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Research article

The vanishing discount problem for monotone systems of Hamilton-Jacobi equations. Part 1: linear coupling†

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Abstract: We establish a convergence theorem for the vanishing discount problem for a weakly coupled system of Hamilton-Jacobi equations. The crucial step is the introduction of Mather measures and their relatives for the system, which we call respectively viscosity Mather and Green-Poisson measures. This is done by the convex duality and the duality between the space of continuous functions on a compact set and the space of Borel measures on it. This is part 1 of our study of the vanishing discount problem for systems, which focuses on the linear coupling, while part 2 will be concerned with nonlinear coupling.

Keywords: systems of Hamilton-Jacobi equations; Mather measures; vanishing discount

Dedicated to Italo Capuzzo Dolcetta with friendship, respect, and admiration on the occasion of his retirement.

1. Introduction

We consider the weakly coupled m-system of Hamilton-Jacobi equations

$$\lambda v^\lambda + Bv^\lambda + H[v^\lambda] = 0 \quad \text{in } \mathbb{T}^n, \tag{P$_\lambda$}$$

where $m \in \mathbb{N}$, λ is a nonnegative constant, called the discount factor in terms of optimal control. Here $\mathbb{T}^n$ denotes the n-dimensional flat torus, $H = (H_i)_{i\in\mathbb{I}}$ is a family of Hamiltonians given by

$$H_i(x, p) = \max_{\xi\in\Xi}[-g_i(x, \xi) \cdot p - L_i(x, \xi)], \tag{H}$$

where $\mathbb{I} = \{1, \dots, m\}$, Ξ is a given compact metric space, $g = (g_i)_{i\in\mathbb{I}} \in C(\mathbb{T}^n \times \Xi, \mathbb{R}^n)^m$ and $L = (L_i)_{i\in\mathbb{I}} \in C(\mathbb{T}^n \times \Xi)^m$. The unknown in (P$_\lambda$) is an $\mathbb{R}^m$-valued function $v^\lambda = (v_i^\lambda)_{i\in\mathbb{I}}$ on $\mathbb{T}^n$, $B : C(\mathbb{T}^n)^m \to C(\mathbb{T}^n)^m$ is a linear map represented by a matrix $B = (b_{ij})_{i,j\in\mathbb{I}} \in C(\mathbb{T}^n)^{m\times m}$, that is,

$$(Bu)_i(x) = (B(x)u(x))_i := \sum_{j\in\mathbb{I}} b_{ij}(x)u_j(x) \quad \text{for } (x, i) \in \mathbb{T}^n \times \mathbb{I}.$$

We use the abbreviated notation $H[v^\lambda]$ to denote $(H_i(x, Dv_i^\lambda(x))_{i\in\mathbb{I}}$. The system is called weakly coupled since the i-th equation depends on Dv^λ only through Dv_i^λ but not on Dv_j^λ, with $j \neq i$. Problem (P$_\lambda$) can be stated in the component-wise style as

$$\lambda v_i^\lambda + \sum_{j\in\mathbb{I}} b_{ij}(x)v_j^\lambda + H_i(x, Dv_i^\lambda) = 0 \quad \text{in } \mathbb{T}^n,\ i \in \mathbb{I}.$$

We are mainly concerned with the asymptotic behavior of the solution v^λ of (P$_\lambda$) as $\lambda \to 0+$. Asymptotic problems in this class are called the vanishing discount problem, in view that the constant λ in (P$_\lambda$) appears as a discount factor in the dynamic programming PDE in optimal control.

Recently, there has been a keen interest in the vanishing discount problem concerned with Hamilton-Jacobi equations and, furthermore, fully nonlinear degenerate elliptic PDEs. We refer to [1, 7, 10, 12, 19, 20, 23–25, 27] for relevant work. The asymptotic analysis in these papers relies heavily on Mather measures or their generalizations and, thus, it is considered part of the Aubry-Mather and weak KAM theories. For the development of these theories we refer to [14, 16, 17] and the references therein.

We are here interested in the case of systems of Hamilton-Jacobi equations and, indeed, Davini and Zavidovique in [12] have established a general convergence result for the vanishing discount problem for (P$_\lambda$). We establish a result (Theorem 9 below) similar to the main result of [12]. In establishing our convergence result, we adapt the argument in [23] (see also [18]) to the case of systems, especially, to construct generalized Mather measures for (P$_\lambda$). Regarding the recent developments of the weak KAM theory and asymptotic analysis in its influence for systems, we refer to [5, 6, 26, 28–30, 33].

The new argument, which is different from that of [12], makes it fairly easy to build a generalized Mather measure for systems in a wide class. One advantage of our argument is that it allows us to treat the case where the coupling matrix B in (P$_\lambda$) depends on the space variable $x \in \mathbb{T}^n$. As in [20, 23], our approach is applicable to the system with nonlinear coupling of fully nonlinear second-order elliptic PDEs, but we restrict ourselves in this paper to the case of the linearly coupled system of first-order Hamilton-Jacobi equations. Another possible approach for constructing generalized Mather measures is the so-called adjoint method (see [5, 15, 19, 27, 33]).

This paper is part 1 of our study of the vanishing discount problem for weakly coupled systems of Hamilton-Jacobi equations and deals only with the linear coupling and with compact control sets Ξ. These restrictions make the presentation of our results clear and transparent. In part 2 [20], we remove these restrictions and establish a general convergence result extending Theorem 9 below. Sections 5 and 6 are devoted to the study of ergodic problems of the form $Bu + H[u] = c$, where $c \in \mathbb{R}^m$ is an unknown as well. Also, thanks to the linearity of the coupling, our results on the ergodic problems are applied to extend the scope of Theorem 9. On the other hand, the role of the ergodic problem, with general right-hand side c, is not clear at least for the author in the vanishing discount problem for the systems with the nonlinear coupling.

In this paper, we adopt the notion of viscosity solution to (P_λ), for which the reader may consult [2, 4, 8, 31].

To proceed, we give our main assumptions on the system (P_λ).

We assume that H is coercive, that is, for any $i \in \mathbb{I}$,

$$\lim_{|p|\to\infty} \min_{x\in\mathbb{T}^n} H_i(x,p) = \infty. \tag{C}$$

This is a convenient assumption, under which any upper semicontinuous subsolution of (P_λ) is Lipschitz continuous on $\mathbb{T}^n$.

We assume that $B(x) = (b_{ij}(x))$ is a monotone matrix for every $x \in \mathbb{T}^n$, that is, it satisfies

$$\text{for any } x \in \mathbb{T}^n, \text{ if } \ u = (u_i)_{i\in\mathbb{I}} \in \mathbb{R}^m \ \text{ and } \ u_k = \max_{i\in\mathbb{I}} u_i \geq 0, \ \text{ then } \ (B(x)u)_k \geq 0. \tag{M}$$

This is a natural assumption that (P_λ) should possess the comparison principle between a subsolution and a supersolution.

In what follows we set, for $\lambda \geq 0$,

$$B^\lambda = \lambda I + B,$$

and (P_λ) can be written as

$$B^\lambda v^\lambda + H[v^\lambda] = 0 \quad \text{in } \mathbb{T}^n.$$

We use the symbol $u \leq v$ (resp., $u \geq v$) for m-vectors $u, v \in \mathbb{R}^n$ to indicate $u_i \leq v_i$ (resp., $u_i \geq v_i$) for all $i \in \mathbb{I}$.

The following theorem is well-known: see [13, 22] for instance.

Theorem 1. *Assume* (C) *and* (M)*. Let $\lambda > 0$. Then the exists a unique solution $v^\lambda \in \mathrm{Lip}(\mathbb{T}^n)^m$ of* (P_λ)*. Also, if $v = (v_i), w = (w_i)$ are, respectively, upper and lower semicontinuous on $\mathbb{T}^n$ and a subsolution and a supersolution of* (P_λ)*, then $v \leq w$ on $\mathbb{T}^n$.*

Henceforth, let $\mathbf{1}$ denote the vector $(1, \ldots, 1) \in \mathbb{R}^m$.

Outline of proof. We follow the line of the arguments in [22]. Although [22] is concerned with the case when the domain is an open subset of a Euclidean space, the results in [22] is valid in the case when the domain is $\mathbb{T}^n$.

Choose a large constant $C > 0$ so that the constant functions $\pm C\mathbf{1}$ are a supersolution and a subsolution of (P_λ), respectively. (See also (2.3) below.) According to [22, Theorems 3.3, Lemma 4.8], there is a function $v^\lambda = (v_i^\lambda)_{i\in\mathbb{I}} : \mathbb{T}^n \to \mathbb{R}^m$ such that the upper and lower semicontinuous envelopes $(v^\lambda)^*$ and v_*^λ are a subsolution and a supersolution of (P_λ), respectively. By the coercivity assumption (C), we find (see [9, Theorem I.14], [21, Example 1]) that the functions $(v_i^\lambda)^*$ are Lipschitz continuous on T^n. Let $R_1 > 0$ be a Lipschitz bound of the functions $(v_i^\lambda)^*$. To take into account the Lipschitz property of $(v_i^\lambda)^*$, we modify the Hamiltonian H. Fix any $M > 0$ so that

$$\max_{(x,\xi,i)\in\mathbb{T}^n\times\Xi\times\mathbb{I}} |g_i(x,\xi)| < M, \tag{1.1}$$

and choose constants $N > 0$ and $R_2 > 0$ so that

$$H_i(x,p) \geq M|p| - N \quad \text{for } (x,p,i) \in \mathbb{T}^n \times B_{R_1} \times \mathbb{I},$$

and, in view of (1.1),

$$H_i(x,p) \le M|p| - N \quad \text{for } (x,p,i) \in \mathbb{T}^n \times B_{R_2} \times \mathbb{I}.$$

Define $G = (G_i)_{i\in\mathbb{I}} \in C(\mathbb{T}^n \times \mathbb{R}^n)^m$ by

$$G_i(x,p) = H_i(x,p) \vee (M|p| - N).$$

By the choice of R_1, it is easy to see that $(v^\lambda)^*$ is a subsolution of

$$\lambda u + Bu + G[u] = 0 \ \ \mathbb{T}^n. \tag{1.2}$$

Also, since $G \ge H$, v^λ_* is a supersolution of (1.2). Observe furthermore that, if $|p| \ge R_2$, then

$$G_i(x,p) = M|p| - N \quad \text{for } (x,i) \in \mathbb{T}^n \times \mathbb{I},$$

the functions G_i are uniformly continuous on $\mathbb{T}^n \times B_{R_2}$, and hence, for some continuous function ω on $[0,\ \infty)$, with $\omega(0) = 0$,

$$|G_i(x,p) - G_i(y,p)| \le \omega(|x-y|) \quad \text{for } (x,y,p) \in (\mathbb{T}^n)^2 \times \mathbb{R}^n,\ i \in \mathbb{I}.$$

The last inequality above shows that G satisfies [22, (A.2)], which allows us to apply [22, Theorem 4.7], to conclude that $(v^\lambda)^* \le v^\lambda_*$ on $\mathbb{T}^n$ and, moreover, that $v^\lambda \in \mathrm{Lip}(\mathbb{T}^n)^*$. Similarly, we deduce that the comparison assertion is valid. Thus, v^λ is a unique solution of (P$_\lambda$). □

Regarding the coercivity (C), the following proposition is well-knwon.

Proposition 2. *The function given by* (H) *satisfies* (C) *if and only if there exists* $\delta > 0$ *such that*

$$B_\delta \subset \mathrm{co}\{g_i(x,\xi)\ :\ \xi \in \Xi\} \quad \text{for } (x,i) \in \mathbb{T}^n \times \mathbb{I}, \tag{1.3}$$

where co *designates "convex hull" and* B_δ *denotes the open ball with origin at the origin and radius* δ.

Outline of proof. Set $C(x,i) = \mathrm{co}\{g_i(x,\xi)\ :\ \xi \in \Xi\}$. Assume that (1.3) is valid for some $\delta > 0$ and observe that

$$\begin{aligned} H_i(x,p,u) &\ge \max_{\xi\in\Xi} -g_i(x,\xi)\cdot p - \max_{(x,i,\xi)\in\mathbb{T}^n\times\mathbb{I}\times\Xi} L_i(x,\xi) \\ &= \max_{q\in C(x,i)} -q\cdot p - \max_{(x,i,\xi)\in\mathbb{T}^n\times\mathbb{I}\times\Xi} L_i(x,\xi) \ge \sup_{q\in B_\delta} -q\cdot p - \max_{(x,i,\xi)\in\mathbb{T}^n\times\mathbb{I}\times\Xi} L_i(x,\xi) \\ &= \delta|p| - \max_{(x,i,\xi)\in\mathbb{T}^n\times\mathbb{I}\times\Xi} L_i(x,\xi), \end{aligned}$$

which shows that (C) holds.

Next, assume that (1.3) does not hold for any $\delta > 0$. Then there exists $(x_k, i_k) \in \mathbb{T}^n \times \mathbb{I}$ for each $k \in \mathbb{N}$ such that

$$B_{1/k} \setminus C(x_k, i_k) \ne \emptyset.$$

For each $k \in \mathbb{N}$ select $q_k \in B_{1/k} \setminus C(x_k, i_k)$ and $r_k \in C(x_k, i_k)$ so that r_k is the point of $C(x_k, i_k)$ closest to q_k. (Notice that $C(x_k, i_k)$ is a compact convex set.) Setting $\nu_k = (q_k - r_k)/|q_k - r_k|$, we find that

$$\nu_k \cdot (q - r_k) \le 0 \quad \text{for } q \in C(x_k, i_k).$$

Sending $k \to \infty$ along an appropriate subsequence, say $(k_j)_{j\in\mathbb{N}}$, we find that there are a unit vector $\nu = \lim_{j\to\infty} \nu_{k_j}$ of $\mathbb{R}^n$, $r = \lim_{j\to\infty} r_{k_j} \in \mathbb{R}^n$ and $(x, i) \in \mathbb{T}^n \times \mathbb{I}$ such that

$$r \in C(x, i) \quad \text{and} \quad \nu \cdot (q - r) \le 0 \quad \text{for } q \in C(x, i).$$

If $r \ne 0$, then we have $\nu = -r/|r|$, since $\lim_{k\to\infty} q_k = 0$, and the inequality above reads

$$\nu \cdot q \le -|r| < 0 \quad \text{for } q \in C(x, i).$$

These observations imply that for $t > 0$,

$$H_i(x, -t\nu) = \max_{\xi\in\Xi} tg_i(x, \xi) \cdot \nu - \min_{\xi\in\Xi} L_i(x, \xi) \le -\min_{\xi\in\Xi} L_i(x, \xi),$$

which shows that (C) does not hold. This completes the proof. □

The rest of this paper is organized as follows. In Section 2, we recall some basic facts concerning monotone matrices. In Section 3, we study viscosity Green-Poisson measures for our system, which are crucial in our asymptotic analysis. We establish the main result for the vanishing discount problem in Section 4. We study the ergodic problem (P_0) in the cases when B is irreducible, and B is a constant matrix, respectively, in Sections 5 and 6, and combine the results with the analysis on the vanishing discount problem of Section 4.

2. Monotone matrices

Here we are concerned with $m \times m$ real matrix $B = (b_{ij})_{i,j\in\mathbb{I}}$.

Let e_i denote the vector $(e_{i1}, \dots, e_{im})$, with $e_{ii} = 1$ and $e_{ij} = 0$ if $i \ne j$.

Lemma 3. *Let $B = (b_{ij})$ be a real $m \times m$ matrix. It is monotone if and only if*

$$b_{ij} \le 0 \quad \text{if } i \ne j \quad \text{and} \quad \sum_{j\in\mathbb{I}} b_{ij} \ge 0 \quad \text{for } i \in \mathbb{I}. \tag{2.1}$$

We remark that if B satisfies (2.1), then

$$b_{ii} = \sum_{j\in\mathbb{I}} b_{ij} - \sum_{j\ne i} b_{ij} \ge 0. \tag{2.2}$$

Proof. We assume first that B is monotone. Since

$$\mathbf{1}_i = 1 = \max_j \mathbf{1}_j > 0,$$

By the monotonicity of B, we have

$$0 \le (B\mathbf{1})_i = \sum_{j=1}^{m} b_{ij}\mathbf{1}_j = \sum_{j=1}^{m} b_{ij} \quad \text{for } i \in \mathbb{I}. \tag{2.3}$$

Similarly, if $i \ne j$ and $t \ge 0$, then we have $1 = (e_i - te_j)_i = \max_{k\in\mathbb{I}}(e_i - te_j)_k$ and hence,

$$0 \le (B(e_i - te_j))_i = b_{ii} - tb_{ij},$$

from which we find by sending $t \to \infty$ that

$$b_{ij} \leq 0.$$

Hence, (2.1) is satisfied.

Next, we assume that (2.1) holds. Let $u \in \mathbb{R}^m$ satisfy

$$u_k = \max_{i \in \mathbb{I}} u_i \geq 0.$$

Then we observe that, since $u_k \geq u_j$ for all $j \in \mathbb{I}$,

$$(Bu)_k = \sum_{j \in \mathbb{I}} b_{kj} u_j = b_{kk} u_k + \sum_{j \neq k} b_{kj} u_j = b_{kk} u_k + \sum_{j \neq k} b_{kj} u_k = u_k \sum_{j \in \mathbb{I}} b_{kj} \geq 0.$$

Thus, B is monotone. □

Lemma 4. *Let $u \in \mathbb{R}^m$ and $C \geq 0$ be a constant. Let B be an $m \times m$ real monotone matrix. Then we have*

$$B(u - C\mathbf{1}) \leq Bu \leq B(u + C\mathbf{1}).$$

Proof. Using Lemma 3, we see that

$$(B\mathbf{1})_i = \sum_{j \in \mathbb{I}} b_{ij} \geq 0 \quad \text{for } i \in \mathbb{I},$$

which states that $B\mathbf{1} \geq 0$. It is then obvious to compute that

$$B(u + C\mathbf{1}) - Bu = CB\mathbf{1}, \quad Bu - B(u - C\mathbf{1}) = CB\mathbf{1} \quad \text{and} \quad CB\mathbf{1} \geq 0$$

and therefore,

$$B(u + C\mathbf{1}) \geq Bu \geq B(u - C\mathbf{1}).$$

□

3. Viscosity Green-Poisson measures

For $\lambda \geq 0$ we write $\mathcal{F}(\lambda)$ for the set of all $(\phi, u) \in C(\mathbb{T}^n \times \Xi)^m \times C(\mathbb{T}^n)^m$ such that u is a subsolution of

$$B^\lambda u + H_\phi[u] = 0 \quad \text{in } \mathbb{T}^n,$$

where $H_\phi = (H_{\phi,i})_{i \in \mathbb{I}}$ and

$$H_{\phi.i}(x, p) = \max_{\xi \in \Xi}(-g_i(x, \xi) \cdot p - \phi_i(x, \xi)).$$

In the above, since ϕ is bounded on $\mathbb{T}^n \times \Xi$, if H satisfies (C), then H_ϕ satisfies (C).

Lemma 5. *The set $\mathcal{F}(\lambda)$ is a convex cone in $C(\mathbb{T}^n \times \Xi)^m \times C(\mathbb{T}^n)^m$ with vertex at the origin.*

Proof. Recall [3, Remark 2.5] that for any $u \in Lip(\mathbb{T}^n)^m$, u is a subsolution of

$$B^\lambda u + H[u] = 0 \quad \text{in } \mathbb{T}^n$$

if and only if for any $i \in \mathbb{I}$,

$$(B^\lambda u)_i(x) + H_i(x, Du_i(x)) \le 0 \quad \text{a.e. in } \mathbb{T}^n,$$

and by the coercivity (C) that for any $(\phi, u) \in \mathcal{F}(\lambda)$, we have $u \in \mathrm{Lip}(\mathbb{T}^n)^m$.

Fix $(\phi, u), (\psi, v) \in \mathcal{F}(\lambda)$ and $t, s \in [0, \infty)$. Fix $i \in \mathbb{I}$ and observe that

$$\begin{aligned}(B^\lambda u)_i(x) + H_{\phi,i}(x, Du_i(x)) \le 0 \quad \text{a.e. in } \mathbb{T}^n,\\ (B^\lambda v)_i(x) + H_{\psi,i}(x, Dv_i(x)) \le 0 \quad \text{a.e. in } \mathbb{T}^n,\end{aligned}$$

which imply that there is a set $N \subset \mathbb{T}^n$ of Lebesgue measure zero such that

$$\begin{aligned}(B^\lambda u)_i(x) \le g(x, \xi) \cdot Du_i(x) + \phi_i(x.\xi) \quad \text{for all } (x, \xi) \in \mathbb{T}^n \setminus N \times \Xi,\\ (B^\lambda v)_i(x) \le g_i(x, \xi) \cdot Dv_i(x) + \psi_i(x, \xi) \quad \text{for all } (x, \xi) \in \mathbb{T}^n \setminus N \times \Xi.\end{aligned}$$

Multiplying the first and second by t and s, respectively, adding the resulting inequalities and setting $w = tu + sv$, we obtain

$$(B^\lambda w)_i(x) \le g(x, \xi) \cdot Dw_i(x) + (t\phi_i + s\psi)(x.\xi) \quad \text{for all } (x, \xi) \in \mathbb{T}^n \setminus N \times \Xi,$$

which readily implies that $t(\phi, u) + s(\psi, v) \in \mathcal{F}(\lambda)$. □

We refer the reader to [23, Lemma 2.2] for another proof of the above lemma.

We establish a representation formula for the solution of (P_λ), with $\lambda > 0$, by modifying the argument in [23] (see also [18]).

For any nonnegative Borel measure ν on $\mathbb{T}^n \times \Xi$ and $\phi \in C(\mathbb{T}^n \times \Xi)$, we write

$$\langle \nu, \phi \rangle = \int_{\mathbb{T}^n \times \Xi} \phi(x, \xi) \nu(dx,\, d\xi).$$

Similarly, for any collection $\nu = (\nu_i)_{i \in \mathbb{I}}$ of nonnegative Borel measures on $\mathbb{T}^n \times \Xi$ and $\phi = (\phi_i) \in C(\mathbb{T}^n \times \Xi)^m$, we write

$$\langle \nu, \phi \rangle = \sum_{i \in \mathbb{I}} \langle \nu_i, \phi_i \rangle \in \mathbb{R}.$$

Note that any collection $\nu = (\nu_i)_{i \in \mathbb{I}}$ of nonnegative Borel measures on $\mathbb{T}^n \times \Xi$ is regarded as a nonnegative Borel measure on $\mathbb{T}^n \times \Xi \times \mathbb{I}$ and vice versa.

We set

$$\rho_i(x) := \sum_{j \in \mathbb{I}} b_{ij}(x) \quad \text{for } i \in \mathbb{I}.$$

Note that

$$B\mathbf{1} = \begin{pmatrix} b_{11}(x) & \cdots & b_{1m}(x) \\ \vdots & & \vdots \\ b_{m1}(x) & \cdots & b_{mm}(x) \end{pmatrix} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} \rho_1(x) \\ \vdots \\ \rho_m(x) \end{pmatrix} \quad \text{and} \quad B^\lambda \mathbf{1} = \begin{pmatrix} \lambda + \rho_1(x) \\ \vdots \\ \lambda + \rho_m(x) \end{pmatrix}. \tag{3.1}$$

By assumption (M) and Lemma 3, we have $\rho_i \ge 0$ on $\mathbb{T}^n$ for all $i \in \mathbb{I}$.

Given a constant $\lambda > 0$, let $\mathbb{P}_{B^\lambda}$ denote the set of of nonnegative Borel measures $\nu = (\nu_i)_{i \in \mathbb{I}}$ on $\mathbb{T}^n \times \Xi \times \mathbb{I}$ such that

$$\langle \nu, B^\lambda \mathbf{1} \rangle = 1.$$

The last condition reads

$$\sum_{i\in\mathbb{I}}(\lambda|\nu_i| + \langle\nu_i, \rho_i\rangle) = 1,$$

where $|\nu_i|$ denotes the total mass of ν_i on $\mathbb{T}^n \times \Xi$. Note as well that $\mathbb{P}_{B^\lambda}$ can be identified with the space of Borel probability measures on $\mathbb{T}^n \times \Xi \times \mathbb{I}$ by the correspondence between $\nu = (\nu_i)_{i\in\mathbb{I}}$ and $\sum_{i\in\mathbb{I}}(\lambda + \rho_i)\nu_i \otimes \delta_i$, where $\otimes$ indicates the product of two measures and δ_i denotes the Dirac measure at i. If we set $\mu := \sum_{i\in\mathbb{I}}(\lambda + \rho_i)\nu_i \otimes \delta_i$ and consider μ as a collection (μ_i) of measures on $\mathbb{T}^n \times \Xi$, then $\nu_i = (\lambda + \rho_i)^{-1}\mu_i$. We denote simply by $\mathbb{P}$ the space of Borel probability measures on $\mathbb{T}^n \times \Xi \times \mathbb{I}$.

For $\lambda \geq 0$ and $(z, k) \in \mathbb{T}^n \times \mathbb{I}$ we set

$$\mathcal{G}(z, k, \lambda) := \{\phi - u_k(z)B^\lambda\mathbf{1} \ : \ (\phi, u) \in \mathcal{F}(\lambda)\} \subset C(\mathbb{T}^n \times \Xi)^m,$$

and

$$\mathcal{G}'(z, k, \lambda) = \{\nu = (\nu_i)_{i\in\mathbb{I}} \in \mathbb{P}_{B^\lambda} \ : \ \langle\nu, f\rangle \geq 0 \quad \text{for} \ \ f = (f_i) \in \mathcal{G}(z, k, \lambda)\}.$$

Theorem 6. *Assume* (H), (C) *and* (M). *Let* $\lambda > 0$ *and* $(z, k) \in \mathbb{T}^n \times \mathbb{I}$. *Let* $v^\lambda \in C(\mathbb{T}^n \times \mathbb{I})$ *be the unique solution of* (P$_\lambda$). *Then there exists a* $\nu^{z,k,\lambda} = (\nu_i^{z,k,\lambda})_{i\in\mathbb{I}} \in \mathcal{G}'(z, k, \lambda)$ *such that*

$$v_k^\lambda(z) = \langle\nu^{z,k,\lambda}, L\rangle. \tag{3.2}$$

We remark that for any $\nu \in \mathcal{G}'(z, k, \lambda)$ we have $\langle\nu, L\rangle \geq v_k^\lambda(z)\langle\nu, B^\lambda\mathbf{1}\rangle = v_k^\lambda(z)$ and, accordingly, in the theorem above, the measures $\nu^{z,k,\lambda}$ has the minimizing property:

$$v_k^\lambda(z) = \langle\nu^{z,k,\lambda}, L\rangle = \min_{\nu\in\mathcal{G}'(z,k,\lambda)}\langle\nu, L\rangle. \tag{3.3}$$

We call any minimizing family $(\nu_i)_{i\in\mathbb{I}} \in \mathbb{P}_{B^\lambda}$ of the optimization problem above a viscosity Green-Poisson measure for (P$_\lambda$).

Proof. Note first that $(L, v^\lambda) \in \mathcal{F}(\lambda)$ and hence, for any $\nu \in \mathcal{G}'(z, k, \lambda)$,

$$0 \leq \langle\nu, L - v_k^\lambda(z)B^\lambda\mathbf{1}\rangle = \langle\nu, L\rangle - v_k^\lambda(z)\langle\nu, B^\lambda\mathbf{1}\rangle = \langle\nu, L\rangle - v_k^\lambda(z). \tag{3.4}$$

Next, we show that

$$\sup_{(\phi,u)\in\mathcal{F}(\lambda)}\ \inf_{\nu\in\mathbb{P}_{B^\lambda}}\langle\nu, L - \phi + (u_k(z) - v_k^\lambda(z))B^\lambda\mathbf{1}\rangle = 0. \tag{3.5}$$

Note that for $z \in \mathbb{T}^n$,

$$\begin{aligned}\sup_{(\phi,u)\in\mathcal{F}(\lambda)}\ &\inf_{\nu\in\mathbb{P}_{B^\lambda}}\langle\nu, L - \phi + (u_k(z) - v_k^\lambda(z))B^\lambda\mathbf{1}\rangle \\ &\geq \inf_{\nu\in\mathbb{P}_{B^\lambda}}\langle\nu, L - \phi + (u_k(z) - v_k^\lambda(z))B^\lambda\mathbf{1}\rangle\Big|_{(\phi,u)=(L,v^\lambda)} = 0.\end{aligned}$$

Hence, in order to prove (3.5), we only need to show that

$$\sup_{(\phi,u)\in\mathcal{F}(\lambda)}\ \inf_{\nu\in\mathbb{P}_{B^\lambda}}\langle\nu, L - \phi + (u_k(z) - v_k^\lambda(z))B^\lambda\mathbf{1}\rangle \leq 0. \tag{3.6}$$

We postpone the proof of (3.6) and, assuming temporarily that (3.5) is valid, we prove that there exists $\nu \in \mathcal{G}'(z,k,\lambda)$ such that

$$v_k^\lambda(z) = \langle \nu, L \rangle, \tag{3.7}$$

which, together with (3.4), completes the proof.

To prove (3.7), we observe that $\mathbb{P}_{B^\lambda}$ and, by Lemma 5, $\mathcal{F}(\lambda)$ are convex,

$$\mathbb{P}_{B^\lambda} \ni \nu \mapsto \langle \nu, L - \phi + (u_k(z) - v_k^\lambda(z))B^\lambda \mathbf{1} \rangle$$

is convex and continuous, in the topology of weak convergence of measures, for any $(\phi, u) \in \mathcal{F}(\lambda)$ and

$$\mathcal{F}(\lambda) \ni (\phi, u) \mapsto \langle \nu, L - \phi + (u_k(z) - v_k^\lambda(z))B^\lambda \mathbf{1} \rangle$$

is concave and continuous for any $\nu \in \mathbb{P}_{B^\lambda}$. Hence, noting moreover that $\mathbb{T}^n \times \Xi \times \mathbb{I}$ is a compact set, we apply the minimax theorem ([32, 34]), to find from (3.5) that

$$\begin{aligned} 0 = \sup_{(\phi,u)\in\mathcal{F}(\lambda)} \min_{\nu\in\mathbb{P}_{B^\lambda}} \langle \nu, L - \phi + (u_k(z) - v_k^\lambda(z))B^\lambda \mathbf{1} \rangle \\ = \min_{\nu\in\mathbb{P}_{B^\lambda}} \sup_{(\phi,u)\in\mathcal{F}(\lambda)} \langle \nu, L - \phi + (u_k(z) - v_k^\lambda(z))B^\lambda \mathbf{1} \rangle. \end{aligned} \tag{3.8}$$

Observe by using the cone property of $\mathcal{F}(\lambda)$ that

$$\sup_{(\phi,u)\in\mathcal{F}(\lambda)} \langle \nu, u_k(z)B^\lambda \mathbf{1} - \phi \rangle = \begin{cases} 0 & \text{if } \nu \in \mathcal{G}'(z,k,\lambda), \\ \infty & \text{if } \nu \in \mathbb{P}_{B^\lambda} \setminus \mathcal{G}'(z,k,\lambda). \end{cases}$$

This and (3.8) yield

$$\begin{aligned} 0 &= \min_{\nu\in\mathbb{P}_{B^\lambda}} \sup_{(\phi,u)\in\mathcal{F}(\lambda)} \langle \nu, L - \phi + (u_k(z) - v_k^\lambda(z)B^\lambda \mathbf{1} \rangle \\ &= \min_{\nu\in\mathcal{G}'(z,k,\lambda)} \sup_{(\phi,u)\in\mathcal{F}(\lambda)} \langle \nu, L - v_k^\lambda(z)B^\lambda \mathbf{1} \rangle \\ &= \min_{\nu\in\mathcal{G}'(z,k,\lambda)} \langle \nu, L - v_k^\lambda(z)B^\lambda \mathbf{1} \rangle = \min_{\nu\in\mathcal{G}'(z,k,\lambda)} (\langle \nu, L \rangle - v_k^\lambda(z)\langle \nu, B^\lambda \mathbf{1} \rangle) \\ &= \min_{\nu\in\mathcal{G}'(z,k,\lambda)} \langle \nu, L \rangle - v_k^\lambda(z), \end{aligned}$$

which proves (3.7).

It remains to show (3.6). For this, we argue by contradiction and thus suppose that (3.6) does not hold. Accordingly, we have

$$\sup_{(\phi,u)\in\mathcal{F}(\lambda)} \inf_{\nu\in\mathbb{P}_{B^\lambda}} \langle \nu, L - \phi + (u_k(z) - v_k^\lambda(z))B^\lambda \mathbf{1} \rangle > \varepsilon$$

for some $\varepsilon > 0$. We may select $(\phi, u) \in \mathcal{F}(\lambda)$ so that

$$\inf_{\nu\in\mathbb{P}_{B^\lambda}} \langle \nu, L - \phi + (u_k(z) - v_k^\lambda(z))B^\lambda \mathbf{1} \rangle > \varepsilon.$$

That is, for any $\nu \in \mathbb{P}_{B^\lambda}$, we have

$$\langle \nu, L - \phi + (u_k(z) - v_k^\lambda(z))B^\lambda \mathbf{1} \rangle > \varepsilon = \langle \nu, \varepsilon B^\lambda \mathbf{1} \rangle.$$

Plugging $\nu = (\lambda + \rho_i)^{-1}\delta_{(x,\xi,i)} \in \mathbb{P}_{B^\lambda}$, with any $(x, \xi, i) \in \mathbb{T}^n \times \Xi \times \mathbb{I}$, into the above, we find that

$$(L_i - \phi_i)(x, \xi) - (v_k^\lambda(z) - u_k(z) - \varepsilon)(B^\lambda \mathbf{1})_i > 0.$$

Hence, we have

$$\phi(x, \xi) < L(x, \xi) + (u_k(z) - v_k^\lambda(z) - \varepsilon)B^\lambda \mathbf{1} \quad \text{for } (x.\xi) \in \mathbb{T}^n \times \mathbb{R}^n.$$

This ensures that u is a subsolution of

$$B^\lambda u + H[u] = (u_k(z) - v_k^\lambda(z) - \varepsilon)B^\lambda \mathbf{1} \quad \text{in } \mathbb{T}^n,$$

which implies that $u - (u_k(z) - v_k^\lambda(z) - \varepsilon)\mathbf{1}$ is a subsolution of (P_λ). By comparison (Theorem 1), we get

$$u(x) - (u_k(z) - v_k^\lambda(z) - \varepsilon) \le v^\lambda(x) \quad \text{for } x \in \mathbb{T}^n.$$

The k-th component of the above, evaluated at $x = z$, yields an obvious contradiction. Thus we conclude that (3.6) holds. □

We have the following characterization of $\mathcal{G}'(z, k, \lambda)$.

Proposition 7. *Assume* (H), (C) *and* (M) *hold. Let* $\nu = (\nu_i)_{i\in\mathbb{I}} \in \mathbb{P}_{B^\lambda}$ *and* $(z, k, \lambda) \in \mathbb{T}^n \times \mathbb{I} \times (0, \infty)$. *Then we have* $\nu \in \mathcal{G}'(z, k, \lambda)$ *if and only if*

$$\sum_{i\in\mathbb{I}} \langle \nu_i, (B^\lambda \psi)_i - g_i \cdot D\psi_i \rangle = \psi_k(z) \quad \text{for } \psi = (\psi_i)_{i\in\mathbb{I}} \in C^1(\mathbb{T}^n)^m. \tag{3.9}$$

Proof. Assume first that $\nu \in \mathcal{G}'(z, k, \lambda)$. Fix any $\psi = (\psi_i)_{i\in\mathbb{I}} \in C^1(\mathbb{T}^n)^m$ and define $\phi = (\phi_i)_{i\in\mathbb{I}} \in C(\mathbb{T}^n \times \mathbb{I})^m$ by

$$\phi_i(x, \xi) = (B^\lambda \psi)_i(x) - g_i(x, \xi) \cdot D\psi_i(x).$$

Observe that $u := \pm\psi$ satisfy, respectively,

$$B^\lambda u + H_{\pm\phi}[u] = 0 \quad \text{in } \mathbb{T}^n,$$

and, hence,

$$\pm(\phi, \psi) \in \mathcal{F}(\lambda).$$

Since $\nu \in \mathcal{G}'(z, k, \lambda)$, we have

$$\pm\psi_k(z) \le \langle \nu, \pm\phi \rangle = \pm\langle \nu, \phi \rangle,$$

respectively, which shows that (3.9) is valid.

Now, assume that (3.9) is satisfied. Fix any $(u, \phi) \in \mathcal{F}(\lambda)$. As noted in the proof of Theorem 1, we have $u \in \mathrm{Lip}(\mathbb{T}^n)$. By the standard mollification technique, given a positive constant $\varepsilon > 0$, we can approximate u by a smooth function u^ε so that

$$\max_{\mathbb{T}^n} |u - u^\varepsilon| < \varepsilon \quad \text{and} \quad B^\lambda u^\varepsilon + H_\phi[u^\varepsilon] \le \varepsilon B^\lambda \mathbf{1} \quad \text{in } \mathbb{T}^n.$$

The last inequality reads

$$B^\lambda u_i^\varepsilon(x) - g_i(x, \xi) \cdot Du_i^\varepsilon(x) - \phi_i(x, \xi) \le \varepsilon(B^\lambda \mathbf{1})_i(x) \quad \text{for } (x, \xi, i) \in \mathbb{T}^n \times \mathbb{R}^n \times \mathbb{I}.$$

Integrating the above by ν_i, summing up in $i \in \mathbb{I}$ and using (3.9), we get

$$u_k^\varepsilon(z) - \langle \nu, \phi \rangle \le \varepsilon\langle \nu, B^\lambda \mathbf{1} \rangle = \varepsilon.$$

Sending $\varepsilon \to 0$ shows that $\nu \in \mathcal{G}'(z, k, \lambda)$. □

It is convenient to restate the theorem above as follows. For $\mu = (\mu_i)_{i\in\mathbb{I}} \in \mathbb{P}$ and $\lambda > 0$, consider $\nu = (\nu_i)_{i\in\mathbb{I}} \in \mathbb{P}_{B^\lambda}$ given by

$$\nu_i := (\lambda + \rho_i)^{-1}\mu_i = \frac{1}{(B^\lambda \mathbf{1})_i}\mu_i.$$

(Notice by the above definition that $\langle \nu, B^\lambda \mathbf{1}\rangle = \langle \mu, \mathbf{1}\rangle = 1$.) Observe that for $\phi = (\phi_i)_{i\in\mathbb{I}} \in C(\mathbb{T}^n \times \Xi)^m$,

$$\langle \nu, \phi\rangle = \sum_{i\in\mathbb{I}} \langle \nu_i, \phi_i\rangle = \sum_{i\in\mathbb{I}} \langle \mu_i, (\lambda+\rho_i)^{-1}\phi_i\rangle,$$

and that for any $(z, k) \in \mathbb{T}^n \times \mathbb{I}$, we have $\nu \in \mathcal{G}'(z, k, \lambda)$ if and only if

$$\sum_{i\in\mathbb{I}} \langle \mu_i, (\lambda+\rho_i)^{-1}\phi_i\rangle \ge u_k(z) \quad \text{for } (\phi, u) \in \mathcal{F}(\lambda). \tag{3.10}$$

The condition above is stated in the spirit of Proposition 7 as

$$\sum_{i\in\mathbb{I}} \langle \mu_i, (\lambda+\rho_i)^{-1}((B^\lambda\psi)_i - g_i \cdot D\psi_i)\rangle = \psi_k(z) \quad \text{for } \ \psi = (\psi_i)_{i\in\mathbb{I}} \in C^1(\mathbb{T}^n)^m.$$

We define

$$\mathbb{P}(z, k, \lambda) = \{\mu = (\mu_i)_{i\in\mathbb{I}} \in \mathbb{P} \,:\, \mu \text{ satisfies (3.10)}\}.$$

The following proposition is an immediate consequence of Theorem 6.

Corollary 8. *Assume* (H)*,* (C) *and* (M)*. Let $\lambda > 0$ and $(z, k) \in \mathbb{T}^n \times \mathbb{I}$. Let $v^\lambda \in C(\mathbb{T}^n \times \mathbb{I})$ be the unique solution of* (P$_\lambda$)*. Then there exists a $\mu^{z,k,\lambda} = (\mu_i^{z,k,\lambda})_{i\in\mathbb{I}} \in \mathbb{P}(z, k, \lambda)$ such that*

$$v_k^\lambda(z) = \sum_{i\in\mathbb{I}} \langle \mu_i^{z,k,\lambda}, (\lambda+\rho_i)^{-1}L_i\rangle = \min_{\mu=(\mu_i)_{i\in\mathbb{I}}\in\mathbb{P}(z,k,\lambda)} \sum_{i\in\mathbb{I}} \langle \mu_i, (\lambda+\rho_i)^{-1}L_i\rangle. \tag{3.11}$$

4. A convergence result for the vanishing discount problem

We study the asymptotic behavior of the solution v^λ of (P$_\lambda$), with $\lambda > 0$, as $\lambda \to 0$.

We make a convenient assumption on the system (P$_0$):

problem (P$_0$) has a solution $v_0 \in \mathrm{Lip}(\mathbb{T}^n)$. (E)

If $\rho_i > 0$ for all $i \in \mathbb{I}$, then Theorem 1 assures that there exists a *unique* solution v_0 of (E). In this situation, it is not difficult to show that the uniform convergence, as $\lambda \to 0+$, of v^λ to the unique solution v_0 on $\mathbb{T}^n$. In general, existence and uniqueness of a solution of (P$_0$) may fail. In fact, one can prove at least in the case when the b_{ij} are constants (see Theorem 18) that there exists $c \in \mathbb{R}^m$ such that

$$Bu + H[u] = c \quad \text{in } \mathbb{T}^n \tag{4.1}$$

has a solution $v_0 \in \mathrm{Lip}(\mathbb{T}^n)$ and possibly multiple solutions. If such a $c = (c_i)$ exists, then the introduction of a new family of Hamiltonians,

$$\widetilde{H} = (\widetilde{H}_i)_{i\in\mathbb{I}}, \qquad \text{with } \widetilde{H}_i(x, p) = H_i(x, p) - c_i,$$

allows us to view (4.1) as in the form of (P$_0$). The link between two vanishing discount problems for the original (P$_\lambda$) and for (P$_\lambda$), with $\widetilde{H}$ in place of H, is discussed in Sections 5 and 6.

Theorem 9. *Assume* (H), (C), (M) *and* (E). *Let v^λ be the unique solution of* (P_λ) *for $\lambda > 0$. Then there exists a solution $v^0 \in \mathrm{Lip}(\mathbb{T}^n)^m$ of* (P_0) *such that the functions v_i^λ converge to v_i^0 uniformly on $\mathbb{T}^n$ as $\lambda \to 0$ for all $i \in \mathbb{I}$.*

Lemma 10. *Under the hypotheses of Theorem 9, there exists a constant $C_0 > 0$ such that for any $\lambda > 0$,*

$$|v_i^\lambda(x)| \le C_0 \quad \text{for } (x, i) \in \mathbb{T}^n \times \mathbb{I}. \tag{4.2}$$

Proof. Let $v_0 = (v_{0,i})_{i\in\mathbb{I}} \in \mathrm{Lip}(\mathbb{T}^n)^m$ be the solution of (P_0). Choose a constant $C_0 > 0$ so that

$$|v_{0,i}(x)| \le C_1 \quad \text{for } (x, i) \in \mathbb{T}^n \times \mathbb{I},$$

and observe by the monotonicity of B (Lemma 4) that $v_0 + C_1\mathbf{1}$ and $v_0 - C_1\mathbf{1}$ are a supersolution and a subsolution of (P_0), respectively. Noting that $v_0 + C_1\mathbf{1} \ge 0$ and $v_0 - C_1\mathbf{1} \le 0$, we deduce that $v_0 + C_1\mathbf{1} \ge 0$ and $v_0 - C_1\mathbf{1} \le 0$ are a supersolution and a subsolution of (P_λ) for any $\lambda > 0$, respectively. By comaprison (Theorem 1), we see that, for any $\lambda > 0$, $v_0 - C_1\mathbf{1} \le v^\lambda \le v_0 + C_1\mathbf{1}$ on $\mathbb{T}^n$ and, moreover, $-2C_1\mathbf{1} \le v^\lambda \le 2C_1\mathbf{1}$ on $\mathbb{T}^n$. Thus, (4.2) holds with $C_0 = 2C_1$. □

Lemma 11. *Under the hypotheses of Theorem 9, the family $(v^\lambda)_{\lambda\in(0,\,1)}$ is equi-Lipschitz continuous on $\mathbb{T}^n$.*

Indeed, the family $(v^\lambda)_{\lambda>0}$ is equi-Lipschitz continuous on $\mathbb{T}^n$, which we do not need here.

Proof. By Lemma 10, there is a constant $C_0 > 0$ such that

$$|(B^\lambda v^\lambda(x))_i| \le C_0 \quad \text{for } (x, i, \lambda) \in \mathbb{T}^n \times \mathbb{I} \times (0,\ 1).$$

Hence, as v^λ is a solution of (P_λ), we deduce by (C) that there exists a constant $C_1 > 0$ such that the v_i^λ are subsolutions of $|Du| \le C_1$ in $\mathbb{T}^n$. It is a standard fact that the v_i^λ are Lipschitz continuous on $\mathbb{T}^n$ with C_1 as their Lipschitz bound. □

In the proof of Theorem 9, Corollary 8 has a crucial role. We need also results for $\lambda = 0$ similar to the corollary.

We consider the condition for $\mu \in \mathbb{P}$,

$$\langle \mu, \phi \rangle \ge 0 \quad \text{for } (\phi, u) \in \mathcal{F}(0). \tag{4.3}$$

We denote by $\mathbb{P}(0)$ the subset of $\mathbb{P}$ consisting of those μ which satisfy (4.3).

Theorem 12. *Assume* (H), (C), (M) *and* (E). *Assume that $\rho_i = 0$ on $\mathbb{T}^n$ for every $i \in \mathbb{I}$. Then there exists a $\mu^0 = (\mu_i^0)_{i\in\mathbb{I}} \in \mathbb{P}(0)$ such that*

$$0 = \langle \mu^0, L \rangle = \min_{\mu\in\mathbb{P}(0)} \langle \mu, L \rangle. \tag{4.4}$$

Proof. We fix a $(z, k) \in \mathbb{T}^n \times \mathbb{I}$. By Corollary 8, for each $\lambda > 0$ there exists $\mu^\lambda = (\mu_i^\lambda)_{i\in\mathbb{I}} \in \mathbb{P}(z, k, \lambda)$ such that

$$\lambda v_k^\lambda(z) = \sum_{i\in\mathbb{I}} \lambda \langle \mu_i^\lambda, \lambda^{-1} L_i \rangle = \langle \mu^\lambda, L \rangle. \tag{4.5}$$

Since $(\mu^\lambda)_{\lambda>0}$ is a family of Borel probability measures on a compact space $\mathbb{T}^n \times \Xi \times \mathbb{I}$, there exists a sequence $(\lambda_j)_{j\in\mathbb{N}} \subset (0,\, 1)$ converging to zero such that the sequence $(\mu^{\lambda_j})_{j\in\mathbb{N}}$ converges weakly in the sense of measures to a Borel probability measure μ^0 on $\mathbb{T}^n \times \Xi \times \mathbb{I}$. It follows from (4.5) and Lemma 10 that

$$0 = \langle \mu^0, L\rangle.$$

Observe that if $(\phi, u) \in \mathcal{F}(0)$, then, for any $\lambda > 0$, u is a subsolution of

$$B^\lambda u + H_\phi[u] = \lambda u \quad \text{in } \mathbb{T}^n,$$

and hence, $(\psi, u) \in \mathcal{F}(\lambda)$, with $\psi(x,\xi) = \phi(x,\xi) + \lambda u(x)$. Hence, the inclusion $\mu^\lambda \in \mathcal{G}'(z,k,\lambda)$ yields

$$u_k(z) \le \sum_{i\in\mathbb{I}} \langle \mu_i^\lambda, \lambda^{-1}(\phi_i + \lambda u_i)\rangle = \lambda\langle \mu^\lambda, \phi\rangle + \langle \mu^\lambda, u\rangle.$$

Multiplying the above by λ and sending $\lambda = \lambda_j \to 0$, in view of Lemma 10, we get

$$0 \le \langle \mu^0, \phi\rangle.$$

This shows that $\mu^0 \in \mathbb{P}(0)$. These observations together with (4.3) for $\mu \in \mathbb{P}(0)$ guarantee that

$$0 = \langle \mu^0, L\rangle = \min_{\mu\in\mathbb{P}(0)} \langle \mu, L\rangle. \qquad \square$$

We state a characterization of $\mathbb{P}(0)$ in the next, similar to Proposition 7, which we leave to the reader to verify.

Proposition 13. *Assume* (H), (C) *and* (M). *Let* $\mu = (\mu_i)_{i\in\mathbb{I}} \in \mathbb{P}$. *We have* $\mu \in \mathbb{P}(0)$ *if and only if*

$$\sum_{i\in\mathbb{I}} \langle \mu_i, (B\psi)_i - g_i \cdot D\psi_i\rangle = 0 \quad \textit{for} \;\; \psi = (\psi_i)_{i\in\mathbb{I}} \in C^1(\mathbb{T}^n)^m.$$

We call any minimizer $\mu \in \mathbb{P}(0)$ of the optimization problem (4.4) a viscosity Mather measure.

We denote by $\mathbb{M}_+$ the set of all Borel nonnegative measures $\mu = (\mu_i)_{i\in\mathbb{I}}$ on $\mathbb{T}^n \times \Xi \times \mathbb{I}$. We set

$$\mathbb{M}_+(0) = \{\mu \in \mathbb{M}_+ \;:\; \mu \text{ satisfies (4.3)}\}.$$

Theorem 14. *Let* $(z,k) \in \mathbb{T}^n \times \mathbb{I}$. *Assume* (H), (C), (M) *and* (E). *For any* $\lambda > 0$, *let* v^λ *be the unique solution of* (P$_\lambda$) *and* $\mu^\lambda \in \mathbb{P}(z,k,\lambda)$ *be a minimizer of* (3.11). *Then there exists a subsequence of* (λ_j), *which is denoted again by the same symbol, such that, as* $j \to \infty$,

$$\frac{\lambda_j}{\lambda_j + \rho_i}\mu_i^{\lambda_j} \to \mu_i^0$$

weakly in the sense of measures for some $\mu^0 = (\mu_i^0)_{i\in\mathbb{I}} \in \mathbb{M}_+(0)$, *and* μ^0 *satisfies*

$$\langle \mu^0, L\rangle = 0. \tag{4.6}$$

In particular,

$$0 = \langle \mu^0, L\rangle = \min_{\mu\in\mathbb{M}_+(0)} \langle \mu, L\rangle. \tag{4.7}$$

Notice that the minimization problem (4.7) is trivial since $\mu^0 = 0$ is a minimizer.

Proof. The proof is similar to that of Theorem 12.

We fix a $(z,k) \in \mathbb{T}^n \times \mathbb{I}$. For each $\lambda > 0$, we have

$$\lambda v_k^\lambda(z) = \sum_{i\in\mathbb{I}} \lambda\langle \mu_i^\lambda, (\lambda+\rho_i)^{-1}L_i\rangle. \tag{4.8}$$

Observe that

$$\langle \lambda(\lambda+\rho_i)^{-1}\mu_i^\lambda, \mathbf{1}\rangle \le \langle \mu_i^\lambda, \mathbf{1}\rangle = \sum_{i\in\mathbb{I}} |\mu_i^\lambda| = 1.$$

Accordingly, since $\mathbb{T}^n \times \Xi \times \mathbb{I}$ is a compact metric space, the families $(\lambda(\lambda+\rho_i)^{-1}\mu_i^\lambda)_{\lambda=\lambda_j, j\in\mathbb{N}}$ have a common subsequence, along which all the families converge to some Borel nonnegative measures μ_i^0 weakly in the sense of measures. We may assume by replacing the original sequence (λ_j) by its subsequence that

$$\frac{\lambda_j}{\lambda_j+\rho_i}\mu_i^{\lambda_j} \to \mu_i^0$$

weakly in the sense of measures. Combining this with (4.8) yields

$$0 = \sum_{i\in\mathbb{I}} \langle \mu_i^0, L_i\rangle = \langle \mu^0, L\rangle.$$

It is obvious to see that $\mu^0 \in \mathbb{M}_+$.

Let $(\phi, u) \in \mathcal{F}(0)$. As before, we have $(\psi, u) \in \mathcal{F}(\lambda)$, with $\psi(x,\xi) = \phi(x,\xi) + \lambda u(x)$ and moreover

$$u_k(z) \le \sum_{i\in\mathbb{I}} \langle \mu_i^\lambda, (\lambda+\rho_i)^{-1}(\phi_i + \lambda u_i)\rangle = \langle \mu^\lambda, (\lambda+\rho_i)^{-1}\phi\rangle + \lambda\langle \mu^\lambda, (\lambda+\rho_i)^{-1}u\rangle.$$

Multiplying the above by λ and sending $\lambda = \lambda_j \to 0$, we get

$$0 \le \langle \mu^0, \phi\rangle.$$

This shows that $\mu^0 \in \mathbb{M}_+(0)$. □

Proof of Theorem 9. Let $\mathcal{V}$ denote the set of accumulation points $v = (v_i) \in C(\mathbb{T}^n)^m$ in the space $C(\mathbb{T}^n)^m$ of v^λ as $\lambda \to 0$. In view of the Ascoli-Arzela theorem, Lemmas 10 and 11 guarantee that the family $(v^\lambda)_{\lambda\in(0,1)}$ is relatively compact in $C(\mathbb{T}^n)^m$. In particular, the set $\mathcal{V}$ is nonempty. Note by the stability of the viscosity property under uniform convergence that any $v \in \mathcal{V}$ is a solution of (P_0).

If $\mathcal{V}$ is a singleton, then it is obvious that the whole family $(v^\lambda)_{\lambda>0}$ converges to the unique element of $\mathcal{V}$ in $C(\mathbb{T}^n)^m$ as $\lambda \to 0$.

We need only to show that $\mathcal{V}$ is a singleton. It is enough to show that for any $v, w \in \mathcal{V}$ and $(z,k) \in \mathbb{T}^n \times \mathbb{I}$, the inequality $w_k(z) \le v_k(z)$ holds.

Fix any $v, w \in \mathcal{V}$ and $(z,k) \in \mathbb{T}^n \times \mathbb{I}$. Select sequences (λ_j) and (δ_j) converging to zero so that

$$v^{\lambda_j} \to v, \quad v^{\delta_j} \to w \quad \text{in } C(T^n)^m \quad \text{as } j \to \infty.$$

By Corollary 8, there exists a sequence $(\mu^j)_{j\in\mathbb{N}}$ such that

$$\mu^j \in \mathcal{G}'(z,k,\lambda_j) \quad \text{and} \quad v_k^{\lambda_j}(z) = \sum_{i\in\mathbb{I}} \langle \mu_i^j, (\lambda_j+\rho_i)^{-1}L_i\rangle \quad \text{for } j \in \mathbb{N}. \tag{4.9}$$

In view of Theorem 14, we may assume by passing to a subsequence if necessary that, as $j \to \infty$,

$$\frac{\lambda_j}{\lambda_j + \rho_i}\mu_i^j \to \mu_i^0 \quad \text{weakly in the sense of measures}$$

for all $i \in \mathbb{I}$ and for some $\mu^0 = (\mu_i^0)_{i\in\mathbb{I}} \in \mathbb{M}_+(0)$ and, moreover,

$$0 = \langle \mu^0, L\rangle. \tag{4.10}$$

Since $(L - \lambda v^\lambda, v^\lambda) \in \mathcal{F}(0)$ and $\mu^0 \in \mathbb{M}_+(0)$, in view of (4.10), we have

$$0 \le \langle \mu^0, L - \lambda v^\lambda\rangle = \langle \mu^0, L\rangle - \langle \mu^0, \lambda v^\lambda\rangle = -\lambda\langle \mu^0, v^\lambda\rangle,$$

which yields after dividing by $\lambda > 0$ and then sending $\lambda \to 0$ along $\lambda = \delta_j$

$$\langle \mu^0, w\rangle \le 0. \tag{4.11}$$

Now, note that w is a solution of

$$B^\lambda w + H[w] = \lambda w \quad \text{in } \mathbb{T}^n,$$

and thus, $(L + \lambda w, w) \in \mathcal{F}(\lambda)$ and infer by (4.9) that

$$w_k(z) \le \sum_{i\in\mathbb{I}} \langle \mu_i^j, (\lambda_j + \rho_i)^{-1}(L_i + \lambda_j w_i)\rangle = v_k^{\lambda_j}(z) + \lambda_j \sum_{i\in\mathbb{I}} \langle \mu_i^j, (\lambda_j + \rho_i)^{-1} w_i\rangle.$$

Sending $j \to \infty$ now yields

$$w_k(z) \le v_k(z) + \langle \mu^0, w\rangle.$$

This together with (4.11) shows that $w_k(z) \le v_k(z)$, which completes the proof. □

5. The ergodic problem for irreducible matrix B

We consider the problem of finding $c = (c_i)_{i\in\mathbb{I}} \in \mathbb{R}^m$ and $v = (v_i)_{i\in\mathbb{I}} \in C(\mathbb{T}^n)^m$ such that v is a solution of

$$Bv + H[v] = c \quad \text{in } \mathbb{T}^n. \tag{5.1}$$

The pair of such c and v is also called a solution of (5.1). This problem is called the *ergodic problem* in this paper although the term, ergodic problem, should be used only when the condition that $\sum_{j\in\mathbb{I}} b_{ij}(x) = 0$ holds for some $(i, x) \in \mathbb{I} \times \mathbb{T}^n$.

Henceforth, $D(x)$ denotes the diagonal matrix

$$D(x) = \operatorname{diag}(\rho_1(x), \ldots, \rho_m(x)) \quad \text{for } x \in \mathbb{T}^n,$$

where, as before, $\rho_i(x) = \sum_{j\in\mathbb{I}} b_{ij}(x)$.

Throughout this section, we treat the case when

$$B(x) \text{ is irreducible.} \tag{5.2}$$

The irreducibility of $B(x)$ is stated as follows: for any nonempty subset I of $\mathbb{I}$, which is not identical to $\mathbb{I}$, there exists a pair of $i \in I$ and $j \in \mathbb{I} \setminus I$ such that $b_{ij}(x) \ne 0$.

The following result has been established in Davini-Zavidovique [11, Theorem 2.10] (see also [6, 30]).

Proposition 15. *Assume* (H), (C), (M), (5.2)*, and that*

$$\sum_{j\in\mathbb{I}} b_{ij}(x) = 0 \quad \textit{for all } (i,x) \in \mathbb{I}\times\mathbb{T}^n. \tag{5.3}$$

Then there exist $c_0 \in \mathbb{R}$ and $v_0 \in \mathrm{Lip}(\mathbb{T}^n)^m$ such that the pair $(c_0\mathbf{1}, v_0)$ is a solution of (5.1).

We remark that (5.3) is satisfied if and only if $B(x)\mathbf{1} = 0$ for all $x \in \mathbb{T}^n$, which holds if and only if $\rho_i(x) = 0$ for all $(i,x) \in \mathbb{I}\times\mathbb{T}^n$.

The next theorem states the central result of this section.

Theorem 16. *Assume* (H), (C), (M), (5.2)*, and* (5.3)*. Let v^λ be the unique solution of* (P$_\lambda$) *for $\lambda > 0$. Then there exists a constant $c^0 \in \mathbb{R}$ and a function $v^0 \in \mathrm{Lip}(\mathbb{T}^n)^m$ such that the functions $v^\lambda + \lambda^{-1}c^0\mathbf{1}$ converge to v^0 uniformly on $\mathbb{T}^n$ as $\lambda \to 0$. Moreover, the pair $(c^0\mathbf{1}, v^0)$ is a solution of* (5.1).

Proof. Thanks to Proposition 15, there exists a solution $(c_0, v_0) \in \mathbb{R}^m \times C(\mathbb{T}^n)^m$ of (5.1). We set $\widetilde{H} = H - c_0\mathbf{1}$, and note that, since $B(x)\mathbf{1} = 0$ for all $x \in \mathbb{T}^n$, the function $w^\lambda := v^\lambda + \lambda^{-1}c_0\mathbf{1}$ satisfies, in the viscosity sense,

$$\lambda w^\lambda + Bw^\lambda + \widetilde{H}[w^\lambda] = \lambda v^\lambda + c_0\mathbf{1} + Bv^\lambda + H[v^\lambda] - c_0\mathbf{1} = 0.$$

By Theorem 9, there exists a solution $v^0 \in \mathrm{Lip}(\mathbb{T}^n)^m$ of $Bv^0 + \widetilde{H}[v^0] = 0$ in $\mathbb{T}^n$ such that, as $\lambda \to 0+$, $w^\lambda \to v^0$ in $C(\mathbb{T}^n)^m$. Noting that $(c_0\mathbf{1}, v^0)$ is a solution of (5.1), we finish the proof. □

The condition (5.3) in Proposition 15 can be removed and the following theorem is valid.

Theorem 17. *Assume* (H), (C), (M), *and* (5.2)*. Then there exist $c^0 \in \mathbb{R}$ and $v^0 = (v_i^0)_{i\in\mathbb{I}} \in \mathrm{Lip}(\mathbb{T}^n)^m$ such that the pair $(c^0\mathbf{1}, v^0)$ is a solution of* (5.1).

Proof. For $x \in \mathbb{I}\times\mathbb{T}^n$, we set

$$B^0(x) = (b_{ij}^0(x)) := B(x) - D(x).$$

and note that $B^0(x)$ is irreducible and (5.3) holds with $b_{ij}(x)$ replaced by $b_{ij}^0(x)$. Note also that $\rho_i(x) \ge 0$ for all $(i,x) \in \mathbb{I}\times\mathbb{T}^n$.

Thanks to Proposition 15, there exist $c^0 \in \mathbb{R}$ and $v = (v_i) \in \mathrm{Lip}(\mathbb{T}^n)^m$ which solve

$$B^0 v + H[v] = c^0\mathbf{1} \quad \text{in } \mathbb{T}^n.$$

We choose a constant $C > 0$ so that $\max_{(i,x)\in\mathbb{I}\times\mathbb{T}^n} |v_i(x)| \le C$ and set $v^\pm(x) = v(x) \pm C\mathbf{1}$, respectively. Observe that, since $v_i^+(x) \ge 0$ and $v_i^-(x) \le 0$ for all $(i,x) \in \mathbb{I}\times\mathbb{T}^n$, the functions $u = v^+$ and $u = v^-$ are a supersolution and subsolution of

$$B^0 u + Pu + H[u] = c^0\mathbf{1} \quad \text{in } \mathbb{T}^n,$$

that is, $Bu + H[u] = c^0\mathbf{1}$ in $\mathbb{T}^n$, respectively. In view of the Perron method, the function $v^0 = (v_i^0)_{i\in\mathbb{I}} \in \mathrm{Lip}(\mathbb{T}^n)$ given by

$$v_i^0(x) = \sup\{u_i(x) \,:\, u = (u_i) \in C(\mathbb{T}^n)^m \text{ is a subsolution of } Bu + H[u] = c^0\mathbf{1} \text{ in } \mathbb{T}^n, \\ v^- \le u \le v^+ \text{ in } \mathbb{T}^n\},$$

is a solution of (5.1), with $c = c^0\mathbf{1}$. □

Even without the assumption (5.3), it is immediate from Theorem 9 that, under the hypotheses of Theorem 17, if $c^0 = 0$, then the convergence holds for the whole family of the solutions v^λ of (P$_\lambda$), with $\lambda > 0$. A typical case when $c^0 = 0$ is given by [6, Theorem 4.2] (see also [11, 28]).

6. The ergodic problem for constant matrix B

Throughout this section we assume that B is a *constant* matrix, that is, independent of $x \in \mathbb{T}^n$.

The main results in this section are as follows.

Theorem 18. *Assume* (H), (C), (M)*, and that B is a constant matrix. Then* (5.1) *has a solution* $(c, v) \in \mathbb{R}^m \times C(\mathbb{T}^n)^m$.

Theorem 19. *Under the same hypotheses of Theorem 18, let* $(c, v_0) \in \mathbb{R}^m \times C(\mathbb{T}^n)^m$ *be a solution of* (5.1) *and let* v^λ *be the unique solution of* (P$_\lambda$) *for* $\lambda > 0$. *Then there exists a function* $v^0 \in C(\mathbb{T}^n)^m$ *such that the functions* $v^\lambda + (\lambda I + B)^{-1}c$ *converge to* v^0 *uniformly on* $\mathbb{T}^n$ *as* $\lambda \to 0$. *Moreover, the pair* (c, v^0) *is a solution of* (5.1).

Proof. It is well-known (and easily checked) that due to the monotonicity of B, $(\lambda I + B)$ is invertible for any $\lambda > 0$. We set $\widetilde{H}(x, p) = H(x, p) - c$ for $(x, p) \in \mathbb{T}^n \times \mathbb{R}^n$ and also $w^\lambda(x) = v^\lambda(x) + (\lambda I + B)^{-1}c$ for $x \in \mathbb{T}^n$. Observe that, in the viscosity sense,

$$\begin{aligned}\lambda w^\lambda(x) + B w^\lambda(x) + \widetilde{H}[w^\lambda] \\ = \lambda v^\lambda + B v^\lambda + H[v^\lambda] - c + \lambda(\lambda I + B)^{-1}c + B(\lambda I + B)^{-1}c = 0 \quad \text{in } \mathbb{T}^n.\end{aligned}$$

It is clear that $\widetilde{H}$ satisfies (H) and (C) and that v_0 is a solution of $Bu + \widetilde{H}[u] = 0$ in $\mathbb{T}^n$. By Theorem 9, we conclude that there exists a solution $v^0 \in C(\mathbb{T}^n)^m$ of $Bu + \widetilde{H}[u] = 0$ in $\mathbb{T}^n$ such that $w^\lambda \to v^0$ in $C(\mathbb{T}^n)^m$ as $\lambda \to 0+$. Noting that (c, v^0) is a solution of (5.1), we finish the proof. □

For the proof of Theorem 18, we begin with a preliminary remark on the permutations.

For a given permutation $\pi : \mathbb{I} \to \mathbb{I}$, we define the $m \times m$ matrix P by

$$P = (\delta_{\pi(i),j})_{i,j \in \mathbb{I}}, \tag{6.1}$$

where $\delta_{ij} = \delta_{i,j} := 1$ if $i = j$ and $= 0$ otherwise. Note that $P^{-1} = (\delta_{i,\pi(j)})_{i,j \in \mathbb{I}} = P^{\mathrm{T}}$ and that for any $u = (u_i)_{i \in \mathbb{I}}$,

$$Pu = P \begin{pmatrix} u_1 \\ \vdots \\ u_m \end{pmatrix} = \begin{pmatrix} u_{\pi(1)} \\ \vdots \\ u_{\pi(m)} \end{pmatrix}.$$

The system of Hamilton-Jacobi equations

$$\lambda u + Bu + H[u] = 0 \tag{6.2}$$

can be written component-wise as

$$\lambda u_{\pi(i)} + (Bu)_{\pi(i)} + H_{\pi(i)}[u_{\pi(i)}] = 0 \quad \text{for } i \in \mathbb{I}.$$

By the use of P, the system above is expressed as

$$\lambda (Pu)_i + (PBu)_i + (PH)_i[(Pu)_i] = 0,$$

and furthermore, if $v = Pu$,

$$\lambda (v)_i + (PBP^{\mathrm{T}} v)_i + (PH)_i[v_i] = 0. \tag{6.3}$$

Set $A = (a_{ij})_{i,j\in\mathbb{I}} = PBP^{\mathrm{T}}$ and observe that if B is monotone, then

$$a_{ij} = \sum_{k,l\in\mathbb{I}} \delta_{i,\pi(k)} b_{kl} \delta_{\pi(l),j} = b_{\pi^{-1}(i),\pi^{-1}(j)} \begin{cases} \geq 0 & \text{if } i = j, \\ \leq 0 & \text{if } i \neq j, \end{cases}$$

and

$$\sum_{j\in\mathbb{I}} a_{ij} = \sum_{j\in\mathbb{I}} b_{\pi^{-1}(i),\pi^{-1}(j)} = \sum_{j\in\mathbb{I}} b_{\pi^{-1}(i),j} \geq 0.$$

Consequently, if B is monotone, then PBP^{T} is monotone as well, and the system (6.2), by using the permutation matrix P, is converted to (6.3).

Proof of Theorem 18. It is well-known (see for instance [35, Section 2.3]) that, given a monotone matrix B, one can find a permutation $\pi : \mathbb{I} \to \mathbb{I}$ such that

$$PBP^{\mathrm{T}} = \begin{pmatrix} B^{(1)} & 0 & \cdots & 0 \\ * & B^{(2)} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ * & \cdots & * & B^{(r_p)} \end{pmatrix}, \tag{6.4}$$

where, P is given by (6.1), $B^{(1)}$ is a diagonal matrix of order r_1 and, for $1 < i \leq p$, $B^{(i)}$ are irreducible matrices of order r_i. In view of the preliminary remark before this proof, to seek for a solution of (5.1), we may and do assume henceforth B has the normal form of the right hand side of (6.4).

Set

$$s_k = \sum_{1\leq i<k} r_i \quad \text{and} \quad \mathbb{I}_k = \{s_k + 1, \ldots, s_k + r_k\} \quad \text{for } k \in \{1, \ldots, p\}.$$

Notice that $s_1 = 0$. If $r_1 \geq 1$, then we first show that there exist an r_1-vector $c^{(1)} = (c_i^{(1)})_{i\in\mathbb{I}_1} \in \mathbb{R}^{r_1}$ and a function $v^{(1)} = (v_i^{(1)})_{i\in\mathbb{I}_1} \in C(\mathbb{T}^n)^{r_1}$ such that $v^{(1)}$ is a solution of

$$B^{(1)} v^{(1)} + H^{(1)}[v^{(1)}] = c^{(1)} \quad \text{in } \mathbb{T}^n, \tag{6.5}$$

where $H^{(1)} = (H_i)_{i\in\mathbb{I}_1}$. The system is, in fact, a collection of single equations

$$b_{ii} v_i^{(1)} + H_i^{(1)}[v_i^{(1)}] = c_i^{(1)} \quad \text{in } \mathbb{T}^n, \text{ with } i \in \mathbb{I}_1, \tag{6.6}$$

and thus the existence of a solution $(c^{(1)}, v^{(1)})$ of (6.5) is a classical result. Indeed, for each $i \in \mathbb{I}_1$, if $b_{ii}^{(1)} > 0$, then (6.6) has a (unique) solution $v_i^{(1)} \in \mathrm{Lip}(\mathbb{T}^n)$ for any choice of $c_i^{(1)}$. If $b_{ii}^{(1)} = 0$, then (6.6) has a solution $(c_i^{(1)}, v_i^{(1)}) \in \mathbb{R} \times \mathrm{Lip}(\mathbb{T}^n)$ (see unpublished work by Lions PL, Papanicolaou G, and Varadhan S: Homogenization of Hamilton-Jacobi equations). If $r_1 = m$, then we are done.

Next, assume that $r_1 < m$ (and equivalently, $1 < p$) and we show that there exist a vector $c^{(2)} = (c_i^{(2)})_{i\in\mathbb{I}_2} \in \mathbb{R}^{r_2}$ and a function $v^{(2)} = (v_i^{(2)})_{i\in\mathbb{I}_2} \in C(\mathbb{T}^n)^{r_2}$ such that $v^{(2)}$ is a solution of the system

$$B^{(2)} v^{(2)} + H^{(2)}[v^{(2)}] = c^{(2)} \quad \text{in } \mathbb{T}^n, \tag{6.7}$$

where

$$H_i^{(2)}(x, p) = H_i(x, p) - \sum_{j\in\mathbb{I}_1} b_{i,j} v_j^{(1)}(x) \quad \text{for } i \in \mathbb{I}_2. \tag{6.8}$$

According to Proposition 15, there exist $c^{(2)} = (c_i^{(2)})_{i\in\mathbb{I}_2} \in \mathbb{R}^{r_2}$ and $v^{(2)} = (v_i^{(2)})_{i\in\mathbb{I}_2} \in C(\mathbb{T}^n)^{r_2}$ which satisfy (6.7). This way (by induction), we find $c^{(1)}, \ldots, c^{(p)}$ and $v^{(1)}, \ldots, v^{(p)}$ such that

$$c^{(k)} \in \mathbb{R}^{r_k} \quad \text{and} \quad v^{(k)} \in C(\mathbb{T}^n)^{r_k} \quad \text{for } k \in \{1, \ldots, p\},$$

and $v^{(k)}$ satisfies

$$B^{(k)}v^{(k)} + H^{(k)}[v^{(p)}] = c^{(k)} \quad \text{in } \mathbb{T}^n, \quad \text{for } k \in \{1, \ldots, p\}. \tag{6.9}$$

where

$$H_i^{(k)}(x, p) = H_i(x, p) - \sum_{1\le j<k} \sum_{q\in\mathbb{I}_j} b_{i,q} v_q^{(j)}(x) \quad \text{for } i \in \mathbb{I}_k. \tag{6.10}$$

We define $c = (c_i)_{i\in\mathbb{I}} \in \mathbb{R}^m$ and $v = (v_i)_{i\in\mathbb{I}} \in C(\mathbb{T}^n)^m$ by setting

$$c_i = c_i^{(k)} \quad \text{and} \quad v_i = v_i^{(k)} \quad \text{for } i \in \mathbb{I}_k,\ k \in \{1, \ldots, p\},$$

and observe that

$$Bv + H[v] = c \quad \text{in } \mathbb{T}^n.$$

This completes the proof. □

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Conflict of interest

The author declares no conflicts of interest in this paper.

References

1. Al-Aidarous ES, Alzahrani EO, Ishii H, et al. (2016) A convergence result for the ergodic problem for Hamilton-Jacobi equations with Neumann-type boundary conditions. *P Roy Soc Edinb A* 146: 225–242.
2. Bardi M, Capuzzo-Dolcetta I (1997) *Optimal Control and Viscosity Solutions of Hamilton-Jacobi-Bellman Equations*, Boston: Birkhäuser Boston, Inc.
3. Barles G (1993) Discontinuous viscosity solutions of first-order Hamilton-Jacobi equations: A guided visit. *Nonlinear Anal* 20: 1123–1134.
4. Barles G (1994) *Solutions de viscosité des équations de Hamilton-Jacobi*, Paris: Springer-Verlag.
5. Cagnetti F, Gomes D, Tran VH (2013) Adjoint methods for obstacle problems and weakly coupled systems of PDE. *ESAIM Contr Optim Ca* 19: 754–779.
6. Camilli F, Ley O, Loreti P, et al. (2012) Large time behavior of weakly coupled systems of first-order Hamilton-Jacobi equations. *NoDEA Nonlinear Diff* 19: 719–749.

7. Chen Q, Cheng W, Ishii H, et al. (2019) Vanishing contact structure problem and convergence of the viscosity solutions. *Commun Part Diff Eq* 44: 801–836.
8. Crandall MG, Ishii H, Lions PL (1992) User's guide to viscosity solutions of second order partial differential equations. *B Am Math Soc* 27: 1–67.
9. Crandall MG, Lions PL (1983) Viscosity solutions of Hamilton-Jacobi equations. *T Am Math Soc* 277: 1–42.
10. Davini A, Fathi A, Iturriaga R, et al. (2016) Convergence of the solutions of the discounted Hamilton-Jacobi equation. *Invent Math* 206: 29–55.
11. Davini A, Zavidovique M (2014) Aubry sets for weakly coupled systems of Hamilton-Jacobi equations. *SIAM J Math Anal* 46: 3361–3389.
12. Davini A, Zavidovique M (2019) Convergence of the solutions of discounted Hamilton-Jacobi systems. *Adv Calc Var*, Available from: `https://doi.org/10.1515/acv-2018-0037`.
13. Engler H, Lenhart SM (1991) Viscosity solutions for weakly coupled systems of Hamilton-Jacobi equations. *P Lond Math Soc* 63: 212–240.
14. Evans LC (2004) A survey of partial differential equations methods in weak KAM theory. *Commun Pure Appl Math* 57: 445–480.
15. Evans LC (2010) Adjoint and compensated compactness methods for Hamilton-Jacobi PDE. *Arch Ration Mech Anal* 197: 1053–1088.
16. Fathi A (1997) Théorème KAM faible et théorie de Mather sur les systèmes lagrangiens. *C R Acad Sci Paris Sér I Math* 324: 1043–1046.
17. Fathi A (2008) *Weak KAM Theorem in Lagrangian Dynamics, Preliminary Version 10*, Available from: `https://www.math.u-bordeaux.fr/~pthieull/Recherche/KamFaible/Publications/Fathi2008_01.pdf`.
18. Gomes DA (2005) Duality principles for fully nonlinear elliptic equations. In: *Trends in Partial Differential Equations of Mathematical Physics*, Basel: Birkhäuser, 125–136.
19. Gomes DA, Mitake H, Tran HV (2018) The selection problem for discounted Hamilton-Jacobi equations: Some non-convex cases. *J Math Soc JPN* 70: 345–364.
20. Ishii H, Jin L (2020) The vanishing discount problem for monotone systems of Hamilton-Jacobi equations. part 2 – Nonlinear coupling. *Calc Var* 59: 140.
21. Ishii H (1987) Perron's method for Hamilton-Jacobi equations. *Duke Math J* 55: 369–384.
22. Ishii H, Koike S (1991) Viscosity solutions for monotone systems of second-order elliptic PDEs. *Commun Part Diff Eq* 16: 1095–1128.
23. Ishii H, Mitake H, Tran HV (2017) The vanishing discount problem and viscosity Mather measures. Part 1: The problem on a torus. *J Math Pure Appl* 108: 125–149.
24. Ishii H, Mitake H, Tran HV (2017) The vanishing discount problem and viscosity Mather measures. Part 2: Boundary value problems. *J Math Pure Appl* 108: 261–305.
25. Ishii H, Siconolfi A (2020) The vanishing discount problem for Hamilton–Jacobi equations in the Euclidean space. *Commun Part Diff Eq* 45: 525–560.

26. Mitake H, Siconolfi A, Tran HV, et al. (2016) A Lagrangian approach to weakly coupled Hamilton-Jacobi systems. *SIAM J Math Anal* 48: 821–846.

27. Mitake H, Tran HV (2017) Selection problems for a discount degenerate viscous Hamilton-Jacobi equation. *Adv Math* 306: 684–703.

28. Mitake H, Tran HV (2012) Remarks on the large time behavior of viscosity solutions of quasi-monotone weakly coupled systems of Hamilton-Jacobi equations. *Asymptot Anal* 77: 43–70.

29. Mitake H, Tran HV (2014) A dynamical approach to the large-time behavior of solutions to weakly coupled systems of Hamilton-Jacobi equations. *J Math Pure Appl* 101: 76–93.

30. Mitake H, Tran HV (2014) Homogenization of weakly coupled systems of Hamilton-Jacobi equations with fast switching rates. *Arch Ration Mech Anal* 211: 733–769

31. Lions PL (1982) *Generalized Solutions of Hamilton-Jacobi Equations*, Boston-London: Pitman.

32. Sion M (1958) On general minimax theorems. *Pacific J Math* 8: 171–176.

33. Terai K (2019) Uniqueness structure of weakly coupled systems of ergodic problems of Hamilton-Jacobi equations. *NoDEA Nonlinear Diff* 26: 44.

34. Terkelsen F (1972) Some minimax theorems. *Math Scand* 31: 405–413.

35. Varga RS (2000) *Matrix Iterative Analysis*, Berlin: Springer-Verlag.