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Research article

The ergodic limit for weak solutions of elliptic equations with Neumann boundary condition†

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Abstract: We consider the so-called *ergodic* problem for weak solutions of elliptic equations in divergence form, complemented with Neumann boundary conditions. The simplest example reads as the following boundary value problem in a bounded domain of $\mathbb{R}^N$:

$$\begin{cases} -\mathrm{div}(A(x)\nabla u) + \lambda = H(x, \nabla u) & \text{in } \Omega, \\ A(x)\nabla u \cdot \vec{n} = 0 & \text{on } \partial\Omega, \end{cases}$$

where $A(x)$ is a coercive matrix with bounded coefficients, and $H(x, \nabla u)$ has Lipschitz growth in the gradient and measurable x-dependence with suitable growth in some Lebesgue space (typically, $|H(x, \nabla u)| \le b(x)|\nabla u| + f(x)$ for functions $b(x) \in L^N(\Omega)$ and $f(x) \in L^m(\Omega)$, $m \ge 1$). We prove that there exists a unique real value λ for which the problem is solvable in Sobolev spaces and the solution is unique up to addition of a constant. We also characterize the ergodic limit, say the singular limit obtained by adding a vanishing zero order term in the equation. Our results extend to weak solutions and to data in Lebesgue spaces $L^m(\Omega)$ (or in the dual space $(H^1(\Omega))'$), previous results which were proved in the literature for bounded solutions and possibly classical or viscosity formulations.

Keywords: elliptic equations; ergodic problem; Neumann condition; additive eigenvalue; weak solutions

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1. Introduction and statement of the main results

Let Ω be a bounded, sufficiently regular, connected domain in $\mathbb{R}^N$, $N \geq 1$, and let $\vec{n}$ denote the outward normal unit vector on the boundary $\partial\Omega$. It is well known (see e.g., [12, 15]) that, if $H(x, p)$ is a Lipschitz function for $(x, p) \in \Omega \times \mathbb{R}^N$, then there is a unique real number λ such that the elliptic problem

$$\begin{cases} -\Delta u + H(x, \nabla u) + \lambda = 0 & \text{in } \Omega, \\ \nabla u \cdot \vec{n} = 0 & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

admits a solution, and this solution is unique up to a constant. The simplest example of this type of problems occurs in the linear case, when $H(x, \nabla u) = b(x) \cdot \nabla u - f(x)$. If $b(x)$ is a Lipschitz continuous, divergence free vector field, then λ is the average value of f; if $b(x)$ is not divergence free, the uniqueness of λ is a consequence of Fredholm theory for linear operators (see e.g., [10]), in which case $\lambda = \int_\Omega f(x)\varphi_1 \, dx$, where φ_1 is the first eigenfunction of the adjoint problem (normalized so that $\int_\Omega \varphi_1 = 1$).

If H is a nonlinear function, the existence and uniqueness of λ was proved in [15] assuming that $H(x, p)$ satisfies fairly general structure conditions with respect to p and enough regularity with respect to x.

The real number λ appearing in this kind of problems is sometimes called *additive eigenvalue* and is definitively a critical value which plays a role in many different contexts. If $H(x, p)$ is convex in p, then λ can be interpreted as the optimal value of an ergodic stochastic control problem; we refer the reader to [2, 3, 12, 15] and especially to [1] for an extensive presentation of the ergodic stochastic control setting. In that framework, it is natural to obtain λ as the limit of $\varepsilon u_\varepsilon$, where u_ε solves the approximating coercive problem

$$\begin{cases} -\Delta u + H(x, \nabla u) + \varepsilon u = 0 & \text{in } \Omega, \\ \nabla u \cdot \vec{n} = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

Indeed, from Bellman's principle, (1.2) is the equation solved by the value function of an infinite horizon stochastic control problem, where ε is the discount factor. The vanishing discount limit leads, through time averaging, to the ergodic control problem represented by (1.1), and $\lambda = \lim_{\varepsilon\to 0} \varepsilon u_\varepsilon$. This interpretation awarded to λ the name of *ergodic constant*, and to the singular limit of solutions of (1.2), as $\varepsilon \to 0$, the name of *ergodic limit*.

The constant λ is also a critical value for the long time behavior of the evolution problem, since it represents the asymptotic speed of time-dependent solutions: typically, a solution of the evolution problem $v_t - \Delta v + H(x, \nabla v) = 0$ (with Neumann conditions) satisfies $v(x, t) \overset{t\to\infty}{\simeq} u(x) + \lambda t$ for some stationary solution u of (1.1). Again, this is consistent with the ergodicity property of the underlying controlled stochastic trajectory, but of course the long time convergence itself does not need the convexity of H, at least for Lipschitz nonlinearities. Finally, λ also plays a crucial role in homogenization problems (in which context problem (1.1) is referred to as the *cell problem*), see [4].

A huge literature has been devoted so far to the existence and characterization of ergodic constants, as well as to the study of ergodic limits and of the long time behavior of evolution problems, at the point that it is impossible here to recall such a long list of contributions. Most papers concerned with the above issues treat the problem in the framework of viscosity solutions' theory, for both second and first order Hamilton-Jacobi-Bellman equations. This explains why this kind of results were proved under many different structure conditions on the "Hamiltonian" $H(x, \nabla u)$ as well as on the second order operator, but mostly assuming a regular dependence with respect to x. This regularity is often required for verification theorems, whenever the application to stochastic control is the main motivation.

The purpose of this note is to give a prototype result of existence, uniqueness of the ergodic constant and a characterization of the ergodic limit under natural assumptions for elliptic operators in divergence form, replacing the L^∞ framework (and most times continuity, needed for viscosity solutions) with the L^2-setting which is natural for weak solutions in the Sobolev space $H^1(\Omega)$. To be precise, we consider the elliptic problem

$$\begin{cases} -\mathrm{div}(A(x)\nabla u) + \lambda = H(x, \nabla u) + \chi \qquad \text{in } \Omega, \\ A(x)\nabla u \cdot \vec{n} = 0 \qquad \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where $\chi \in \left(H^1(\Omega)\right)'$ (the dual space of $H^1(\Omega)$), $A(x)$ is a measurable matrix such that

$$A(x) \in L^\infty(\Omega)^{N\times N}, \qquad A(x)\xi \cdot \xi \geq \alpha|\xi|^2, \tag{1.4}$$

for some $\alpha > 0$, and where $H(x, p)$ is a Carathéodory function (measurable in x, continuous in p) satisfying the following linear growth condition (this is for $N \geq 3$)

$$|H(x, p)| \leq b(x)|p| + f(x), \qquad \text{for some } b(x) \in L^N(\Omega),\ f(x) \in L^{\frac{2N}{N+2}}(\Omega), \tag{1.5}$$

for almost every $x \in \Omega$ and for every $p \in \mathbb{R}^N$.

Let us stress that $f(x)$ (and therefore $H(x, 0)$) is not assumed to be bounded, and only belongs to $L^{(2^*)'}(\Omega)$, where $2^* = \frac{2N}{N-2}$ is the Sobolev exponent for $N \geq 3$. The case $N = 1, 2$ is mentioned later, see Remark 2.4. Due to Sobolev embedding, the condition $b \in L^N(\Omega)$ is the usual threshold for Lebesgue summability of drift terms in elliptic equations, see e.g., [11]; in fact $b(x)|\nabla u| \in L^{\frac{2N}{N+2}}(\Omega)$ whenever $\nabla u \in L^2(\Omega)$ and $b(x) \in L^N(\Omega)$.

Here and below, we assume that Ω is a Lipschitz bounded and connected domain in $\mathbb{R}^N$; the Lipschitz regularity being just one possible condition which ensures that the Sobolev embedding (and the Poincaré-Wirtinger inequality) hold true.

Eventually, for the purpose of uniqueness, we will also assume the following Lipschitz condition upon H, namely that

$$|H(x, p) - H(x, q)| \leq b(x)\,|p - q|, \qquad b(x) \in L^N(\Omega), \tag{1.6}$$

for almost every $x \in \Omega$ and every $p, q \in \mathbb{R}^N$.

The first main result that we prove in this note is the following.

Theorem 1.1. *Let $N \geq 3$. Assume that $A(x)$ satisfies (1.4) and that $H(x, p)$ satisfies (1.5). Then there exist a constant $\lambda \in \mathbb{R}$ and a function $u \in H^1(\Omega)$ which satisfy the elliptic problem (1.3) in the weak sense, i.e.,*

$$\int_\Omega A(x)\nabla u \nabla\varphi + \lambda \int_\Omega \varphi = \int_\Omega H(x, \nabla u)\varphi + \langle \chi, \varphi \rangle \qquad \forall \varphi \in H^1(\Omega)\,.$$

In addition, if H satisfies (1.6), problem (1.3) is solvable for a unique constant λ and the corresponding weak solution u is unique up to addition of a constant.

Even if the result of Theorem 1.1 is quite simple, it seems new to the best of our knowledge, except for the linear case, which was treated in [10] through Fredholm theory. As usual, the existence of the constant λ is proved by considering the singular *ergodic limit*, as $\varepsilon \to 0$, of solutions u_ε of

$$\begin{cases} -\mathrm{div}(A(x)\nabla u^\varepsilon) + \varepsilon\, u^\varepsilon = H(x, \nabla u^\varepsilon) + \chi & \text{in } \Omega, \\ A(x)\nabla u^\varepsilon \cdot \vec{n} = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.7}$$

Here the main difference, compared to the classical case ([15]), is that the uniform estimate of $\varepsilon u^\varepsilon$, usually given by the maximum principle, is not available because of the more singular x-dependence of the Hamiltonian. In fact, we directly estimate $\|\nabla u^\varepsilon\|_{L^2(\Omega)}$ as a first, and then crucial, step; this is done with a similar strategy as suggested in [20] for the Dirichlet problem.

As a consequence of Theorem 1.1, and of our structure conditions, we eventually give a complete description of the limit of u^ε, assuming further that $H(x, p)$ is differentiable with respect to p. In that case, the limit of u^ε can be fully characterized in terms of the *additive eigenvalue* of the linearized problem: this is the (non homogeneous) linear problem

$$\begin{cases} -\mathrm{div}(A(x)\nabla w) + \theta = H_p(x, \nabla \bar{u}) \cdot \nabla w - \bar{u} & \text{in } \Omega, \\ A(x)\nabla w \cdot \vec{n} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.8}$$

where $\bar{u}$ is the unique solution of (1.3) such that $\int_\Omega \bar{u} = 0$.

Theorem 1.2. *Assume that $H(x, p)$ satisfies (1.5), (1.6) and H is differentiable with respect to p with $H_p(x, p) := \frac{\partial H(x,p)}{\partial p}$ being continuous in p, for a.e. $x \in \Omega$. Let u^ε be the unique solution of (1.7) and $(\lambda, \bar{u})$ be the unique solution of (1.3) such that $\int_\Omega \bar{u} = 0$. Then we have*

$$\lim_{\varepsilon\to 0} \left(u^\varepsilon - \frac{\lambda}{\varepsilon} - \bar{u} \right) = \theta$$

where the limit is in $H^1(\Omega)$, and θ is the unique constant for which problem (1.8) is solvable. Moreover, we also have (in $L^2(\Omega)$)

$$\nabla u^\varepsilon = \nabla \bar{u} + \varepsilon \nabla w + o(\varepsilon) \qquad \text{as } \varepsilon \to 0,$$

where w is the unique solution of (1.8) with zero average.

The equivalent of this result for much more general diffusion operators and Hamiltonians, but with a smooth dependence on x, is proved in many recent papers through viscosity solutions' methods, see e.g., [13, 17] for second order problems, where this is called *the selection problem*, since the constant θ selects the limit of u^ε among all possible solutions of the ergodic problem (1.3). Of course, this is much simpler for elliptic equations rather than for first order (or degenerate elliptic) problems, as treated e.g., in [9] or in the pioneering paper [8].

The proofs of Theorem 1.1 and Theorem 1.2 are given in the next Section. Later we briefly address some extension of our results to nonlinear divergence form operators (Theorem 2.5) and to more singular x-dependence, including the case of data $f(x)$ in Lebesgue spaces $L^m(\Omega)$, $m \geq 1$ (see Theorem 2.9 and Theorem 2.10).

2. Proof of the results

Let us recall that, in the following, Ω is a bounded connected domain in $\mathbb{R}^N$, $N \geq 3$, with Lipschitz boundary, and we denote by $\vec{n}$ the outward normal unit vector to the boundary $\partial\Omega$. The Sobolev space is denoted by $H^1(\Omega)$ and its dual by $\left(H^1(\Omega)\right)'$. We start with a characterization of all possible weak *subsolutions* of a Neumann elliptic problem.

Lemma 2.1. *Let $w \in H^1(\Omega)$ and $\lambda \in \mathbb{R}$ satisfy*

$$\begin{cases} -\mathrm{div}(A(x)\nabla w) = B(x) + \lambda \qquad \text{in } \Omega, \\ A(x)\nabla w \cdot \vec{n} = 0 \qquad \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where the matrix $A(x)$ satisfies (1.4) and the function $B(x) \in L^{\frac{2N}{N+2}}(\Omega)$ satisfies

$$|B(x)| \leq b(x)\,|\nabla w(x)| \qquad \text{for some } b \in L^N(\Omega). \tag{2.2}$$

Then, we have $\lambda = 0$ and the function $w(x)$ is constant in Ω.

Proof. We divide the proof in three steps.

Step 1. We prove that $w \in L^\infty(\Omega)$.

This is standard, but we sketch the argument for the reader's convenience, following [21]. For $k > 0$ we use $G_k(w) := (w - k)_+$ as test function in problem (2.1). Then, using the ellipticity of $A(x)$ and condition (2.2), we get

$$\alpha \int_\Omega |\nabla G_k(w)|^2 dx \leq \int_\Omega [b(x)|\nabla w| + \lambda]\, G_k(w)\, dx \tag{2.3}$$

Let us set $A_k := \{x \in \Omega \,:\, w(x) > k\}$. Since the integral in the right-hand side is restricted in the set A_k, we deduce, using Hölder inequality,

$$\alpha \int_\Omega |\nabla G_k(w)|^2\, dx \leq \left(\int_{A_k \cap \{|\nabla w| \neq 0\}} |b|^N\right)^{\frac{1}{N}} \|\nabla G_k(w)\|_{L^2(\Omega)} \|G_k(w)\|_{L^{2^*}(\Omega)} + |\lambda|\ \|G_k(w)\|_{L^{2^*}(\Omega)} |A_k|^{1-\frac{1}{2^*}}\,.$$

This readily implies

$$\int_\Omega |\nabla G_k(w)|^2\, dx \leq c\ \left(\int_{A_k \cap \{|\nabla w| \neq 0\}} |b|^N\right)^{\frac{2}{N}} \|G_k(w)\|^2_{L^{2^*}(\Omega)} + c\,|\lambda|\ \|G_k(w)\|_{L^{2^*}(\Omega)} |A_k|^{1-\frac{1}{2^*}}\,, \tag{2.4}$$

where, here and below, c denotes possibly different numbers only depending on α, N, Ω. By Poincaré-Wirtinger inequality we deduce

$$\|G_k(w) - \fint_\Omega G_k(w)\|^2_{L^{2^*}(\Omega)} \leq c\left(\int_{A_k} |b|^N\right)^{\frac{2}{N}} \|G_k(w)\|^2_{L^{2^*}(\Omega)} + c\ |\lambda| \|G_k(w)\|_{L^{2^*}(\Omega)} |A_k|^{1-\frac{1}{2^*}}$$

which implies

$$\begin{aligned}||G_k(w)||^2_{L^{2^*}(\Omega)} \le c\left(\int_{A_k}|b|^N\right)^{\frac{2}{N}}||G_k(w)||^2_{L^{2^*}(\Omega)}\\ + c\,|\lambda|||G_k(w)||_{L^{2^*}(\Omega)}|A_k|^{1-\frac{1}{2^*}} + c\left|\int_\Omega G_k(w)\right|^2 .\end{aligned} \tag{2.5}$$

We estimate last term as

$$\left|\int_\Omega G_k(w)\right|^2 \le ||G_k(w)||^2_{L^{2^*}(\Omega)}|A_k|^{2-\frac{2}{2^*}} \le ||G_k(w)||^2_{L^{2^*}(\Omega)}\left(\frac{||w||_{L^1(\Omega)}}{k}\right)^{2-\frac{2}{2^*}} .$$

Using this estimate in (2.5), we obtain that

$$||G_k(w)||^2_{L^{2^*}(\Omega)}\left[1 - c\left(\int_{A_k}|b|^N\right)^{\frac{2}{N}} - c\left(\frac{||w||_{L^1(\Omega)}}{k}\right)^{2-\frac{2}{2^*}}\right] \le c\,|\lambda|||G_k(w)||_{L^{2^*}(\Omega)}|A_k|^{1-\frac{1}{2^*}} .$$

Let us take k_0 sufficiently large such that, for every $k \ge k_0$,

$$1 - c\left(\int_{A_k}|b|^N\right)^{\frac{2}{N}} - c\left(\frac{||w||_{L^1(\Omega)}}{k}\right)^{2-\frac{2}{2^*}} \ge \frac{1}{2} .$$

Then we have

$$||G_k(w)||_{L^{2^*}(\Omega)} \le 2c\,|\lambda|\,|A_k|^{1-\frac{1}{2^*}} \qquad \forall k \ge k_0 .$$

Hence

$$\int_\Omega G_k(w) \le ||G_k(w)||_{L^{2^*}(\Omega)}|A_k|^{1-\frac{1}{2^*}} \le 2c\,|\lambda|\,|A_k|^{2-\frac{2}{2^*}} .$$

Recall that $2 - \frac{2}{2^*} = 1 + \frac{2}{N}$, and, for a.e. k, we have $\frac{d}{dk}\int_\Omega G_k(w) = -|A_k|$. This means that the function $\varphi(k) := \int_\Omega G_k(w)$ is a non increasing function which satisfies $\varphi \le 2c\,|\lambda|(-\varphi')^{1+\frac{2}{N}}$ for all $k \ge k_0$. It follows that $\varphi(k_1) = 0$ for some $k_1 > k_0$. Hence $w(x) \le k_1$ a.e. in Ω. Repeating the argument for $-w$, we conclude that $w \in L^\infty(\Omega)$.

Step 2. We prove that $\lambda = 0$.

To this purpose, we reason by contradiction. Suppose that $\lambda < 0$. Since $w \in L^\infty(\Omega)$, for ε sufficiently small we have $\lambda + \varepsilon w \le 0$. Hence w satisfies

$$\begin{cases} -\mathrm{div}(A(x)\nabla w) + \varepsilon w \le b(x)|\nabla w| , & \text{in } \Omega,\\ A(x)\nabla w \cdot \vec{n} = 0 & \text{on } \partial\Omega. \end{cases}$$

This implies (with the same proof as e.g., [20, Proposition 2.1]) that $w \le 0$. Since $w + c$ is still a solution of (2.1), whatever is the constant $c \in R$, we easily get a contradiction. Of course the same argument applies if $\lambda > 0$. We conclude that $\lambda = 0$.

Step 3. We now prove that w is a constant. To this purpose, we recall that the *median* of a function $u \in H^1(\Omega)$ is defined as

$$\mathrm{med}\,(u) := \sup\{k \in \mathbb{R} \,:\, \mathrm{meas}(\{u > k\}) \ge \frac{|\Omega|}{2}\} .$$

As for the average of u, a Poincaré type inequality holds for $u - \mathrm{med}\,(u)$, see e.g., [22]. Namely there exists a constant C (only depending on N, Ω) such that

$$\|u - \mathrm{med}\,(u)\|_{L^2(\Omega)} \le C\|\nabla u\|_{L^2(\Omega)} \qquad \forall u \in H^1(\Omega)$$

and therefore, by Sobolev inequality, we have, for a possibly different constant C,

$$\|u - \mathrm{med}\,(u)\|_{L^{2^*}(\Omega)} \le C\|\nabla u\|_{L^2(\Omega)} \qquad \forall u \in H^1(\Omega)\,. \tag{2.6}$$

We now normalize our solution w so that

$$\mathrm{med}\,(w) = 0\,.$$

This implies that $\mathrm{med}\,(\psi(w)) = 0$ for every nondecreasing Lipschitz function $\psi : \mathbb{R} \to \mathbb{R}$ such that $\psi(0) = 0$. In particular, we have that $\mathrm{med}\,(G_k(w)) = 0$ for all $k > 0$, where now $G_k(s) = (|s|-k)_+\mathrm{sign}(s)$. Defining $A_k = \{x : |w(x)| > k\}$, we obtain as in (2.4) (with the additional information now that $\lambda = 0$):

$$\int_\Omega |\nabla G_k(w)|^2 \le c\left(\int_{A_k\cap\{|\nabla w|\neq 0\}} |b|^N\right)^{\frac{2}{N}} \|G_k(w)\|^2_{L^{2^*}(\Omega)}$$

and so by (2.6) we deduce

$$\|G_k(w)\|^2_{L^{2^*}(\Omega)} \le c\left(\int_{A_k\cap\{|\nabla w|\neq 0\}} |b|^N\right)^{\frac{2}{N}} \|G_k(w)\|^2_{L^{2^*}(\Omega)}\,.$$

This inequality now yields that $w = 0$. In fact, assume by contradiction that $M := \sup|w| > 0$ and take a sequence of $k < M$, $k \uparrow M$. Since $k < M$, we have $\|G_k(w)\|^2_{L^{2^*}(\Omega)} \neq 0$ (otherwise this contradicts the definition of M), so the previous inequality implies

$$1 \le c\left(\int_{A_k\cap\{|\nabla w|\neq 0\}} |b|^N\right)^{\frac{2}{N}}\,.$$

But, as $k \uparrow M$, we have $|A_k \cap \{|\nabla w| \neq 0\}| \to 0$, because $\nabla w = 0$ a.e. in the set $\{|w| = M\}$. Hence we reach a contradiction. This implies that $w = 0$, so w coincides with its median, and it is a constant. □

Remark 2.2. *We notice that this lemma remains true for a nonlinear divergence form operator* $-\mathrm{div}(a(x, \nabla u))$ *which satisfies*

$$a(x, p) \cdot p \ge \alpha|p|^2 \qquad \textit{for some } \alpha > 0. \tag{2.7}$$

We now analyze the limit, as $\varepsilon \to 0$, of the elliptic problem (1.7). For the purposes of Theorem 1.2, it is convenient to state this kind of result in a slightly more general form, where the Hamiltonian $H(x, p)$ may possibly depend on ε as well.

Lemma 2.3. *Assume that* $A(x)$ *satisfies (1.4), and that* $H_\varepsilon(x, p)$ *is a sequence of Carathéodory functions satisfying*

$$|H_\varepsilon(x, p)| \le b(x)|p| + f(x)\,, \qquad \textit{for some } b(x) \in L^N(\Omega),\ f(x) \in L^{\frac{2N}{N+2}}(\Omega), \tag{2.8}$$

for almost every $x \in \Omega$ and for every $p \in \mathbb{R}^N$.

For $\chi \in \left(H^1(\Omega)\right)'$, and $\varepsilon > 0$, let u^ε be solutions of the elliptic problem

$$\begin{cases} -\mathrm{div}(A(x)\nabla u^\varepsilon) + \varepsilon\, u^\varepsilon = H_\varepsilon(x, \nabla u^\varepsilon) + \chi & \text{in } \Omega, \\ A(x)\nabla u^\varepsilon \cdot \vec{n} = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.9}$$

and assume that there exists a function $H(x, p)$ such that

$$H_\varepsilon(x, p) \to H(x, p) \qquad \text{for every } p \in \mathbb{R}^N, \text{ and almost every } x \in \Omega.$$

Then there exist a constant $\lambda \in \mathbb{R}$ and a function $u \in H^1(\Omega)$ such that, up to a subsequence,

$$\varepsilon u^\varepsilon \to \lambda \qquad \text{and} \qquad u^\varepsilon - \fint_\Omega u^\varepsilon \to u$$

where the limits are in the (strong) topology of $H^1(\Omega)$. Moreover, (λ, u) solve problem (1.3).

Proof. We first claim that

$$\exists\ K > 0 \text{ (independent of } \varepsilon\text{)}: \quad \|\nabla u^\varepsilon\|_{L^2(\Omega)} \le K \qquad \forall \varepsilon > 0\,. \tag{2.10}$$

We proceed by contradiction and suppose that (2.10) is not true. This implies the existence of a subsequence of u^ε, not relabeled, such that $\|\nabla u^\varepsilon\|_{L^2(\Omega)} \to \infty$. We set

$$w_\varepsilon := \frac{u^\varepsilon - \fint_\Omega u^\varepsilon}{\|\nabla u^\varepsilon\|_{L^2(\Omega)}}\,.$$

Since w_ε has zero average and $\|\nabla w_\varepsilon\|_{L^2(\Omega)} = 1$, we deduce that w_ε is weakly relatively compact in $H^1(\Omega)$ and strongly in $L^2(\Omega)$ by Rellich's theorem. We observe that w_ε satisfies

$$-\mathrm{div}(A(x)\nabla w_\varepsilon) + \varepsilon\, w_\varepsilon = \frac{1}{\|\nabla u^\varepsilon\|_{L^2(\Omega)}}\,\left[H_\varepsilon(x, \nabla u^\varepsilon) + \chi\right] - \frac{\varepsilon}{|\Omega|}\frac{\int_\Omega u^\varepsilon}{\|\nabla u^\varepsilon\|_{L^2(\Omega)}} \qquad \text{in } \Omega. \tag{2.11}$$

Last term is a sequence of real numbers that we estimate, integrating (2.9), as

$$\begin{aligned} \varepsilon\frac{\int_\Omega u^\varepsilon}{\|\nabla u^\varepsilon\|_{L^2(\Omega)}} &= \frac{1}{\|\nabla u^\varepsilon\|_{L^2(\Omega)}}\int_\Omega H_\varepsilon(x, \nabla u^\varepsilon) + \frac{1}{\|\nabla u^\varepsilon\|_{L^2(\Omega)}}\langle \chi, 1\rangle \\ &\le \frac{1}{\|\nabla u^\varepsilon\|_{L^2(\Omega)}}\int_\Omega [b(x)|\nabla u^\varepsilon| + f(x)] + \frac{\|\chi\|_{(H^1(\Omega))'}|\Omega|^{\frac12}}{\|\nabla u^\varepsilon\|_{L^2(\Omega)}} \\ &\le \frac{1}{\|\nabla u^\varepsilon\|_{L^2(\Omega)}}[\|b\|_{L^2(\Omega)}\|\nabla u^\varepsilon\|_{L^2(\Omega)} + \|f\|_{L^1(\Omega)} + \|\chi\|_{(H^1(\Omega))'}\,|\Omega|^{\frac12}]\,, \end{aligned}$$

where we used (2.8) as well. Since $b \in L^N(\Omega)$ ($N \ge 2$) and $f \in L^1(\Omega)$, and since $\|\nabla u^\varepsilon\|_{L^2(\Omega)}$ diverges, the right-hand side is bounded. Hence $\frac{\varepsilon}{|\Omega|}\frac{\int_\Omega u^\varepsilon}{\|\nabla u^\varepsilon\|_{L^2(\Omega)}}$ is a bounded sequence and there exists a real value $\lambda \in \mathbb{R}$ such that, up to subsequences (not relabeled here), we have

$$\frac{\varepsilon}{|\Omega|}\frac{\int_\Omega u^\varepsilon}{\|\nabla u^\varepsilon\|_{L^2(\Omega)}} \to \lambda\,. \tag{2.12}$$

Now we use once more (2.8) to estimate

$$\frac{1}{\|\nabla u^\varepsilon\|_{L^2(\Omega)}}\,|H_\varepsilon(x,\nabla u^\varepsilon)| \le b(x)|\nabla w_\varepsilon| + \frac{f(x)}{\|\nabla u^\varepsilon\|_{L^2(\Omega)}}\,. \tag{2.13}$$

Since $b \in L^N(\Omega)$ and w_ε is bounded in $H^1(\Omega)$, the product $b(x)|\nabla w_\varepsilon|$ is bounded in $L^{\frac{2N}{N+2}}(\Omega)$. In addition, it is also equi-integrable: indeed, for any set E, one has

$$\begin{aligned}\int_E |b(x)\nabla w_\varepsilon|^{\frac{2N}{N+2}} &\le \|\nabla w_\varepsilon\|_{L^2(\Omega)}^{\frac{2N}{N+2}}\left(\int_E b(x)^N\right)^{\frac{2}{N+2}}\\ &\le C\left(\int_E b(x)^N\right)^{\frac{2}{N+2}}\end{aligned}$$

and last term (independent of ε) goes to zero as $|E| \to 0$. Thus, in view of (2.13), the term $\frac{1}{\|\nabla u^\varepsilon\|_{L^2(\Omega)}}\,H_\varepsilon(x,\nabla u^\varepsilon)$ is bounded and equi-integrable in $L^{\frac{2N}{N+2}}(\Omega)$. Therefore, we have

$$\begin{cases} w_\varepsilon \text{ is bounded in } H^1(\Omega),\\ -\mathrm{div}(A(x)\nabla w_\varepsilon) + \varepsilon\, w_\varepsilon = h_\varepsilon + \frac{\chi}{\|\nabla u^\varepsilon\|_{L^2(\Omega)}}\end{cases} \tag{2.14}$$

for some h_ε bounded and equi-integrable in $L^{\frac{2N}{N+2}}(\Omega)$.

This implies that w_ε is actually strongly compact in $H^1(\Omega)$. We recall the argument for the reader's convenience. The key point is that, if w is a weak limit of w_ε, then

$$\int_\Omega h_\varepsilon(w_\varepsilon - w) \overset{\varepsilon\to0}{\to} 0\,. \tag{2.15}$$

In fact, we have

$$\int_\Omega h_\varepsilon(w_\varepsilon - w) = \int_\Omega h_\varepsilon T_k(w_\varepsilon - w) + \int_\Omega h_\varepsilon G_k(w_\varepsilon - w)$$

where $T_k(s) = \max(\min(s,k),-k)$ is the truncation function and $G_k(s)$ is the difference $s - T_k(s)$. By Hölder inequality, since $w_\varepsilon - w$ is bounded in $L^{2^*}(\Omega)$ one has, by definition of $G_k(\cdot)$:

$$\left|\int_\Omega h_\varepsilon G_k(w_\varepsilon - w)\right| \le C\left(\int_{\{|w_\varepsilon - w|>k\}} |h_\varepsilon|^{\frac{2N}{N+2}}\right)^{\frac{N+2}{2N}}$$

where last term vanishes as $k \to \infty$, uniformly with respect to ε, since h_ε is equi-integrable in $L^{\frac{2N}{N+2}}(\Omega)$. Therefore one has

$$\int_\Omega h_\varepsilon(w_\varepsilon - w) = \int_\Omega h_\varepsilon T_k(w_\varepsilon - w) + r_k$$

where $r_k \overset{k\to\infty}{\to} 0$ uniformly with respect to ε. But h_ε is bounded in $L^{\frac{2N}{N+2}}(\Omega)$ while $T_k(w_\varepsilon - w) \to 0$ strongly in $L^p(\Omega)$ for any $p < \infty$ due to Rellich theorem; hence the first integral in the right-hand side converges to zero as $\varepsilon \to 0$. Therefore, by letting $\varepsilon \to 0$ and then $k \to \infty$, we deduce that (2.15) holds true. With (2.15) in hands, it is now easy to deduce from (2.14) that

$$\|\nabla w_\varepsilon - \nabla w\|_{L^2(\Omega)}^2 \le \frac{1}{\alpha}\int_\Omega A(x)\nabla(w_\varepsilon - w)\cdot\nabla(w_\varepsilon - w) \overset{\varepsilon\to0}{\to} 0.$$

We have obtained so far that there exists a function $w \in H^1(\Omega)$ such that, up to subsequences,

$$w_\varepsilon \to w \quad \text{strongly in } H^1(\Omega).$$

In particular, this implies, up to subsequences, that $\nabla w_\varepsilon \to \nabla w$ almost everywhere in Ω. Therefore, as a consequence of (2.13), the term $\frac{1}{\|\nabla u^\varepsilon\|_{L^2(\Omega)}} H_\varepsilon(x, \nabla u^\varepsilon)$ weakly converges in $L^{\frac{2N}{N+2}}(\Omega)$ towards some function $B(x)$ which satisfies $|B(x)| \le b(x)|\nabla w(x)|$ a.e. in Ω.

Finally, due to (2.11), (2.12), (2.13), w satisfies (in weak sense)

$$\begin{cases} -\operatorname{div}(A(x)\nabla w) \le b(x)|\nabla w| + \lambda & \text{in } \Omega, \\ A(x)\nabla w \cdot \vec{n} = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.16}$$

By Lemma 2.1 we deduce that w is constant, and since $\int_\Omega w = 0$ this means $w(x) \equiv 0$. But this is a contradiction with the fact that w_ε strongly converges to w in $H^1(\Omega)$ and $\|\nabla w_\varepsilon\|_{L^2(\Omega)} = 1$ for every ε. The contradiction proves that the assertion (2.10) is true.

Thanks to (2.10), now we integrate (2.9) and we get

$$\varepsilon \left| \int_\Omega u^\varepsilon \right| = \left| \int_\Omega H_\varepsilon(x, \nabla u^\varepsilon) + \langle \chi, 1 \rangle \right| \le \int_\Omega b(x)|\nabla u_\varepsilon| + f(x) + |\langle \chi, 1 \rangle| \le C\,,$$

due to the growth condition (2.8). Therefore, $\varepsilon \int_\Omega u^\varepsilon$ is a bounded sequence, and there exists a real value $\lambda \in \mathbb{R}$ such that, up to subsequences,

$$\varepsilon \fint_\Omega u^\varepsilon \to \lambda\,.$$

We now set $\hat{u^\varepsilon} : u^\varepsilon - \fint_\Omega u^\varepsilon$. Then $\hat{u^\varepsilon}$ is bounded in $H^1(\Omega)$ and solves

$$-\operatorname{div}(A(x)\nabla \hat{u^\varepsilon}) + \varepsilon \hat{u^\varepsilon} = H_\varepsilon(x, \nabla \hat{u^\varepsilon}) + \chi - \varepsilon \fint_\Omega u^\varepsilon \qquad \text{in } \Omega. \tag{2.17}$$

With the same arguments used before for w_ε, we can show that $\hat{u^\varepsilon}$ is strongly compact in $H^1(\Omega)$, so there exists a function $u \in H^1(\Omega)$ such that $\hat{u^\varepsilon} \to u$ in $H^1(\Omega)$. We may also assume that $\nabla \hat{u^\varepsilon} \to \nabla u$ almost everywhere in Ω, and therefore almost uniformly as well; since $H_\varepsilon(x, \nabla \hat{u^\varepsilon})$ is equi-integrable and $H_\varepsilon(x, p) \to H(x, p)$, we can deduce that $H_\varepsilon(x, \nabla \hat{u^\varepsilon})$ converges to $H(x, \nabla u)$ (which belongs to $L^{\frac{2N}{N+2}}(\Omega)$ due to (2.8)). Finally, passing to the limit in (2.17), we obtain that (u, λ) is a solution to problem (1.3). □

The proof of Theorem 1.1 immediately follows from the above two results.

Proof of Theorem 1.1. For $\varepsilon > 0$, let u^ε be the solution of the elliptic problem (1.7). Applying Lemma 2.3, we have that, up to subsequences, $\varepsilon u^\varepsilon \to \lambda$ and $u^\varepsilon - \fint_\Omega u^\varepsilon \to u$, where (λ, u) gives a solution to problem (1.3).

Now we assume further the condition (1.6) and we prove uniqueness of λ and u (up to a constant). This is a straightforward consequence of Lemma 2.1. Indeed, let (u_1, λ_1) and (u_2, λ_2) be solutions to problem (1.3). Then $w := u_1 - u_2$ is a solution to

$$\begin{cases} -\operatorname{div}(A(x)\nabla w) = H(x, \nabla u_1) - H(x, \nabla u_2) + \lambda_1 - \lambda_2 & \text{in } \Omega, \\ A(x)\nabla w \cdot \vec{n} = 0 & \text{on } \partial\Omega, \end{cases}$$

where we have, due to (1.6),

$$|H(x,\nabla u_1)-H(x,\nabla u_2)|\le b(x)|\nabla w|\,.$$

From Lemma 2.1 we deduce that $\lambda_1=\lambda_2$ and that u_1-u_2 is a constant. □

Given the solution u^ε of (1.7), we now investigate the behavior of $u^\varepsilon-\frac{\lambda}{\varepsilon}-\bar u$ where $(\lambda,\bar u)$ is a solution of the ergodic problem (1.3). In order to fix a reference solution, we normalize $\bar u$ so that $\int_\Omega \bar u=0$. So $(\lambda,\bar u)$ is uniquely defined from Theorem 1.1, provided (1.6) holds. The proof of Theorem 1.2 now follows as a Corollary of the previous results.

Proof of Theorem 1.2. Let us define

$$v^\varepsilon:=\frac{1}{\varepsilon}\left(u^\varepsilon-\frac{\lambda}{\varepsilon}-\bar u\right).$$

One can check that v^ε solves

$$\begin{cases}-\mathrm{div}(A(x)\nabla v^\varepsilon)+\varepsilon\, v^\varepsilon=\frac{1}{\varepsilon}\,[H(x,\nabla u^\varepsilon)-H(x,\nabla\bar u)]-\bar u & \text{in }\Omega,\\ A(x)\nabla v^\varepsilon\cdot\vec n=0 \qquad \text{on }\partial\Omega.\end{cases}\tag{2.18}$$

We notice that, thanks to (1.6), the function

$$\tilde H_\varepsilon(x,p):=\frac{1}{\varepsilon}\,[H(x,\varepsilon p+\nabla\bar u)-H(x,\nabla\bar u)]$$

satisfies

$$|\tilde H_\varepsilon(x,p)|\le b(x)|p|\,,\qquad |\tilde H_\varepsilon(x,p)-\tilde H_\varepsilon(x,q)|\le b(x)|p-q|$$

and, from the differentiability of H, we have

$$\tilde H_\varepsilon(x,p)\to H_p(x,\nabla\bar u)\cdot p\qquad\text{for every } p\in\mathbb{R}^N \text{ and a.e. } x\in\Omega.$$

Therefore Lemma 2.3 applies and we deduce that there exists a constant $\theta\in\mathbb{R}$ and a function $w\in H^1(\Omega)$ such that

$$\varepsilon v^\varepsilon\to\theta\,,\qquad v^\varepsilon-⨍_\Omega v^\varepsilon\to w$$

where the limits are in the strong topology of $H^1(\Omega)$ and (θ,w) is the unique couple which satisfies the linear elliptic problem (1.8) with the normalized condition $\int_\Omega w=0$. We stress that the uniqueness of the limit couple implies that the whole sequence v^ε converges.

Coming back from v^ε to u^ε, this means that

$$\lim_{\varepsilon\to0}\left(u^\varepsilon-\frac{\lambda}{\varepsilon}-\bar u\right)=\theta$$

where the limit is meant in $H^1(\Omega)$, and in addition

$$\frac{1}{\varepsilon}\left(u^\varepsilon-⨍_\Omega u^\varepsilon-\bar u\right)\to w\,.$$

This latter convergence yields the first order expansion for the gradient in $L^2(\Omega)$:

$$\nabla u^\varepsilon=\nabla\bar u+\varepsilon\nabla w+o(\varepsilon)\qquad\text{as }\varepsilon\to0.$$

□

Remark 2.4. The case $N = 2$ can be dealt with in exactly the same way as before, except that the threshold summability of the drift term should be adapted to the Sobolev embedding of dimension two. Since for $N = 2$ the space $H^1(\Omega)$ is embedded in $L^p(\Omega)$ for every $p < \infty$, but not in $L^\infty(\Omega)$, one needs here $b(x) \in L^q(\Omega)$ for some $q > 2$ in order that the product $b(x)|\nabla u|$ belongs to the dual space $\left(H^1(\Omega)\right)'$. Therefore, conditions (1.5) and (1.6) should be changed into

$$|H(x, p)| \le b(x)|p| + f(x), \quad \text{for some } b(x) \in L^q(\Omega),\, q > 2, \text{ and } f(x) \in L^m(\Omega),\, m > 1, \tag{2.19}$$

and respectively,

$$|H(x, p) - H(x, q)| \le b(x)\,|p - q|, \qquad b(x) \in L^q(\Omega),\, q > 2, \tag{2.20}$$

for almost every $x \in \Omega$ and for every $p, q \in \mathbb{R}^N$.

Replacing assumptions (1.5) and (1.6) with, respectively, (2.19) and (2.20), the results stated in Theorem 1.1 and Theorem 1.2 are true for the dimension $N = 2$, and the proof remains the same up to the obvious modifications in the Lebesgue spaces which are involved.

A similar remark holds true for the case of dimension $N = 1$; in that case it is enough to assume $b(x) \in L^2(\Omega)$ and $f(x) \in L^1(\Omega)$.

2.1. Extensions to nonlinear operators

In this subsection we give a short extension of the result of Theorem 1.1 to the case of nonlinear operators. To this purpose, we introduce a function $a(x, p) : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ which is assumed to be measurable with respect to x, for every $p \in \mathbb{R}^N$, and continuous with respect to p, for a.e. $x \in \Omega$. We assume that $a(x, p)$ satisfies the following monotonicity and growth conditions:

$$(a(x, p) - a(x, q)) \cdot (p - q) \ge \alpha\,|p - q|^2, \quad \text{for some } \alpha \in \mathbb{R},\, \alpha > 0, \tag{2.21}$$

and

$$|a(x, p)| \le \beta\,(|p| + k(x)) \qquad \text{for some } \beta \in \mathbb{R},\, k(x) \in L^2(\Omega), \tag{2.22}$$

for almost every $x \in \Omega$ and every $p, q \in \mathbb{R}^N$.

Notice that (2.21)–(2.22) imply also the coercivity condition

$$a(x, p) \cdot p \ge \alpha|p|^2 - \tilde{k}(x), \qquad \text{where } \alpha \in \mathbb{R},\, \alpha > 0,\, \tilde{k}(x) \in L^1(\Omega). \tag{2.23}$$

We also assume that $a(x, p)$ satisfies the following asymptotic condition for $|p| \to \infty$:

$$\text{for a.e. } x \in \Omega, \text{ every } p \in \mathbb{R}^N,\ \exists \quad \lim_{t\to\infty} \frac{a(x, tp)}{t}\ . \tag{2.24}$$

Of course, the case $a(x, p) = A(x) \cdot p + K(x)$, for a matrix $A(x)$ satisfying (1.4) and a vector field $K \in L^2(\Omega)$, is the simplest example where conditions (2.21)–(2.24) are satisfied.

Thanks to (2.21)–(2.24), we can extend Theorem 1.1 to a nonlinear setting. Notice that additional terms in the equation, belonging to $\left(H^1(\Omega)\right)'$, can be included here in the vector field $a(x, p)$ or in the local function $H(x, p)$.

Theorem 2.5. *Assume that $a(x, p)$ satisfies (2.21)–(2.24), and that $H(x, p)$ satisfies (1.5) (or it satisfies (2.19) if $N = 2$). Then there exist a constant $\lambda \in \mathbb{R}$ and a function $u \in H^1(\Omega)$ which solve, in the weak sense, the elliptic problem*

$$\begin{cases} -\mathrm{div}(a(x, \nabla u)) + \lambda = H(x, \nabla u) & \text{in } \Omega, \\ a(x, \nabla u) \cdot \vec{n} = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.25}$$

In addition, if (1.6) holds true (respectively, (2.20) if $N = 2$), then λ is unique and u is unique up to addition of a constant.

Proof. For $\varepsilon > 0$, we consider the approximating problem

$$\begin{cases} -\mathrm{div}(a(x, \nabla u^\varepsilon)) + \varepsilon\, u^\varepsilon = H(x, \nabla u^\varepsilon) & \text{in } \Omega, \\ a(x, \nabla u^\varepsilon) \cdot \vec{n} = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.26}$$

Then we aim at showing that the a priori estimate (2.10) holds true and we proceed, as in Lemma 2.3, by contradiction. This allows us to build a sequence w_ε such that $\|\nabla w_\varepsilon\|_{L^2(\Omega)} = 1$, w_ε has zero average and satisfies

$$-\,\mathrm{div}\left(\frac{a(x, \|\nabla u^\varepsilon\|_{L^2(\Omega)} \nabla w_\varepsilon)}{\|\nabla u^\varepsilon\|_{L^2(\Omega)}}\right) + \varepsilon\, w_\varepsilon = \frac{H(x, \nabla u^\varepsilon)}{\|\nabla u^\varepsilon\|_{L^2(\Omega)}} - \frac{\varepsilon}{|\Omega|} \frac{\int_\Omega u^\varepsilon}{\|\nabla u^\varepsilon\|_{L^2(\Omega)}} \qquad \text{in } \Omega, \tag{2.27}$$

where $\|\nabla u^\varepsilon\|_{L^2(\Omega)} \to \infty$. As in Lemma 2.3, last term is a relatively compact sequence of real numbers, and we observe that the right-hand side is bounded and equi-integrable in $L^{\frac{2N}{N+2}}(\Omega)$, so that w_ε satisfies

$$\begin{cases} w_\varepsilon \text{ is bounded in } H^1(\Omega), \\ -\mathrm{div}(\hat{a}_\varepsilon(x, \nabla w_\varepsilon)) + \varepsilon\, w_\varepsilon = h_\varepsilon \end{cases} \tag{2.28}$$

for some h_ε bounded and equi-integrable in $L^{\frac{2N}{N+2}}(\Omega)$, where

$$\hat{a}_\varepsilon(x, p) := \frac{a(x, \|\nabla u^\varepsilon\|_{L^2(\Omega)} p)}{\|\nabla u^\varepsilon\|_{L^2(\Omega)}}\,.$$

Now, let w be a weak limit of w_ε in $H^1(\Omega)$, and a strong limit in $L^2(\Omega)$. Notice that we have that

$$\int_\Omega \hat{a}_\varepsilon(x, \nabla w) \nabla(w_\varepsilon - w) \to 0$$

because $\hat{a}_\varepsilon(x, \nabla w)$ is strongly convergent in $L^2(\Omega)$ due to (2.24) and (2.22), by Lebesgue's theorem. Therefore, using (2.28) and (2.15) as well, we deduce that

$$\lim_{\varepsilon \to 0} \int_\Omega (\hat{a}_\varepsilon(x, \nabla w_\varepsilon) - \hat{a}_\varepsilon(x, \nabla w))\, \nabla(w_\varepsilon - w) = 0\,. \tag{2.29}$$

Notice that $\hat{a}_\varepsilon$ satisfies (2.21) for the same $\alpha > 0$, and for every $\varepsilon > 0$. Then (2.29) implies that $w_\varepsilon \to w$ strongly in $H^1(\Omega)$ (and, up to a subsequence, $\nabla w_\varepsilon \to \nabla w$ almost everywhere in Ω). Now, for every $\varphi \in H^1(\Omega)$, we set

$$\hat{a}(x, \nabla\varphi(x)) := \lim_{\varepsilon \to 0} \hat{a}_\varepsilon(x, \nabla\varphi(x)) \tag{2.30}$$

which exists after (2.24) and belongs to $L^2(\Omega)$ due to (2.22). It is a consequence of (2.29) and a standard monotonicity argument (sometimes known as Minty's argument) that $\hat{a}_\varepsilon(x, \nabla w_\varepsilon(x))$ converges to $\hat{a}(x, \nabla w(x))$ weakly in $L^2(\Omega)$. Finally, passing to the limit in (2.27), we obtain that w is a weak solution of

$$\begin{cases} -\mathrm{div}(\hat{a}(x, \nabla w(x))) + \lambda = B(x) & \text{in } \Omega, \\ \hat{a}(x, \nabla w(x)) \cdot \vec{n} = 0 & \text{on } \partial\Omega \end{cases}$$

for some $\lambda \in \mathbb{R}$ and some $B(x) \in L^{\frac{2N}{N+2}}(\Omega)$ satisfying condition (2.2). Since $\hat{a}(x, p)$ satisfies (2.7) (because $\hat{a}(x, 0) = 0$ and (2.21) holds), we deduce from Lemma 2.1, Remark 2.2, that $\lambda = 0$ and that w is constant (hence $w = 0$ because it has zero average). We get a contradiction with $\|\nabla w\|_{L^2(\Omega)} = 1$. This proves that the a priori estimate (2.10) holds true, that is $\|\nabla u^\varepsilon\|_{L^2(\Omega)}$ is uniformly bounded. Then the compactness of u^ε in $H^1(\Omega)$ follows in a similar way as before, using (2.21), and we conclude as in Theorem 1.1 the existence of $\lambda \in \mathbb{R}$, $u \in H^1(\Omega)$ which solve problem (2.25).

By using (1.6) and (2.21), the uniqueness of λ and the uniqueness of u, up to a constant, are proved exactly as in Theorem 1.1. Even if the problem is nonlinear, and Lemma 2.1 cannot be literally applied, the arguments are exactly the same as used in Lemma 2.1, now applied to $w := u_1 - u_2$; the nonlinearity of the operator is readily handled with assumption (2.21). □

Remark 2.6. *Condition (2.24) is assumed here in order that we can follow the same approach used before in the proof of Theorem 1.1. We believe this condition to be unnecessary for a similar result to hold for general nonlinear operators, however removing this condition would need a substantial change in the method of proof (e.g. using symmetrization methods), which is beyond the scope of this note.*

Remark 2.7. *The result remains true for the case of dimension $N = 1$, up to requiring $b \in L^2(\Omega)$ and $f \in L^1(\Omega)$ in assumptions (1.5) and (1.6).*

2.2. Extensions to more singular data

The same approach as before can be used in case of more singular dependence with respect to x. We have in mind here that the assumption (1.5) is replaced by

$$|H(x, p)| \le b(x)|p| + f(x), \qquad \text{for some } b(x) \in L^N(\Omega),\ f(x) \in L^m(\Omega), \tag{2.31}$$

for almost every $x \in \Omega$ and for every $p \in \mathbb{R}^N$, where $m \ge 1$, so that data can belong to Lebesgue spaces of any order.

Of course, if $m \ge \frac{2N}{N+2}$, assumption (2.31) implies (1.5), so there is nothing new to be proved. By contrast, if $m < \frac{2N}{N+2}$, we cannot expect the solutions to belong to $H^1(\Omega)$ anymore; however it is still possible to obtain similar results in a setting of generalized solutions. We recall below the notion of renormalized solutions. The truncation function is denoted, as before, by $T_k(s) := \max(\min(s, k), -k)$. In what follows we suppose that $N > 1$.

Definition 2.8. *A function u, belonging to $W^{1,q}(\Omega)$ for every $q < \frac{N}{N-1}$, is a renormalized solution of problem (2.25) if:*

(i) $T_k(u) \in H^1(\Omega)$ for every $k > 0$, and $H(x, \nabla u) \in L^1(\Omega)$.

(ii) For every function $h:\mathbb{R}\to\mathbb{R}$ which is C^1 with compact support, it holds

$$\int_\Omega a(x,\nabla u)\nabla(h(u)\varphi)+\lambda\int_\Omega h(u)\varphi=\int_\Omega H(x,\nabla u)h(u)\varphi \qquad \forall\varphi\in H^1(\Omega)\cap L^\infty(\Omega)\,. \tag{2.32}$$

(iii)

$$\lim_{n\to\infty}\ \frac{1}{n}\int\limits_{\{n<|u|<2n\}}|\nabla u|^2=0\,. \tag{2.33}$$

We recall that this notion of solution, introduced by P. L. Lions and F. Murat (in a joint paper unpublished, whose content can be found in [16, 18]) is nowadays currently used as a formulation of elliptic problems with L^1-data, or even Radon measures. For similar problems with first order terms having linear growth, we refer e.g., to [5].

It is to be noted that, in the renormalized formulation, the truncations of the solution belong to the energy space $H^1(\Omega)$, so that (2.32) makes sense for any h with compact support. Condition (2.33), in turn, implies that renormalized solutions are also distributional solutions. To this purpose one can take $h=S\left(\frac{r}{n}\right)$ where $S(t)$ is a piecewise linear function, supported in $[-2,2]$, such that $S(t)\equiv 1$ for $t\in[-1,1]$; then $S\left(\frac{r}{n}\right)\to 1$ as $n\to\infty$ and the distributional formulation (with test functions $\varphi\in C^1(\overline{\Omega})$) is recovered thanks to (2.33).

Up to replacing the $\left(H^1(\Omega)\right)'$ formulation with the renormalized setting, Theorem 2.5 admits a natural extension which is the following one. We recall (see [7]) that elliptic equations (including possibly nonlinear divergence form operators with discontinuous coefficients) with source terms in $L^m(\Omega)$, $m<\frac{2N}{N+2}$, admit solutions which belong to $W^{1,m^*}(\Omega)$, where m^* is the Sobolev exponent $(Nm)/(N-m)$.

Theorem 2.9. *Let $N\geq 3$. Assume that $a(x,p)$ satisfies (2.21)–(2.24) and that $H(x,p)$ satisfies (2.31) for some $1<m<\frac{2N}{N+2}$. Then there exists a constant $\lambda\in\mathbb{R}$ and a function $u\in W^{1,m^*}(\Omega)$ which is a renormalized solution of (2.25). In addition, if (1.6) holds true, λ is unique and u is unique up to addition of a constant.*

Proof. We only sketch the main steps, and the main differences with the arguments of Theorem 2.5. We start with the solutions u^ε of the approximating problem

$$\begin{cases}-\mathrm{div}(a(x,\nabla u^\varepsilon))+\varepsilon\, u^\varepsilon=H_\varepsilon(x,\nabla u^\varepsilon) & \text{in } \Omega,\\ a(x,\nabla u^\varepsilon)\cdot\vec{n}=0 \quad \text{on } \partial\Omega,\end{cases} \tag{2.34}$$

where $H_\varepsilon(x,p)=T_{\frac{1}{\varepsilon}}(H(x,p))$. Here the truncation of H is only needed if one wants to work, at fixed ε, with the more comfortable setting of finite energy solutions. Then we claim that there exists $K>0$ independent of ε, such that

$$\|\nabla u^\varepsilon\|_{L^{m^*}(\Omega)}\leq K \qquad \forall\varepsilon>0\,. \tag{2.35}$$

The proof of (2.35) is done, as before, by contradiction. In this case we build a subsequence, not relabeled, such that $\|\nabla u^\varepsilon\|_{L^{m^*}(\Omega)}\to\infty$ and we define

$$w_\varepsilon=\frac{u^\varepsilon-\fint_\Omega u^\varepsilon}{\|\nabla u^\varepsilon\|_{L^{m^*}(\Omega)}}$$

which solves

$$- \operatorname{div}(\hat{a}(x, \nabla w_\varepsilon)) + \varepsilon\, w_\varepsilon = \frac{H_\varepsilon(x, \nabla u^\varepsilon)}{\|\nabla u^\varepsilon\|_{L^{m^*}(\Omega)}} - \frac{\varepsilon}{|\Omega|} \frac{\int_\Omega u^\varepsilon}{\|\nabla u^\varepsilon\|_{L^{m^*}(\Omega)}} \qquad \text{in } \Omega, \tag{2.36}$$

where $\hat{a}_\varepsilon(x, p)$ is defined now as

$$\hat{a}_\varepsilon(x, p) := \frac{a(x, \|\nabla u^\varepsilon\|_{L^{m^*}(\Omega)} p)}{\|\nabla u^\varepsilon\|_{L^{m^*}(\Omega)}} .$$

Compared to Theorem 2.5, we now have that the right-hand side is bounded and equi-integrable in $L^m(\Omega)$, hence w_ε satisfies

$$\begin{cases} w_\varepsilon \text{ is bounded in } W^{1,m^*}(\Omega), \\ -\operatorname{div}(\hat{a}_\varepsilon(x, \nabla w_\varepsilon)) + \varepsilon\, w_\varepsilon = h_\varepsilon \end{cases} \tag{2.37}$$

for some h_ε which is bounded and equi-integrable in $L^m(\Omega)$. One can prove now that w_ε is actually strongly compact in $W^{1,m^*}(\Omega)$. The argument needs a modification of what is done in Lemma 2.3: first of all, by only using that h_ε is weakly converging in $L^1(\Omega)$, one can prove (see e.g., [14, 18]) that

$$\lim_{\varepsilon \to 0} \int_\Omega (\hat{a}_\varepsilon(x, \nabla T_k(w_\varepsilon)) - \hat{a}_\varepsilon(x, \nabla T_k(w)))\, \nabla(T_k(w_\varepsilon) - T_k(w)) = 0$$

which yields, thanks to (2.21),

$$T_k(w_\varepsilon) \to T_k(w) \quad \text{in } H^1(\Omega), \text{ for every } k > 0. \tag{2.38}$$

To estimate $G_k(w_\varepsilon) = w_\varepsilon - T_k(w_\varepsilon)$, we take

$$[(\sigma + |G_k(w_\varepsilon)|)^\gamma - \sigma^\gamma]\operatorname{sign}(w_\varepsilon)$$

as test function in (2.37), for $\sigma, \gamma > 0$. We get

$$\gamma \int_\Omega \hat{a}_\varepsilon(x, \nabla w_\varepsilon) \nabla G_k(w_\varepsilon)\, (\sigma + |G_k(w_\varepsilon)|)^{\gamma - 1} \le \int_\Omega h_\varepsilon [(\sigma + |G_k(w_\varepsilon)|)^\gamma - \sigma^\gamma]\operatorname{sign}(w_\varepsilon)$$

which implies, using (2.23) and the definition of $\hat{a}$,

$$\begin{aligned} \gamma\, \alpha \int_\Omega |\nabla G_k(w_\varepsilon)|^2\, (\sigma + |G_k(w_\varepsilon)|)^{\gamma-1} \le & \int_\Omega h_\varepsilon |(\sigma + |G_k(w_\varepsilon)|)^\gamma - \sigma^\gamma| \\ & + \frac{\gamma}{\|\nabla u^\varepsilon\|^2_{L^{m^*}(\Omega)}} \int_\Omega \tilde{k}(x)\, (\sigma + |G_k(w_\varepsilon)|)^{\gamma-1} . \end{aligned} \tag{2.39}$$

We choose $\gamma = \frac{N(m-1)}{N-2m}$, so that $|w_\varepsilon|^\gamma = |w_\varepsilon|^{\frac{m^{**}}{m'}}$ is bounded in $L^{m'}(\Omega)$ due to the bound of w_ε in $W^{1,m^*}(\Omega)$ and Sobolev embedding. Moreover, we have $\gamma < 1$ since $m < \frac{2N}{N+2}$. Taking for instance $\sigma = 1$, the previous inequality implies

$$\gamma\, \alpha \int_\Omega |\nabla G_k(w_\varepsilon)|^2\, (1 + |G_k(w_\varepsilon)|)^{\gamma-1} \le 2 \int_{\{|w_\varepsilon| > k\}} |h_\varepsilon|(1 + |w_\varepsilon|)^\gamma + \frac{\gamma}{\|\nabla u^\varepsilon\|^2_{L^{m^*}(\Omega)}} \|\tilde{k}\|_{L^1(\Omega)}$$

$$\leq C \left(\int_{\{|w_\varepsilon|>k\}} |h_\varepsilon|^m \right)^{\frac{1}{m}} + \frac{C}{\|\nabla u^\varepsilon\|^2_{L^{m^*}(\Omega)}} .$$

Due to the equi-integrability of h_ε in $L^m(\Omega)$ we deduce

$$\int_\Omega |\nabla G_k(w_\varepsilon)|^2 \, (1 + |G_k(w_\varepsilon)|)^{\gamma-1} \leq \delta_k + \frac{C}{\|\nabla u^\varepsilon\|^2_{L^{m^*}(\Omega)}} \tag{2.40}$$

where δ_k denotes a quantity which vanishes as $k \to \infty$ uniformly with respect to ε. From Hölder inequality, since $m^* < 2$ we have

$$\int_\Omega |\nabla G_k(w_\varepsilon)|^{m^*} \leq \left(\int_\Omega |\nabla G_k(w_\varepsilon)|^2 \, (1 + |G_k(w_\varepsilon)|)^{\gamma-1} \right)^{\frac{m^*}{2}} \left(\int_\Omega (1 + |G_k(w_\varepsilon)|)^{\frac{(1-\gamma)m^*}{2-m^*}} \right)^{1-\frac{m^*}{2}} .$$

The precise value of γ yields $\frac{(1-\gamma)m^*}{2-m^*} = m^{**}$, so last term is bounded and from (2.40) we deduce

$$\int_\Omega |\nabla G_k(w_\varepsilon)|^{m^*} \leq C \, \delta_k^{\frac{m^*}{2}} + \frac{C}{\|\nabla u^\varepsilon\|^{m^*}_{L^{m^*}(\Omega)}} .$$

Putting together this information with (2.38), it follows that $w_\varepsilon \to w$ strongly in $W^{1,m^*}(\Omega)$. In particular, up to a subsequence, $\nabla w_\varepsilon \to \nabla w$ almost everywhere in Ω. This implies that $\frac{H_\varepsilon(x,\nabla u^\varepsilon)}{\|\nabla u^\varepsilon\|_{L^{m^*}(\Omega)}}$ weakly converges to some function $B(x)$ in $L^m(\Omega)$, and, because of (2.31) and the a.e. convergence of ∇w_ε, it follows that B satisfies (2.2). If the vector filed $\hat{a}(x, p)$ is defined as in (2.30), then by passing to the limit we obtain a function w such that

$$\begin{cases} w \in W^{1,m^*}(\Omega), \\ -\mathrm{div}(\hat{a}(x, \nabla w)) + \lambda = B(x) \quad \text{with } B(x) \in L^m(\Omega) \text{ satisfying (2.2).} \end{cases}$$

Now we get a contradiction by showing that $w = 0$. This needs a slight refinement of the argument of Lemma 2.1; indeed, w does not belong a priori to $H^1(\Omega)$. However, from (2.39) one obtains, letting first $\varepsilon \to 0$ and then $\sigma \to 0$, the inequality

$$\gamma \, \alpha \int_\Omega |\nabla G_k(w)|^2 \, |G_k(w)|^{\gamma-1} \leq \int_\Omega [b(x) \, |\nabla w| + |\lambda|] \, |G_k(w)|^\gamma . \tag{2.41}$$

This replaces now the starting inequality (2.3) of Lemma 2.1; using that

$$\begin{aligned} \int_\Omega |\nabla G_k(w)|^{m^*} &\leq \left(\int_\Omega |\nabla G_k(w)|^2 \, |G_k(w)|^{\gamma-1} \right)^{\frac{m^*}{2}} \left(\int_\Omega |G_k(w)|^{\frac{(1-\gamma)m^*}{2-m^*}} \right)^{1-\frac{m^*}{2}} \\ &\leq C \left(\int_\Omega [b(x) \, |\nabla G_k(w)| + |\lambda|] \, |G_k(w)|^\gamma \right)^{\frac{m^*}{2}} \left(\int_\Omega |G_k(w)|^{\frac{(1-\gamma)m^*}{2-m^*}} \right)^{1-\frac{m^*}{2}} \end{aligned}$$

and the equalities $\gamma \, m' = m^{**}$ and $\frac{(1-\gamma)m^*}{2-m^*} = m^{**}$, one can prove with similar steps as in Lemma 2.1 that $w \in L^\infty(\Omega)$. Once w is proved to be bounded, then it has also finite energy and the conclusion of Lemma 2.1 applies. This completes the contradiction argument and concludes the proof of the a priori estimate (2.35). Once u^ε is proved to be bounded in $W^{1,m^*}(\Omega)$, we repeat the same argument to deduce that $u^\varepsilon - \fint_\Omega u^\varepsilon$ is relatively compact and converges, up to subsequences, to some $u \in W^{1,m^*}(\Omega)$. Due to (2.31), this implies that $H(x, \nabla u^\varepsilon) \to H(x, \nabla u)$ in $L^m(\Omega)$. By well-known results (see e.g., [5, 18]), the limit function u is a weak and a renormalized solution of the limiting problem (2.25).

Let us now sketch the uniqueness part. Assume that λ_1, u_1 and λ_2, u_2 are (renormalized) solutions of the problem. Then it is possible to prove, using renormalization arguments (e.g., the arguments used for the obtention of inequality (3.14) in [19, Theorem 3.1]), that u_1, u_2 satisfy

$$\begin{aligned}\gamma \int_\Omega (a(x, \nabla u_1) - a(x, \nabla u_2))\nabla G_k(u_1 - u_2)\,(\sigma + |G_k(w)|)^{\gamma-1} \\ \leq \int_\Omega |H(x, \nabla u_1) - H(x, \nabla u_2)|\,|(\sigma + |G_k(w)|)^\gamma - \sigma^\gamma| + |\lambda_1 - \lambda_2| \int_\Omega |(\sigma + |G_k(w)|)^\gamma - \sigma^\gamma|\end{aligned}$$

where $w = u_1 - u_2$ and $\gamma = \frac{N(m-1)}{N-2m}$ as before. The reader may check that all terms here are well defined because of the precise value of γ and since $u_1, u_2 \in W^{1,m^*}(\Omega)$. Using (2.21) and (1.6) it follows that

$$\begin{aligned}\gamma\,\alpha \int_\Omega |\nabla G_k(w)|^2\,(\sigma + |G_k(w)|)^{\gamma-1} \leq \int_\Omega b(x)\,|\nabla w|\,|(\sigma + |G_k(w)|)^\gamma - \sigma^\gamma| \\ + |\lambda_1 - \lambda_2| \int_\Omega |(\sigma + |G_k(w)|)^\gamma - \sigma^\gamma|\,.\end{aligned}$$

By letting $\sigma \to 0$ one gets inequality (2.41) with $\lambda = \lambda_1 - \lambda_2$. This inequality allows us to show that $w \in L^\infty(\Omega)$ as we said before. Now, let us assume for instance that $\lambda_1 > \lambda_2$; since $u_1 - u_2 \in L^\infty(\Omega)$, then $\varepsilon(u_1 - u_2) < \lambda_1 - \lambda_2$ for ε sufficiently small. Therefore, we get that u_1, u_2 are, respectively, a renormalized sub and super solution of the same equation

$$-\mathrm{div}(a(x, \nabla u)) + \varepsilon u = H(x, \nabla u) + \varepsilon u_2 - \lambda_2\,.$$

This implies (e.g., as in [19, Theorem 3.1], or in [6]) that $u_1 \leq u_2$, which yields a contradiction, because u_1, u_2 are solutions up to addition of any constant. We notice here that, if u is a renormalized solution of (2.25), then it can be easily proved that $u + c$ is also a renormalized solution, whenever $c \in \mathbb{R}$. The contradiction obtained proves that $\lambda_1 = \lambda_2$. Then, we replace u_1 with $u_1 - \mathrm{med}\,(u_1 - u_2)$; now again from inequality (2.41) (obtained before for $w = u_1 - u_2 - \mathrm{med}\,(u_1 - u_2)$ and $\lambda = \lambda_1 - \lambda_2 = 0$) we can deduce that $w = 0$, i.e., $u_1 - u_2 = \mathrm{med}\,(u_1 - u_2)$. Hence the two solutions differ by a constant. □

Finally, we conclude by observing that a similar result can be proved in the limiting case $m = 1$, up to requiring that $b(x) \in L^q(\Omega)$ for some $q > N$. More precisely, we have the following result, whose proof can be done with similar arguments as indicated above.

Theorem 2.10. *Let $N \geq 2$. Assume that $a(x, p)$ satisfies (2.21)–(2.24) and that $H(x, p)$ satisfies (2.31) with $m = 1$ and $b(x) \in L^q(\Omega)$ for some $q > N$. Then there exists a constant $\lambda \in \mathbb{R}$ and a renormalized solution u of (2.25). In addition, if (1.6) holds true with $b(x) \in L^q(\Omega)$ for some $q > N$, then λ is unique and u is unique up to addition of a constant.*

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Conflict of interest

The authors declare no conflict of interest.

References

1. Arisawa M, Lions PL (1998) On ergodic stochastic control. *Commun Part Diff Eq* 23: 2187–2217.
2. Bensoussan A, Frehse J (1987) On Bellman equation of ergodic type with quadratic growth Hamiltonian, In: *Contributions to Modern Calculus of Variations*, Wiley, 13–26.
3. Bardi M, Capuzzo-Dolcetta I (1997) *Optimal Control and Viscosity Solutions of Hamilton-Jacobi-Bellman Equations*, Boston: Birkhäuser.
4. Bensoussan A, Lions JL, Papanicolaou G (1978) *Asymptotic Analysis for Periodic Structures*, Amsterdam: North-Holland.
5. Betta F, Mercaldo A, Murat F, et al. (2003) Existence of renormalized solutions to nonlinear elliptic equations with a lower-order term and right-hand side a measure. *J Math Pure Appl* 82: 90–124.
6. Betta F, Mercaldo A, Murat F, et al. (2005) Uniqueness results for nonlinear elliptic equations with a lower order term. *Nonlinear Anal* 63: 153–170.
7. Boccardo L, Gallouët T (1992) Nonlinear elliptic equations with right hand side measures. *Commun Part Diff Eq* 17: 641–655.
8. Capuzzo Dolcetta I, Menaldi JL (1988) Asymptotic behavior of the first order obstacle problem. *J Differ Equations* 75: 303–328.
9. Davini A, Fathi A, Iturriaga R, et al. (2016) Convergence of the solutions of the discounted equation. *Invent Math* 206: 29–55.
10. Droniou J, Vazquez JL (2009) Noncoercive convection-diffusion elliptic problems with Neumann boundary conditions. *Calc Var* 34: 413–434.
11. Gilbarg D, Trudinger N (1983) *Partial Differential Equations of Second Order*, 2 Eds., Berlin-New-York: Springer–Verlag.
12. Gimbert F (1985) Problèmes de Neumann quasilineaires. *J Funct Anal* 68: 65–72.
13. Ishii H, Mitake H, Tran HV (2017) The vanishing discount problem and viscosity Mather measures. Part 1: the problem on a torus. *J Math Pure Appl* 108: 125–149.
14. Leone C, Porretta A (1998) Entropy solutions for nonlinear elliptic equations in L^1. *Nonlinear Anal Theor* 32: 325–334.
15. Lions PL (1985) Quelques remarques sur les problèmes elliptiques quasilinéaires du second ordre. *J Anal Math* 45: 234–254.
16. Lions PL (1996) *Mathematical Topics in Fluid Mechanics. Vol. 1. Incompressible Models*, New York: The Clarendon Press.
17. Mitake H, Tran HV (2017) Selection problems for a discounted degenerate viscous Hamilton-Jacobi equation. *Adv Math* 306: 684–703.

18. Murat F, Soluciones renormalizadas de EDP elipticas no lineales, *Preprint 93023 of Laboratoire d'Analyse Numérique, Université Paris VI*, 1993. Available from: `http://archive.schools.cimpa.info/anciensite/NotesCours/PDF/2009/Alexandrie_Murat_2.pdf`.

19. Porretta A (2008) On the comparison principle for p-Laplace type operators with first order terms, In: *On the Notion of Solutions to Nonlinear Elliptic Problems: Results and Developments*, Quaderni di Matematica 23, Dept. Math. Seconda Univ. Napoli, Caserta, 459–497.

20. Porretta A, Elliptic equations with first-order terms, notes of the CIMPA School, Alexandria, 2009. Available from: `http://archive.schools.cimpa.info/anciensite/NotesCours/PDF/2009/Alexandrie_Porretta.pdf`.

21. Stampacchia G (1965) Le problème de Dirichlet pour les équations elliptiques du second ordre à coefficients discontinus, *Ann I Fourier* 15: 189–258.

22. Ziemer WP (1989) *Weakly Differentiable Functions*, New York: Springer-Verlag.