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Editorial

Foreword to the special issue "Contemporary PDEs between theory and modeling"†

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† **This contribution is part of the Special Issue:** Contemporary PDEs between theory and modeling—Dedicated to Sandro Salsa, on the occasion of his 70th birthday
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I first met Sandro Salsa in the spring of 1999. In those days, I was finishing my PhD under the supervision of Susanna Terracini. Together with Susanna and Monica Conti, we were trying to extend the so-called Nehari method, a broken geodesic argument for oscillatory solutions to nonlinear ODEs, to the context of PDEs. Passing to higher dimensions, the moving zeros of the solution to the ODE become free boundaries; thus, it was inevitable for us to ask for a hearing to the greatest expert in Milano. I will never forget that first meeting, with Sandro at the blackboard, introducing us to the secrets of the Alt-Caffarelli-Friedmann monotonicity formula. A few months later, he recruited me as a teaching assistant for one of the courses he was in charge of, starting a collaboration which is lasting for more than 20 years.

Consequently, for me it was a great pleasure to accept the invitation from Enrico Valdinoci, Editor in chief of *Mathematics in Engineering*, to act as guest editor of this journal for a special issue dedicated to Sandro Salsa.

Sandro's scientific activity is rich and vast, ranging over diverse aspects of nonlinear, nonlocal, singular or degenerate PDEs of elliptic and parabolic type, such as free boundary problems with sharp interface, obstacle problems for the fractional Laplacian, free boundary problems from Finance, boundary behavior of solutions to degenerate and singular type operators. Salsa has made essential contributions to an extensive range of topics, and it would be a daunting task for me to compile a complete list here. Rather, I will put in prominence just some of them, together with a few comments about his service to the mathematical community.

After his degree in 1972 at the University of Milano, the first point of departure of Sandro's career was the meeting with Giorgio Talenti in Cortona, for the Summer Course organized by the Scuola Matematica Interuniversitaria (SMI) in 1975. This meeting fostered his interests in symmetrizzation

and inverse problems, leading to a series of papers with Carla Maderna and later also with Carlo Pagani, see e.g., [24, 25]. The other – crucial – cornerstone was the meeting with Eugene Fabes, during the year spent by Sandro in Minneapolis, in 1978/79. The collaboration with Fabes, which lasted for many years, concerned different aspects of the boundary behavior of solutions to elliptic and parabolic equations, (backward) Harnack inequalities and Fatou theorems. Such collaboration involved several other coauthors, among which Nicola Garofalo, leading to significant results, as for instance those contained in [9, 17–19]. It was during that year spent in Minneapolis that Sandro met Luis Caffarelli for the first time, even though their real collaboration started only in the 90's. In such years, Athanasopoulos and Salsa had just proved the $C^{1,\alpha}$ regularity of the free boundary for the classical one-phase Stefan problem in [8]. Eager to deal with the two-phase problem, they joined forces with Caffarelli. The result of this effort is the trilogy [1, 2, 4], which completes the study of the regularity of parabolic free boundaries of Stefan type, closing a problem which was open since the fifties. This collaboration also provided striking results in other free boundary problems, such as [3], also with Kenig, and [5,6], finally leading to the breakthrough in [7]: the optimal regularity of the free boundary and the classification of non-degenerate blow-up profiles for lower dimensional (boundary) obstacle problems. Thanks to this, and to the connection between boundary diffusion and fractional Laplacians (now well known as the "Caffarelli-Silvestre extension" [11]), Sandro was also on the front line of the study of regularity issues for nonlocal operators, with his celebrated paper with Caffarelli and Silvestre [12]. In the meanwhile, Salsa has provided relevant contributions on free boundary problems for equations with non-homogeneous coefficients and/or distributed sources, with his former students Fausto Ferrari and Maria Cristina Cerutti [13, 20–22], and more recently also with Daniela De Silva [14–16]. To conclude this short overview, I would like also to mention Sandro's ubiquitous taste for applications, as testified for instance by the very nice paper [23], written with Peter Laurence, about the regularity properties of the free boundary of American options with convex payoff, in the classical Black-Scholes multidimensional setting.

His scientific production is just a part of Sandro's versatile mathematical attitude. Two remarkable examples are provided by his role of co-founder of the Educational program in Mathematical Engineering and of the Lab in Modeling and Scientific Computing (MOX), both at Politecnico di Milano. In particular, the former is pervaded by his vision of the synergy between Applied Mathematics and Engineering. But in the first place, Salsa is an outstanding professor and teacher: this is primarily witnessed by the high regard that all of his students tributes him, together with the success of the many monographs and textbooks he has written, such as the introduction to free boundary problems [10], written with Caffarelli, or the modern approach to PDEs developed in [28]. In this respect, it is impossible not to mention the generations of Italian students in Engineering, Mathematics and Physics who learned the basics of Mathematical Analysis thanks to his nowadays classical textbooks, written with Pagani [26, 27]. Among the many services that Salsa has provided to the mathematical community, I would like just to recall that he has recently been the Director of the SMI, organizing in particular the Summer Courses in Cortona for some years. This seems to me a nice detail, in a journey that has seen him being in Cortona twice as a student and twice as a professor.

To conclude, this project has gathered together 21 research papers, filling two full issues of *Mathematics in Engineering* with contributions on current and future trends in PDEs, as a tribute to Sandro's influence on the field; most of them have been written by Sandro's friends and collaborators, all of them by mathematicians who deeply respect Sandro's work.

I want to thank all the authors and the reviewers that made this special issue possible. A special thank goes to Xiang Li and Cheng Bi at AIMS Press for their constant support. Lastly, I am deeply indebted to the Editors in Chief, Antonio De Simone and Enrico Valdinoci. Their help was essential to the success of this special issue.

Conflict of interest

The author declares no conflict of interest.

References

1. Athanasopoulos I, Caffarelli L, Salsa S (1996) Caloric functions in Lipschitz domains and the regularity of solutions to phase transition problems. *Ann Math*, 143: 413–434.
2. Athanasopoulos I, Caffarelli L, Salsa S (1996) Regularity of the free boundary in parabolic phase-transition problems. *Acta Math* 176: 245–282.
3. Athanasopoulos I, Caffarelli LA, Kenig C, et al. (2001) An area-Dirichlet integral minimization problem. *Commun Pure Appl Math* 54: 479–499.
4. Athanasopoulos I, Caffarelli LA, Salsa S (1998) Phase transition problems of parabolic type: flat free boundaries are smooth. *Commun Pure Appl Math* 51: 77–112.
5. Athanasopoulos I, Caffarelli LA, Salsa S (2001) The free boundary in an inverse conductivity problem. *J Reine Angew Math* 534: 1–31.
6. Athanasopoulos I, Caffarelli LA, Salsa S (2003) Stefan-like problems with curvature. *J Geom Anal* 13: 21–27.
7. Athanasopoulos I, Caffarelli LA, Salsa S (2008) The structure of the free boundary for lower dimensional obstacle problems. *Am J Math* 130: 485–498.
8. Athanasopoulos I, Salsa S (1991) An application of a parabolic comparison principle to free boundary problems. *Indiana Univ Math J* 40: 29–32.
9. Caffarelli L, Fabes E, Mortola S, et al. (1981) Boundary behavior of nonnegative solutions of elliptic operators in divergence form. *Indiana Univ Math J* 30: 621–640.
10. Caffarelli L, Salsa S (2005) *A Geometric Approach to Free Boundary Problems*, Providence, RI: American Mathematical Society.
11. Caffarelli L, Silvestre L (2007), An extension problem related to the fractional Laplacian. *Commun Part Diff Eq* 32: 1245–1260.
12. Caffarelli LA, Salsa S, Silvestre L (2008) Regularity estimates for the solution and the free boundary of the obstacle problem for the fractional Laplacian. *Invent Math* 171: 425–461.
13. Cerutti MC, Ferrari F, Salsa S (2004) Two-phase problems for linear elliptic operators with variable coefficients: Lipschitz free boundaries are $C^{1,\gamma}$. *Arch Ration Mech Anal* 171: 329–348.
14. De Silva D, Ferrari F, Salsa S (2014) Two-phase problems with distributed sources: regularity of the free boundary. *Anal PDE* 7: 267–310.

15. De Silva D, Ferrari F, Salsa S (2015) Free boundary regularity for fully nonlinear non-homogeneous two-phase problems. *J Math Pure Appl* 103: 658–694.
16. De Silva D, Ferrari F, Salsa S (2019) Regularity of higher order in two-phase free boundary problems. *T Am Math Soc* 371: 3691–3720.
17. Fabes E, Garofalo N, Marín-Malave S, et al. (1988) Fatou theorems for some nonlinear elliptic equations. *Rev Mat Iberoamericana* 4: 227–251.
18. Fabes E, Salsa S (1983) Estimates of caloric measure and the initial-Dirichlet problem for the heat equation in Lipschitz cylinders. *T Am Math Soc* 279: 635–650.
19. Fabes EB, Garofalo N, Salsa S (1986) A backward Harnack inequality and Fatou theorem for nonnegative solutions of parabolic equations. *Illinois J Math* 30: 536–565.
20. Ferrari F, Salsa S (2007) Regularity of the free boundary in two-phase problems for linear elliptic operators. *Adv Math* 214: 288–322.
21. Ferrari F, Salsa S (2010) Regularity of the solutions for parabolic two-phase free boundary problems. *Commun Part Diff Eq* 35: 1095–1129.
22. Ferrari F, Salsa S (2014) Two-phase free boundary problems for parabolic operators: smoothness of the front. *Commun Pure Appl Math* 67: 1–39.
23. Laurence P, Salsa S (2009) Regularity of the free boundary of an American option on several assets. *Commun Pure Appl Math* 62: 969–994.
24. Maderna C, Pagani CD, Salsa S (1986) Existence results in an inverse problem of potential theory. *Nonlinear Anal* 10: 277–298.
25. Maderna C, Salsa S (1979) Symmetrization in Neumann problems. *Appl Anal* 9: 247–256.
26. Pagani CD, Salsa S (1990) *Analisi Matematica 1*, Zanichelli.
27. Pagani CD, Salsa S (1990) *Analisi Matematica 2*, Zanichelli.
28. Salsa S (2016) *Partial Differential Equations in Action: From Modelling to Theory*, 3 Eds., Springer, 2016.

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