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Research article

Asymptotic analysis for non-local curvature flows for plane curves with a general rotation number

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Abstract: Several non-local curvature flows for plane curves with a general rotation number are discussed in this work. The types of flows include the area-preserving flow and the length-preserving flow. We have a relatively good understanding of these flows for plane curves with the rotation number one. In particular, when the initial curve is strictly convex, the flow exists globally in time, and converges to a circle as time tends to infinity. Even if the initial curve is not strictly convex, a global solution, if it exists, converges to a circle. Here, we deal with curves with a general rotation number, and show, not only a similar result for global solutions, but also a blow-up criterion, upper estimates of the blow-up time, and blow-up rate from below. For this purpose, we use a geometric quantity which has never been considered before.

Keywords: non-local curvature flow; rotation number; blow-up; asymptotic behavior; the isoperimetric inequality; the isoperimetric deficit

1. Introduction

In this paper, we deal with curvature flows comprising non-local terms for plane curves with a general rotation number. Let $\boldsymbol{f}$ be an $\mathbb{R}^2$-valued function on $\mathbb{R}/L(t)\mathbb{Z} \times [0, T)$ such that for a fixed $t \in [0, T)$, it is an arc-length parametrization of a closed plane curve with total length $L(t)$. In the following text, we simply denote $L(t)$ as L in many cases. To explain the curvature flow that is considering in this work, we introduce a certain geometric quantity. For a fixed $t \in [0, T)$, $s \in \mathbb{R}/L\mathbb{Z}$ is an arc-length parameter. Then, $\boldsymbol{\tau} = \partial_s \boldsymbol{f}$ and $\boldsymbol{\kappa} = \partial_s^2 \boldsymbol{f}$ are the unit tangent vector and the curvature vector respectively.

The vector $\boldsymbol{\nu}$ is a unit normal vector given by rotating $\boldsymbol{\tau}$ counter-clockwise by $\frac{\pi}{2}$. The curvature κ and its deviation $\tilde{\kappa}$ are given by

$$\kappa = \boldsymbol{\kappa} \cdot \boldsymbol{\nu}, \quad \tilde{\kappa} = \kappa - \frac{1}{L}\int_0^L \kappa\, ds.$$

Here, $\tilde{\kappa}$ is a non-local quantity. The equation we consider is of the following form:

$$\partial_t \boldsymbol{f} = \left(\tilde{\kappa} - \frac{g}{L}\right)\boldsymbol{\nu}.$$

Here, we assume that the function g is a scale-invariant non-local quantity determined by $\boldsymbol{f}$. That is, set $\boldsymbol{f}_\lambda(s) = \frac{1}{\lambda}\boldsymbol{f}(\lambda s)$ $(s \in \mathbb{R}/\lambda^{-1}L\mathbb{Z})$, then,

$$g(\boldsymbol{f}_\lambda) = g(\boldsymbol{f}).$$

Here we study three cases of g:

(AP) If we set $g \equiv 0$, then our equation represents the area-preserving flow. In fact, we set A as

$$A = -\frac{1}{2}\int_0^L \boldsymbol{f} \cdot \boldsymbol{\nu}\, ds$$

which is the enclosed area when $\mathrm{Im}\boldsymbol{f}$ is a simple curve. Consequently, it holds that

$$\frac{dA}{dt} = 0.$$

(LP) Let $g = L\left(\int_0^L \kappa\, ds\right)^{-1}\int_0^L \tilde{\kappa}^2 ds$. Here, the equation represents the length-preserving flow:

$$\frac{dL}{dt} = 0.$$

(JP) Jiang-Pan considered an equation with $g = \frac{L^2}{2A} - \int_0^L \kappa\, ds$ in [5]. Here, the isoperimetric ratio does not increase along with the flow:

$$\frac{d}{dt}\frac{L^2}{A} = -\frac{2L}{A}\int_0^L \|\partial_t \boldsymbol{f}\|^2 ds.$$

Let

$$n = \frac{1}{2\pi}\int_0^L \kappa\, ds$$

be the rotation number. For classical solutions, the rotation number n is independent of t. There are a multitude of literature available considering the case when $n = 1$ in the above equations. First of all, we should mention Gage's result [3]. Assume that $\mathrm{Im}\boldsymbol{f}(0)$ is a strictly convex, closed curve with a rotation number equal to 1 in the class of C^2. Then, the solution $\boldsymbol{f}$ with the initial data $\boldsymbol{f}(0)$ exists globally in time, and $\mathrm{Im}\boldsymbol{f}(t)$ converges to a circle with a surrounding area $A(0)$ as $t \to \infty$. Similar results for (LP) and (JP) were proved by [6] and [5] respectively under the convexity condition. The authors considered flows without the convexity condition in [7, 8]. Instead of convexity, we assume the global existence of the solution. Then the solution of (AP), (LP), or (JP) converges to a circle as $t \to \infty$ exponentially. As a result, the curvature uniformly converges to a positive constant, and thus, the curve becomes convex in finite time. In our previous works, the isoperimetric deficit

$$I_{-1} = 1 - \frac{4\pi A}{L^2}$$

played an important role. First, we show the decay of I_{-1}. Set

$$I_\ell = L^{2\ell+1} \int_0^L \left|\tilde{\kappa}^{(\ell)}\right|^2 ds \text{ for } \ell \in \{0\} \cup \mathbb{N}.$$

In [7], we showed the inequality

$$I_j \leqq C \left(I_{-1}^{\frac{\ell-j}{2}} I_\ell + I_{-1}^{\frac{\ell-j}{\ell+1}} I_\ell^{\frac{j+1}{\ell+1}} \right) \tag{1.1}$$

for an integer $j \in [0, \ell]$ with a positive constant $C = C(j, \ell)$ independent of the total length of curve. Since I_{-1} is small for a sufficiently large t, we can regard this inequality as an embedding with a small embedding constant. We showed the exponential decay of I_ℓ using the standard energy method, combining the above inequality. Finally, using the decay of I_ℓ, we showed the convergence of $\mathrm{Im}\boldsymbol{f}$ to a circle.

In this paper we study the case of $n > 1$, when the isomerimetric deficit is

$$I_{-1} = 1 - \frac{4n\pi A}{L^2}.$$

The isoperimetric inequality shows $I_{-1} \geqq 0$ when $n = 1$. However, I_{-1} is not necessarily non-negative for $n > 1$. This implies the technique used in [7, 8] is not applicable for $n > 1$. In spite of this, I_{-1} gives us some useful information. For example, we can show that if I_{-1} is negative for $t = 0$, then the solution blows up in finite time. See our first main result, Theorem 3.1. This implies $I_{-1} \geqq 0$ for global solutions, and that sounds a good information. However, the inequality (1.1) does not hold for $n > 1$. There are at least two approaches for dealing with this difficulty. One is to give a proof without using (1.1), and another is to show an alternative inequality to (1.1). In this paper, we show that both are in success. For the second approach, we use a geometric quantity which has never been considered before, given as follows:

$$\widetilde{I}_{-1} = \frac{1}{L} \left\| \frac{2\pi n}{L} \left(\boldsymbol{f} - \frac{1}{L} \int_0^L \boldsymbol{f}\, ds \right) + \boldsymbol{\nu} \right\|_{L^2}^2 .$$

Then we can show

$$I_j \leqq C \left(\widetilde{I}_{-1}^{\frac{\ell-j}{2}} I_\ell + \widetilde{I}_{-1}^{\frac{\ell-j}{\ell+1}} I_\ell^{\frac{j+1}{\ell+1}} \right). \tag{1.2}$$

We prepare several inequalities and estimates for closed curves with a rotation number n, in § 2.1. And we describe some basic properties of the flows (AP), (LP) and (JP), in § 2.2. Using these, in § 3, we discuss blow-up solutions with blow-up time estimates, blow-up quantities, and blow-up rates. In § 4, the convergence to an n-fold circle of global solutions is proved without using (1.2). Finally, we show (1.2) in the final section.

2. Preliminaries

In this section, we provide several estimates and inequalities for plane curves. Those in § 2.1 hold for curves which are not necessarily solutions of the flows. We derive the basic properties of flows in § 2.2.

2.1. Estimates for plane curves

Let $\boldsymbol{f} = (f_1, f_2)$ be an arc-length parametrization of a plane curve with the rotation number $n \geqq 1$. Set

$$f = f_1 + if_2, \quad \nu = \nu_1 + i\nu_2 = -f_2' + if_1' = if'.$$

The functions $\varphi_k = \dfrac{1}{\sqrt{L}} \exp\left(\dfrac{2\pi iks}{L}\right)$ for $k \in \mathbb{Z}$ generate the standard complete orthogonal system of $L^2(\mathbb{R}/L\mathbb{Z})$. Let $\hat{f}(k)$ be the Fourier coefficient of f. Subsequently, we can derive the following relations in a manner similar to [7, Corollary 2.1], where we dealt with the case of $n = 1$. The difference is just "n" in (2.3) which comes exactly from the definition of the rotation number. We can find similar argument in [1, 10]

Lemma 2.1.

$$\sum_{k\in\mathbb{Z}} k|\hat{f}(k)|^2 = \frac{LA}{\pi}, \tag{2.1}$$

$$\sum_{k\in\mathbb{Z}} k^2|\hat{f}(k)|^2 = \left(\frac{L}{2\pi}\right)^2 \int_0^L \kappa^0 ds = \frac{L^3}{4\pi^2}, \tag{2.2}$$

$$\sum_{k\in\mathbb{Z}} k^3|\hat{f}(k)|^2 = \left(\frac{L}{2\pi}\right)^3 \int_0^L \kappa\, ds = \frac{nL^3}{4\pi^2}, \tag{2.3}$$

$$\sum_{k\in\mathbb{Z}} k^4|\hat{f}(k)|^2 = \left(\frac{L}{2\pi}\right)^4 \int_0^L \kappa^2 ds, \tag{2.4}$$

$$\sum_{k\in\mathbb{Z}} k^5|\hat{f}(k)|^2 = \left(\frac{L}{2\pi}\right)^5 \int_0^L \kappa^3 ds, \tag{2.5}$$

$$\sum_{k\in\mathbb{Z}} k^6|\hat{f}(k)|^2 = \left(\frac{L}{2\pi}\right)^6 \int_0^L \left\{\kappa^4 + (\kappa')^2\right\} ds. \tag{2.6}$$

Note that we have

$$\sum_{k\in\mathbb{Z}} k^2(k-n)|\hat{f}(k)|^2 = 0 \tag{2.7}$$

from (2.2) and (2.3). The above is very useful for our analysis.

Lemma 2.2. *We have*

$$I_0 = \frac{16\pi^4}{L^3} \sum_{k\in\mathbb{Z}} k^3(k-n)|\hat{f}(k)|^2 \tag{2.8}$$

$$= \frac{16\pi^4}{L^3} \sum_{k\in\mathbb{Z}} k^2(k-n)^2|\hat{f}(k)|^2. \tag{2.9}$$

Proof. We obtain (2.8) as

$$I_0 = L\int_0^L \tilde{\kappa}^2 ds = L\int_0^L \tilde{\kappa}\kappa\, ds = L\left(\int_0^L \kappa^2 ds - \frac{2\pi n}{L}\int_0^L \kappa\, ds\right)$$
$$= \frac{16\pi^4}{L^3}\sum_{k\in\mathbb{Z}} k^3(k-n)|\hat{f}(k)|^2$$

from (2.4) and (2.3). Combining this with (2.7), we obtain (2.9). □

Though I_0 must be non-negative by the definition, it is not obvious to see that from the first expression (2.8). However, it can be seen from the second one (2.9). Furthermore, we see from (2.9) that $I_0 = 0$ if and only if Im$\boldsymbol{f}$ is an n-fold circle.

The isoperimetric inequality holds even if n is not 1.

Lemma 2.3. *We have* $L^2 - 4\pi A \geqq 0$.

Proof. It follows from (2.2) and (2.1) that

$$L^2 - 4\pi A = \frac{4\pi^2}{L}\left(\frac{L^3}{4\pi^2} - \frac{LA}{\pi}\right) = \frac{4\pi^2}{L}\sum_{k\in\mathbb{Z}} k(k-1)|\hat{f}(k)|^2 \geqq 0.$$

□

Similarly, I_{-1} has two expressions.

Lemma 2.4. *We have*

$$I_{-1} = \frac{4\pi^2}{L^3}\sum_{k\in\mathbb{Z}} k(k-n)|\hat{f}(k)|^2 = -\frac{4\pi^2}{nL^3}\sum_{k\in\mathbb{Z}\setminus\{0\}} k(k-n)^2|\hat{f}(k)|^2.$$

Proof. It follows from (2.2) and (2.1) that

$$I_{-1} = 1 - \frac{4\pi nA}{L^2} = \frac{4\pi^2}{L^3}\left(\frac{L^3}{4\pi^2} - \frac{nLA}{\pi}\right) = \frac{4\pi^2}{L^3}\sum_{k\in\mathbb{Z}} k(k-n)|\hat{f}(k)|^2.$$

The second expression of I_{-1} is obtained from the above and (2.7). □

Since $k(k-n)$ is not necessarily non-negative when $n > 1$, we know the same holds for I_{-1}. However, the modulus of I_{-1} can be estimated by I_0 for $n \geqq 1$ as follows. This is Wirtinger's inequality when $n = 1$.

Lemma 2.5. *It holds that* $4\pi^2 n|I_{-1}| \leqq I_0$.

Proof. From Lemmas 2.2–2.4 we obtain

$$I_0 \pm 4\pi^2 nI_{-1} = \frac{16\pi^4}{L^3}\sum_{k\in\mathbb{Z}}\left\{k^2(k-n)^2 \mp k(k-n)^2\right\}|\hat{f}(k)|^2$$
$$= \frac{16\pi^4}{L^3}\sum_{k\in\mathbb{Z}} k(k\mp 1)(k-n)^2|\hat{f}(k)|^2 \geqq 0.$$

Here, we use $k(k\mp 1) \geqq 0$ for $k \in \mathbb{Z}$. □

2.2. Estimates for flows

In this subsection, we derive the basic properties of the flows, which we use in following sections. Let $\boldsymbol{f}$ be a classical solution of one of (AP), (LP), or (JP) on $[0, T)$, and let T be the maximum existence time. Since $\dfrac{dL}{dt} = -\displaystyle\int_0^L \partial_t \boldsymbol{f} \cdot \boldsymbol{\kappa}\, ds$, we have

$$\frac{dL^2}{dt} = -2L \int_0^L \left(\tilde{\kappa} - \frac{g}{L}\right) \kappa\, ds = -2L \int_0^L \tilde{\kappa}^2 ds + 4\pi n g,$$

that is,

$$\frac{dL^2}{dt} + 2I_0 = 4\pi n g. \tag{2.10}$$

Similarly, we have

$$\frac{dA}{dt} = -\int_0^L \partial_t \boldsymbol{f} \cdot \boldsymbol{\nu}\, ds = -\int_0^L \left(\tilde{\kappa} - \frac{g}{L}\right) ds = g. \tag{2.11}$$

It follows from the above that

$$\frac{d}{dt}\left(L^2 I_{-1}\right) + 2I_0 = \frac{d}{dt}\left(L^2 - 4\pi n A\right) + 2I_0 = 0. \tag{2.12}$$

From these, we summarize the basic properties of each solution as follows.

Proposition 2.1. *Assume that the initial curve is smooth, and that $A(0)$ is positive. Let $\boldsymbol{f}$ be a classical solution of one of (AP), (LP), or (JP) on $[0, T)$ and let T be the maximum existence time. Then, the following holds for $t \in (0, T)$.*

1). For solutions of (AP),

$$\frac{dA}{dt} = 0, \quad A \equiv A(0) > 0, \quad \frac{dL^2}{dt} \leqq 0, \quad \frac{dI_{-1}}{dt} \leqq 0.$$

2). For solutions of (LP),

$$\frac{dA}{dt} \geqq 0, \quad A \geqq A(0) > 0, \quad \frac{dL^2}{dt} = 0, \quad \frac{dI_{-1}}{dt} \leqq 0.$$

3). For solutions of (JP),

$$A > 0, \quad \frac{dI_{-1}}{dt} \leqq 0.$$

4). For solutions of (AP), (LP), (JP),

$$1 - n \leqq I_{-1} \leqq I_{-1}(0).$$

In other words,

$$4\pi \leqq \frac{L^2}{A} \leqq \frac{L(0)^2}{A(0)}.$$

Proof. In the cases of (AP) and (LP), the signs of $\frac{dA}{dt}$ and $\frac{dL^2}{dt}$ immediately follow from (2.11) and (2.10). Therefore, $A > 0$ and

$$\frac{dI_{-1}}{dt} = -\frac{d}{dt}\frac{4\pi nA}{L^2} = -\frac{4\pi n}{L^2}\frac{dA}{dt} + \frac{4\pi nA}{L^4}\frac{dL^2}{dt} \leqq 0.$$

In the case of (JP), we prove the positivity of A by applying the contradiction argument. In this case,

$$g = \frac{L^2 I_{-1}}{2A}. \tag{2.13}$$

It follows from (2.11) that

$$\frac{dA^2}{dt} = 2Ag = L^2 I_{-1}. \tag{2.14}$$

Assume that $A(t_0)^2 = 0$ for some first time $t_0 \in (0, T)$. Since $A^2 \geqq 0$, we have

$$\frac{dA^2}{dt}(t_0) = 0. \tag{2.15}$$

Since $A(0)^2 > 0$, there exists $t_1 \in (0, t_0)$ such that

$$\frac{dA^2}{dt}(t_1) < 0. \tag{2.16}$$

It follows from (2.14) and (2.12) that

$$\frac{d^2A^2}{dt^2} = \frac{d}{dt}\left(L^2 I_{-1}\right) = -2I_0 \leqq 0.$$

Therefore, by (2.16)

$$\frac{dA^2}{dt}(t_0) = \frac{dA^2}{dt}(t_1) + \int_{t_1}^{t_0} \frac{d^2A^2}{dt^2}\, dt \leqq \frac{dA^2}{dt}(t_1) < 0.$$

This contradicts (2.15). Hence, $A > 0$ on $(0, T)$. Using (2.12), (2.10), $I_{-1} - 1 = -\frac{4\pi nA}{L^2}$, and (2.13), we have

$$\begin{aligned} L^2\frac{dI_{-1}}{dt} &= -I_{-1}\frac{dL^2}{dt} - 2I_0 \\ &= -I_{-1}(4\pi ng - 2I_0) - 2I_0 = -4\pi ngI_{-1} + 2(I_{-1} - 1)I_0 \\ &= -\frac{4\pi n}{L^2}\left(L^2 gI_{-1} + 2AI_0\right) = -\frac{4\pi n}{L^2}\left(\frac{L^4 I_{-1}^2}{2A} + 2AI_0\right) \leqq 0. \end{aligned}$$

Since I_{-1} is non-increasing, we have $I_{-1} \leqq I_{-1}(0)$. Lemma 2.3 gives us

$$I_{-1} = 1 - \frac{4\pi nA}{L^2} = 1 - n + n\left(1 - \frac{4\pi A}{L^2}\right) \geqq 1 - n.$$

□

3. Blow-up solutions

The non-positivity of $I_{-1}(0)$ implies that the blow-up phenomena occurs in finite time.

Theorem 3.1. *Let f be a classical solution of one of (AP), (LP), or (JP) on $[0, T)$ and let T be the maximum existence time. Assume that the initial curve is smooth, and satisfies $A(0) > 0$, $I_{-1}(0) < 0$. Then, the solution blows up in finite time. The blow-up time T is estimated from above as follows:*

(AP) $T \leqq \dfrac{L(0)^2 - 4\pi A(0)}{-8\pi^2 n I_{-1}(0)}$,

(LP) $T \leqq \dfrac{L(0)^2 - 4\pi A(0)}{-8\pi^2 I_{-1}(0)}$,

(JP) $T \leqq \dfrac{L(0)^2}{-8\pi^2 n I_{-1}(0)}$.

Proof. In the case of (AP), $g \equiv 0$. It follows from Proposition 2.1 that $I_{-1}(t) \leqq I_{-1}(0) < 0$. By (2.10) and Lemma 2.5, we have

$$\frac{dL^2}{dt} = -2I_0(t) \leqq 8\pi^2 n I_{-1}(t) \leqq 8\pi^2 n I_{-1}(0).$$

Integrating this from 0 to $t \in (0, T)$, and using Lemma 2.3, we obtain

$$4\pi A(0) - L^2(0) = 4\pi A(t) - L^2(0) \leqq L^2(t) - L^2(0) \leqq 8\pi^4 n I_{-1}(0)t.$$

Since the first side is non-positive by the isoperimetric inequality (Lemma 2.3), t must satisfy

$$t \leqq \frac{L(0)^2 - 4\pi A(0)}{-8\pi^2 n I_{-1}(0)}.$$

In the case of (LP), $g = \dfrac{I_0}{2\pi n} \geqq 0$. Proposition 2.1 shows $I_{-1}(t) \leqq I_{-1}(0) < 0$. From (2.11) and Lemma 2.5, we have

$$-\frac{dA}{dt} = -\frac{1}{2\pi n} I_0(t) \leqq 2\pi I_{-1}(t) \leqq 2\pi I_{-1}(0).$$

We integrate this from 0 to $t \in (0, T)$. Using Lemma 2.3, we obtain

$$4\pi A(0) - L(0)^2 = 4\pi A(0) - L(t)^2 \leqq 4\pi(A(0) - A(t)) \leqq 8\pi I_{-1}(0)t.$$

Consequently, t must satisfy

$$t \leqq \frac{L(0)^2 - 4\pi A(0)}{-8\pi^2 I_{-1}(0)}.$$

In the case of (JP), $g = \dfrac{L^2 I_{-1}}{2A}$. It follows from (2.10), Proposition 2.1, and Lemma 2.5 that

$$\frac{dL^2}{dt} = -2I_0(t) + \frac{2\pi n L(t)^2}{A(t)} I_{-1}(t) \leqq -2I_0(t) \leqq 8\pi^2 n I_{-1}(t) \leqq 8\pi^2 n I_{-1}(0).$$

We integrate this from 0 to $t \in (0, T)$. Using Lemma 2.3, we obtain

$$-L(0)^2 \leqq L(t)^2 - L(0)^2 \leqq 8\pi^2 n I_{-1}(0)t.$$

Consequently t must satisfy

$$t \leqq \frac{L(0)^2}{-8\pi^2 n I_{-1}(0)}.$$

□

Corollary 3.1. *Let* $\boldsymbol{f}$ *be a classical solution of one of (AP), (LP), or (JP) on* $[0, T)$ *and let* T *be the maximum existence time. Assume that the initial curve is smooth, and that satisfies* $A(0) > 0$*, and* $I_{-1}(0) = 0$*, but it is not an* n*-fold circle. Then,* $T < \infty$*.*

Proof. Assume $T = \infty$. Then, Theorem 3.1 implies that $I_{-1}(t) \geqq 0$ for all $t \in [0, \infty)$. On the other hand, (2.12) with $I_{-1}(0) = 0$ shows that $I_{-1}(t) \leqq 0$. Hence, $I_{-1}(t) \equiv 0$. When $t > 0$,

$$\int_0^L \tilde{\kappa}^2 ds = \frac{I_0}{L} = -\frac{1}{2L}\frac{d}{dt}\left(L^2 I_{-1}\right) = 0.$$

Combining this with the rotation number n, we find that $\mathrm{Im}\boldsymbol{f}(t)$ is an n-fold circle. However, this does not satisfy the initial condition. □

Corollary 3.2. $\boldsymbol{f}$ *is a classical stationary solution of one of (AP), (LP), or (JP), if and only of it is an* n*-fold circle.*

Proof. Assume that $\mathrm{Im}\boldsymbol{f}$ is an n-fold circle. Then, $\tilde{\kappa} \equiv 0$. Since $f = \hat{f}(0)\varphi_0 + \hat{f}(n)\varphi_n$, we see $I_0 = I_{-1} = 0$ by Lemmas 2.2 and 2.4. Hence, $\tilde{\kappa} - \dfrac{g}{L} \equiv 0$ for each case. Consequently, it is a stationary solution.

Conversely, assume that $\boldsymbol{f}$ is a stationary solution. It follows from (2.12) that $I_0(t) \equiv 0$. Hence, we can conclude that $\mathrm{Im}\boldsymbol{f}(t)$ is an n-fold circle in a manner similar to the proof of the previous corollary. □

Suppose now $\boldsymbol{f}$ blows up as $t \nearrow T < \infty$. Then, we have

$$\limsup_{t \nearrow T} I_0(t) = \infty.$$

Indeed, if $\limsup_{t \nearrow T} I_0(t) < \infty$, then $\sup_{t \in (0,T)} I_0(t)$ is bounded. We can show the boundedness of $\sup_{t \in (0,T)} I_\ell(t)$ by the standard energy method. Using this and the equation of the flow, we can see that $\boldsymbol{f}(t)$ converges to a smooth function as $t \nearrow T$. Consequently, the solution can be expanded beyond T. This is a contradiction.

Set

$$W = \int_0^L \kappa^2 ds.$$

We will show the blow-up of W and its blow-up rate. Firstly, we consider the limit supremum of W.

Lemma 3.1. *It holds that* $\limsup_{t \nearrow T} W(t) = \infty$.

Proof. Set

$$R = \int_0^L \kappa \, ds,$$

and we have

$$LW = L \int_0^L \left\{ \tilde{\kappa}^2 + \left(\frac{R}{L} \right)^2 \right\} ds = I_0 + R^2.$$

Hence,

$$\limsup_{t \nearrow T} L(t) W(t) = \infty.$$

Therefore, the assertion immediately follows in the case of (LP).

In the case of (AP), L is non-increasing by Proposition 2.1. Lemma 2.3 implies that $L \geqq \sqrt{4\pi A} = \sqrt{4\pi A_0}$. Consequently, $L(t)$ converges to a positive constant as $t \nearrow T$, and the assertion follows.

We show that $L(t)$ converges to a positive constant in the case of (JP) as well. We assume that $\liminf_{t \nearrow T} A(t) = 0$. I_{-1} is monotone by Proposition 2.1. Therefore, it follows from

$$\frac{dA}{dt} = \frac{L^2}{2A} I_{-1}$$

that A does not oscillate near $t = T$. Hence, we may assume $\lim_{t \nearrow T} A(t) = 0$. From the above relation and Proposition 2.1, we find that $\dfrac{dA}{dt}$ is bounded. Consequently, the estimate

$$0 < A(t) \leqq C(T - t)$$

holds. Thus, we have

$$0 \leqq \frac{A(t)^2}{T - t} \leqq \frac{C(T - t)^2}{T - t} \to 0 \text{ as } t \nearrow T,$$

and therefore,

$$\lim_{t \nearrow T} \frac{A(T - 0)^2 - A(t)^2}{T - t} = 0.$$

This implies that the left derivative of A^2 at T vanishes:

$$\frac{dA^2}{dt}(T - 0) = 0. \tag{3.1}$$

However, $A(0)^2 > 0$ and $A(T - 0)^2 = 0$ show the existence of $t_* \in (0, T)$ such that

$$\frac{dA^2}{dt}(t_*) < 0.$$

Since

$$\frac{d^2 A^2}{dt^2} = -2I_0 \leqq 0,$$

we have

$$\frac{dA^2}{dt}(t) < \frac{dA^2}{dt}(t_*) < 0$$

for $t \in (t_*, T)$. This contradicts (3.1). Now, we prove $\liminf_{t \nearrow T} A(t) > 0$. Since

$$\frac{dA}{dt} = \frac{L^2}{2A} I_{-1}$$

has a constant sign near T, we conclude that $\lim_{t \nearrow T} A(t) > 0$. The convergence of $\lim_{t \nearrow T} L(t)$ follows from the convergence of A, and the monotonicity and boundedness of I_{-1}. Since $\dfrac{L^2}{A}$ is strictly positive by Proposition 2.1, the limit of L is positive. □

Next, we derive the time derivative of W. Set

$$J_p = L^{p-1} \int_0^L \tilde{\kappa}^p \, ds \quad (p \in \mathbb{N} \setminus \{1\}),$$

which are scale-invariant quantities. Note that $I_0 = J_2$.

Lemma 3.2. *It holds that*

$$\frac{dW}{dt} = \frac{1}{L^3} \left\{ -2I_1 + J_4 + (3R - g)J_3 + 3R(R - g)J_2 - R^3 g \right\}.$$

Proof. The proof is a direct calculation:

$$\begin{aligned}
\frac{dW}{dt} &= \int_0^L \partial_t \boldsymbol{f} \cdot \left(2\partial_s^2 \kappa + \kappa^3 \right) ds = \int_0^L \left(\tilde{\kappa} - \frac{g}{L} \right) \left(2\partial_s^2 \kappa + \kappa^3 \right) ds \\
&= -2 \int_0^L (\partial_s \tilde{\kappa})^2 \, ds + \int_0^L \left(\tilde{\kappa} - \frac{g}{L} \right) \left(\tilde{\kappa} + \frac{R}{L} \right)^3 ds \\
&= -\frac{2I_1}{L^3} + \int_0^L \left(\tilde{\kappa}^3 + \frac{3R\tilde{\kappa}^2}{L} + \frac{3R^2 \tilde{\kappa}}{L^2} + \frac{R^3}{L^3} \right) \left(\tilde{\kappa} - \frac{g}{L} \right) ds \\
&= -\frac{2I_1}{L^3} + \int_0^L \left\{ \tilde{\kappa}^4 + \left(\frac{3R}{L} - \frac{g}{L} \right) \tilde{\kappa}^3 + \left(\frac{3R^2}{L^2} - \frac{3Rg}{L^2} \right) \tilde{\kappa}^2 - \frac{R^3 g}{L^4} \right\} ds \\
&= \frac{1}{L^3} \left\{ -2I_1 + J_4 + (3R - g)J_3 + 3R(R - g)J_2 - R^3 g \right\}.
\end{aligned}$$

□

Thirdly, we estimate $\frac{dW}{dt}$ from above.

Lemma 3.3. *We have*

$$\frac{dW}{dt} \leqq \frac{W^3}{2M^2}.$$

Here,

$$M = \begin{cases} C & \textit{for (AP) and (LP)}, \\ C \left\{ 1 + \left(\dfrac{L_0^2}{A_0} \right)^{\frac{4}{3}} \right\}^{-\frac{1}{2}} & \textit{for (JP)} \end{cases}$$

with the constant C being independent of the initial curve and the rotation number.

Proof. Here, we use Lemma 3.2. In the case of (AP), since $g = 0$, we have

$$\frac{dW}{dt} + \frac{2I_1}{L^3} = \frac{1}{L^3}\left(J_4 + 3RJ_3 + 3R^2 J_2\right).$$

Set $\theta = \frac{1}{2} - \frac{1}{p}$. Then, Gagliardo-Nirenberg's inequality yields

$$|J_p| \leqq C\left(I_0^{1-\theta} I_1^{\theta}\right)^{\frac{p}{2}} = C I_0^{\frac{p}{4}+\frac{1}{2}} I_1^{\frac{p}{4}-\frac{1}{2}}.$$

Hence,

$$\begin{aligned}\frac{dW}{dt} + \frac{2I_1}{L^3} &\leqq \frac{C}{L^3}\left(I_0^{\frac{3}{2}} I_1^{\frac{1}{2}} + R I_0^{\frac{5}{4}} I_1^{\frac{1}{4}} + R^2 I_0\right) \\ &\leqq \frac{I_1}{L^3} + \frac{C}{L^3}\left(I_0^3 + R^{\frac{4}{3}} I_0^{\frac{5}{3}} + R^2 I_0\right).\end{aligned}$$

Since $0 \leqq I_0 \leqq LW$ and $R^2 \leqq LW$, we obtain

$$\begin{gathered} I_0^3 \leqq L^3 W^3, \quad I_0^{\frac{5}{3}} \leqq L^{\frac{5}{3}} W^{\frac{5}{3}} = (LW)^{-\frac{4}{3}} L^3 W^3 \leqq R^{-\frac{8}{3}} L^3 W^3, \\ I_0 \leqq LW = (LW)^{-4} L^3 W^3 \leqq R^{-8} L^3 W^3. \end{gathered}$$

Furthermore,

$$R = 2\pi n \geqq 2\pi.$$

Consequently, we conclude that

$$\frac{dW}{dt} \leqq C\left(1 + R^{-\frac{4}{3}} + R^{-6}\right) W^3 \leqq C W^3.$$

In the case of (LP), since $g = \dfrac{I_0}{R}$, we have

$$\begin{aligned}\frac{dW}{dt} + \frac{1}{L^3}\left(2I_1 + 3I_0^2 + R^2 I_0\right) &= \frac{1}{L^3}\left\{J_4 + \left(3R - \frac{I_0}{R}\right) J_3 + 3R^2 I_0\right\} \\ &\leqq \frac{C}{L^3}\left(I_0^{\frac{3}{2}} I_1^{\frac{1}{2}} + R I_0^{\frac{5}{4}} I_1^{\frac{1}{4}} + R^{-1} I_0^{\frac{9}{4}} I_1^{\frac{1}{4}} + R^2 I_0\right) \\ &\leqq \frac{I_1}{L^3} + \frac{C}{L^3}\left(I_0^3 + R^{\frac{4}{3}} I_0^{\frac{5}{3}} + R^{-\frac{4}{3}} I_0^3 + R^2 I_0\right) \\ &\leqq \frac{I_1}{L^3} + C\left(1 + R^{-\frac{4}{3}} + R^{-6}\right) W^3 \\ &\leqq \frac{I_1}{L^3} + C W^3.\end{aligned}$$

In the case of (JP), since $g = \dfrac{L^2}{2A} - R$, we have

$$\frac{dW}{dt} + \frac{1}{L^3}\left(2I_1 + \frac{3RL^2}{2A} I_0 + \frac{R^3 L^2}{2A}\right)$$

$$
\begin{aligned}
&= \frac{1}{L^3}\left\{J_4 + \left(3R - \frac{L^2}{2A} + R\right)J_3 + 6R^2 J_2 + R^4\right\} \\
&\leqq \frac{C}{L^3}\left\{I_0^{\frac{3}{2}} I_1^{\frac{1}{2}} + R I_0^{\frac{5}{4}} I_1^{\frac{1}{4}} + \frac{L^2}{A} I_0^{\frac{5}{4}} I_1^{\frac{1}{4}} + R^2 I_0 + R^{-2}(LW)^3\right\} \\
&\leqq \frac{I_1}{L^3} + \frac{C}{L^3}\left[I_0^3 + \left\{R + \left(\frac{L^2}{A}\right)\right\}^{\frac{4}{3}} I_0^{\frac{5}{3}} + R^2 I_0 + R^{-2} L^3 W^3\right] \\
&\leqq \frac{I_1}{L^3} + C\left[1 + \left\{\frac{1}{R} + \left(\frac{L^2}{R^2 A}\right)\right\}^{\frac{4}{3}} + R^{-6} + R^{-2}\right] W^3 \\
&\leqq \frac{I_1}{L^3} + C\left\{1 + \left(\frac{L^2}{A}\right)^{\frac{4}{3}}\right\} W^3.
\end{aligned}
$$

By Proposition 2.1, we have

$$
\left(\frac{L^2}{A}\right)^{\frac{4}{3}} \leqq \left(\frac{L_0^2}{A_0}\right)^{\frac{4}{3}}.
$$

Consequently, we can conclude that

$$
\frac{dW}{dt} \leqq C\left\{1 + \left(\frac{L_0^2}{A_0}\right)^{\frac{4}{3}}\right\} W^3.
$$

□

Now, we prove the following theorem.

Theorem 3.2. *Let $T \in (0, T)$ be the blow-up time for a solution of one of (AP), (LP), or (JP). Then, $W(t)$ blows up as*

$$
W(t) \geqq \frac{M}{\sqrt{T - t}},
$$

where

$$
M = \begin{cases} C & \textit{for (AP) and (LP)}, \\ C\left\{1 + \left(\frac{L_0^2}{A_0}\right)^{\frac{4}{3}}\right\}^{-\frac{1}{2}} & \textit{for (JP)} \end{cases}
$$

with a constant C that is independent of the initial curve and the rotation number.

Proof. It follows from Lemma 3.3 that

$$
\frac{d}{dt} W^{-2} \geqq -M^{-2}.
$$

Due to Lemma 3.1, there exists a sequence $\{t_n\}$ such that $t_n t \nearrow T$ and $W(t_n)^{-2} \to 0$ as $n \to \infty$. Integrating the differential inequality from t to t_n, we have

$$
W(t)^{-2} - W(t_n)^{-2} \leqq M^{-2}(t_n - t).
$$

Therefore, we obtain the theorem as $n \to \infty$. □

The curve Im$\boldsymbol{f}$ may have several loops. When the orientation of a loop is counter-clockwise as s increases, it is called a *positive* loop. Otherwise, it is called a *negative* loop. It has already been shown that $L(t)$ converges to a positive constant as $t \to \infty$. Therefore, from the above theorem we know that

$$\lim_{t \nearrow T} \max_{s \in \mathbb{R}/L(t)\mathbb{Z}} \kappa(s,t) = \infty$$

or

$$\lim_{t \nearrow T} \min_{s \in \mathbb{R}/L(t)\mathbb{Z}} \kappa(s,t) = -\infty.$$

If a positive/negative loop of Im$\boldsymbol{f}$ shrinks as $t \nearrow T$, the maximum/minimum value of the curvature may not remain bounded. On the other hand, there is a possibility of the maximum or minimum remaining bounded as $t \nearrow T$. For example, if a negative loop shrinks as $t \nearrow T$ before the positive loops shrink, the minimum value of the curvature goes to $-\infty$, but the maximum remains bounded. In the last part of this section, we discuss the blow-up of the maximum and minimum.

Theorem 3.3. *Let $T \in (0,\infty)$ be the blow-up time for a solution of one of (AP), (LP), or (JP). Assume that*

$$\limsup_{t \nearrow T} \max_{s \in \mathbb{R}/L(t)\mathbb{Z}} \kappa(s,t) = \infty,$$

then it satisfies

$$\max_{s \in \mathbb{R}/L(t)\mathbb{Z}} \kappa(s,t) \geqq \frac{1}{\sqrt{2(T-t)}}.$$

Proof. Set

$$K(t) = \max_{s \in \mathbb{R}/L(t)\mathbb{Z}} \kappa(s,t),$$
$$\frac{d^+K}{dt}(t) = \limsup_{h \to +0} \frac{K(t+h)-K(t)}{h}.$$

Define the set S_t by $S_t = \{s \in \mathbb{R}/L(t)\mathbb{Z} \,|\, \kappa(s,t) = K(t)\}$. After re-parametrizing $\boldsymbol{f}(\cdot,t)$ by a new parameter that is independent of t, we apply [2, Lemma B.40]. Consequently, we can conclude that K is a continuous function of t, and that

$$\frac{d^+K}{dt}(t) = \max_{s \in S_t} \partial_t \kappa(s,t).$$

κ satisfies the equation

$$\partial_t \kappa = \partial_s^2 \kappa + \kappa^2 \left(\tilde{\kappa} - \frac{g}{L} \right) = \partial_s^2 \kappa + \kappa^2 \left(\kappa - \frac{R+g}{L} \right).$$

For the cases of (AP) and (LP), $R+g>0$ as $R>0$ and $g \geqq 0$. In the case of (JP),

$$R+g = \frac{L^2}{A} \geqq 0.$$

$\partial_s^2 \kappa \leqq 0$ holds for $s \in S_t$. Hence, we have

$$\partial_s^2 \kappa + \kappa^2 \left(\kappa - \frac{R+g}{L} \right) \leqq \kappa^3 = K^3$$

for $s \in S_t$, and

$$\frac{d^+K}{dt}(t) \leqq \max_{s\in S_t} \partial_t\kappa \leqq K^3(t).$$

We calculate Dini's derivative of K^{-2} as

$$\begin{aligned}
\frac{d^+}{dt}K^{-2}(t) &= \limsup_{h\to+0} \frac{K^{-2}(t+h)-K^{-2}(t)}{h} \\
&= \limsup_{h\to+0} \frac{(K(t)+K(t+h))(K(t)-K(t+h))}{K^2(t+h)K^2(t)h} \\
&= -2K^{-3}(t)\liminf_{h\to+0} \frac{K(t+h)-K(t)}{h} \\
&\geqq -2K^{-3}(t)\limsup_{h\to+0} \frac{K(t+h)-K(t)}{h} \\
&= -2K^{-3}(t)\frac{d^+K}{dt}(t) \geqq -2.
\end{aligned}$$

According to the assumption of the theorem, there exists a sequence $\{t_k\}_{k\in\mathbb{N}}$ such that $t_k \nearrow T$ and $K(t_k)^{-2} \to 0$ as $k \to \infty$. Using [4, Theorem 3], we have

$$K^{-2}(t_k) - K^{-2}(t) \geqq \underline{\int_t^{t_k}} \frac{d^+}{dt}K^{-2}(t)\,dt \geqq -2(t_k - t)$$

for $t_k \in (t, T)$. Therefore, we can conclude that

$$K^{-2}(t) \leqq 2(T-t)$$

by $k \to \infty$ □

Theorem 3.4. *Let $T \in (0,\infty)$ be the blow-up time for a solution of one of (AP), (LP), or (JP). Assume that*

$$\sup_{t\in[0,T)} \max_{s\in\mathbb{R}/L(t)\mathbb{Z}} \kappa(s,t) < \infty.$$

For the solution of (AP),

$$\min_{s\in\mathbb{R}/L(t)\mathbb{Z}} \kappa(s,t) \leqq -\frac{1}{\sqrt{4(T-t)}}$$

holds.

For the solution of (LP),

$$\min_{s\in\mathbb{R}/L(t)\mathbb{Z}} \kappa(s,t) \leqq -\left\{\frac{2\pi n}{9L(0)(T-t)}\right\}^{\frac{1}{3}}$$

holds.

For the solution of (JP), there exists a time $T_* \in [0,T)$ such that

$$-\min_{s\in\mathbb{R}/L(t)\mathbb{Z}} \kappa(s,t) \geqq \max_{s\in\mathbb{R}/L(t)\mathbb{Z}} \kappa(s,t)$$

holds for $t \in [T_*, T)$. Additionally, it holds that

$$\min_{s\in\mathbb{R}/L(t)\mathbb{Z}} \kappa(s,t) \leqq -\frac{1}{\sqrt{2C_*(T-t)}},$$

where

$$C_* = 1 + \frac{L(T_*)^2}{4\pi n A(T_*)}.$$

Remark 3.1. *The time T_* above exists for all cases. And for the proof, it does not need to be the first or last such time.*

Proof. Here, we set

$$K(t) = -\min_{s\in\mathbb{R}/L(t)\mathbb{Z}} \kappa(s,t),$$
$$\frac{d^+K}{dt}(t) = \limsup_{h\to+0} \frac{K(t+h)-K(t)}{h}.$$

Define the set S_t by $S_t = \{s \in \mathbb{R}/L(t)\mathbb{Z} \mid -\kappa(s,t) = K(t)\}$. As shown before, it holds that

$$\frac{d^+K}{dt}(t) = \max_{s\in S_t} \partial_t(-\kappa).$$

$-\kappa$ satisfies

$$\partial_t(-\kappa) = \partial_s^2(-\kappa) + (-\kappa)^2 \left\{(-\kappa) + \frac{R+g}{L}\right\}.$$

Since $\partial_s^2(-\kappa) \leqq 0$ and $-\kappa = K$ for $s \in S_t$,

$$\partial_t(-\kappa) \leqq K^3 + \frac{(R+g)K^2}{L}.$$

If $\kappa \leqq C < \infty$ holds on $[0,T)$, then,

$$L\max\{C^2 + K^2\} \geqq \int_0^L \kappa^2 ds = W \to \infty \text{ as } t \nearrow T$$

by Theorem 3.2. Since L is bounded, we conclude that $K \to \infty$ as $t \nearrow T$. Therefore, $|\kappa| \leqq \max\{C, K\} \leqq K$ near T. Hence, there exists $T_* \in [0,T)$ as mentioned in the statement. Considering $t \geqq T_*$, we may assume that $|\kappa| \leqq K$.

In the case of (AP), since $g = 0$,

$$\frac{(R+g)K^2}{L} = \frac{RK^2}{L}.$$

Using this and

$$R = \int_0^L \kappa\, ds \leqq \int_0^L |\kappa|\, ds \leqq LK,$$

we have $\partial_t(-\kappa) \leqq 2K^3$ on S_t, *i.e.*,

$$\frac{d^+K}{dt}(t) \leqq 2K^3.$$

Consequently, we obtain the assertion as before.

In the case of (LP),

$$\frac{K^2 g}{L} = \frac{K^2 I_0}{RL} = \frac{K^2}{R}\int_0^L \tilde{\kappa}^2 ds \leqq \frac{K^2}{R}\int_0^L \kappa^2 ds \leqq \frac{LK^4}{R}.$$

The estimate $\dfrac{R}{L} \leqq K$ holds for all cases. Hence,

$$K^3 = \frac{L}{R}\cdot\frac{R}{L}\cdot K^3 \leqq \frac{LK^4}{R}, \qquad \frac{K^2 R}{L} = \left(\frac{R}{L}\right)^2 \frac{LK^2}{R} \leqq \frac{LK^4}{R}.$$

Consequently, we have

$$\frac{d^+K}{dt}(t) \leqq \frac{3LK^4}{R} = \frac{3L(0)K^4}{2\pi n}.$$

Here, we use $L \equiv L(0)$. The statement follows from the above, as shown before.

In the case of (JP), using $R + g = \dfrac{L^2}{2A}$ and Lemma 2.1, we have

$$\frac{K^2(R+g)}{L} = \frac{K^2 L}{2A} = \frac{L^2}{2A}\cdot\frac{R}{L}\cdot\frac{K^2}{R} \leqq \frac{L(T_*)^2}{2A(T_*)}\cdot\frac{K^3}{R} = \frac{L(T_*)^2 K^3}{4\pi n A(T_*)}.$$

Hence, it holds that

$$\frac{d^+K}{dt}(t) \leqq \left(1 + \frac{L(T_*)^2}{4\pi n A(T_*)}\right) K^3,$$

which leads to the required conclusion and ends the proof. □

Remark 3.2. *At a glance, the power $\frac{1}{3}$ of blow-up rate in (LP) seems to be curious. The difference with other cases is that there is the length $L(0)$ in the braces. If an estimate*

$$\min_{s\in\mathbb{R}/L(t)\mathbb{Z}} \kappa(s,t) \leqq -\left\{\frac{2\pi n}{9L(0)(T-t)}\right\}^p$$

holds, then the power p must be $\frac{1}{3}$. To see this, assume that $\boldsymbol{f}$ is a solution of (LP) which blows up at $T < \infty$. For a positive constant λ, set

$$\boldsymbol{f}_\lambda(s,t) = \lambda^{-1}\boldsymbol{f}(\lambda s, \lambda^2 t).$$

We denote quantities of $\boldsymbol{f}_\lambda$ the notation with the suffix λ; for example κ_λ is its curvature. Then, $\boldsymbol{f}_\lambda$ satisfies (LP) with the length $L_\lambda = \lambda^{-1}L(0)$, and blows up at $T_\lambda = \lambda^{-2}T$. The minimum of curvature is

$$\min_{s\in\mathbb{R}/L_\lambda(t)\mathbb{Z}} \kappa_\lambda(s,t) = \lambda \min_{\lambda s\in\mathbb{R}/L(\lambda^2 t)\mathbb{Z}} \kappa(\lambda s, \lambda^2 t) \leqq -\lambda\left\{\frac{2\pi n}{9L(0)(T-\lambda^2 t)}\right\}^p.$$

Using $L(0) = \lambda L_\lambda(0)$ and $T = \lambda^2 T_\lambda$, we have

$$-\lambda\left\{\frac{2\pi n}{9L(0)(T-\lambda^2 t)}\right\}^p = -\lambda^{1-3p}\left\{\frac{2\pi n}{9L_\lambda(0)(T_\lambda - t)}\right\}^p.$$

Hence, p must be $\frac{1}{3}$. The $L(0)$ in braces comes from the estimate $\frac{K^2 g}{L} \leqq \frac{LK^4}{R}$ in the proof. If we can improve this as $\frac{K^2 g}{L} \leqq CK^3$, then the blow-up rate coincides with other cases.

4. Convergence of global solutions

In this section, we assume that $\boldsymbol{f}$ is a classical global solution of one of (AP), (LP), or (JP), and that the initial curve satisfies $A(0) > 0$. We prove that $\mathrm{Im}\boldsymbol{f}$ converges to an n-fold circle exponentially as $t \to \infty$.

Remark 4.1. *However, this conclusion is meaningless if n-fold circles are only global solutions. At least, in the case of (AP), under suitable assumptions on the initial curve, regarding symmetry and convexity, solutions exist globally in time even if* $n > 1$. *See [9].*

Firstly we prove the decay of I_{-1}.

Lemma 4.1. *For the global solution above,* $I_{-1}(t)$ *fulfills*

$$0 \leqq I_{-1}(t) \leqq \frac{L(0)^2 I_{-1}(0)}{L(t)^2} \exp\left(- \int_0^t \frac{8\pi^2 n}{L(\tau)^2}\, d\tau \right).$$

In particular, the estimate

$$0 \leqq I_{-1}(t) \leqq \frac{L(0)^2 I_{-1}(0)}{4\pi n A(0)} \exp\left(- \frac{8\pi^2 n}{L(0)^2} t \right)$$

is satisfied with respect to the global solution for (AP); the estimate

$$0 \leqq I_{-1}(t) \leqq I_{-1}(0) \exp\left(- \frac{8\pi^2 n}{L(0)^2} t \right)$$

for the global solution of (LP). In the case of (JP), setting $\bar{L} = \sup_{t \in [0,\infty)} L(t)$, *we have* $\bar{L} < \infty$, *and*

$$0 \leqq I_{-1}(t) \leqq \frac{L(0)^2 I_{-1}(0)}{4\pi n A(0)} \exp\left(- \frac{8\pi^2 n}{\bar{L}^2} t \right).$$

Proof. For global solutions, we know, from Theorem 3.1, that $I_{-1}(t) \geqq 0$. Hence, we have

$$4\pi^2 n I_{-1}(t) \leqq I_0(t) \tag{4.1}$$

by Lemma 2.5. Consequently, (2.12) becomes

$$\frac{d}{dt}\left(L^2 I_{-1} \right) + \frac{8\pi^2 n}{L^2}\left(L^2 I_{-1} \right) \leqq 0.$$

Solving this differential inequality, we obtain the first assertion.

We use $\sqrt{4n\pi A(0)} \leqq L(t) \leqq L(0)$ for (AP), and $L(t) \equiv L(0)$ for (LP). Then, the second assertion follows for these two cases.

Now, we consider the case of (JP). Integrating (2.12), we have

$$L^2 I_{-1} + 2\int_0^t I_0 d\tau = L_0^2 I_{-1}(0).$$

$\frac{L^2}{A}$ is uniformly positive and bounded by Proposition 2.1. From this, (2.10) with $g = \frac{L^2 I_{-1}}{2A}$ and (4.1), we have

$$\frac{dL^2}{dt} + 2I_0 = \frac{2\pi n L^2}{A} I_{-1} \leqq \frac{L^2}{2\pi A} I_0 \leqq C I_0.$$

Integrating this, we have

$$L^2 + 2\int_0^t I_0(\tau)\, d\tau \leqq L_0^2 + C\int_0^t I_0(\tau)\, d\tau \leqq L_0^2 \left(1 + C I_{-1}(0)\right).$$

Hence, $\bar{L} < \infty$. It follows from (2.11) and $g = \frac{L^2 I_{-1}}{2A} \geqq 0$ that

$$\frac{dA^2}{dt} = L^2 I_{-1} \geqq 0.$$

Therefore, the lower bound L follows from $L(t)^4 \geqq (4\pi n A(t))^2 \geqq (4\pi n A(0))^2$. Consequently, we obtain the second assertion for (JP). □

We denote the relevant statement of Lemma 4.1 as

$$I_{-1}(t) \leqq C e^{-\lambda_{-1} t}.$$

Corollary 4.1. *For the global solution above, there exists $L_\infty > 0$ and $A_\infty > 0$ such that*

$$|L - L_\infty| + |A - A_\infty| \leqq C e^{-\lambda_{-1} t}.$$

Proof. In the case of (AP), by Proposition 2.1, we have $\frac{dL}{dt} \leqq 0$. Hence, we conclude the convergence of $\lim_{t\to\infty} L(t)$. Set the limit value as L_∞. Since $A(t) \equiv A(0)$, and since $\lim_{t\to\infty} I_{-1}(t) = 0$, it holds that

$$L_\infty^2 = \lim_{t\to\infty} 4\pi n A(t) = 4\pi n A(t) = 4\pi n A(0) > 0$$

and $L_\infty \leqq L \leqq L(0)$. Therefore,

$$\begin{aligned} 0 \leqq L - L_\infty &= \frac{L^2 - L_\infty^2}{L + L_\infty} = \frac{L^2 - 4\pi n A}{L + L_\infty} = \frac{L^2 I_{-1}}{L + L_\infty} \\ &\leqq \frac{L(0)^2 I_{-1}}{2L_\infty} = \frac{L(0)^2 I_{-1}}{4\sqrt{\pi n A(0)}} \leqq C e^{-\lambda_{-1} t}. \end{aligned}$$

In the case of (LP), since $\frac{dA}{dt} \geqq 0$ and since $4\pi A \leqq L^2 = L(0)^2$, we conclude the convergence of $\lim_{t\to\infty} A(t)$. Set the limit value as A_∞. Since $L(t) \equiv L(0)$, and $\lim_{t\to\infty} I_{-1}(t) = 0$, it holds that $4\pi n A_\infty = L(0)^2$. Consequently, (2.11) with $g = \frac{I_0}{2\pi n}$ yields

$$0 \leqq A_\infty - A = \int_t^\infty \frac{I_0}{2\pi n}\, dt = \frac{L_0^2}{4\pi n} I_{-1}(t) \leqq C e^{-\lambda_{-1} t}.$$

Here, we use (2.12) and Lemma 4.1.

In the case of (JP), $\frac{dA}{dt} = \frac{L^2 I_{-1}}{2A} \geqq 0$. By Proposition 2.1, $\frac{A}{L^2}$ is uniformly positive and bounded. Combining the above two statements with Lemma 4.1, we conclude

$$0 \leqq A_\infty - A = \int_t^\infty \frac{L^2 I_{-1}}{2A}\, dt \leqq C \int_t^\infty I_{-1} dt \leqq Ce^{-\lambda_{-1} t}.$$

Furthermore, we estimate that

$$\begin{aligned} |L - L_\infty| &= \frac{|L^2 - L_\infty^2|}{L + L_\infty} = \frac{|L^2 I_{-1} + 4\pi n A - 4\pi n A_\infty|}{L + L_\infty} \\ &\leqq \frac{L^2 I_{-1} + 4\pi n |A - A_\infty|}{L_\infty} \leqq Ce^{-\lambda_{-1} t}. \end{aligned}$$

□

Corollary 4.2. *For the global solution above, it holds that*

$$\int_t^\infty I_0 dt \leqq Ce^{-\lambda_{-1} t}.$$

Proof. We know that L is uniformly bounded for all cases. Therefore, (2.12) implies that

$$\int_t^\infty I_0\, dt = \frac{L^2 I_{-1}}{2} \leqq Ce^{-\lambda_{-1} t}.$$

□

Lemma 4.2. *For the global solution above, there exists $\lambda_0 > 0$ such that*

$$I_0 \leqq Ce^{-\lambda_0 t}.$$

Proof. As in Section 3, we set

$$W = \int_0^L \kappa^2 ds, \quad R = \int_0^L \kappa\, ds, \quad J_p = L^{p-1} \int_0^L \tilde{\kappa}^p ds.$$

As we know that $L \to L_\infty > 0$ as $t \to \infty$, it is enough to show that

$$L^2 I_0 \leqq Ce^{-\lambda_0 t}.$$

Since $I_0 = J_2 = LW - R^2$, we have from (2.10) and Lemma 3.2

$$\begin{aligned} \frac{d}{dt}\left(L^2 I_0\right) &= \frac{d}{dt}\left(L^3 W - R^2 L^2\right) = L^3 \frac{dW}{dt} + \left(\frac{3}{2} LW - R^2\right) \frac{dL^2}{dt} \\ &= -2I_1 + J_4 + (3R - g)J_3 + 3R(R - g)J_2 - R^3 g \end{aligned}$$

$$
\begin{aligned}
&+ \left(\frac{3}{2}I_0 + \frac{1}{2}R^2\right)(-2I_0 + 2Rg) \\
= & -2I_1 - 3I_0^2 + J_4 + (3R - g)J_3 + 2R^2 J_2.
\end{aligned}
$$

We obtain

$$
\frac{d}{dt}\left(L^2 I_0\right) + I_1 + 3I_0^2 \leqq C\left(I_0^3 + I_0 + I_0^{\frac{5}{3}} + |g|^{\frac{4}{3}} I_0^{\frac{5}{3}}\right)
$$

in a manner similar to the proof of Lemma 3.3.

Since $g = 0$ in (AP), and $g = \dfrac{L^2 I_{-1}}{2A}$ in (JP), $|g|$ is uniformly bounded for these cases. In (LP), $g = R^{-1} I_0$. Hence, it holds for every case that

$$
\frac{d}{dt}\left(L^2 I_0\right) + I_1 + 3I_0^2 \leqq C\left(I_0 + I_0^3\right).
$$

This can be presented as

$$
\frac{d}{dt}\left(L^2 I_0\right) + I_1 + I_0^2\,(3 - CI_0) \leqq CI_0.
$$

By Corollary 4.2, there exists $t_0 > 0$ such that

$$
I_0(t_0) \leqq \frac{1}{C}, \qquad \int_{t_0}^{\infty} I_0 dt \leqq \frac{L^2}{C}.
$$

Set

$$
t_1 = \sup\left\{ t \in [t_0, \infty) \,\middle|\, I_0(t) < \frac{3}{C} \quad (t \in [t_0, \infty)) \right\}.
$$

If $t_1 < \infty$, then,

$$
\limsup_{t \to t_1 - 0} I_0(t) = \frac{3}{C} < \infty.
$$

For $t \in (t_0, t_1)$, we have

$$
\frac{d}{dt}\left(L^2 I_0\right) \leqq CI_0,
$$

and therefore,

$$
I_0(t) \leqq I_0(t_0) + \frac{1}{L^2}\int_{t_0}^{t} I_0 dt \leqq \frac{2}{C} = \frac{2}{3}\limsup_{t \to t_1} I_0(t).
$$

Letting $t \nearrow t_1$, we obtain a contradiction. Consequently, $t_1 = \infty$, that is, $I_0(t) < \dfrac{3}{C}$ for $t \in [t_0, \infty)$. Since we know that I_0 is uniformly bounded, we obtain

$$
\frac{d}{dt}\left(L^2 I_0\right) + I_1 + 3I_0^2 \leqq CI_0.
$$

It follows from Wirtinger's inequality and the uniform estimate of L^2 that

$$
\frac{d}{dt}\left(L^2 I_0\right) + 2\lambda L^2 I_0 \leqq CI_0
$$

for some constant $\lambda > 0$. Multiplying both sides by $e^{2\lambda t}$, and integrating from $\frac{t}{2}$ to t, we have

$$\begin{aligned} e^{2\lambda t} L(t)^2 I_0(t) &\leqq C e^{\lambda t} L\left(\frac{t}{2}\right)^2 I_0\left(\frac{t}{2}\right) + C \int_{\frac{t}{2}}^{t} e^{2\lambda\tau} I_0(\tau)\, d\tau \\ &\leqq C e^{\lambda t} + C e^{2\lambda t} \int_{\frac{t}{2}}^{\infty} I_0(\tau)\, d\tau. \end{aligned}$$

That is, we have

$$L(t)^2 I_0(t) \leqq C e^{-\lambda t} + C \int_{\frac{t}{2}}^{\infty} I_0(\tau)\, d\tau.$$

Using the uniform estimate of L and the exponential decay of $\int_{\frac{t}{2}}^{\infty} I_0 dt$, we finally obtain the exponential decay of I_0. □

Once we obtain the exponential decay of $\widetilde{I}_{-1}$ and I_0, we can obtain the convergence of Im$\boldsymbol{f}$ to an n-fold circle as $t \to \infty$.

Theorem 4.1. *Let $\boldsymbol{f}$ be a classical global solution of one of (AP), (LP), or (JP), with the smooth initial curve satisfying $A(0) > 0$. Then,* Im$\boldsymbol{f}$ *converges to an n-fold circle with centre $\boldsymbol{c}_\infty$, and radius $r_\infty = \frac{L_\infty}{2\pi n}$ in the following sense. Set*

$$\boldsymbol{f}(s,t) = \boldsymbol{c}(t) + r(t)\left(\cos\frac{2\pi n(s+\sigma(t))}{L(t)}, \sin\frac{2\pi n(s+\sigma(t))}{L(t)}\right) + \boldsymbol{\rho}(s,t),$$

$$\boldsymbol{c}(t) = \frac{1}{L(t)} \int_0^{L(t)} \boldsymbol{f}(s,t)\, ds, \quad r(t) = \frac{L(t)}{2\pi n},$$

with the $\mathbb{R}/L(t)\mathbb{Z}$-valued function σ defined by

$$\hat{f}(n)(t) = \sqrt{L(t)}\, r(t) \exp\left(\frac{2\pi i n \sigma(t)}{L(t)}\right).$$

Then, there exist $\boldsymbol{c}_\infty \in \mathbb{R}^2$, $r_\infty = \frac{L_\infty}{2\pi n} > 0$, $\sigma_\infty \in \mathbb{R}/L_\infty\mathbb{Z}$, $\lambda > 0$, and $C > 0$ such that

$$\|\boldsymbol{c}(t) - \boldsymbol{c}_\infty\| + |r(t) - r_\infty| + \left|\frac{\sigma(t)}{L(t)} - \frac{\sigma_\infty}{L_\infty}\right| \leqq C e^{-\lambda t}.$$

Furthermore, for $k \in \{0\} \cup \mathbb{N}$, there exist $\gamma_k > 0$ and $C_k > 0$ such that

$$\|\boldsymbol{\rho}(\cdot, t)\|_{C^k(\mathbb{R}/L(t)\mathbb{Z})} \leqq C_k e^{-\gamma_k t}.$$

When $n = 1$, we used (1.1) for the proof of this theorem in [7, § 4], and [8, § 2.2]. The most crucial part is to show the decay of I_0. As above, we have already obtained a decay estimate of I_0 without using (1.1) for $n \geqq 1$. Once we obtain it, to show the theorem, we can perform the standard energy method with help of usual Gagliardo-Nirenberg's inequality rather than (1.1) as the previous papers. In this sense, (1.1) is not absolutely necessary, however, we need several modification of argument. Using (1.2) which is an alternative inequality to (1.1), we can develop the argument almost word to word as the previous papers. Thus, we deal with (1.2) in the next section.

5. Interpolation inequalities

We discuss (1.2) in this section. Set

$$\widetilde{I}_{-1} = \frac{4\pi^2}{L^3} \sum_{k\in\mathbb{Z}\setminus\{0\}} (k-n)^2 |\hat{f}(k)|^2.$$

Proposition 5.1. *We have*

$$\widetilde{I}_{-1} = \frac{1}{L} \left\| \frac{2\pi n}{L} \left(\boldsymbol{f} - \frac{1}{L} \int_0^L \boldsymbol{f}\, ds \right) + \boldsymbol{\nu} \right\|_{L^2}^2 .$$

$\widetilde{I}_{-1}$ *vanishes if and only if* $\mathrm{Im}\boldsymbol{f}$ *is an n-fold circle.*

Proof. Setting

$$\tilde{f} = f - \frac{1}{L} \int_0^L f\, ds,$$

we have

$$\|\tilde{f}\|_{L^2}^2 = \sum_{k\in\mathbb{Z}\setminus\{0\}} |\hat{f}(k)|^2.$$

The squared L^2-norm of ν is

$$\|\nu\|_{L^2}^2 = \|f'\|_{L^2}^2 = \sum_{k\in\mathbb{Z}} \left(\frac{2\pi k}{L}\right)^2 |\hat{f}(k)|^2 = \frac{4\pi^2}{L^2} \sum_{k\in\mathbb{Z}\setminus\{0\}} k^2 |\hat{f}(k)|^2.$$

On the other hand, we have

$$\langle \tilde{f}, \nu \rangle_{L^2} = \langle \tilde{f}, if' \rangle = -\sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{2\pi k}{L} |\hat{f}(k)|^2 = -\frac{4\pi^2}{L^2} \sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{kL}{2\pi} |\hat{f}(k)|^2.$$

Since the last right-hand side expression is a real number, it holds that

$$\frac{4\pi^2}{L^2} \sum_{k\in\mathbb{Z}\setminus\{0\}} k |\hat{f}(k)|^2 = -\frac{2\pi}{L} \Re \langle \tilde{f}, \nu \rangle_{L^2}.$$

Consequently, we obtain

$$\begin{aligned}
\frac{4\pi^2}{L^2} \sum_{k\in\mathbb{Z}\setminus\{0\}} (k-n)^2 |\hat{f}(k)|^2 &= \|\nu\|_{L^2}^2 + \frac{4n\pi}{L} \Re \langle \tilde{f}, f' \rangle_{L^2} + \left(\frac{2\pi n}{L}\right)^2 \|\tilde{f}\|_{L^2}^2 \\
&= \left\| \frac{2\pi n}{L} \tilde{f} + \nu \right\|_{L^2}^2 \\
&= \left\| \frac{2\pi n}{L} \left(\boldsymbol{f} - \frac{1}{L} \int_0^L \boldsymbol{f}\, ds \right) + \boldsymbol{\nu} \right\|_{L^2}^2 .
\end{aligned}$$

$\widetilde{I}_{-1}$ vanishes if and only if

$$f = \hat{f}(0)\varphi_0 + \hat{f}(n)\varphi_n.$$

Hence, $\mathrm{Im}\boldsymbol{f}$ is an n-fold circle. □

An estimate similar to Lemma 2.5 holds for $\widetilde{I}_{-1}$ as well.

Lemma 5.1. *It holds that* $4\pi^2\widetilde{I}_{-1} \leqq I_0$.

Proof. Since $k^2 - 1 \geqq 0$ for $k \in \mathbb{Z} \setminus \{0\}$, we have

$$I_0 - 4\pi^2\widetilde{I}_{-1} = \frac{16\pi^4}{L^3} \sum_{k\in\mathbb{Z}\setminus\{0\}} (k^2 - 1)(k-n)^2|\hat{f}(k)|^2 \geqq 0.$$

□

The next proposition corresponds to [7, Theorem 2.2].

Proposition 5.2. *It holds that*

$$I_0 \leqq \widetilde{I}_{-1}^{\frac{1}{2}} \left[\int_0^L L^3 \left\{ \kappa^4 + (\kappa')^2 \right\} ds. \right]$$

Proof. It follows from Lemma 2.2, Schwarz' inequality, and (2.6) that

$$\begin{aligned} I_0 &= \frac{16\pi^4}{L^3} \sum_{k\in\mathbb{Z}\setminus\{0\}} k^3(k-n)|\hat{f}(k)|^2 \\ &\leqq \frac{8\pi^3}{L^{\frac{3}{2}}} \left\{ \frac{4\pi^2}{L^3} \sum_{k\in\mathbb{Z}\setminus\{0\}} (k-n)^2|\hat{f}(k)|^2 \right\}^{\frac{1}{2}} \left\{ \sum_{k\in\mathbb{Z}\setminus\{0\}} k^6|\hat{f}(k)|^2 \right\}^{\frac{1}{2}} \\ &= \frac{8\pi^3}{L^{\frac{3}{2}}} \widetilde{I}_{-1}^{\frac{1}{2}} \left\{ \sum_{k\in\mathbb{Z}\setminus\{0\}} k^6|\hat{f}(k)|^2 \right\}^{\frac{1}{2}} \\ &= \widetilde{I}_{-1}^{\frac{1}{2}} \left[\int_0^L L^3 \left\{ \kappa^4 + (\kappa')^2 \right\} ds \right]. \end{aligned}$$

□

Using this proposition, we can prove the following estimates.

Theorem 5.1. *Let $j \in [0, \ell]$ be an integer. Then, there exists a positive constant $C = C(j, \ell)$ independent of L such that*

$$I_j \leqq C \left(\widetilde{I}_{-1}^{\frac{\ell-j}{2}} I_\ell + \widetilde{I}_{-1}^{\frac{\ell-j}{\ell+1}} I_\ell^{\frac{j+1}{\ell+1}} \right).$$

Proof. Since the assertion can be proven in a manner similar to the proof of [7, Theorem 3.1], we give only the sketch. Firstly, we derive

$$I_0 \leqq C\widetilde{I}_{-1}^{\frac{1}{2}} \left(I_1 + \widetilde{I}_1^{\frac{1}{2}} \right) \tag{5.1}$$

from Proposition 5.2 and Gagliardo-Nirenberg's inequality

$$\left(L^{(j+1)p-1} \int_0^L |\widetilde{\kappa}^{(j)}|^p ds \right)^{\frac{1}{p}} \leqq C(j, m, p) I_m^{\frac{1}{2m}\left(j-\frac{1}{p}+\frac{1}{2}\right)} I_0^{\frac{1}{2}\left\{1-\frac{1}{m}\left(j-\frac{1}{p}+\frac{1}{2}\right)\right\}} \tag{5.2}$$

for $p \geqq 2$ and $j \leqq m$. Here $C(j, m, p)$ is independent of L. It follows from (5.2) that

$$I_j \leqq C(j, n) I_n^{\frac{j}{m}} I_0^{1-\frac{j}{m}}. \tag{5.3}$$

Combining this together with (5.1), we obtain the assertion for $j = 0$. It gives also the assertion for $j \geqq 1$ with help of (5.3). □

For the proof of convergence of global flow to a circle, we use in [7] the following properties of I_{-1}:

(i) $I_{-1} \geqq 0$,
(ii) $I_{-1} = 0$ holds if and if the image of $\boldsymbol{f}$ is a circle,
(iii) $C^{-1} I_{-1} \leqq I_0$ (an inequality of Wirtinger's type).

These are satisfied when $n = 1$, but not when $n > 1$. The quantity $\widetilde{I}_{-1}$ satisfies

(i) $\widetilde{I}_{-1} \geqq 0$,
(ii) $\widetilde{I}_{-1} = 0$ holds if and if the image of $\boldsymbol{f}$ is an n-fold circle,
(iii) $C^{-1} \widetilde{I}_{-1} \leqq I_0$ (an inequality of Wirtinger's type).

Hence, it is an alternative quantity to I_{-1}.

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Conflict of interest

The authors declare no conflict of interest.

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