation roots in [21], is presented in [12, 13], where we do not need any derivatives beyond second order. A different analytic approach, based on a weak form of condition A3w corresponding to a sharpening of the quasi-convexity property (2.11), is developed in [2].

We will return to the convexity theory in Section 4, in conjunction with consideration of the strict convexity and continuous differentiability results foreshadowed in [18], which are applications of Lemma 2.3.

3. Existence and regularity

Since the arguments in Section 3 of [18] are largely independent of condition A4w, they extend readily to the more general case as a consequence of Lemma 2.2. First we recall from [18] the definition of generalized solution. For convenience we let Ω and Ω^* be bounded domains in $\mathbb{R}^n$, whose closures lie in the domains U and V respectively, as introduced in Section 1, and $u \in C^0(\overline{\Omega})$ be g-convex in Ω, with $Tu(\Omega) \subset\subset V$ and conditions A1, A2, A1* satisfied. Then there is a measure $\mu = \mu[u] = \mu(u, f^*)$ on Ω, for $f^* \geq 0 \in L^1(V)$, such that for any Borel set $E \subset \Omega$,

$$\mu(E) = \int_{Tu(E)} f^* \tag{3.1}$$

which is also weakly continuous with respect to local uniform convergence. A g-convex function u on Ω is now defined to be a generalized solution of the second boundary value problem (1.5) for Eqs (1.1) and (1.6), under the mass balance condition (1.7), if

$$\mu[u] = \nu_f \tag{3.2}$$

where $\nu_f = f\mathrm{d}x$ and f^* is extended to vanish outside Ω^*. We then have the following extension of Theorem 3.1 in [18].

Theorem 3.1. *Let Ω and Ω^* be domains satisfying $\bar{\Omega} \subset U$ and $\bar{\Omega}^* \subset V$ and let g be a generating function satisfying A1, A2, A1* and A5. Suppose that f and f^* are positive densities in $L^1(\Omega)$ and $L^1(\Omega^*)$ satisfying the mass balance condition* (1.7)*. Then for any $x_0 \in \Omega$ and $u_0 > m_0 + K_1$, where $K_1 = K_0 diam(\Omega)$, there exists a generalized solution of* (1.2), (1.5) *satisfying $u(x_0) = u_0$. Furthermore if Ω^* is g^*-convex with respect to u, then any generalized solution of* (1.5) *satisfies $Tu(\Omega) \subset \bar{\Omega}^*$.*

Using Lemma 2.2 in place of the corresponding Lemma 2.2 in [18], we then have the following extension of the local regularity result in Theorem 4.6 in [18].

Theorem 3.2. *Let $u \in C^0(\bar{\Omega})$ be a generalized solution of* (1.5) *with positive densities $f \in C^{1,1}(\Omega)$, $f^* \in C^{1,1}(\bar{\Omega}^*)$ with $f, 1/f \in L^\infty(\Omega), f^*, 1/f^* \in L^\infty(\Omega^*)$ and with generating function g satisfying conditions A1, A2, A1* and A3. Suppose that $u(\Omega) \subset\subset J_0$, Ω^* is g^*-convex with respect to u and Ω is sub g-convex with respect to the dual function $v = u^*_g$. Then $u \in C^3(\Omega)$ and is an elliptic solution of* (1.2), (1.10)*. Furthermore if Ω is g-convex with respect to v, then Tu is also a diffeomorphism from Ω to Ω^*, with v an elliptic solution of the dual boundary value problem.*

For the explicit form of the dual boundary value problem, the reader is referred to Eqs (3.5) and (3.7) in [18].

Theorem 1.1 now follows as a consequence of Theorems 3.1 and 3.2, (and approximating f^* near $\partial\Omega^*$ if only locally $C^{1,1}$). From Rankin [15], the solution in Theorem 1.1 is unique. As foreshadowed in [18], we will also consider the extension of Theorem 3.2 to $f^* \in C^{1,1}(\Omega^*)$ in Section 4, by adapting the strict convexity argument in [21], as this also relates to our extension to A3w.

In the rest of this section we will treat the extension to A3w which follows for strictly convex generalized solutions from modification of the Pogorelov estimates in [10, 20]. For this we consider classical elliptic solutions u of the Monge-Ampère type Eq (1.2) in sections $\Omega = S(u, g_0)$, with respect to a g-affine function g_0 on $\bar{\Omega}$, so that we have a Dirichlet boundary condition $u = g_0$ on $\partial\Omega$. Accordingly we assume A and B are C^2 smooth on an open set $\mathcal{U} \subset \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n$, with A regular and B positive. We may also assume $\mathcal{U}$ is convex in p, for fixed x, u. We then have the following estimate:

Lemma 3.3. *Let $u \in C^4(\Omega) \cap C^{0,1}(\bar{\Omega})$ be an elliptic solution of* (1.2) *in Ω and $u_0 \in C^2(\Omega) \cap C^{0,1}(\bar{\Omega})$ a degenerate elliptic solution of the homogeneous equation with $B = 0$, such that $J_1[u], J_1[u_0] \subset \mathcal{U}$ and $u = u_0$ on $\partial\Omega$, $u < u_0$ in Ω. Then there exist positive constants β, δ and C depending on $n, \mathcal{U}, A, B, u_0$ and $|u|_1 = \sup_\Omega(|u| + |Du|)$, such that if $d := diam(\Omega) < \delta$,*

$$\sup_\Omega (u_0 - u)^\beta |D^2 u| \leq C. \tag{3.3}$$

For generated Jacobian equations, satisfying A1, A2, A1*, A3w and A4w, Lemma 3.3 corresponds to Theorem 1.2 in [5] and does not need the smallness condition on Ω. We can prove Lemma 3.3 by modification of the global estimate Theorem 3.1 in [20], incorporating an appropriate cut-off function, or by modification of the interior estimate Theorem 2.1 in [10], extending to u dependence in A. Dealing with the general u dependence is the critical issue in the treatment of the cut-off function so we will largely focus on this in the proof. For this it is more convenient for us to begin with the calculations in [20]. Accordingly with u and u_0 satisfying the hypotheses of Lemma 2.3, we consider, as in [10] an auxiliary function

$$v = v(\cdot, \xi) = \log(w_{ij}\xi_i\xi_j) + \tau|Du|^2 + \kappa\varphi + \beta\log(u_0 - u) \tag{3.4}$$

where $|\xi| = 1$, τ, κ and β are positive constants to be chosen, $w = D^2u - A(\cdot, u, Du)$ and $\varphi \in C^2(\bar{\Omega})$ satisfies the global barrier condition,

$$[D_{ij}\varphi - D_{p_k}A_{ij}(\cdot, u, Du)D_k\varphi]\xi_i\xi_j \geq |\xi|^2 \tag{3.5}$$

Using the smallness condition on Ω we can fix such a barrier by taking $\varphi = |x - x_1|^2$ for some point $x_1 \in \Omega$.

For the linearized operator, L given by

$$L = L[u] = w^{ij}[D_{ij} - D_{p_k}A_{ij}(\cdot, u, Du)D_k] - D_{p_k}\log B(\cdot, u, Du)D_k, \tag{3.6}$$

with $[w^{ij}]$ denoting the inverse of $[w_{ij}]$, we can now follow the calculations in [20] to obtain, from inequalities (3.11) and (3.12) there, at a maximum point x_0 and vector $\xi = e_1$ of v in Ω,

$$Lv \geq \tau w_{ii} + \kappa w^{ii} - C(\tau + \kappa) + \frac{1}{2w_{11}^2}\sum_{i>1} w^{ii}(D_i w_{11})^2 + \beta L\log(u_0 - u), \tag{3.7}$$

provided $\tau \geq C$ and $\kappa \geq C\tau$. Here, as is customary, we use C to denote a constant depending on the same quantities as in the estimate being proved, (3.3). We also note here that a term τw^{ii} is missing in the bracketed terms in (3.11), (3.12) in [20], (which is controlled, as in (3.7), by taking $\kappa \geq C\tau$). To handle the last term in (3.7), we now need to estimate $L\eta$, where $\eta = u_0 - u$ by extending analogous estimates in the special cases of optimal transportation in [10] and generated Jacobian equations satisfying A4w in [5]. We then have in our general case, at $x = x_0 \in \Omega$, with similar computation to that underlying our differential inequality(2.4), using condition A3w,

$$\begin{aligned} L\eta = {} & w^{ij}[-w_{ij} + A_{ij}(x, u_0, Du_0) - A_{ij}(x, u, Du) - D_{p_k}A_{ij}(x, u, Du)D_k\eta] \\ & \qquad\qquad\qquad\qquad\qquad\qquad -D_{p_k}(\log B)(x, u, Du)D_k\eta \\ \geq {} & -C(1 + w^{ii}|\eta| + w^{ii}|D_i\eta|), \end{aligned} \tag{3.8}$$

Substituting in (3.7), we then obtain, with $[w_{ij}]$ assumed diagonal at x_0,

$$Lv \geq \tau w_{ii} + \kappa w^{ii} - C(\tau + \kappa) + \frac{1}{2w_{11}^2}\sum_{i>1} w^{ii}(D_i w_{11})^2 - \frac{C\beta}{\eta} - \frac{C\beta}{\eta^2}w^{ii}|D_i\eta|^2, \tag{3.9}$$

provided also $\kappa \geq C\beta$. Now we follow [10] to control the last term in (3.9). Using $Dv(x_0) = 0$, together with $|D\varphi| \leq d$, we then estimate, for each $i = 1, \cdots n$,

$$\beta^2\left(\frac{D_i\eta}{\eta}\right)^2 \leq 4\left(\frac{D_i w_{11}}{w_{11}}\right)^2 + C\tau^2(w_{ii}^2 + 1) + 4\kappa^2 d^2. \tag{3.10}$$

Assuming $\eta w_{11}(x_0) \geq \beta$, we can now estimate the last term in (3.9),

$$\frac{\beta}{\eta^2}w^{ii}|D_i\eta|^2 \leq \frac{C}{\eta} + \frac{4}{\beta w_{11}^2}\sum_{i>1} w^{ii}(D_i w_{11})^2 + C\frac{\tau^2}{\beta}(w_{ii} + w^{ii}) + \frac{4}{\beta}\kappa^2 d^2 w^{ii}. \tag{3.11}$$

We now conclude the proof of Lemma 3.3 by choosing $\tau \geq C$, $\kappa \geq C\beta, \beta \geq C\tau^2$ and $d \leq 1/\kappa$.

From Lemma 3.3 we now have the following extension of Theorem 3.2 to A3w.

Theorem 3.4. *Let u be a strictly g-convex generalized solution of* (1.5) *with positive densities $f \in C^{1,1}(\Omega), f^* \in C^{1,1}(\Omega^*)$ with $f, 1/f \in L^\infty(\Omega), f^*, 1/f^* \in L^\infty(\Omega^*)$ and with generating function g satisfying conditions, A1, A2, A1* and A3w. Suppose that $u(\Omega) \subset\subset J_0$, Ω^* is g^*-convex with respect to u and Ω is sub g-convex with respect to $v = u_g^*$. Then $u \in C^3(\Omega)$ and is an elliptic solution of* (1.2), (1.10). *Furthermore if Ω is g-convex with respect to $v = u_g^*$, then Tu is also a diffeomorphism from Ω to Ω^*, with v an elliptic solution of the dual boundary value problem.*

To prove Theorem 3.4, we need to adjust the regularity argument in [18] by using the interior estimate (3.3) for the solutions w of the approximating Dirichlet problems in Lemma 4.6, in place of the estimate (4.12) in [18], when the strong condition A3 is satisfied. To be more specific using Lemma 4.6 in [18] we construct a sequence $\{\bar{u}_m\}$ of smooth elliptic solutions of the Dirichlet problem for Eq (1.2) in a small ball of radius r, $B_r \subset \Omega' \subset\subset \Omega$, with boundary condition $\bar{u}_m = u_m$ on ∂B_r, where $\{u_m\}$ is an appropriately chosen sequence of smooth functions converging uniformly to u in Ω', uniformly bounded in C^1 and uniformly semi-convex. Here the strict g-convexity of u ensures $Tu_m(\Omega') \subset \tilde{\Omega}^*$ for some $\tilde{\Omega}^* \subset\subset \Omega^*$. Otherwise we need to assume $f^* \in C^{1,1}(\bar{\Omega}^*)$, as done for Theorem 3.2. The sequence $\{\bar{u}_m\}$ will also be uniformly bounded in C^1 and moreover with r sufficiently small, $\{\bar{u}_m\}$ is g-convex in

B_r. Then using the strict g-convexity of u and Lemma 3.3, we can follow the proof of Theorem 3.1 in [9] to obtain uniform interior second derivative bounds for $\bar{u}_m$ in B_r. Subsequently, returning to the proof of Theorem 4.6 in [18] and using Lemmas 4.3 and 4.4 in [18], we conclude that $\bar{u}_m$ converges to u in B_r. From standard elliptic regularity theory [1], we then infer $u \in C^3(\Omega)$, is an elliptic solution of (1.2), while from the mass balance condition (1.7), $Tu(\Omega) = \Omega^* - E$, for some closed null set E. Now letting T^*v denote the g^*-normal mapping of v on Ω^*, we then have, (as in the A3 case), from the g-convexity of Ω and smoothness of u, that $T^*v(\Omega^*) = \Omega$. Hence Tu is a diffeomorphism from Ω to Ω^* with $(Tu)^{-1} = T^*v$ and we conclude from [18], Section 3 that v is a C^3 elliptic solution of the dual boundary value problem.

In order to apply the strict convexity result of Kitagawa and Guillen [2], it will be convenient to strengthen our domain conditions by assuming that the domain U is g-convex with respect to all $y \in V$ and $z \in g^*(\cdot, y, J_0)(U)$ and V is g^*-convex with respect to $U \times J_0$. Then we have the following corollaries of Theorem 3.4.

Corollary 3.5. *Let u be a g-convex generalized solution of* (1.5) *with positive densities $f \in C^{1,1}(\Omega), f^* \in C^{1,1}(\Omega^*)$ with $f, 1/f \in L^\infty(\Omega), f, 1/f^* \in L^\infty(\Omega^*)$, $\bar{\Omega} \subset U$, $\bar{\Omega}^* \subset V$ and with generating function g satisfying conditions, A1, A2, A1* and A3w. Suppose that $u(\Omega) \subset\subset J_0$, together with its g-affine supports, and Ω^* is g^*-convex with respect to u on Ω. Then $u \in C^3(\Omega)$ and is an elliptic solution of* (1.2) *and* (1.10). *Furthermore if Ω is g-convex with respect to $v = u^*_g$, then Tu is also a diffeomorphism from Ω to Ω^*, with v an elliptic solution of the dual boundary value problem.*

Note that from Theorems 2.3 and 2.4 in [2] we have, in Corollary 3.5, that $u \in C^1(\Omega)$ is strictly g-convex in Ω, using just the boundedness of the densities f, f^* and their reciprocals.

From Theorem 3.1 and Corollary 3.5, and taking account of the uniqueness result of Rankin [15], we now have the following extension of Theorem 1.1 to the situation when A3 is weakened to A3w.

Corollary 3.6. *Let Ω and Ω^* be bounded domains in $\mathbb{R}^n$, with closures $\bar{\Omega} \subset U$ and $\bar{\Omega}^* \subset V$ and let g be a generating function satisfying A1, A2, A1*, A3 and A5. Suppose that $f > 0, \in C^{1,1}(\Omega), f^* > 0, \in C^{1,1}(\Omega^*)$, with $f, 1/f \in L^\infty(\Omega), f^*, 1/f^* \in L^\infty(\Omega^*)$ and that f and f^* satisfy the mass balance condition* (1.7). *Then for any $x_0 \in \Omega$ and u_0 satisfying $m_0 + 2K_1 < u_0 < M_0 - K_1$, there exists a unique g-convex, elliptic solution $u \in C^3(\Omega)$ of the second boundary value problem* (1.2), (1.5), *satisfying $u(x_0) = u_0$, provided Ω^* is g^*-convex with respect to $\Omega \times J_1$, where $J_1 = (u_0 - K_1, u_0 + K_1)$, and Ω is g-convex with respect to all $y \in \Omega^*$ and $z \in g^*(\cdot, y, J_1)(\Omega)$.*

To conclude this section we remark that we can also use Lemma 3.3 to extend the second derivative estimates in [5,9] and consequently the classical existence theory in [6] to generated Jacobian equations under just conditions A1, A1*, A2 and A3w.

4. More on convexity

4.1. Strict convexity

In [18] we also mentioned that the optimal transportation strict convexity and C^1 regularity results in [21] extended readily to the generated Jacobian case. The key convexity lemmas for this result are Lemmas 2.3 and its predecessor, Lemma 2.3 in [18], on the g-convexity of contact sets.

Lemma 4.1. *Let u and g satisfy the hypotheses of Lemma 2.3 with $g_0 = g(\cdot, y_0, z_0)$ a g-support at $x_0 \in \Omega$ and condition A3w strengthened to A3. Suppose for some ball $B = B_R(x_0) \subset\subset \Omega$, there exists a positive constant λ_0 such that*

$$|Tu(\omega)| \geq \lambda_0|\omega|, \tag{4.1}$$

for all open subsets $\omega \subset B$. Then u is strictly g-convex at x_0. Alternatively suppose that $Tu(\Omega)$ is g^*-convex with respect to u on Ω, where Ω is sub g-convex with respect to $g^*(x, \cdot, u(x))$ on $Tu(\Omega)$ for all $x \in \Omega$, and

$$|Tu(\omega)| \leq \Lambda_0|\omega| \tag{4.2}$$

for all open subsets $\omega \subset \Omega$ and some positive constant Λ_0. Then $u \in C^1(\Omega)$.

To prove Lemma 4.1, we suppose that u is not strictly g-convex at x_0 so that by Lemma 2.3 there must exist a g-segment γ, with respect to y_0, z_0 joining x_0 to another point $x_1 \in B$ and lying in B. For any sufficiently small radius ρ, there then exists a ball $B_\rho \subset\subset B$ of radius ρ intersecting γ, with $x_0, x_1 \notin B_\rho$. Now let $\tilde{u} \in C^3(B_\rho) \cap C^0(\bar{B}_\rho)$ be the unique elliptic solution of the Dirichlet problem,

$$\det[D^2\tilde{u} - A(\cdot, \tilde{u}, D\tilde{u})] = \frac{\lambda_0}{2} \text{ in } B_\rho, \quad \tilde{u} = u \text{ on } \partial B_\rho, \tag{4.3}$$

which, as in the proofs of regularity, is well defined for sufficiently small ρ. By our previous comparison arguments, we also have $\tilde{u} > u \geq g_0$ in B_ρ and $\tilde{u} = u = g_0$ on $\gamma \cap \partial B_\rho$. This can be seen readily from the finer inequality $\tilde{u} > \tilde{u}_0 \geq u$, where $\tilde{u}_0$ solves the Dirichlet problem (4.3) with $\lambda_0/2$ replaced by λ_0. Again Lemma 4.6 in [18] is also crucial here as in the regularity theory in Section 3. Applying now Lemma 2.3 again in B_ρ, with z_0 slightly reduced, or more simply the proof of Lemma 2.1, we thus reach a contradiction. The second assertion in Lemma 4.1 on C^1 regularity now follows by duality.

Note that in the above argument we have taken advantage of the g-convexity of the contact set S_0 to simplify the technicalities in [21], where just the connectedness of S_0 is used, although the basic approach though reduction to a Dirichlet problem in small balls is the same. Also, from either the ellipticity of (4.3) or condition A3, we only need the strict differential inequality (2.5) in replicating the differential inequality argument in Lemma 2.1.

For application to regularity we need to enhance our domain sub convexity conditions. Namely we will call Ω sub g-convex in U, with respect to (y_0, z_0), if $z_0 \in I(U, y_0)$ and the convex hull of $Q(\Omega, y_0, z_0) \subset Q(U, y_0, z_0)$ and Ω^* sub g^*-convex in V, with respect to (x_0, u_0), if $u_0 \in J(x_0, V)$ and the convex hull of $P(x_0, \Omega^*, u_0) \subset P(x_0, V, u_0)$. Then we can weaken the g-convexity of Ω in Lemma 2.3, in the case of arbitrary contact sets S_0, by Ω being sub g-convex in a larger domain U, with respect to (y, z), for all g-supports, $g(\cdot, y, z)$ to u, or equivalently with respect to the dual function $v = u^*_g$. In this case we need to strengthen condition (ii) so that $Tu(\Omega)$ is sub g^*-convex in a larger domain V with respect to $g(\cdot, y, z_y)$ on $\hat{\Omega}_y$, for all $y \in Tu(\Omega), z_y = v(y)$, where $\hat{\Omega}_y$ denotes the $Q(\cdot, y, z_y)$ convex hull of Ω. To be more explicit, we define $\hat{\Omega}_y$ as the image under the inverse mapping $Q^{-1}(\cdot, y, z_y)$ of the convex hull of $Q(\Omega, y, z_y)$.

Then using duality we have the following extension of Theorem 3.2 to non smooth densities.

Theorem 4.2. *Let u be a generalized solution of* (1.5) *with positive densities f and f^*, with $f, 1/f \in L^\infty(\Omega), f^*, 1/f^* \in L^\infty(\Omega^*)$ and with generating function g satisfying conditions A1, A2, A1* and A3. Suppose that $\Omega \subset\subset U$ is sub g-convex in U with respect to $g^*(x, \cdot, u(x)$ on $Tu(\Omega)$, for all $x \in \Omega$ and $\Omega^* \subset\subset V$ is sub g^*-convex in V with respect to $g(\cdot, y, z_y)$ on $\hat{\Omega}_y$ for all $y \in \Omega^*, z_y = v(y)$. Then $u \in C^1(\bar{\Omega})$*

is strictly g-convex in Ω if Ω^* is g^*-convex with respect to u and Tu is a homeomorphism from Ω to Ω^* if also Ω is g-convex with respect to v. Furthermore if $f \in C^{1,1}(\Omega)$, $f^* \in C^{1,1}(\Omega^*)$, then $u \in C^3(\Omega)$.

We can also simplify the statement of Theorem 4.2, (as well as earlier results), by replacing Γ^* by $U \times V \times J_0$ for sufficiently large domains U and V and interval J_0. We note here also that the approach of Loeper [11] to local $C^{1,\alpha}$ regularity in optimal transportation under A3 was used in near field geometric optics in [3] and has been recently extended to general generated Jacobian equations in [4]. As a byproduct of our local convexity theory in [13], these results also extend to C^2 cost and generating functions.

4.2. Other convexity results

We prove here a version of Lemma 2.2 which does not use duality and corresponds in some sense to Lemma 2.3. We use the same notation as in Lemma 2.2.

Lemma 4.3. *Assume g satisfies A1, A2, A1* and A3w and suppose $u \in C^0(\Omega)$ is g-convex in Ω. Assume also:*

(i) Ω is g-convex with respect to some $\tilde{y} \in \Sigma_0$, $\tilde{z} = g^*(x_0, \tilde{y}, u_0)$ for some x_0 in Ω;

(ii) $Tu(x_0)$ is sub g^*-convex with respect to $\tilde{g} = g(\cdot, \tilde{y}, \tilde{z})$ in Ω.

Then $\tilde{y} \in Tu(x_0)$.

To prove Lemma 4.3 we first choose two extreme points p_0 and p_1 in $\partial u(x_0)$ such that $p_\theta = \theta p_0 + (1-\theta)p_1 = g_x(x_0, \tilde{y}, \tilde{z})$ for some $\theta \in (0, 1)$. Setting $y_\theta, z_\theta = Y, Z(x_0, u_0, p_\theta)$, and $g_\theta = g(\cdot, y_\theta, z_\theta)$ for any $\theta \in [0, 1]$, it follows that for either $i = 0$ or 1, $\tilde{h} = g_i - \tilde{g} > 0$ at some point on the g-segment I_g, with respect to $\tilde{y}, \tilde{z}$, joining x_0 and a point $x \in \Omega$, provided $D_\eta g_0(x_0) \neq D_\eta g_1(x_0)$, where $\eta_j = E^{i,j}(x_0, \tilde{y}, \tilde{z})(q_i - q_{0i})$, $q = Q(x, \tilde{y}, \tilde{z}), q_0 = Q(x_0, \tilde{y}, \tilde{z})$. By decreasing $\tilde{z}$ we can then apply the argument of Lemma 2.3 in the domain Ω to obtain a contradiction if $\max\{g_0, g_1\}(x) < \tilde{g}(x)$. On the other hand, if $D_\eta g_0(x_0) = D_\eta g_1(x_0)$, then the function f, given by $f(t) = \tilde{h}(x_t)$, satisfies $f(0) = f'(0) = 0$ for both $i = 0$ and 1 so that if also $f \le 0$ on $[0, 1]$, $f(1) < 0$, we also obtain a contradiction with the differential inequality (2.7). Alternatively, we can approximate x to reduce to the case $D_\eta g_0(x_0) \neq D_\eta g_1(x_0)$, as in [18]. Consequently $\tilde{g} \le \max\{g_0, g_1\}$ on Ω whence $\tilde{y} \in Tu(x_0)$ as asserted.

Note that the domain Ω in Lemma 4.3 can be made arbitrary by replacing it by its $Q(\cdot, \tilde{y}, \tilde{z})$ convex hull in condition (ii). By extending to all $\tilde{y} \in \Sigma_0$, we can then obtain another version of Lemma 2.2, though under stricter sub convexity hypotheses.

We also have another version of Lemma 2.3 from our earlier drafts, which essentially follows from the proof of Lemma 2.1.

Lemma 4.4. *Assume g satisfies A1, A2, A1* and A3w and suppose $u \in C^0(\Omega)$ is g-convex in Ω and $g_0 = g(\cdot, y_0, z_0)$ is a g-affine function on Ω. Assume also:*

(i) Ω is g-convex with respect to y_0 and all $z \in g^*(\cdot, y_0, \sup\{u, g_0\})(\Omega)$;

(ii) $Tu(\Omega) \cup \{y_0\}$ is sub g^*-convex with respect to $\sup\{u, g_0\}$ in Ω.

Then the section $S = S(u, g_0)$ is g-convex with respect to (y_0, z_0), while if g_0 is a g-support to u, the contact set $S_0 = S_0(u, g_0)$ is g-convex with respect to (y_0, z_0).

Note that when we extend Lemma 4.4 to general domains, when g_0 is an arbitrary g-support to u, we end up with the same sub convexity hypotheses as the corresponding extension above of Lemma 2.3.

Finally we also indicate that by modification of the proof of Lemma 4.3, we may obtain a refinement of the local version of the Loeper maximum principle (2.11) under condition A3, corresponding to those used to show $C^{1,\alpha}$ regularity in [3, 4, 11]. This is also extended to $g \in C^2$, through the geometric approach in [13].

Lemma 4.5. *Assume g satisfies A1, A2, A1* and A3 and* $(x_0, u_0, [p_0, p_1]) \subset \mathcal{U}' \subset\subset \mathcal{U}$*, where* $[p_0, p_1]$ *denotes the straight line joining* p_0 *and* p_1 *in* $\mathbb{R}^n$*. Then there exist small positive constants,* ϵ_0 *and* γ_0*, depending on g and* $\mathcal{U}'$*, such that*

$$g(x, y_\theta, z_\theta) \leq \max\{g_0(x), g_1(x)\} - \gamma_0[\theta(1-\theta)|p_1 - p_0||x - x_0|]^2 \tag{4.4}$$

for all $|x - x_0| \leq \epsilon_0$ *and* $\theta \in [0, 1]$.

To prove (4.4), we replace $\tilde{g} = g_\theta$ in the proof of Lemma 4.3 by the function $\tilde{g}_\gamma$ given by

$$\tilde{g}_\gamma = g_\theta + \gamma[\theta(1-\theta)|p_1 - p_0||x - x_0|]^2 \tag{4.5}$$

for sufficiently small $\gamma > 0$ and distance $|x - x_0|$, using more explicitly the differential inequalities (2.9) and (2.10) in [18], in conjunction with condition A3 in $\mathcal{U}'$, to extend (2.7) to $\tilde{g}_\gamma$.

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Conflict of interest

The author declares no conflict of interest.

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