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Editorial

Variational models in elasticity[†]

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A short overview on the contributions to the special issue "Variational models in elasticity"

Variational approaches are ubiquitous in Mathematics as well as in several models arising in Physics, Engineering and Materials Science.

In elasticity, equilibria may be described as ground states or, more in general, critical points of suitable energy functionals, taking into account both elastic and inelastic phenomena. According to Hooke's law, classical linear theories of hyperelasticity are based on the minimization of a quadratic elastic energy; more recently, the variational approach to elasticity has been extended to describe also non-linear effects by J. Ball [5] and his collaborators. Furthermore, well established relaxation and De Giorgi's Γ-convergence methods [12] turned out to be efficient and elegant tools to describe complex phenomena such as homogenization and shape optimization [2, 9] (see also [23] for a criticism on purely variational approaches to homogenization), formation and evolution of microstructures [6], energy concentration [1, 8]. One of the features of the variational approach is its capability to deal with intrinsic lack of regularity, describing failure phenomena where PDE based formulations are either unavailable or not easy to handle with standard methods. Plasticity, damage and crack formation and propagation are prototypical examples where variational methods had a terrific impact [15] on modeling aspects, stimulating as well the development of new functional spaces [4, 7, 13, 22], methods and tools, promoting a proficuous and still active dialogue among pure and applied mathematicians, engineers and physicists.

This lucky combination is striking in [14], where harmonic analysis, elliptic PDE tools and Geometric Measure Theory merge together to provide a deep understanding of the space of functions of bounded deformation, laying the foundation of the rigorous functional description of formation and evolution of plasticity and microstructures.

Characterization of admissible configurations is often a matter of understanding ridigity properties coming from PDE constraints [17]; in [14], the rigidity comes from the Saint-Venant principle. Clearly, the lack of regularity represents a major difficulty in the analysis. In this respect, fracture mechanics seems to be a hostile environment for rigidity estimates. Nevertheless, in [16] it is shown that brittle bodies, even if with dense cracks, obey to some quantitative Korn type inequalities. This remarkable fact allows to derive brittle linearized Griffith theories from nonlinear models.

Griffith theory [19] provides criteria for crack propagation based on the balance between stored energy gain and energy spent to enlarge the crack. While Griffith assumes that the crack path is straight, his seminal variational principle was pushed forward in [15] to also select the path of the growing cracks. To this purpose, in the recent past much effort has been devoted to provide energy release rate formulas for (non-straight) growing cracks. A relevant progress is done in [3], where energy release rates are analyzed for a general class of cracks in planar elasticity, and vanishing viscosity solutions satisfying Griffith type criteria are derived.

The variational approach to fracture turned out to be very robust, and its field of possible applications has been largely extended in the recent past to model, together with crack growth, also damage and plasticity, all these effects possibly interacting each other. Clearly, understanding compactness and semicontinuity properties in the proper functional spaces is crucial to build up the appropriate variational models. In [11], lower semicontinuity properties for a total energy functional governing elasto-plasticity coupled with damage have been derived for a sequence of Sobolev functions converging to a *BD* function, for sub-critical Sobolev exponents.

The variational model [15] is also capable to deal with crack growing in composite and homogenized materials [18]. From an analytical point of view, this requires the understanding of total energy functionals with highly oscillating and discontinuous coefficients. Allowing the toughness and elastic coefficients to degenerate in the limit, new unexpected effects may appear in the homogenized material. This analysis has been developed in the recent past by several authors, with relevant contributions by C. I. Zeppieri; in [24], she reviews this literature with some new original progresses.

The homogenization theory is useful also to describe and optimize the effective behavior of complex materials taking into account inclusions and mixtures of different components. In [10], a Landau-de Gennes model for a polydisperse, inhomogeneous suspension of colloidal inclusions in a nematic host has been built up and analyzed in details: Using a Γ-convergence approach, it is proven that, in the dilute regime, the colloidal nematics behave like a homogenized, standard nematic material, with better properties than those of the original nematic host.

Shape optimization problems in elasticity can be understood by means of variational principles looking at those configurations (either microscopic or macroscopic) minimizing suitable total energy functionals. A well established theory of elastic sheets with a geometrical flavor is based on the minimization of the Willmore functional. In [21], fine properties of minimizers for a Plateau type problem involving the Willmore energy, are analyzed; in particular, under suitable energy bounds, minimizers are shown to be connected and, in the large diameter limit, to converge (after scaling) to a round sphere. In [20], shape optimization for non-local energies related to dislocations is studied, revisiting semi-circle type laws derived by the same authors for a general class of anisotropic interaction kernels.

We hope that this collection of results may introduce the interested readers to this active, wide and fascinating research field, also inspiring new future progresses.

Conflict of interest

The authors declare no conflict of interest.

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