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Research article

Weak maximum principle for Dirichlet problems with convection or drift terms†

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Abstract: In this paper, dedicated to Italo Capuzzo Dolcetta, a maximum principle for some linear boundary value problems with lower order terms of order one is proved: the aim of this paper is the proof that the solutions can be zero at most in a zero measure set, if we assume that the data are greater or equal than zero (but not identically zero).

Keywords: weak maximum principle; Dirichlet problems with convection terms; Dirichlet problems with drift terms; very singular terms; very singular solutions

1. Introduction

During the period 1968–1970, in the "Dipartimento di Matematica dell'Università di Roma", I recall two persons: Guido Stampacchia, my teacher for the last two university years, and Italo Capuzzo Dolcetta, to whom this paper is dedicated.

A common point connecting them is the maximum principle: one of the scientific interests of Italo (see [1–6],) and one of the subjects of the courses taught by Guido Stampacchia, in the classical framework in the first course and in the Sobolev framework (see [15]) in the second one. For recent results on maximum principle see also [13, 16].

In this paper, I study the positivity, up to a zero measure set, of the solutions of Dirichlet problems having a first order term (either of convection type or of drift type).

2. Setting

Let Ω be a bounded, open subset of $\mathbb{R}^N$, $N > 2$ and $M : \Omega \to \mathbb{R}^{N^2}$, be a bounded and measurable matrix such that

$$\alpha|\xi|^2 \le M(x)\xi \cdot \xi, \quad |M(x)| \le \beta, \quad \text{a.e. } x \in \Omega, \quad \forall\ \xi \in \mathbb{R}^N. \tag{2.1}$$

We assume that $E(x)$ is a vector field and $f(x)$, $g(x)$ are functions such that

$$E \in (L^N(\Omega))^N, \tag{2.2}$$

$$f,\ g \in L^m(\Omega), \quad m > 1, \tag{2.3}$$

and we consider the following boundary value problems with a lower order convection term or with a lower order drift term:

$$\begin{cases} -\text{div}(M(x)\nabla\psi) = -\text{div}(\psi\, E(x)) + g(x) & \text{in } \Omega, \\ \psi = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.4}$$

$$\begin{cases} -\text{div}(M(x)\nabla u) = E(x) \cdot \nabla u + f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.5}$$

We will also consider the two above boundary value problems if a zero order term is present; that is

$$\begin{cases} -\text{div}(M(x)\nabla\Psi) + \Psi = -\text{div}(\Psi\, E(x)) + g(x) & \text{in } \Omega, \\ \Psi = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.6}$$

$$\begin{cases} -\text{div}(M(x)\nabla z) + z = E(x) \cdot \nabla z + f(x) & \text{in } \Omega, \\ z = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.7}$$

For the existence and properties of solutions we refer to [7] and to the references therein (see also [11], [14]) for ψ, Ψ, to [9] (and to the references therein) for u and to [10] for z; see also the references of [7–10] for bibliographic informations. We only point out that in [10] the existence of a weak, bounded solution z of (2.7) is proved assuming on E no more than $E \in (L^2(\Omega))^N$, instead of (2.2).

2.1. *Positivity of solutions*

We recall that positivity (that is, greater or equal than zero, but not identically zero) of solutions of the above boundary value problems holds in the case of data f or g positive (that is, greater or equal than zero, but not identically zero): see [7, 9].

2.2. *Positivity of solutions up a zero measure set*

The aim of this paper is the proof that the solutions of the above boundary value problems can be zero at most in a zero measure set, if we assume that the data f or g are greater or equal than zero, but not identically zero.

3. Dirichlet problems with convection terms

This section deals with the boundary value problem (2.4). Of course the solution ψ (or Ψ) is understood in weak (or distributional) sense.

3.1. $|E| \in L^N$

If we assume (2.1), (2.2), (2.3) with $m \geq \frac{2N}{N+2}$ (or with $1 < m < \frac{2N}{N+2}$), in [7], it is proved

i the existence of $\psi \in W_0^{1,2}(\Omega)$ (or $\psi \in W_0^{1,m^*}(\Omega)$) such

$$\int_\Omega M(x)\nabla\psi\nabla v = \int_\Omega \psi(E(x)\cdot\nabla v) + \int_\Omega g(x)\, v(x), \tag{3.1}$$

for every $v \in W_0^{1,2}(\Omega)$ (or v smooth);

ii moreover it is proved that $\psi \geq 0$, if $g \geq 0$ (of course not zero a.e.);

iii

$$\int_{\{k\leq|\psi|\}} \frac{|\nabla\psi|^2}{(1+|\psi|)^2} \leq \frac{2}{\alpha^2}\int_{\{k\leq|\psi|\}} |E|^2 + \frac{2}{\alpha}\int_{\{k\leq|\psi|\}} |g|, \quad k \geq 0; \tag{3.2}$$

iv

$$\int_\Omega |\nabla T_k(\psi)|^2 \leq \frac{k^2}{\alpha^2}\int_\Omega |E|^2 + k\,\frac{2}{\alpha}\int_\Omega |g|, \quad k \geq 0, \tag{3.3}$$

where T_k is the Stampacchia truncation:

$$T_k(s) = \begin{cases} s, & \text{if } |s| \leq k, \\ k\frac{s}{|s|}, & \text{if } |s| > k. \end{cases}$$

The next theorem improves the statement (ii); the proof hinges on the approach of [12].

Theorem 3.1. *Assume (2.1), (2.2), (2.3) with* $m \geq \frac{2N}{N+2}$ *and* $g \geq 0$ *(of course not zero a.e.). Then the solution* $\psi \in W_0^{1,2}(\Omega)$ *is positive and it is zero at most on a set of zero Lebesgue measure.*

Proof. In subsection 2.1 we recalled that $\psi(x) \geq 0$; thus we can use

$$v = \frac{\phi^2}{h+\psi}, \quad 0 \leq \phi \leq 1, \ \phi \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega), \ h \in \mathbb{R}^+,$$

as test function in (3.1). Then we have

$$\left[\begin{array}{l} 2\displaystyle\int_\Omega M(x)\nabla\psi\nabla\phi\frac{\phi}{h+\psi} + 2\int_\Omega \frac{\psi}{h+\psi}\,E(x)\nabla\phi\,\phi - \int_\Omega \frac{\psi}{(h+\psi)}\,E(x)\nabla\psi\frac{\phi^2}{(h+\psi)} \\ = \displaystyle\int_\Omega M(x)\nabla\psi\nabla\psi\frac{\phi^2}{(h+\psi)^2} + \int_\Omega g(x)\frac{\phi^2}{h+\psi} \end{array}\right.$$

so that, using (2.1),

$$\left[\begin{array}{l} 2\beta\displaystyle\int_\Omega |\nabla\phi|\frac{|\nabla\psi|}{h+\psi}\,\phi + 2\int_\Omega \phi|E(x)||\nabla\phi| + \int_\Omega \phi|E(x)|\frac{|\nabla\psi|}{(h+\psi)}\,\phi \\ \geq \alpha\displaystyle\int_\Omega \frac{|\nabla\psi|^2}{(h+\psi)^2}\,\phi^2 + \int_\Omega g(x)\frac{\phi^2}{h+\psi}. \end{array}\right.$$

Now we use twice the Young inequality (with $0 < B < \frac{\alpha}{2}$) and we deduce that (recalling that $0 \le \phi \le 1$)

$$\left[\begin{array}{l} \frac{1}{4B}4\beta^2 \int_\Omega |\nabla \phi|^2 + \int_\Omega |\nabla \phi|^2 + \int_\Omega |E(x)|^2 + \frac{1}{4B}\int_\Omega |E(x)|^2 \\ \ge (\alpha - 2B)\int_\Omega \frac{|\nabla\psi|^2}{(h+\psi)^2}\,\phi^2 + \int_\Omega g(x)\frac{\phi^2}{h+\psi} \end{array}\right.$$

which implies, thanks to the positivity of $g(x)$, that

$$\left[\begin{array}{l} \frac{1}{4B}4\beta^2 \int_\Omega |\nabla \phi|^2 + \int_\Omega |\nabla \phi|^2 + \int_\Omega |E(x)|^2 + \frac{1}{4B}\int_\Omega |E(x)|^2 \\ \ge (\alpha - 2B)\int_\Omega \frac{|\nabla\psi|^2}{(h+\psi)^2}\,\phi^2 \end{array}\right.$$

The last inequality implies that

$$\Big(\frac{\beta^2}{B}+1\Big)\int_\Omega |\nabla \phi|^2 + \Big(\frac{1}{4B}+1\Big)\int_\Omega |E(x)|^2 \ge (\alpha-2B)\int_\Omega \frac{|\nabla\psi|^2}{(h+\psi)^2}\,\phi^2, \tag{3.4}$$

that is

$$\Big(\frac{\beta^2}{B}+1\Big)\int_\Omega |\nabla \phi|^2 + \Big(\frac{1}{4B}+1\Big)\int_\Omega |E(x)|^2 \ge (\alpha-2B)\int_\Omega \Big|\nabla \log\Big(1+\frac{\psi}{h}\Big)\Big|^2\,\phi^2. \tag{3.5}$$

By contradiction, we assume that $\psi(x) = 0$ on a subset of positive measure of Ω.

Let $\omega \subset\subset \Omega$ be such that $Z = \omega \cap \{x :\ \psi(x) = 0\}$ has positive measure. Let $\phi \equiv 1$ on ω. Then (3.5) becomes

$$\Big(\frac{\beta^2}{B}+1\Big)\int_\Omega |\nabla \phi|^2 + \Big(\frac{1}{4B}+1\Big)\int_\Omega |E(x)|^2 \ge (\alpha-2B)\int_\omega \Big|\nabla \log\Big(1+\frac{\psi}{h}\Big)\Big|^2$$

Since $\psi(x) = 0$ on a subset of positive measure of ω, we can use Poincaré inequality in ω so that, for $t \in \mathbb{R}^+$,

$$\left[\begin{array}{l} \Big(\frac{\beta^2}{B}+2\Big)\int_\Omega |\nabla \phi|^2 + \Big(\frac{1}{4B}+1\Big)\int_\Omega |E(x)|^2 \\ \ge\ C_P\,(\alpha-2B)\int_\omega \Big[\log\Big(1+\frac{\psi}{h}\Big)\Big]^2 \\ \ge (\alpha-2B)C_P\Big[\log\Big(1+\frac{t}{h}\Big)\Big]^2 \int_{\omega\cap\{\psi>t\}} 1 \end{array}\right.$$

which implies

$$0 = \lim_{h\to 0} \frac{C_1 + \int_\Omega |E(x)|^2}{C_2\Big[\log\Big(1+\frac{t}{h}\Big)\Big]^2} \ge \int_{\omega\cap\{\psi>t\}} 1. \tag{3.6}$$

Thus we proved that

$$\psi \equiv 0 \ \text{ in } \ \omega. \tag{3.7}$$

Since ω is an arbitrary subset of Ω , we conclude that $\psi = 0$ a.e. in Ω and then $g(x)$ should be equal to zero a.e. in Ω; which is a contradiction.

Remark 3.2. Note that in (3.6) we only need $E \in (L^2(\Omega))^N$, which is a weaker demand with respect to assumption (2.2).

Now we discuss the case of infinite energy solutions, which appears if we assume $1 < m < \frac{2N}{N+2}$ in (2.3).

Remark 3.3. Since the existence of a solution ψ is proved in [7] as limit of a sequence $\{\psi_n\}$ (every $\psi_n \geq 0$) of solutions of approximating problems, a possible approach is to repeat the proof of the previous theorem on the sequence $\{\psi_n\}$ in order to prove that ψ_n satisfies inequality (3.4).

In [7], not only the estimates (3.2) and (3.3) are proved, but also the following estimates on the sequence $\{\psi_n\}$:

$$\int_{\{k\leq\psi_n\}} \frac{|\nabla\psi_n|^2}{(1+\psi_n)^2} \leq \frac{2}{\alpha^2}\int_{\{k\leq\psi_n\}} |E|^2 + \frac{2}{\alpha}\int_{\{k\leq\psi_n\}} g \leq C_0, \tag{3.8}$$

where

$$C_0 = \frac{2}{\alpha^2}\int_\Omega |E|^2 + \frac{2}{\alpha}\int_\Omega g,$$

$$\int_\Omega |\nabla T_k(\psi_n)|^2 \leq \frac{k^2}{\alpha^2}\int_\Omega |E|^2 + k\,\frac{2}{\alpha}\int_\Omega g, \quad k \geq 0. \tag{3.9}$$

The inequality (3.8), for $k = 0$, gives

$$\mathbb{S}\Big[\int_\Omega [\log(1+\psi_n)]^{2^*}\Big]^{\frac{2}{2^*}} \leq \int_\Omega \frac{|\nabla\psi_n|^2}{(1+\psi_n)^2} \leq \frac{2}{\alpha^2}\int_\Omega |E|^2 + \frac{2}{\alpha}\int_\Omega g, \tag{3.10}$$

which implies that the sequence $\{\log(1+\psi_n)\}$ is bounded in $W_0^{1,2}(\Omega)$. Then there exist a positive function $w \in W_0^{1,2}(\Omega)$ and a subsequence, still denoted by $\{\psi_n\}$, such that $\log(1+\psi_n)$ converges a.e. to w. Thus $\psi_n(x) \to e^{w(x)} - 1$ a.e. Define $\psi(x) = e^{w(x)} - 1$.

Similarly, we prove that

$$\forall\, k > 0, \text{ the sequence } \{T_k(\psi_n)\} \text{ converges weakly to } T_k(\psi) \text{ in } W_0^{1,2}(\Omega). \tag{3.11}$$

Now we prove that

$$\text{the sequence } \{\tfrac{\nabla\psi_n}{1+\psi_n}\} \text{ converges weakly to } \tfrac{\nabla\psi}{1+\psi} \text{ in } L^2. \tag{3.12}$$

Indeed, for every $\Phi \in (L^2(\Omega))^N$,

$$\int_\Omega \Big[\frac{\nabla\psi_n\,\Phi}{1+\psi_n} - \frac{\nabla\,\psi\,\Phi}{1+\psi}\Big]$$

$$= \int_\Omega \Big[\frac{\nabla T_k(\psi_n)\,\Phi}{1+\psi_n} - \frac{\nabla\,T_k(\psi)\,\Phi}{1+\psi}\Big] + \int_{\{k\leq\psi_n\}} \frac{\nabla\psi_n\,\Phi}{1+\psi_n} - \int_{\{k\leq\psi\}} \frac{\nabla\psi\,\Phi}{1+\psi}$$

Observe that, in the first integral, $\nabla T_k(\psi_n)$ converges weakly in L^2 and $\frac{\Phi}{1+\psi_n}$ converges strongly in L^2; moreover

$$\Big|\int_{\{k\leq\psi_n\}} \frac{\nabla\psi_n\,\Phi}{1+\psi_n}\Big| \leq \Big[\int_\Omega \frac{|\nabla\psi_n|^2}{(1+\psi_n)^2}\Big]^{\frac{1}{2}}\Big[\int_{\{k\leq\psi_n\}} |\Phi|^2\Big]^{\frac{1}{2}} \leq C_0\Big[\int_{\{k\leq\psi_n\}} |\Phi|^2\Big]^{\frac{1}{2}}$$

and

$$\Big|\int_\Omega \frac{\nabla\psi\,\Phi}{1+\psi}\Big| \leq \Big[\int_\Omega \frac{|\nabla\psi_n|^2}{(1+\psi)^2}\Big]^{\frac{1}{2}}\Big[\int_{\{k\leq\psi\}} |\Phi|^2\Big]^{\frac{1}{2}} \leq \sqrt{C_0}\Big[\int_{\{k\leq\psi\}} |\Phi|^2\Big]^{\frac{1}{2}}$$

The estimate (3.10) says that the last integral is uniformly (with respect to n) small for k large. Thus (3.12) is proved, so that we pass to the limit, by weak L^2 lower semicontinuity in the inequality (3.4) for the sequence $\{\psi_n\}$, that is

$$\Big(\frac{\beta^2}{B}+1\Big)\int_\Omega |\nabla\phi|^2+\Big(\frac{1}{4B}+1\Big)\int_\Omega |E(x)|^2 \geq (\alpha-2B)\int_\Omega \frac{|\nabla\psi_n|^2}{(h+\psi_n)^2}\,\phi^2,$$

and we have

$$\Big(\frac{\beta^2}{B}+1\Big)\int_\Omega |\nabla\phi|^2+\Big(\frac{1}{4B}+1\Big)\int_\Omega |E(x)|^2 \geq (\alpha-2B)\int_\Omega \frac{|\nabla\psi|^2}{(h+\psi)^2}\,\phi^2, \tag{3.13}$$

Thus it is possible to prove the following theorems, where ψ is a solution obtained as said above.

Theorem 3.4. *Assume (2.1), (2.2), (2.3) with* $1<m<\frac{2N}{N+2}$ *and* $g\geq 0$ *(of course not zero a.e.). Then there exists a solution* $\psi\in W_0^{1,m^*}(\Omega)$ *which is positive and it is zero at most on a set of zero measure.*

In [8], it is proved the existence of a very weak solution (entropy solution) if the data are very singular (e.g., $E\in L^2$). Even for this solution it is possible to prove the maximum principle, thanks to the above discussion of Remark 3.3, as stated in the following theorem.

Theorem 3.5. *Assume (2.1),* $E\in(L^2(\Omega))^N$, $g\in L^1(\Omega)$, $g\geq 0$ *(of course not zero a.e.). Then the entropy solution* ψ *is positive and it is zero at most on a set of zero measure.*

4. Dirichlet problems with drift terms

This section deals with the boundary value problem (2.5). Of course the solution u (or z) is understood in weak (or distributional) sense.

4.1. $|E|\in L^N$

If we assume (2.1), (2.2), (2.3) with $m\geq\frac{2N}{N+2}$ (or with $1<m<\frac{2N}{N+2}$), in [9], it is proved

1). the existence of $u\in W_0^{1,2}(\Omega)$ (or $u\in W_0^{1,m^*}(\Omega)$) such

$$\int_\Omega M(x)\nabla u\nabla v=\int_\Omega v(E(x)\cdot\nabla u)+\int_\Omega f(x)\,v(x), \tag{4.1}$$

for every $v\in W_0^{1,2}(\Omega)$ (or v smooth);

2). moreover $u\geq 0$, if $f\geq 0$ (of course not zero a.e.)

Theorem 4.1. *Assume (2.1), (2.2), (2.3) with* $m\geq\frac{2N}{N+2}$ *and* $f\geq 0$ *(of course not zero a.e.). Then* $u(x)\geq 0$ *and it is zero at most on a set of zero measure.*

Proof. As in the proof of Theorem 3.1, we can use

$$v=\frac{\phi^2}{h+u},\quad 0\leq\phi\leq 1,\ \ \phi\in W_0^{1,2}(\Omega)\cap L^\infty(\Omega),\ \ h\in\mathbb{R}^+,$$

as test function in (4.1). Then we have

$$\left[\begin{array}{l} 2\displaystyle\int_\Omega M(x)\nabla u\nabla\phi\frac{\phi}{h+u}+\int_\Omega \phi\, E(x)\frac{\nabla u}{(h+u)}\,\phi \\ =\displaystyle\int_\Omega M(x)\nabla u\nabla u\frac{\phi^2}{(h+u)^2}+\int_\Omega f\frac{\phi^2}{h+u}\end{array}\right.$$

and, thanks to the positivity to the fact that $f(x) \geq 0$,

$$2\beta\int_\Omega |\nabla\,\phi|\frac{|\nabla u|}{h+u}\,\phi+\int_\Omega \phi|E(x)|\frac{|\nabla u|}{(h+u)}\,\phi \geq \alpha\int_\Omega \frac{|\nabla u|^2}{(h+u)^2}\,\phi^2.$$

The use of the Young inequality twice gives ($0 < B < \frac{\alpha}{2}$)

$$\frac{1}{4B}4\beta^2\int_\Omega |\nabla\,\phi|^2+\frac{1}{4B}\int_\Omega \phi^2|E(x)|^2 \geq (\alpha-2B)\int_\Omega \frac{|\nabla u|^2}{(h+u)^2}\,\phi^2, \tag{4.2}$$

which implies (3.5) (for the solution u instead of ψ), so that we conclude as in the proof of Theorem 3.1.

If we repeat the discussion in Remark 3.3, it is possible to state the following theorem (similar to Theorem 3.4).

Theorem 4.2. *Assume (2.1), (2.2), (2.3) with* $1 < m < \frac{2N}{N+2}$ *and* $f \geq 0$ *(of course not zero a.e.). Then there exists a solution* $u \in W_0^{1,m^*}(\Omega)$, $u(x) \geq 0$, *which is zero at most on a set of zero measure.*

4.1.1. A duality approach

Here we give a different proof of Theorem 4.1. In this section we studied the Dirichlet problem

$$u \in W_0^{1,2}(\Omega) : -\mathrm{div}(M(x)\nabla u) = E(x)\cdot\nabla u + f(x),$$

whereas, in the previous section, the Dirichlet problems studied include

$$\psi \in W_0^{1,2}(\Omega) : -\mathrm{div}(M(x)\nabla\psi) = -\mathrm{div}(\psi\, E(x)) + \chi_{\{u(x)=0\}}.$$

If the measure $\{u(x) = 0\}$ is zero, then $\chi_{\{u(x)=0\}} = 0$ and in [7] is proved that $\psi = 0$.

Thus (by contradiction) assume that the measure of $\{u(x) = 0\}$ is strictly positive. Then the duality (i.e., use ψ as test function in the first problem and u in the second problem) gives

$$\int_\Omega \psi(x)\, f(x) = \int_\Omega u(x)\chi_{\{u(x)=0\}} = 0,$$

that is

$$0 = \int_\Omega \psi(x)\, f(x).$$

But the result of Theorem 3.1 says that, if $\chi_{\{u(x)=0\}} \geq 0$ (of course not zero a.e.), then $\psi(x) \geq 0$ and it is zero at most on a set of zero measure: this property and the assumption on $f(x)$ yield $\int_\Omega \psi(x)\, f(x) > 0$: a contradiction. Thus the measure of $\{u(x) = 0\}$ is zero.

4.2. $|E| \in L^2$

Remark 4.3. If we assume (2.1), $E \in (L^2(\Omega))^N$, $f \in L^\infty(\Omega)$, in [10], it is proved the existence of a weak, bounded solution z of (2.7), that is

$$\int_\Omega M(x)\nabla z \nabla v + \int_\Omega z(x)\, v(x) = \int_\Omega v(E(x)\cdot \nabla z) + \int_\Omega f(x)\, v(x), \tag{4.3}$$

for every $v \in W_0^{1,2}(\Omega)$.

If we consider the boundary value problem (2.7), (in place of (2.5)) with the same test function used in the proof of Theorem 4.1, the new proof changes slightly: instead of (4.2), we have (since $0 \leq \frac{z}{h+z} \leq 1$)

$$\frac{1}{4B} 4\beta^2 \int_\Omega |\nabla \phi|^2 + \frac{1}{4B}\int_\Omega \phi^2 |E(x)|^2 + \int_\Omega \phi^2 \geq (\alpha - 2B)\int_\Omega \frac{|\nabla z|^2}{(h+z)^2}\phi^2$$

and the conclusion on the solution z is again the positivity up, at most, a set of zero measure.

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Conflict of interest

The author declares no conflict of interest.

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