



Research article

Relapse-driven bistability in a nonlinear model of gang dynamics with deterrence and reformed-led advocacy

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Abstract: A central question in gang-intervention modeling is whether reducing recruitment below an invasion threshold is enough to eliminate gang activity. Here, we show that relapse from a reformed class can make it insufficient. We developed a deterministic compartmental SGR model with susceptible, active gang, and reformed classes, in which recruitment was reduced by perceived-risk deterrence and by an advocacy interaction between reformed and susceptible individuals. We established positive invariance and an invasion threshold for a weighted-contact formulation, and showed that its reciprocity-consistent proportional-mixing reduction exhibits backward bifurcation when relapse is sufficiently strong. The backward bifurcation generates a subthreshold positive gang equilibrium even when the simplified reproduction number is below one; for the worked parameter regime, this branch turns at a saddle-node and yields bistability, so the long-term outcome is set by basin membership rather than by the threshold alone. In dimensionless form, the backward-bifurcation condition reduces, under typical parameter regimes, to the requirement that the dimensionless relapse rate exceed the natural outflow rate of the active gang class. A worked instance at $\mathcal{R}_0^{\text{simp}} = 0.5$ yields gang-free and high-gang attractors separated by a low-prevalence saddle with a substantial reformed-class reservoir; numerical continuation places the saddle-node at $\mathcal{R}_0^{\text{simp}} \approx 0.08755$, so the bistable interval spans approximately the upper 91% of the unit interval. Numerical bifurcation diagrams and a baseline-centered PRCC sensitivity analysis identify relapse and reformation as the dominant opposing drivers of long-term gang prevalence in the sampled regime, with recruitment a smaller positive driver. These results suggest that gang reduction may require coordinated strategies that reduce recruitment, prevent relapse, strengthen exit pathways, and sustain reformed-led prevention efforts.

Keywords: gang dynamics; nonlinear dynamical systems; relapse; perceived-risk deterrence; reformed-led advocacy; backward bifurcation; bistability; intervention modeling

Abbreviations: *SGR*: susceptible–gang–reformed; *ODE*: ordinary differential equation; *LHS*: Latin hypercube sampling; *PRCC*: partial rank correlation coefficient; *CI*: confidence interval; *G.R.E.A.T.*: Gang Resistance Education and Training

1. Introduction

Gang activity remains a serious public-safety and community-development concern in many parts of the world. Street gangs are associated with violence, territorial conflict, drug markets, firearm exposure, social disruption, and elevated risks for youth and young adults who become embedded in gang networks [1–4]. In the United States, gang activity continues to pose challenges for law enforcement, schools, families, and community organizations. Estimates from national reporting systems vary across time, definitions, and reporting methods. For example, the National Gang Center’s National Youth Gang Survey estimated approximately 30,000 gangs and 850,000 gang members in 2012, its most recent survey year [5], while the 2011 National Gang Threat Assessment reported approximately 33,000 gangs and 1.4 million gang members [6]. Although these estimates are not directly comparable, they underscore the scale and continuing relevance of gang involvement as a social and public-safety problem.

Gang involvement is shaped by individual, peer, family, school, neighborhood, and structural factors. Street gangs are often associated with violence and competition for respect, identity, protection, and credibility [7–12]. Empirical studies show that many gang members are adolescents or young adults, making youth especially vulnerable to recruitment [13–17]. Risk factors associated with gang joining include family instability, school disengagement, neighborhood disadvantage, prior delinquency, and association with gang-involved peers [16, 18, 19]. Gang membership can also intensify criminal behavior: youth who join gangs often show increased delinquency during membership, followed by reduced delinquency after leaving gangs [20–22]. These observations suggest that gang involvement is not a static individual condition, but rather a dynamic social process involving recruitment, escalation, exit, relapse, reintegration, and prevention.

Mathematical modeling provides a useful framework for studying such dynamic processes. Because gang recruitment can spread through peer influence and social exposure, several authors have drawn analogies between gang involvement, crime transmission, and infectious-disease dynamics [23–25]. Other mathematical approaches have modeled crime hotspots, gang territories, organized crime, policing, and intervention strategies. Brantingham et al. [26] used ecological and territorial models to study gang boundary dynamics and violence, while Short et al. [27] used reaction-diffusion equations to investigate crime hotspots. Gonzalez et al. [28] developed a nonlinear compartmental crime model, and Sooknanan et al. [29, 30] studied organized crime and policing through coupled nonlinear systems and predator-prey-type formulations. Agent-based approaches have also been used to study gang rivalry and spatial interactions [31]. Moreover, optimal-control and compartmental models have been proposed to examine gang recruitment reduction, prevention, intervention, rehabilitation, and enforcement [32–35].

Despite these advances, two mechanisms remain underrepresented in many quantitative models of gang dynamics. The first is the role of perceived-risk deterrence. Individuals at risk of recruitment

may be discouraged from joining gangs by awareness of violence, legal consequences, incarceration, social stigma, or visible community-prevention efforts. In this paper, we use the term *fear* only in a modeling sense to represent perceived-risk deterrence: A behavioral reduction in recruitment caused by the perceived costs and dangers of gang involvement. This interpretation is related to fear-modified interaction terms used in ecological models, where perceived threats can alter contact behavior and reduce risky engagement [36]. In this social context, this mechanism is not intended to advocate fear as a policy goal; rather, it provides a phenomenological way to represent risk awareness, deterrence messaging, and public-safety visibility.

The second mechanism is the role of reformed individuals. Former gang members who exit active gang involvement may participate in reform programs, mentoring, outreach, public testimony, reintegration activities, or community-based prevention. Because of their lived experience, reformed individuals may provide credible counter-narratives to gang involvement and help disrupt recruitment pathways [37–39]. At the same time, reformation is not necessarily permanent. Individuals in the reformed class may relapse into gang involvement because of continued peer exposure, limited employment opportunities, neighborhood pressure, or insufficient reintegration support [40]. They may also return to a susceptible or neutral community status after program completion, reintegration, or reduced engagement in active advocacy. Thus, reformed individuals can play multiple roles in gang dynamics: They may represent successful exit from gang involvement, help suppress recruitment through advocacy, return to a neutral at-risk status, or relapse into active gang activity. Capturing these roles is important for understanding long-term gang reduction.

To address these gaps, we formulate a deterministic compartmental model for an at-risk population divided into three classes: susceptible individuals S , active gang members G , and reformed individuals R . Susceptible individuals may be recruited into the gang class through contact with active gang members. Active gang members may leave gang life and enter the reformed class. Reformed individuals may either relapse into gang involvement or return to a susceptible or neutral community class. Recruitment is further modified by a nonlinear reduction factor that represents perceived-risk deterrence and the recruitment-suppressing influence of reformed individuals. In this formulation, reformed-led advocacy is treated phenomenologically as an effect arising through interaction between reformed and susceptible individuals during reintegration or community reentry.

Our main objective of this paper is to understand how recruitment, deterrence, reformation, relapse, and reformed-led advocacy interact to shape threshold behavior in gang dynamics. We derive the gang-free equilibrium and invasion threshold for a formal weighted-contact extension, then use its reciprocity-consistent proportional-mixing reduction to obtain explicit equilibrium and bifurcation results. This reduction enables us to identify relapse as a mechanism capable of generating backward bifurcation, the regime in which a gang-free equilibrium and a positive gang equilibrium coexist below the invasion threshold, producing bistability and a hysteresis effect. A consequence is that reducing recruitment below this threshold need not eliminate gang activity, an obstacle that a threshold-only analysis would miss. The detailed mathematical form of the recruitment-reduction function is introduced in the model formulation section.

The closest prior compartmental treatment of gang dynamics with backward-bifurcation behavior is

that of Nyabadza et al. [37], who introduced a gang reproduction number R_g and showed that their correctional-services model exhibits backward bifurcation, so that gangs may persist in the population even when $R_g < 1$. Their backward bifurcation arises from a critical correctional-services efficacy parameter ϵ^* and is driven by the interplay between recruitment imitation and correctional-services rehabilitation efficacy. We extend this line of investigation in three directions. First, the backward bifurcation in our SGR model is driven by the reformed-class relapse rate β_{rg} rather than by correctional-services efficacy, identifying a distinct social mechanism, post-exit relapse rather than in-program rehabilitation failure, as a source of bistability. Second, we incorporate two mechanisms absent from [37]: A perceived-risk deterrence term that captures behavioral recruitment suppression, and a reformed-led advocacy term that captures the recruitment-suppressing influence of community-based prevention by former gang members. Third, our analysis derives an explicit dimensionless characterization of the bifurcation condition (Corollary 3.5) and decomposes the threshold into a structural baseline plus two perturbative drag terms, surfacing the relapse-to-decay ratio as the primary control on whether bistability can occur in this model class. The accompanying numerical continuation quantifies the width of the bistable wedge for an illustrative parameter set (approximately $0.0876 < \mathcal{R}_0^{\text{simp}} < 1$, an order of magnitude in $\mathcal{R}_0^{\text{simp}}$) with substantive policy implications for whether threshold-lowering alone is sufficient for elimination from arbitrary initial conditions. We view our paper as complementary to [37] rather than competing: Where their model emphasizes correctional-services interventions, ours emphasizes community-based prevention through reformed-led advocacy and the structural role of relapse.

We address the following research questions:

- (1) How does perceived-risk deterrence affect gang recruitment and long-term gang prevalence?
- (2) How do reformed individuals influence gang dynamics through advocacy, reintegration, and possible relapse?
- (3) Under what conditions can relapse generate bistability between a gang-free state and a positive gang equilibrium?
- (4) What relapse-driven bifurcation mechanisms are revealed by the homogeneous-contact reduction, viewed as the reciprocity-consistent proportional-mixing reduction of the weighted-contact formulation?
- (5) Which parameters most strongly influence long-term active gang prevalence and the relapse threshold for backward bifurcation?

The remainder of the paper is organized as follows: Section 2, we present the model formulation, contact structure, recruitment-reduction mechanism, and normalized system. In Section 3, we establish positive invariance, derives the gang-free equilibrium and reproduction threshold, and analyzes positive gang equilibria and bifurcation behavior. In Section 4, we present bifurcation diagrams and sensitivity analysis illustrating the effects of recruitment, relapse, deterrence, return from reformed status, and reformed-led advocacy. Finally, Section 5 discusses the qualitative implications, limitations, and possible extensions of the model.

2. Formulation of the mathematical model

We formulate a deterministic compartmental model for gang dynamics in an at-risk population. The model captures four core social transitions: recruitment into active gang involvement, exit from gang involvement, relapse from the reformed class, and return from the reformed class to a susceptible or neutral community class. In addition, the model includes a nonlinear recruitment-reduction mechanism representing perceived-risk deterrence and reformed-led advocacy.

Let $N(t)$ denote the total at-risk population. This population is divided into three mutually exclusive compartments:

- $S(t)$: Susceptible individuals who are at risk of gang recruitment but are not currently gang-involved.
- $G(t)$: active gang members who are involved in gang activities.
- $R(t)$: Reformed individuals who have left active gang involvement and may influence others through reintegration, mentoring, outreach, or advocacy. Individuals in this class may remain engaged in reform-oriented activities, return to the susceptible or neutral community class, or relapse into gang activities.

Thus,

$$N(t) = S(t) + G(t) + R(t).$$

Class $S(t)$ represents the general at-risk population, not a group of individuals who are inevitably recruited. Class $R(t)$ represents individuals who have exited active gang involvement and are currently reformed, reform-engaged, or connected to reintegration or advocacy processes. The transition $R \rightarrow G$ represents relapse into gang activity, whereas the transition $R \rightarrow S$ represents return to the susceptible or neutral community class after program completion, reintegration, or reduced engagement in active reform or advocacy activities. Thus, $R \rightarrow S$ is distinct from relapse.

2.1. Balanced population assumption

We assume demographically balanced turnover in the at-risk population at a constant reference size N_0 . Births and other entries into the at-risk pool feed the susceptible class at the constant rate μN_0 , while individuals in all three compartments leave the at-risk population at per-capita rate μ . Therefore,

$$\dot{N}(t) = \mu N_0 - \mu(S(t) + G(t) + R(t)).$$

If the initial condition satisfies $S(0) + G(0) + R(0) = N_0$, then

$$S(t) + G(t) + R(t) = N_0 \quad \text{for all } t \geq 0,$$

and the total at-risk population remains constant at N_0 . Writing the recruitment input as μN_0 rather than $\mu N(t)$ makes explicit that the entry rate is set by the (fixed) reference population size rather than by the (time-dependent) compartmental sum, removing the tautological appearance of the closure relation.

This balanced-population assumption allows us to focus on transitions among S , G , and R within a fixed at-risk pool. It is appropriate when the size of the at-risk population changes more slowly than the social-transition processes being modeled, or when entry and exit from the at-risk pool are approximately balanced over the time horizon of interest. The assumption is not meant to imply that real populations are perfectly closed; rather, it provides a tractable baseline model for studying recruitment, exit, relapse, and reformed-led prevention mechanisms. Recruitment in this model is frequency-dependent: per-capita recruitment rate depends on gang prevalence rather than absolute count, consistent with saturated contact networks in moderate-to-large at-risk populations.

2.2. Dimensional model

The dimensional model is

$$\begin{aligned}\dot{S} &= \mu N_0 - \rho(S, G, R)\lambda_{sg}S + \lambda_{rs}R - \mu S, \\ \dot{G} &= \rho(S, G, R)\lambda_{sg}S + \lambda_{rg}R - (\gamma + \mu)G, \\ \dot{R} &= \gamma G - \lambda_{rg}R - \lambda_{rs}R - \mu R.\end{aligned}\tag{2.1}$$

The first equation describes the susceptible class. Susceptible individuals enter the at-risk population at rate μN_0 , are recruited into active gang involvement at rate $\rho(S, G, R)\lambda_{sg}S$, receive individuals returning from the reformed class at rate $\lambda_{rs}R$, and leave the at-risk population at rate μS .

The second equation describes the active gang class. Individuals enter this class through recruitment from S , and reformed individuals may relapse into gang involvement at rate $\lambda_{rg}R$. Active gang members leave gang involvement and enter the reformed class at rate γG , and they leave the at-risk population at rate μG . Thus, $1/\gamma$ represents the average duration of active gang involvement in the absence of relapse and demographic exit.

The third equation describes the reformed class. Active gang members become reformed at rate γG . Individuals in R may relapse into gang involvement through contact with active gang members, represented by the transition $R \rightarrow G$ at rate $\lambda_{rg}R$. They may also leave the active reformed/advocacy class and return to the susceptible or neutral community class, represented by the transition $R \rightarrow S$ at rate $\lambda_{rs}R$. This latter transition does not represent relapse; rather, it represents reintegration, program completion, or reduced engagement in active advocacy while remaining non-gang-involved. Reformed individuals also leave the at-risk population at rate μR .

2.3. Contact structure and transition rates

Transitions between classes are generated by social contact. We define

$$\lambda_{ij} = \beta_{ij}c_{ij},$$

where c_{ij} is the fraction of relevant contacts made by individuals in class i with individuals in class j , and β_{ij} is an effective transition-rate coefficient.

For dimensional consistency, we interpret β_{ij} as an effective rate that incorporates the frequency of meaningful contacts and the probability that such contacts produce a transition. Thus, β_{ij} has units of time^{-1} . This interpretation avoids mixing a dimensionless probability-per-contact formulation with a weekly transition-rate formulation.

The contact fractions are defined by

$$c_{rs} = \frac{a_{rs}S}{a_{rs}S + a_{rg}G + a_{rr}R}, \quad (2.2)$$

$$c_{sg} = \frac{a_{sg}G}{a_{ss}S + a_{sg}G + a_{sr}R}, \quad (2.3)$$

$$c_{rg} = \frac{a_{rg}G}{a_{rs}S + a_{rg}G + a_{rr}R}. \quad (2.4)$$

Here, $a_{ij} \geq 0$ is a contact preference or mixing weight. Larger a_{ij} indicates stronger relative contact tendency from class i toward class j . The denominators represent the weighted contact environment experienced by the corresponding class. In the mathematical analysis below, strict positivity of the relevant contact weights is assumed when needed to ensure that contact denominators remain positive and that persistence arguments are well defined.

When $a_{ij} \equiv 1$, the contact fractions reduce to $c_{rs} = S/N$, $c_{sg} = G/N$, and $c_{rg} = G/N$, so each contact fraction equals the population share of the destination class. This is the canonical proportional-mixing structure of heterogeneous-population epidemic theory [41], and it automatically satisfies the reciprocity balance equation $c_{ij}N_i = c_{ji}N_j$ that any physically consistent mixing matrix must obey.

For the general weighted form $a_{ij} \neq 1$, reciprocity is not automatic. Substituting our weighted definition $c_{ij} = a_{ij}N_j / \sum_k a_{ik}N_k$ into $c_{ij}N_i = c_{ji}N_j$ and simplifying gives the parametric constraint

$$a_{ij} \left(\sum_k a_{jk}N_k \right) = a_{ji} \left(\sum_k a_{ik}N_k \right), \quad i, j \in \{S, G, R\}. \quad (2.5)$$

The admissible class of weighted contact matrices is therefore the set of $\{a_{ij}\}$ satisfying (2.5). The proportional-mixing case $a_{ij} \equiv 1$ lies in this class trivially: both sides reduce to $\sum_k N_k = N$. A symmetric weighted matrix ($a_{ij} = a_{ji}$) lies in this class under the additional condition that the weighted contact-environment sums $\sum_k a_{ik}N_k$ are equal across classes i ; this is automatic under proportional mixing but not in general.

Throughout the analytical and numerical results of this paper, we work with the proportional-mixing case $a_{ij} \equiv 1$, which is balance-consistent by construction. Characterizing the full admissible class (2.5) for arbitrary population partitions, and analyzing how the bifurcation structure of Theorem 3.4 depends on the admissible weights, is a substantive direction for future work; as discussed in the remark following Theorem 3.4, the reciprocity balance is restrictive away from proportional mixing, which makes this extension genuinely open rather than a routine perturbation.

The three transition rates are therefore

$$\lambda_{sg} = \beta_{sg}c_{sg}, \quad \lambda_{rg} = \beta_{rg}c_{rg}, \quad \lambda_{rs} = \beta_{rs}c_{rs}.$$

Parameter β_{sg} measures the effective recruitment strength from susceptible individuals to active gang membership. Parameter β_{rg} measures the effective relapse strength from the reformed class back to active gang membership. The parameter β_{rs} measures the effective transition strength from the reformed class to the susceptible or neutral class, representing reintegration, program completion, or reduced engagement in active reform or advocacy activities. The transition $R \rightarrow S$ is distinct from relapse, which is represented by $R \rightarrow G$.

2.4. Recruitment reduction through perceived-risk deterrence and reformed-led advocacy

Gang recruitment is not determined solely by contact between susceptible individuals and active gang members. Recruitment may also be reduced by perceived risk, public-safety awareness, fear of violence or incarceration, social stigma, and messages from reformed individuals who discourage gang involvement. To represent these mechanisms, we introduce a state-dependent recruitment-reduction function

$$\rho(S, G, R) = \frac{1}{1 + \eta Rc_{rs} + \epsilon G}. \quad (2.6)$$

Parameter $\epsilon \geq 0$ measures perceived-risk deterrence associated with active gang prevalence. Larger values of ϵ correspond to stronger recruitment reduction due to perceived consequences of gang involvement, such as violence, legal consequences, incarceration, or social stigma. In this manuscript, “fear” is used only in this modeling sense as perceived-risk deterrence, not as a policy prescription.

Parameter $\eta \geq 0$ is a dimensionless coefficient that measures the per-contact effectiveness of reformed-led advocacy on recruitment suppression. The product Rc_{rs} measures the rate of contact between reformed and susceptible individuals, so the term

$$\eta Rc_{rs}$$

is a phenomenological proxy for advocacy, mentoring, public testimony, outreach, credible counternarratives, or other recruitment-suppressing influence generated through contact between reformed and susceptible individuals. Here, c_{rs} is the reformed–susceptible contact fraction, and η scales the per-contact effectiveness of the resulting advocacy.

This formulation separates two conceptually distinct quantities: The durability of reformed status, governed by the transition rate β_{rs} at which reformed individuals return to the susceptible or neutral class, and the per-contact effectiveness of advocacy on recruitment, governed by the independent coefficient η . An alternative single-parameter form sets the advocacy coefficient to $\alpha\beta_{rs}$, tying advocacy effectiveness to the reintegration rate; because reintegration is typically slow relative to demographic and desistance turnover, that coupling renders advocacy numerically inert. Treating η as independent of β_{rs} keeps the model parsimonious while enabling advocacy effectiveness to be calibrated against intervention data on community-based programs separately from the reintegration rate inferred from desistance data, and it preserves the closed-form bifurcation analysis of Theorem 3.4. As Corollary 3.5 shows, advocacy effectiveness enters the backward-bifurcation threshold additively and at first order, so η is a primary lever rather than a secondary modulation of the threshold.

Remark (Interpretation of ρ). *The function $\rho(S, G, R)$ satisfies*

$$0 < \rho(S, G, R) \leq 1.$$

Thus, it reduces recruitment but never reverses the direction of recruitment. When perceived-risk deterrence and reformed-led advocacy are absent, $\epsilon = \eta = 0$, and $\rho = 1$, so recruitment occurs at its baseline contact-driven rate. The reciprocal form is chosen because it is positive, bounded above by one, monotonically decreasing in deterrence and advocacy intensity, and cannot produce negative recruitment rates. Alternative functional forms, such as exponential or multiplicative deterrence functions, could be explored in future robustness analysis.

Remark (Model scope). *The model does not include a direct $S \rightarrow R$ transition. Thus, the reformed class represents individuals who have exited active gang involvement, not individuals who become protected through upstream prevention without prior gang involvement. A future extension could include a protected or prevention-engaged class to represent direct prevention among susceptible individuals.*

2.5. Normalized system

Because the total at-risk population is constant under the balanced-turnover assumption, we normalize the system by defining

$$s(t) = \frac{S(t)}{N}, \quad g(t) = \frac{G(t)}{N}, \quad r(t) = \frac{R(t)}{N}.$$

Then

$$s(t) + g(t) + r(t) = 1.$$

In terms of normalized variables, the contact fractions become

$$c_{rs} = \frac{a_{rs}s}{a_{rs}s + a_{rg}g + a_{rr}r}, \quad (2.7)$$

$$c_{sg} = \frac{a_{sg}g}{a_{ss}s + a_{sg}g + a_{sr}r}, \quad (2.8)$$

$$c_{rg} = \frac{a_{rg}g}{a_{rs}s + a_{rg}g + a_{rr}r}. \quad (2.9)$$

After rescaling with the constant population size N , the normalized parameters $\eta' := \eta N$ and $\epsilon' := \epsilon N$ are both dimensionless, so the denominator terms $\eta'rc_{rs}$ and $\epsilon'g$ are dimensionless and the recruitment-reduction factor ρ is dimensionless as required. To keep notation consistent with the dimensional form, we continue to write η and ϵ for these rescaled constants in the normalized model. The normalized model is

$$\dot{s} = \mu - \frac{\beta_{sg}sc_{sg}}{1 + \eta rc_{rs} + \epsilon g} + \beta_{rs}rc_{rs} - \mu s, \quad (2.10)$$

$$\dot{g} = \frac{\beta_{sg}sc_{sg}}{1 + \eta rc_{rs} + \epsilon g} + \beta_{rg}rc_{rg} - (\gamma + \mu)g, \quad (2.11)$$

$$\dot{r} = \gamma g - \beta_{rg}rc_{rg} - \beta_{rs}rc_{rs} - \mu r. \quad (2.12)$$

The feasible region for the normalized model is

$$\mathcal{D} = \{(s, g, r) \in \mathbb{R}_+^3 : s + g + r = 1\}.$$

The model is therefore effectively two-dimensional on $\mathcal{D}$, although the three-compartment representation is retained for interpretability. Using $s = 1 - g - r$, the dynamics can be written as

$$\dot{g} = \frac{\beta_{sg}(1 - g - r)c_{sg}}{1 + \eta rc_{rs} + \epsilon g} + \beta_{rg}rc_{rg} - (\gamma + \mu)g, \quad (2.13)$$

$$\dot{r} = \gamma g - \beta_{rg}rc_{rg} - \beta_{rs}rc_{rs} - \mu r, \quad (2.14)$$

where, after substituting $s = 1 - g - r$, the contact fractions are

$$c_{rs} = \frac{a_{rs}(1 - g - r)}{a_{rs}(1 - g - r) + a_{rg}g + a_{rr}r},$$

$$c_{sg} = \frac{a_{sg}g}{a_{ss}(1 - g - r) + a_{sg}g + a_{sr}r},$$

and

$$c_{rg} = \frac{a_{rg}g}{a_{rs}(1 - g - r) + a_{rg}g + a_{rr}r}.$$

2.6. Qualitative model properties

The right-hand side of Eqs (2.10)–(2.12) is continuously differentiable wherever the denominators in Eqs (2.7)–(2.9) are positive. Hence, for feasible initial conditions and positive contact denominators, local existence and uniqueness follow from standard ODE theory [42]. Since $s + g + r = 1$, the dynamics reduce to the two-dimensional autonomous system Eqs (2.13)–(2.14) on the feasible simplex. By the Poincaré–Bendixson theorem, a bounded planar continuous-time flow of this kind cannot possess chaotic compact invariant sets; its bounded trajectories converge to equilibria, periodic orbits, or connections between them, so the long-term dynamics are governed by the equilibria and their stability, while multiple equilibria and bistability remain possible [43].

Table 1. Parameter descriptions, units, baseline values or ranges, and sources. Some values used in the numerical bifurcation diagrams are illustrative and are chosen to demonstrate nonlinear threshold behavior rather than to provide calibrated estimates for a specific community.

Parameter	Description	Units	Baseline/range	Source
μ	Turnover rate in the at-risk population; $1/\mu$ is the mean residence time in the at-risk population	week^{-1}	1/500	[34]
γ	Reformation rate from active gang membership to the reformed class; $1/\gamma$ is the mean duration of active gang involvement in the absence of relapse and demographic exit	week^{-1}	varied/ illustrative	[37]
β_{sg}	Effective recruitment-rate coefficient from susceptible individuals to active gang membership	week^{-1}	varied	[18, 34, 44]
β_{rg}	Effective relapse-rate coefficient from the reformed class back to active gang membership	week^{-1}	varied	Informed by [40]
β_{rs}	Effective transition-rate coefficient from the reformed class to the susceptible/neutral class, representing reintegration, program completion, or reduced engagement in active reform/advocacy activities	week^{-1}	varied/assumed	Assumed
ϵ	Strength of perceived-risk deterrence in suppressing recruitment	dimensionless (normalized); person^{-1} (dimensional)	varied/assumed	Assumed
η	Dimensionless effectiveness coefficient for reformed-led advocacy through the interaction term ηrc_{rs}	dimensionless	varied/assumed	Assumed
a_{ij}	Contact preference or mixing weight from class i toward class j ; assumed positive in analytical results requiring nonzero denominators	dimensionless	nonnegative; positive when required	Assumed

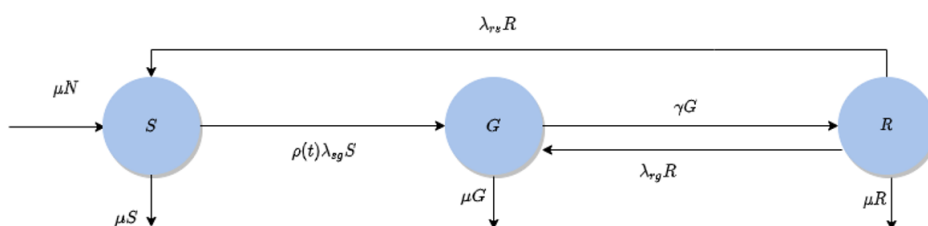


Figure 1. Schematic representation of the SGR gang-dynamics model. Susceptible individuals S may be recruited into active gang membership G . Active gang members may exit gang involvement and enter the reformed class R . Reformed individuals may relapse into active gang membership or return to a susceptible/neutral community class. Perceived-risk deterrence and reformed-led advocacy reduce the effective recruitment flow from S to G .

In summary, Eqs (2.10)–(2.12) define the normalized SGR gang-dynamics model used throughout the analysis. The schematic model structure is shown in Figure 1, and parameter definitions are summarized in Table 1.

2.7. Nondimensional formulation

The numerical values reported in Table 1 are illustrative weekly rates. To make the structural dependencies of the model more transparent and to express results in a form independent of any particular time unit, we introduce a dimensionless rescaling. Let

$$\tau = (\mu + \gamma) t$$

denote dimensionless time, scaled by the natural turnover rate of active gang membership. The corresponding dimensionless parameters are

$$\tilde{\beta}_{sg} = \frac{\beta_{sg}}{\mu + \gamma}, \quad \tilde{\beta}_{rg} = \frac{\beta_{rg}}{\mu + \gamma}, \quad \tilde{\beta}_{rs} = \frac{\beta_{rs}}{\mu + \gamma}, \quad \tilde{\mu} = \frac{\mu}{\mu + \gamma},$$

with η and ϵ already dimensionless under the rescalings $\eta' := \eta N$ and $\epsilon' := \epsilon N$ established in the normalized model of Section 2.5. The dimensionless deterrence/advocacy denominator is $1 + \eta'rc_{rs} + \epsilon'$, preserving the structure of the recruitment-reduction factor.

A direct consequence of this rescaling is that the simplified reproduction number coincides exactly with the dimensionless recruitment ratio:

$$\tilde{\beta}_{sg} = \frac{\beta_{sg}}{\mu + \gamma} = \mathcal{R}_0^{\text{simp}}.$$

This identification gives an immediate operational meaning to the dimensionless recruitment parameter and provides a natural choice of bifurcation-parameter axis for the diagrams in Section 4.

The dimensionless form provides the same dynamical content as the dimensional system but makes the structural inequalities governing threshold behavior easier to interpret. In particular, the bifurcation

condition derived in Theorem 3.4 below can be expressed entirely in terms of dimensionless ratios (Corollary 3.5), which removes the dependence on the choice of weekly time units that prompted concern about empirical interpretability. We retain the dimensional form for the analytical derivations and figures to maintain continuity with the model formulation, and refer to the dimensionless form for interpretation.

3. Mathematical analysis

In this section, we analyze the normalized SGR model Eqs (2.10)–(2.12). We first establish positive invariance of the feasible region, identify the gang-free equilibrium, and derive the invasion threshold for the formal weighted-contact extension. We then study positive gang equilibria and bifurcation structure in its reciprocity-consistent proportional-mixing reduction.

Throughout this section, we distinguish between two threshold quantities:

$$\mathcal{R}_0^{\text{full}} \quad \text{and} \quad \mathcal{R}_0^{\text{simp}}.$$

The quantity $\mathcal{R}_0^{\text{full}}$ is the invasion threshold for the formal fixed-weight (weighted-contact) extension. The quantity $\mathcal{R}_0^{\text{simp}}$ is the corresponding threshold for the homogeneous-contact reduction, obtained by setting all contact-preference weights equal to one. The full model is used to derive local stability, persistence, and a sufficient global-stability criterion for the gang-free equilibrium. The homogeneous-contact reduction is used for explicit positive-equilibrium calculations and center-manifold bifurcation analysis.

The weighted-contact system should be understood as a formal fixed-weight extension rather than a genuinely heterogeneous balance-consistent model: As shown in the remark following Theorem 3.4, is required to the reciprocity balance to hold at all population state force proportional mixing, so the proportional-mixing (homogeneous-contact) reduction is the physically reciprocity-consistent model analyzed here. The formal fixed-weight extension is retained because it makes the invasion threshold $\mathcal{R}_0^{\text{full}}$ explicit and shows how the contact weights enter it; a backward-bifurcation analysis for genuinely heterogeneous, balance-consistent mixing remains open.

3.1. Feasible region and positive invariance

Theorem 3.1 (Full model: Positive invariance). *Assume that all parameters are nonnegative and that the contact denominators in Eqs (2.7)–(2.9) are positive on the feasible state space. Then the normalized model Eqs (2.10)–(2.12) is positively invariant in the region*

$$\mathcal{D} = \{(s, g, r) \in \mathbb{R}_+^3 : s + g + r = 1\}. \quad (3.1)$$

The proof is given in Appendix A.

Remark (Interpretation). *Theorem 3.1 shows that the model is well posed as a population-fraction model. If the initial fractions are nonnegative and sum to one, then the solution remains nonnegative and the total fraction remains one for all future time.*

3.2. Gang-free equilibrium and invasion threshold

The gang-free equilibrium is obtained by setting $g = 0$ and $r = 0$. Since $s + g + r = 1$, we have

$$\mathcal{E}_0 = (1, 0, 0).$$

To derive the invasion threshold, we examine the linearized growth of a small active gang population introduced into a fully susceptible population. Although the reformed class contributes to gang activity through relapse, the relapse term does not contribute to the linear invasion threshold at $\mathcal{E}_0$. The contribution

$$\beta_{rg}rc_{rg}$$

is nonlinear near $\mathcal{E}_0$, since c_{rg} is proportional to g . Thus, $\beta_{rg}rc_{rg}$ is of order rg near the gang-free equilibrium. The first-order invasion threshold is therefore determined by recruitment from S into G .

At $\mathcal{E}_0 = (1, 0, 0)$,

$$c_{sg} = \frac{a_{sg}g}{a_{ss}s + a_{sg}g + a_{sr}r} = \frac{a_{sg}}{a_{ss}}g + O(g^2, gr),$$

and the recruitment term satisfies

$$\frac{\beta_{sg}sc_{sg}}{1 + \eta rc_{rs} + \epsilon g} = \frac{a_{sg}\beta_{sg}}{a_{ss}}g + O(g^2, gr).$$

Therefore, the linearized g -equation near $\mathcal{E}_0$ is

$$\dot{g} = \left(\frac{a_{sg}\beta_{sg}}{a_{ss}} - (\mu + \gamma) \right) g.$$

This gives the full-model reproduction number

$$\mathcal{R}_0^{\text{full}} = \frac{a_{sg}\beta_{sg}}{a_{ss}(\mu + \gamma)}. \quad (3.2)$$

For the homogeneous-contact reduction, where

$$a_{ss} = a_{sg} = a_{sr} = a_{rs} = a_{rg} = a_{rr} = 1,$$

the corresponding reproduction number is

$$\mathcal{R}_0^{\text{simp}} = \frac{\beta_{sg}}{\mu + \gamma}. \quad (3.3)$$

Comparing (3.2) and (3.3), the two thresholds are related by

$$\mathcal{R}_0^{\text{full}} = \frac{a_{sg}}{a_{ss}} \mathcal{R}_0^{\text{simp}},$$

so the contact-preference ratio a_{sg}/a_{ss} acts as a multiplicative correction that adjusts the homogeneous-contact threshold for relative mixing intensity between susceptible individuals and active gang members. Because the analytical and numerical work below is carried out in the proportional-mixing case, $\mathcal{R}_0^{\text{simp}}$ is the relevant threshold for all explicit results. The factor a_{sg}/a_{ss} in $\mathcal{R}_0^{\text{full}}$ encodes the multiplicative effect of contact-preference heterogeneity within a balance-preserving weighted family; characterizing the admissible a_{ij} that preserve balance, and analyzing the resulting parametric thresholds, is an extension we leave for future work.

Remark (Next-generation interpretation). *The same threshold can be obtained through the next-generation matrix method [45]. At the gang-free equilibrium, the only compartment that generates new active gang members at first order is G . The relapse term from R to G does not contribute to the first-order next-generation matrix at $\mathcal{E}_0$, because $r = 0$ and c_{rg} is proportional to g . Thus, the new-recruitment and transition terms for the active gang class are*

$$\mathcal{F}(g) = \frac{\beta_{sg} s c_{sg}}{1 + \eta r c_{rs} + \epsilon g}, \quad \mathcal{V}(g) = (\mu + \gamma)g.$$

Evaluating the derivatives at $\mathcal{E}_0$ gives

$$F = \frac{\partial \mathcal{F}}{\partial g}(\mathcal{E}_0) = \frac{a_{sg} \beta_{sg}}{a_{ss}}, \quad V = \frac{\partial \mathcal{V}}{\partial g}(\mathcal{E}_0) = \mu + \gamma.$$

Hence,

$$\sigma(FV^{-1}) = \frac{a_{sg} \beta_{sg}}{a_{ss}(\mu + \gamma)} = \mathcal{R}_0^{\text{full}},$$

where $\sigma(\cdot)$ denotes the spectral radius (distinct from the recruitment-reduction function ρ).

Remark (Interpretation of the reproduction number). *The quantity $\mathcal{R}_0^{\text{full}}$ represents the expected number of new active gang members generated by one active gang member introduced into an otherwise susceptible at-risk population over the expected duration of active gang involvement. The factor a_{sg}/a_{ss} adjusts this threshold by the relative contact preference between susceptible individuals and active gang members. If $\mathcal{R}_0^{\text{full}} < 1$, a small gang introduction decays near the gang-free state. If $\mathcal{R}_0^{\text{full}} > 1$, a small gang introduction can grow.*

Theorem 3.2 (Full model: Stability, persistence, and global stability of the gang-free equilibrium). *Assume that all contact weights appearing in the denominators of Eqs (2.7)–(2.9) are positive, with $\mu > 0$ and $\gamma > 0$. For the normalized SGR model Eqs (2.10)–(2.12), the following statements hold.*

(a) *The gang-free equilibrium*

$$\mathcal{E}_0 = (1, 0, 0)$$

is locally asymptotically stable if

$$\mathcal{R}_0^{\text{full}} < 1,$$

and unstable if

$$\mathcal{R}_0^{\text{full}} > 1.$$

(b) If

$$\mathcal{R}_0^{\text{full}} > 1,$$

then the active-gang compartment is uniformly persistent. That is, there exists $\delta > 0$, such that every solution with $g(0) > 0$ satisfies

$$\liminf_{t \rightarrow \infty} g(t) \geq \delta.$$

(c) If

$$\max \left\{ \frac{\beta_{sg} a_{sg}}{(\mu + \gamma) \min\{a_{ss}, a_{sg}, a_{sr}\}}, \frac{\beta_{rg} a_{rg}}{(\mu + \gamma) \min\{a_{rs}, a_{rg}, a_{rr}\}} \right\} < 1, \quad (3.4)$$

then $\mathcal{E}_0 = (1, 0, 0)$ is globally asymptotically stable in $\mathcal{D}$.

The proof is given in Appendix A.

Implication of Theorem 3.2. The threshold $\mathcal{R}_0^{\text{full}}$ provides the invasion criterion for the formal fixed-weight extension. If $\mathcal{R}_0^{\text{full}} < 1$, small introductions of active gang members decay near the gang-free state. If $\mathcal{R}_0^{\text{full}} > 1$, the active gang class is uniformly persistent under the theorem assumptions, meaning that gang activity cannot approach zero along trajectories starting with $g(0) > 0$. The global-stability condition in part (c) is stronger and more conservative; it ensures convergence to the gang-free equilibrium from every feasible initial condition. The bifurcation results below show why this distinction matters: Even when a local threshold suggests elimination, a positive gang equilibrium can coexist with the gang-free state in a backward-bifurcation regime.

3.3. Gang-present equilibria in the homogeneous-contact reduction

The full normalized model is algebraically difficult to solve for positive gang equilibria under arbitrary heterogeneous contact weights. To obtain explicit equilibrium conditions, we consider the homogeneous-contact reduction

$$a_{ss} = a_{sg} = a_{sr} = a_{rs} = a_{rg} = a_{rr} = 1.$$

Then

$$c_{sg} = g, \quad c_{rs} = s, \quad c_{rg} = g.$$

The reduced system becomes

$$\dot{s} = \mu(1 - s) - \frac{\beta_{sg} s g}{1 + \eta r s + \epsilon g} + \beta_{rs} r s, \quad (3.5)$$

$$\dot{g} = \frac{\beta_{sg} s g}{1 + \eta r s + \epsilon g} + \beta_{rg} r g - (\gamma + \mu)g, \quad (3.6)$$

$$\dot{r} = \gamma g - \beta_{rg} r g - \mu r - \beta_{rs} r s. \quad (3.7)$$

Remark (Relation to the full model). *The homogeneous-contact reduction is the proportional-mixing case obtained by setting all contact-preference weights equal to one; because requiring reciprocity*

at all population states forces proportional mixing, it is the reciprocity-consistent model, whereas the weighted system is a formal fixed-weight extension. The reduced-system analysis therefore identifies relapse-driven mechanisms, including backward bifurcation and bistability, in the reciprocity-consistent model, and extending the bifurcation analysis to genuinely heterogeneous balance-consistent mixing is left open (see the remark following Theorem 3.4). The explicit equilibrium and bifurcation calculations are carried out in the homogeneous-contact case because this reduction permits tractable algebraic and center-manifold analysis.

A gang-present equilibrium is an equilibrium

$$(s^*, g^*, r^*) \in \mathcal{D}$$

with $g^* > 0$. At steady state,

$$\begin{cases} \mu(1-s) - \frac{\beta_{sg}sg}{1+\eta rs + \epsilon g} + \beta_{rs}rs = 0, \end{cases} \quad (3.8)$$

$$\begin{cases} \frac{\beta_{sg}sg}{1+\eta rs + \epsilon g} + \beta_{rg}rg - (\gamma + \mu)g = 0, \end{cases} \quad (3.9)$$

$$\begin{cases} \gamma g - \beta_{rg}rg - \mu r - \beta_{rs}rs = 0. \end{cases} \quad (3.10)$$

Using $s = 1 - g - r$, (3.10) gives

$$g = \frac{r(\mu + \beta_{rs}(1-r))}{(\beta_{rs} - \beta_{rg})r + \gamma}. \quad (3.11)$$

For a positive root $r^* > 0$, the corresponding values are

$$g^* = \frac{r^*(\mu + \beta_{rs}(1-r^*))}{(\beta_{rs} - \beta_{rg})r^* + \gamma}, \quad s^* = 1 - g^* - r^*.$$

The equilibrium is feasible only if

$$s^* \geq 0, \quad g^* > 0, \quad r^* > 0.$$

Substitution of (3.11) into (3.9) yields the quartic equation

$$\begin{aligned} f(r) = & \beta_{rg}^2 \eta r^4 - \beta_{rg} (\beta_{rg} \eta + \beta_{rs} \epsilon + 2\eta \gamma + 2\eta \mu) r^3 \\ & + [\beta_{rg} (\beta_{sg} - \beta_{rg} + \mu \epsilon + \beta_{rs} + \beta_{rs} \epsilon + \eta \mu + 2\eta \gamma) + \beta_{rs} \epsilon (\gamma + \mu) + \eta (\gamma + \mu)^2] r^2 \\ & + [\beta_{rg} (2\gamma + \mu - \beta_{sg}) - \beta_{rs} (\gamma + \mu) (1 + \epsilon) - \beta_{sg} (\gamma + \mu) - \epsilon \mu (\gamma + \mu) - \eta \gamma (\gamma + \mu)] r \\ & - \gamma (\mu - \beta_{sg} + \gamma) = 0. \end{aligned} \quad (3.12)$$

Since Eq (3.12) is quartic in general, the homogeneous-contact system may have multiple feasible positive gang equilibria; when advocacy is absent ($\eta = 0$), the leading coefficient vanishes and the equation reduces to a cubic. Such multiplicity is the algebraic structure underlying bistability and backward bifurcation. We emphasize that the quartic Eq (3.12) may admit up to four real roots, but only those satisfying the feasibility constraints

$$r^* > 0, \quad g^* = \frac{r^*(\mu + \beta_{rs}(1-r^*))}{(\beta_{rs} - \beta_{rg})r^* + \gamma} > 0, \quad s^* = 1 - g^* - r^* \geq 0$$

correspond to feasible gang-present equilibria of the model. Algebraic roots that fail any of these feasibility tests are discarded as biologically and socially meaningless. Throughout this paper we distinguish carefully between (i) the algebraic roots of Eq (3.12), which depend only on the model parameters, and (ii) the feasible positive gang equilibria, which additionally satisfy the population-fraction constraints above.

For example, using

$$\beta_{sg} = 0.855, \quad \beta_{rg} = 1.2, \quad \mu = 0.005, \quad \epsilon = 0.95, \quad \gamma = 0.86, \quad \beta_{rs} = 0.01, \quad \eta = 0,$$

the system has two feasible positive equilibria:

$$(s, g, r) \approx (0.9678, 0.0006, 0.0317),$$

and

$$(s, g, r) \approx (0.0278, 0.2672, 0.7051).$$

This illustrates that low-gang and high-gang states can coexist for the same parameter set.

These parameter values are illustrative dimensional (weekly) rates chosen to expose the backward-bifurcation regime rather than to reproduce empirically calibrated rates for a specific community; the corresponding dimensionless quantities are obtained by dividing the rates by $\mu + \gamma$, as in the nondimensionalization of Table 2. The present triple is a separate illustrative point, chosen to display two coexisting positive equilibria in closed form; in particular its $\beta_{rg} = 1.2$ and $\beta_{rs} = 0.01$ differ from the baseline set of Table 2 ($\beta_{rg} = 1.63$, $\beta_{rs} = 0.0001$) used for the bifurcation diagrams and threshold decomposition.

Table 2. Illustrative parameter values in weekly (dimensional) and dimensionless form. Conversions use $\mu + \gamma = 0.865 \text{ week}^{-1}$ for the baseline value $\gamma = 0.86$, $\mu = 0.005$. The dimensionless form expresses each rate relative to the natural turnover scale and is interpretable independent of the choice of time unit.

Parameter	Weekly	Dimensionless	Interpretation
β_{sg} (illustrative)	0.855	$\tilde{\beta}_{sg} \approx 0.99$	recruitment marginally below turnover
β_{rg} (illustrative)	1.63	$\tilde{\beta}_{rg} \approx 1.88$	relapse exceeds turnover by $\sim 90\%$
β_{rs} (illustrative)	0.0001	$\tilde{\beta}_{rs} \approx 1.2 \times 10^{-4}$	reintegration much slower than turnover
μ	0.005	$\tilde{\mu} \approx 0.0058$	demographic exit negligible relative to desistance
ϵ (illustrative)	0.95	0.95 (unchanged)	perceived-risk deterrence strength
η (illustrative)	0	0 (unchanged)	advocacy off in the baseline
γ	0.86	n/a	sets the time unit (absorbed)

Remark (Stability of positive equilibria). *Analytical stability of positive gang equilibria is not fully characterized for the general quartic case. In the numerical bifurcation diagrams, stability of positive equilibria is determined from the eigenvalues of the Jacobian matrix of the two-dimensional reduced system on $\mathcal{D}$ (defined in Eq (3.1)). Thus, blue and green branches in the diagrams represent numerically*

classified locally stable equilibria and saddle equilibria, respectively. Near the transcritical bifurcation point, local stability is determined by the center-manifold normal form used in Theorem 3.4.

3.4. A tractable case: $\beta_{rs} = 0$

To obtain explicit root conditions, we consider the special case

$$\beta_{rs} = 0.$$

Under the decoupled formulation the advocacy term ηrs does not vanish when $\beta_{rs} = 0$, since advocacy effectiveness η is independent of the reintegration rate. For the explicit root conditions below we additionally set $\eta = 0$ (advocacy absent), the baseline regime used throughout the numerical illustrations. This case may be interpreted as a structured reentry regime in which individuals who have exited gang involvement remain in the reformed class over the modeled time horizon, so return from the reformed class to the susceptible or neutral class is negligible.

When $\beta_{rs} = 0$, (3.12) reduces to

$$r^2 + a_1 r + a_2 = 0, \quad (3.13)$$

where

$$a_1 = \frac{(-\mu\epsilon - \beta_{sg} + 2\beta_{rg})\gamma + (\mu - \beta_{sg})\beta_{rg} - \mu^2\epsilon - \mu\beta_{sg}}{\beta_{rg}(\mu\epsilon + \beta_{sg} - \beta_{rg})}, \quad (3.14)$$

$$a_2 = -\frac{\gamma(\mu - \beta_{sg} + \gamma)}{\beta_{rg}(\mu\epsilon + \beta_{sg} - \beta_{rg})} = -\frac{\gamma(\mu + \gamma)(1 - \mathcal{R}_0^{\text{simp}})}{\beta_{rg}(\mu\epsilon + \beta_{sg} - \beta_{rg})}. \quad (3.15)$$

Theorem 3.3 (Homogeneous reduction, strong-relapse case: Positive gang equilibria when $\beta_{rs} = 0$ and $\mu\epsilon + \beta_{sg} < \beta_{rg}$). Assume

$$\beta_{rs} = 0, \quad \beta_{rg} > 0,$$

and

$$\mu\epsilon + \beta_{sg} < \beta_{rg}.$$

Then the following statements hold for the homogeneous-contact system.

- (a) If $\mathcal{R}_0^{\text{simp}} > 1$, then Eq (3.13) has exactly one positive root. This root corresponds to a feasible positive gang equilibrium if

$$g^* > 0, \quad s^* = 1 - g^* - r^* \geq 0.$$

- (b) If $\mathcal{R}_0^{\text{simp}} < 1$, $a_1 < 0$, and $a_1^2 - 4a_2 > 0$, then Eq (3.13) has two distinct positive roots. These roots correspond to two feasible positive gang equilibria if

$$g_i^* > 0, \quad s_i^* = 1 - g_i^* - r_i^* \geq 0, \quad i = 1, 2.$$

(c) If $\mathcal{R}_0^{\text{simp}} < 1$ and either $a_1^2 - 4a_2 < 0$, or the real roots fail the feasibility conditions above, then no feasible positive gang equilibrium is obtained from Eq (3.13).

For each positive root r^* , the corresponding value of g^* is

$$g^* = \frac{\mu r^*}{\gamma - \beta_{rg} r^*},$$

and

$$s^* = 1 - g^* - r^*.$$

The proof is given in Appendix A.

Implications of Theorem 3.3. This theorem gives explicit algebraic conditions under which relapse can generate multiple positive gang equilibria. In particular, two feasible positive equilibria may occur when $\mathcal{R}_0^{\text{simp}} < 1$, which is the equilibrium structure associated with backward bifurcation. Thus, even when the gang-free equilibrium is locally stable, a positive gang state may also exist. The condition

$$\mu\epsilon + \beta_{sg} < \beta_{rg}$$

shows that sufficiently strong relapse can create nonlinear threshold behavior that is not predicted by the local invasion criterion alone.

3.5. Backward and forward bifurcations

Forward bifurcation means that, as $\mathcal{R}_0^{\text{simp}}$ passes through one, the gang-free equilibrium changes stability and a positive gang equilibrium emerges for $\mathcal{R}_0^{\text{simp}} > 1$. Backward bifurcation means that a positive gang equilibrium can coexist with a locally stable gang-free equilibrium for some values with $\mathcal{R}_0^{\text{simp}} < 1$. This bistability makes control more difficult because reducing the invasion threshold below one may not be sufficient to eliminate an already established positive gang state.

Following Castillo-Chavez and Song [46], we use center-manifold theory to determine the bifurcation direction at

$$\mathcal{R}_0^{\text{simp}} = 1.$$

Let β_{sg} be the bifurcation parameter and define

$$\beta_{sg}^* = \mu + \gamma.$$

Then

$$\mathcal{R}_0^{\text{simp}} = 1 \iff \beta_{sg} = \beta_{sg}^*.$$

Theorem 3.4 (Homogeneous reduction: Bifurcation type). *For the homogeneous-contact SGR system Eqs (3.5)–(3.7), and provided the nondegeneracy condition $a \neq 0$ holds (where a is the center-manifold coefficient computed in Appendix A), the bifurcation at*

$$\mathcal{R}_0^{\text{simp}} = 1$$

is backward if

$$\beta_{rg} > \frac{\beta_{sg}^*}{\gamma} [(\mu + \beta_{rs})(1 + \epsilon) + \gamma(1 + \eta)], \quad (3.16)$$

and forward if the inequality in Eq (3.16) is reversed. At equality in Eq (3.16) the coefficient a vanishes identically, and higher-order terms (beyond the quadratic normal form used here) determine the bifurcation direction; this degenerate case is not analyzed.

The proof is given in Appendix A.

Implications of Theorem 3.4. Theorem 3.4 identifies relapse as the key mechanism capable of generating backward bifurcation. If β_{rg} is sufficiently large, then a positive gang equilibrium may coexist with the gang-free equilibrium even when $\mathcal{R}_0^{\text{simp}} < 1$. This does not mean persistence in the uniform dynamical-systems sense; rather, it means bistability: The long-term outcome depends on initial conditions and basin membership. This result suggests that relapse prevention and reintegration support are structurally important, not merely secondary interventions.

Remark (Consistency of the $\beta_{rs} = 0$ equilibria with the backward-bifurcation condition). *Theorem 3.3 and Theorem 3.4 describe the same subthreshold coexistence from complementary viewpoints. In the case $\beta_{rs} = 0$ with advocacy absent ($\eta = 0$), the backward-bifurcation condition Eq (3.16) specializes to*

$$\beta_{rg} > \frac{\beta_{sg}^*}{\gamma} [\mu(1 + \epsilon) + \gamma],$$

and the equilibrium equation reduces to the quadratic of Theorem 3.3. As $\mathcal{R}_0^{\text{simp}} \rightarrow 1^-$, the two-positive-equilibrium regime of Theorem 3.3(b) emerges precisely at this threshold, so the two explicit positive equilibria there realize the subthreshold coexistence whose local emergence Theorem 3.4 characterizes through the center-manifold coefficient. The two results are therefore consistent: Theorem 3.3 exhibits the two positive equilibria explicitly in a tractable case without classifying their stability, while Theorem 3.4 establishes the local bifurcation direction; that this subthreshold branch yields two coexisting stable attractors (bistability) is established numerically in Section 4.1.

Remark (Contact heterogeneity and the limits of the reciprocity balance). *The backward-bifurcation analysis in Theorem 3.4 is carried out in the proportional-mixing case $\mathbf{a} = \mathbf{1}$, which is the contact structure consistent with the reciprocity balance Eq (2.5) at every population state. Extending the analysis to genuinely heterogeneous contact weights is more delicate than a direct perturbation of $\mathbf{a}$ would suggest. Requiring the reciprocity balance $a_{ij} \sum_k a_{jk} N_k = a_{ji} \sum_k a_{ik} N_k$ to hold for all population partitions (S, G, R) , rather than only at a fixed reference partition, equates the coefficient of each N_k and forces $a_{ij} a_{jk} = a_{ji} a_{ik}$ for all i, j, k ; taking $k = i$ gives $a_{ij} = a_{ji}$, so each row of $\mathbf{a}$ is constant and the contact fractions reduce to $c_{ij} = N_j/N$. Proportional mixing is therefore the only fixed-weight contact structure that remains balance-consistent across the evolving state space; genuinely heterogeneous fixed weights cannot satisfy the reciprocity balance globally. A backward-bifurcation analysis for genuinely heterogeneous, balance-consistent mixing consequently requires a state-dependent or relaxed-reciprocity contact construction, which we identify as an open problem in Section 5. The center-manifold coefficient $a(\mathbf{a})$ of Theorem 3.4 is continuous in $\mathbf{a}$, so the backward-bifurcation behavior is not knife-edge*

under small perturbations of the weights that maintain reciprocity at the gang-free reference state; we emphasize, however, that such perturbations do not preserve the balance along trajectories, so this continuity alone does not establish a globally balance-consistent heterogeneous result.

Corollary 3.5 (Dimensionless form of the bifurcation condition). *Using the dimensionless parameters introduced in Section 2.7, the backward-bifurcation condition Eq (3.16) takes the equivalent form*

$$\tilde{\beta}_{rg} > \frac{(\tilde{\mu} + \tilde{\beta}_{rs})(1 + \epsilon)}{1 - \tilde{\mu}} + (1 + \eta), \quad (3.17)$$

where $\tilde{\beta}_{ij} = \beta_{ij}/(\mu + \gamma)$, $\tilde{\mu} = \mu/(\mu + \gamma)$, and η, ϵ are dimensionless. In the demographically negligible limit $\tilde{\mu} \rightarrow 0$, the condition reduces to

$$\tilde{\beta}_{rg} > \tilde{\beta}_{rs}(1 + \epsilon) + (1 + \eta).$$

Proof of Corollary 3.5. Substitute $\beta_{rg} = \tilde{\beta}_{rg}(\mu + \gamma)$, $\beta_{rs} = \tilde{\beta}_{rs}(\mu + \gamma)$, $\mu = \tilde{\mu}(\mu + \gamma)$, and $\beta_{sg}^* = \mu + \gamma$ into (3.16), and divide both sides by $\mu + \gamma$. The factor $\beta_{sg}^*/\gamma = (\mu + \gamma)/\gamma = 1/(1 - \tilde{\mu})$, where $1 - \tilde{\mu} = \gamma/(\mu + \gamma)$. Routine algebra then yields (3.17). $\square$

Interpretation of Corollary 3.5. The dimensionless form makes the structure of the bifurcation criterion transparent. Backward bifurcation requires that the dimensionless relapse rate exceed the sum of two contributions: A “reformed-class drag” term $(\tilde{\mu} + \tilde{\beta}_{rs})(1 + \epsilon)/(1 - \tilde{\mu})$, inflated by perceived-risk deterrence and by the demographic-exit fraction, plus a structural baseline of unity together with the advocacy contribution η . The threshold is therefore controlled by ratios of rates rather than by the rates themselves; in particular, the bifurcation criterion does not depend on the choice of time unit, addressing the empirical-interpretability concern that the raw weekly rates raise. For the illustrative parameter values in Table 2, the right-hand side evaluates to approximately $(0.0058 + 0.00012)(1 + 0.95)/(1 - 0.0058) + 1 \approx 1.012$ at the baseline $\eta = 0$, so backward bifurcation occurs once $\tilde{\beta}_{rg} > 1.012$; the illustrative value $\tilde{\beta}_{rg} \approx 1.88$ lies well inside the backward-bifurcation regime, consistent with the bistability visible in Figure 3.

Numerical decomposition of the threshold. For interpretive clarity, it is useful to separate the structural and parameter-dependent contributions to the threshold:

$$\tilde{\beta}_{rg}^* = \underbrace{1}_{\text{structural baseline}} + \underbrace{\frac{(\tilde{\mu} + \tilde{\beta}_{rs})(1 + \epsilon)}{1 - \tilde{\mu}}}_{\text{reformed-class drag}} + \underbrace{\eta}_{\text{advocacy contribution}}, \quad (3.18)$$

where the structural baseline of unity reflects the irreducible algebraic floor below which backward bifurcation cannot occur, regardless of intervention parameters. For the illustrative parameter values in Table 2 with advocacy off ($\eta = 0$), the three contributions evaluate to 1.000, 0.011, and 0, respectively, summing to $\tilde{\beta}_{rg}^* \approx 1.012$. Unlike the reformed-class drag and the deterrence terms, which act on the threshold only through the small dimensionless quantities $\tilde{\mu}$ and $\tilde{\beta}_{rs}$, the advocacy contribution enters additively as η , at leading algebraic order. The baseline $\eta = 0$ isolates the relapse mechanism;

the algebraic effect of activating advocacy is quantified by the sensitivity $\partial\tilde{\beta}_{rg}^*/\partial\eta = 1$ below. In the illustrative regime the relapse rate is what places the system well inside the backward-bifurcation region, and advocacy effectiveness enters the threshold directly rather than only through the small quantities $\tilde{\mu}$ and $\tilde{\beta}_{rs}$.

Interpretation for the illustrative regime. Treating $\tilde{\beta}_{rg}$ as the principal lever and holding the remaining illustrative parameters fixed, the illustrative value $\tilde{\beta}_{rg} \approx 1.88$ is approximately $1.86\times$ the threshold $\tilde{\beta}_{rg}^* \approx 1.012$. For this illustrative parameter set, leaving the backward-bifurcation regime with all other parameters held fixed would correspond to reducing the dimensionless relapse rate by approximately 46%; in dimensional terms, β_{rg} would fall from $\approx 1.63/\text{week}$ to below $\approx 0.87/\text{week}$. These figures are properties of the selected numerical example, not estimates of an intervention requirement, and carry no calibration to an observed community baseline. The bifurcation analysis does not indicate whether such a reduction is achievable through realistic interventions; that judgement requires community-specific calibration data outside the scope of this paper. What the analysis provides is a transparent, dimensionless benchmark, through Corollary 3.5 and the decomposition in Eq (3.18), that makes the relative algebraic roles of relapse, deterrence, and advocacy explicit for a given parameter set, rather than a calibrated intervention target. Within this benchmark, changes in advocacy effectiveness (η) enter the threshold at leading algebraic order ($\partial\tilde{\beta}_{rg}^*/\partial\eta = 1$) and also modulate equilibrium gang prevalence within each branch (see Section 4.7), while reductions in β_{rg} or increases in γ act on the threshold through $\tilde{\beta}_{rg}$ and $\tilde{\mu}$.

Analytical sensitivity of the threshold. Differentiating equation (3.18) yields closed-form partial derivatives of the threshold with respect to each dimensionless parameter:

$$\begin{aligned}\frac{\partial\tilde{\beta}_{rg}^*}{\partial\tilde{\mu}} &= \frac{(1+\epsilon)(1+\tilde{\beta}_{rs})}{(1-\tilde{\mu})^2}, & \frac{\partial\tilde{\beta}_{rg}^*}{\partial\tilde{\beta}_{rs}} &= \frac{1+\epsilon}{1-\tilde{\mu}}, \\ \frac{\partial\tilde{\beta}_{rg}^*}{\partial\epsilon} &= \frac{\tilde{\mu}+\tilde{\beta}_{rs}}{1-\tilde{\mu}}, & \frac{\partial\tilde{\beta}_{rg}^*}{\partial\eta} &= 1.\end{aligned}\tag{3.19}$$

Evaluated at the illustrative parameter values from Table 2, these partials (listed in the order $\tilde{\mu}, \tilde{\beta}_{rs}, \epsilon, \eta$) are approximately 1.97, 1.96, 0.0059, and 1, respectively. These are algebraic sensitivities of the threshold with respect to parameters that differ in meaning and in plausible range, so the bare partial derivatives should not be read as a ranking of intervention effect sizes. The advocacy partial $\partial\tilde{\beta}_{rg}^*/\partial\eta = 1$ is of order unity rather than the near-zero value $\tilde{\beta}_{rs} \approx 10^{-4}$ that the advocacy term took in the earlier coupled form, so decoupling advocacy effectiveness from the reintegration rate is what lets η enter the threshold at leading algebraic order at all; whether a given change in η is achievable, and how it compares with feasible changes in the other parameters, depends on their empirical ranges and is not determined by the derivatives alone. The deterrence partial $\partial/\partial\epsilon \propto \tilde{\mu} + \tilde{\beta}_{rs}$ remains small because it acts only through coupling to the small quantities $\tilde{\mu}$ and $\tilde{\beta}_{rs}$. This analytical sensitivity is complementary to the baseline-centered PRCC analysis in Section 4.7: the PRCC quantifies how parameters affect long-term gang prevalence at fixed bifurcation structure, whereas Eq (3.19) quantifies how parameters shift the threshold itself.

Table 3. Summary of key mathematical results and qualitative implications.

Mathematical result	Condition	Qualitative implication
Local stability of gang-free equilibrium	$\mathcal{R}_0^{\text{full}} < 1$	Small introductions of active gang members decay near the gang-free state.
Uniform persistence of active gang class	$\mathcal{R}_0^{\text{full}} > 1$, under the assumptions of Theorem 3.2	Active gang involvement persists away from zero for trajectories with $g(0) > 0$.
Global stability of gang-free equilibrium	Eq (3.4)	Recruitment and relapse are uniformly too weak to sustain a positive gang state from any feasible initial condition.
Backward bifurcation in homogeneous-contact reduction	$\mathcal{R}_0^{\text{simp}} = 1$ and Eq (3.16)	A local backward branch permits a positive gang equilibrium to coexist with the gang-free state for some $\mathcal{R}_0^{\text{simp}} < 1$; stability and bistability are established numerically.
Forward bifurcation in homogeneous-contact reduction	Reverse inequality in Eq (3.16)	The transition through $\mathcal{R}_0^{\text{simp}} = 1$ follows the standard forward-threshold scenario.

Remark (Limiting cases of the threshold). *The decomposition in Eq (3.18) exposes the structural role of each parameter through its limiting behavior. In the strong-relapse limit $\tilde{\beta}_{rs} \rightarrow 0$ (no transitions back to the susceptible class once an individual is reformed), the threshold reduces to*

$$\tilde{\beta}_{rg}^* \rightarrow 1 + \frac{\tilde{\mu}(1 + \epsilon)}{1 - \tilde{\mu}},$$

so that the reformed-class drag reduces to its demographic-exit part. The advocacy contribution η is unaffected by this limit, since advocacy effectiveness is independent of the reintegration rate; this is the key structural difference from the earlier coupled form, in which the advocacy term $\tilde{\alpha}\tilde{\beta}_{rs}$ vanished together with β_{rs} . The case $\beta_{rs} = 0$ with advocacy also absent ($\eta = 0$) is treated directly in Theorem 3.3. In the demographically negligible limit $\tilde{\mu} \rightarrow 0$ (reformation much faster than demographic turnover, the typical regime for short-timescale gang dynamics), the threshold becomes

$$\tilde{\beta}_{rg}^* \rightarrow 1 + \tilde{\beta}_{rs}(1 + \epsilon) + \eta,$$

which is the form most directly comparable across empirical settings with similar reformation timescales. In the no-deterrence limit $\epsilon \rightarrow 0$, the perceived-risk component drops out of the reformed-class drag, and in the no-advocacy limit $\eta \rightarrow 0$ the advocacy contribution vanishes; in all four cases the structural baseline of unity remains, reflecting that backward bifurcation requires the dimensionless relapse rate

to exceed the natural decay rate of the active gang class regardless of which auxiliary mechanisms are present or absent.

The principal analytical results developed in this section are collected for reference in Table 3.

3.6. The no-relapse case

The bifurcation condition in Theorem 3.4 clarifies the role of relapse. If relapse is removed by setting

$$\beta_{rg} = 0,$$

then the center-manifold coefficient determining bifurcation direction is negative. Thus, the backward-bifurcation mechanism disappears.

Theorem 3.6 (Homogeneous reduction: No-relapse case). *Assume*

$$\beta_{rg} = 0.$$

Then the homogeneous-contact system Eqs (3.5)–(3.7) has the following properties:

(a) *If*

$$\mathcal{R}_0^{\text{simp}} < 1,$$

then the gang-free equilibrium $\mathcal{E}_0 = (1, 0, 0)$ is globally asymptotically stable in $\mathcal{D}$.

(b) *If*

$$\mathcal{R}_0^{\text{simp}} > 1,$$

then the gang-free equilibrium is unstable, and the system undergoes a forward bifurcation at $\mathcal{R}_0^{\text{simp}} = 1$. A positive gang equilibrium emerges locally from the gang-free equilibrium.

The proof is given in Appendix A.

Implications of Theorem 3.6. Theorem 3.6 identifies relapse as the critical mechanism responsible for nonstandard threshold behavior. When $\beta_{rg} = 0$, individuals in the reformed class do not return to active gang involvement, and the model no longer supports relapse-driven backward bifurcation. In this case, the threshold

$$\mathcal{R}_0^{\text{simp}} = 1$$

fully separates the gang-elimination and gang-invasion regimes. If $\mathcal{R}_0^{\text{simp}} < 1$, the gang-free equilibrium is globally asymptotically stable; if $\mathcal{R}_0^{\text{simp}} > 1$, the gang-free equilibrium is unstable and a positive gang equilibrium emerges through a forward bifurcation. Therefore, without relapse, recruitment reduction and reformation are sufficient to restore standard threshold dynamics. Comparing this theorem with Theorem 3.4 shows that relapse prevention is structurally important because sufficiently strong relapse can create bistability and make gang elimination more difficult even when the reproduction number is below one.

4. Numerical illustration of thresholds, bistability, and intervention mechanisms

In this section, we use bifurcation diagrams and sensitivity analysis to illustrate the theoretical results established in Section 3. The simulations are based on the homogeneous-contact reduction Eqs (3.5)–(3.7), for which

$$\mathcal{R}_0^{\text{simp}} = \frac{\beta_{sg}}{\mu + \gamma}.$$

This reduced system is the proportional-mixing (reciprocity-consistent) case obtained by setting all contact-preference weights equal to one; the weighted system from which it descends is a formal fixed-weight extension. The numerical results therefore illustrate mechanisms of the reciprocity-consistent model, and the explicit bifurcation conditions are those derived for that reduction.

The simulations focus on five mechanisms: recruitment into gang involvement, perceived-risk deterrence, relapse from the reformed class, return from the reformed class to a susceptible or neutral status, and reformed-led advocacy. Unless otherwise stated, the parameter values are illustrative and chosen to demonstrate nonlinear threshold behavior, including bistability and backward bifurcation. They are not calibrated estimates for a specific community.

The numerical results are organized around Theorems 3.2 to 3.4 and 3.6. First, Theorem 3.2 gives the full-model invasion threshold and persistence criterion. Second, Theorem 3.4 shows that sufficiently strong relapse can generate backward bifurcation in the homogeneous-contact reduction. Third, Theorem 3.6 shows that removing relapse restores a standard forward-threshold scenario. Thus, the simulations below illustrate how recruitment, relapse, deterrence, reintegration, and advocacy shape the threshold and bistability structures predicted by the analysis.

In the bifurcation diagrams, solid blue curves denote locally stable equilibria and dashed green curves denote saddle equilibria. Each branch was computed by root-finding over a grid in the bifurcation parameter: At every grid value all feasible equilibria are located and their stability classified from the eigenvalues of the Jacobian matrix of the two-dimensional reduced system, so the diagrams do not depend on initial conditions. Complete code that reproduces every branch and every saddle-node location is available in the reproducibility repository (see the data and code availability statement). Because the model can exhibit bistability, the long-term dynamics from a given initial condition depend on basin membership as well as on parameter values.

4.1. A worked instance of bistability

Before sweeping across parameter regimes in the bifurcation diagrams below, it is useful to display a concrete numerical instance of the bistability that the analysis predicts. We fix the illustrative parameter values from Tables 1 and 2 and set the recruitment rate to $\beta_{sg} = 0.5(\mu + \gamma) \approx 0.432/\text{week}$, so that $\mathcal{R}_0^{\text{simp}} = 0.5$, well below the standard invasion threshold of unity. By Theorem 3.4 and the threshold value $\tilde{\beta}_{rg}^* \approx 1.012$ derived in Corollary 3.5, the illustrative value $\tilde{\beta}_{rg} \approx 1.88$ places the system inside the backward-bifurcation regime, so a positive gang equilibrium is expected to coexist with the locally

stable gang-free equilibrium.

Numerical root-finding on the steady-state system identifies three feasible equilibria:

$$\begin{aligned}\mathcal{E}_0 &= (1.000, 0.000, 0.000), & (\text{gang-free; stable}) \\ \mathcal{E}_S &= (0.628, 0.007, 0.365), & (\text{low-prevalence saddle}) \\ \mathcal{E}_+ &= (0.036, 0.440, 0.524), & (\text{high-prevalence; stable}),\end{aligned}\tag{4.1}$$

where each triple gives (s^*, g^*, r^*) . The Jacobian eigenvalues at $\mathcal{E}_0$ are $\{-0.0050, -0.4325, -0.0051\}$ (all negative; $\mathcal{E}_0$ is a sink); at $\mathcal{E}_S$, the spectrum is $\{+0.0369, -0.0050, -0.0581\}$ (one positive eigenvalue); and at $\mathcal{E}_+$ the spectrum is $\{-0.1315, -0.0050, -0.7283\}$ (all negative; $\mathcal{E}_+$ is a sink). In each spectrum, the eigenvalue $-0.0050 \approx -\mu$ is the direction transverse to the invariant simplex $s + g + r = 1$; the remaining two eigenvalues are those of the two-dimensional reduced Jacobian on $\mathcal{D}$ and govern the dynamics within the simplex, consistent with the two-dimensional reduction used for the bifurcation diagrams. Within the feasible simplex, the reduced Jacobian at $\mathcal{E}_S$ has eigenvalues $\{+0.0369, -0.0581\}$, so $\mathcal{E}_S$ is a saddle with a one-dimensional stable manifold and a one-dimensional unstable manifold; the transverse eigenvalue -0.0050 adds a stable direction off the simplex that does not affect the basin boundary. The two stable equilibria therefore coexist at the same parameter set, and the one-dimensional stable manifold of the saddle $\mathcal{E}_S$ forms the separatrix between their basins of attraction (Figure 2). The equilibria were located by polynomial (resultant) elimination on the steady-state system, which returns all feasible roots, including the saddle, without dependence on an initial guess; each root satisfies $\|F(\mathcal{E})\|_\infty < 10^{-8}$, and the Jacobian eigenvalues were then obtained by standard numerical eigendecomposition.

The qualitative structure of the saddle (Eq (4.1)) is worth noting. The saddle has a very small active-gang fraction ($g_S \approx 0.7\%$) but a substantial reformed-class fraction ($r_S \approx 37\%$), with the susceptible class taking up the remaining $\approx 62\%$. The interpretation is that the saddle represents a transitional community state in which gang activity is nearly extinguished but a large reformed population maintains a steady inflow of potential relapses; whether this state tips toward elimination or toward re-establishment of substantial gang activity depends on the perturbation direction along the saddle's unstable manifold. This worked example makes a structural feature of the model concrete: relapse-driven bistability is not merely a theoretical possibility at $\mathcal{R}_0^{\text{simp}} = 1$ but persists at recruitment levels substantially below the standard invasion threshold, and the basin separator can sit at low active-gang prevalence when the reformed-class reservoir is large. Interventions that reduce active gang prevalence below some small threshold are therefore not, on their own, sufficient to guarantee convergence to the gang-free state; the reformed-class reservoir and the relapse rate jointly determine which basin is reached.

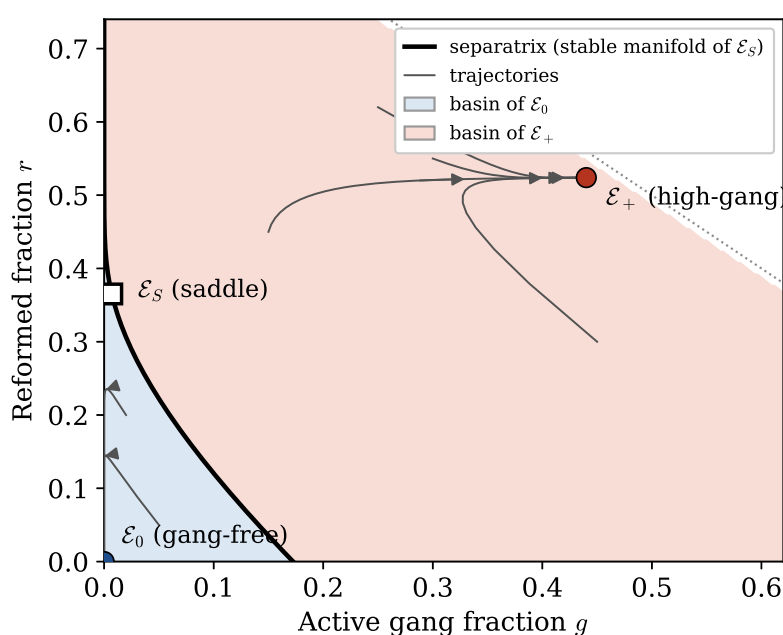


Figure 2. Basin structure of the worked bistable instance at $\mathcal{R}_0^{\text{simp}} = 0.5$ (homogeneous-contact reduction; illustrative parameters from Table 2), projected onto the (g, r) plane with $s = 1 - g - r$. The gang-free sink $\mathcal{E}_0$ and the high-prevalence sink $\mathcal{E}_+$ coexist at the same parameter set; the stable manifold of the low-prevalence saddle $\mathcal{E}_S$ (solid curve) separates their basins of attraction. Trajectories from comparable initial conditions on opposite sides of this separatrix approach different long-term states, so that basin membership, rather than the invasion threshold alone, determines the outcome. The dotted line marks the feasibility boundary $s = 0$.

The worked instance above fixes the recruitment rate at $\mathcal{R}_0^{\text{simp}} = 0.5$, but the bistability structure persists over a much wider range of β_{sg} . Numerical continuation along the same illustrative parameter set, with β_{sg} treated as the bifurcation parameter, locates the saddle-node bifurcation at $\beta_{sg}^{\text{SN}} \approx 0.07573/\text{week}$, corresponding to $\mathcal{R}_0^{\text{SN}} \approx 0.08755$. For $\beta_{sg} < \beta_{sg}^{\text{SN}}$ (equivalently $\mathcal{R}_0^{\text{simp}} < \mathcal{R}_0^{\text{SN}}$) only the gang-free equilibrium is feasible; as β_{sg} increases past β_{sg}^{SN} , the saddle $\mathcal{E}_S$ and the high-prevalence sink $\mathcal{E}_+$ emerge as a real-saddle / sink pair, and for the interval $\beta_{sg}^{\text{SN}} < \beta_{sg} < \mu + \gamma$ (i.e., $0.0876 < \mathcal{R}_0^{\text{simp}} < 1$), the system exhibits bistability between $\mathcal{E}_0$ and $\mathcal{E}_+$. At the saddle-node, the two upper equilibria collide at approximately $(s^*, g^*, r^*) \approx (0.307, 0.174, 0.518)$, with the active-gang prevalence at the collision point an order of magnitude higher than at the saddle in the bistable interior. The width of the bistable wedge for this parameter set is therefore approximately $1.0 - 0.0876 \approx 0.91$ in $\mathcal{R}_0^{\text{simp}}$; for this illustrative set, threshold-lowering alone corresponds to reducing recruitment by approximately 91% below the standard invasion threshold before it guarantees elimination. Within the bistable interior, by contrast, elimination requires either driving $\mathcal{R}_0^{\text{simp}}$ below $\mathcal{R}_0^{\text{SN}}$ or direct intervention on the (g, r) state to cross the saddle's stable manifold. This is a substantively different policy posture from the standard forward-threshold case, in which any reduction of $\mathcal{R}_0$ below unity is sufficient asymptotically.

4.2. Recruitment-driven backward bifurcation

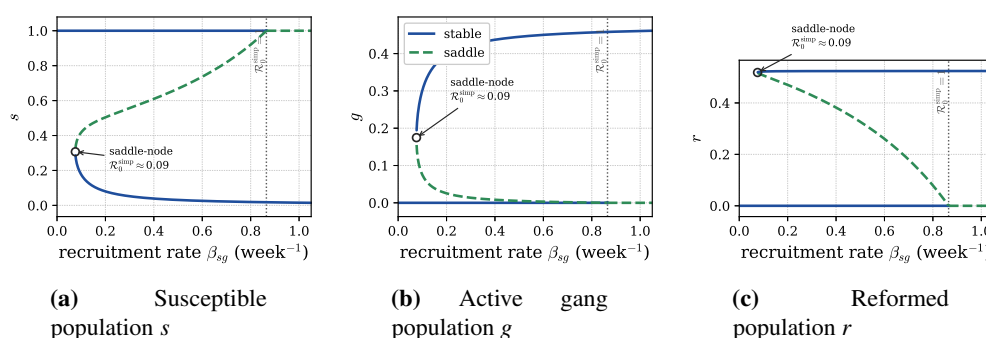


Figure 3. Bifurcation diagrams for the homogeneous-contact SGR system Eqs (3.5)–(3.7) with the recruitment parameter β_{sg} as the bifurcation parameter. Solid blue curves denote locally stable equilibria, and dashed green curves denote saddle equilibria; open circles mark saddle-node points. The diagrams illustrate bistability over an intermediate range of β_{sg} , where the gang-free equilibrium and a positive gang equilibrium coexist. Parameter values are $\mu = 0.005$, $\gamma = 0.86$, $\beta_{rg} = 1.63$, $\beta_{rs} = 0.0001$, $\eta = 0$, and $\epsilon = 0.95$. In dimensionless form, the horizontal axis β_{sg} (in week⁻¹) corresponds to $\tilde{\beta}_{sg} = \beta_{sg}/(\mu + \gamma) = \mathcal{R}_0^{\text{simp}}$, so the threshold $\mathcal{R}_0^{\text{simp}} = 1$ corresponds to $\beta_{sg} = \mu + \gamma \approx 0.865$ week⁻¹, marked by the vertical dotted line. The branches were obtained by solving the steady-state equations $\dot{s} = \dot{g} = \dot{r} = 0$ and classifying stability from the eigenvalues of the Jacobian; the diagrams do not depend on initial conditions.

Figure 3 illustrates the backward-bifurcation mechanism identified in Theorem 3.4. Since

$$\mathcal{R}_0^{\text{simp}} = \frac{\beta_{sg}}{\mu + \gamma},$$

varying β_{sg} directly varies the simplified reproduction number. For sufficiently small β_{sg} , trajectories approach the gang-free equilibrium $(s, g, r) = (1, 0, 0)$. For an intermediate range of recruitment values, the system exhibits bistability: the gang-free equilibrium and a positive gang equilibrium are locally stable, separated by a saddle equilibrium.

The active-gang panel in Figure 3(b) illustrates why a local threshold alone is insufficient in a backward-bifurcation regime. Even when $\mathcal{R}_0^{\text{simp}} < 1$, a sufficiently large initial gang presence can converge to a positive gang equilibrium. Thus, Figure 3 complements Theorem 3.2: The local threshold determines whether small introductions grow or decay near the gang-free equilibrium, while the bifurcation structure determines whether additional positive equilibria coexist away from the gang-free state.

The reformed class in Figure 3(c) follows the same bistable structure because individuals enter R through exit from the active gang class. When the active gang equilibrium is high, the reformed class can also remain large, reflecting continued cycling among gang involvement, reformation, and relapse. The

susceptible-class panel (a) mirrors this inversely: when gang prevalence is high, the susceptible fraction is depleted.

4.3. Effect of perceived-risk deterrence

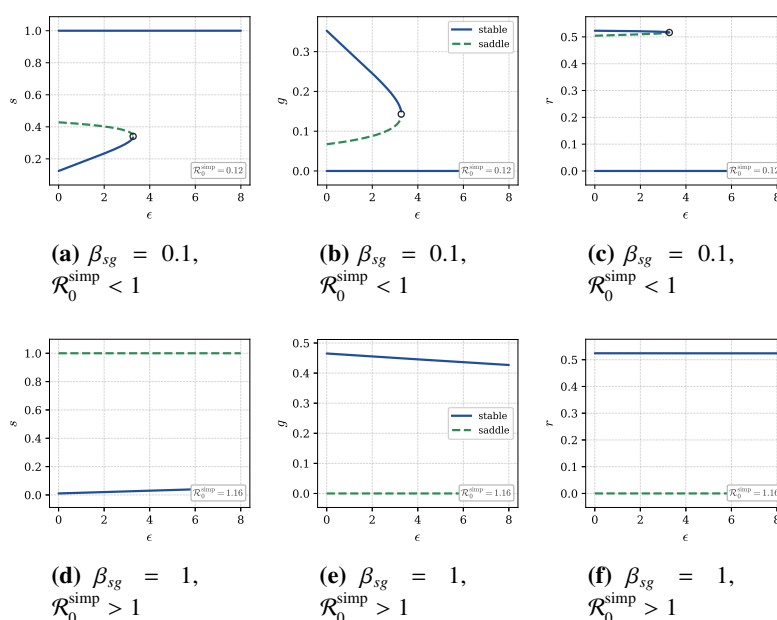


Figure 4. Bifurcation diagrams for the homogeneous-contact SGR system with perceived-risk deterrence ϵ as the bifurcation parameter. The top row corresponds to a lower recruitment regime, and the bottom row corresponds to a higher recruitment regime. Solid blue curves denote locally stable equilibria, and dashed green curves denote saddle equilibria; open circles mark saddle-node points. Fixed parameters are $\mu = 0.005$, $\gamma = 0.86$, $\beta_{rg} = 1.63$, $\beta_{rs} = 0.0001$, and $\eta = 0$. The fixed value of $\mathcal{R}_0^{\text{simp}}$ is annotated in each panel. Branches are obtained by solving the steady-state equations and classifying stability from the Jacobian eigenvalues, so the diagrams do not depend on initial conditions.

Figure 4 illustrates the effect of perceived-risk deterrence. In the homogeneous-contact recruitment term,

$$\frac{\beta_{sg}sg}{1 + \eta rs + \epsilon g},$$

larger ϵ reduces the effective flow from S to G . Thus, ϵ represents recruitment suppression associated with perceived risks such as violence, legal consequences, incarceration, social stigma, or public awareness of gang-related harm.

This effect connects directly to the backward-bifurcation condition in Theorem 3.4. Increasing ϵ raises the relapse threshold required for backward bifurcation because ϵ appears in the term

$$(\mu + \beta_{rs})(1 + \epsilon)$$

in (3.16). Thus, stronger perceived-risk deterrence can make relapse-driven bistability harder to sustain. However, once the system is in the basin of attraction of a positive gang equilibrium, increasing ϵ may reduce the equilibrium level without immediately eliminating gang activity.

4.4. Effect of relapse from the reformed class

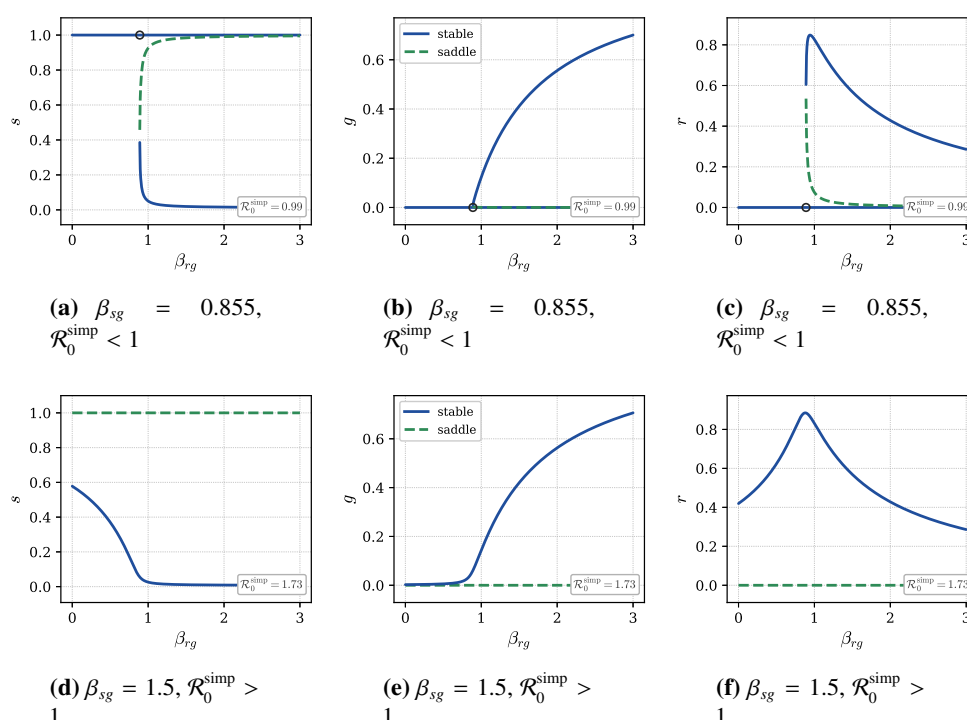


Figure 5. Bifurcation diagrams for the homogeneous-contact SGR system with relapse rate β_{rg} as the bifurcation parameter. The top row corresponds to a lower recruitment regime, and the bottom row corresponds to a higher recruitment regime. Solid blue curves denote locally stable equilibria, and dashed green curves denote saddle equilibria; open circles mark saddle-node points. Fixed parameters are $\mu = 0.005$, $\gamma = 0.86$, $\beta_{rs} = 0.0001$, $\eta = 0$, and $\epsilon = 0.95$. The fixed value of $\mathcal{R}_0^{\text{simp}}$ is annotated in each panel. Branches are obtained by solving the steady-state equations and classifying stability from the Jacobian eigenvalues, so the diagrams do not depend on initial conditions.

Figure 5 shows that relapse is the dominant mechanism behind the nonstandard threshold behavior. This directly supports Theorem 3.4, where backward bifurcation occurs when

$$\beta_{rg} > \frac{\beta_{sg}^*}{\gamma} [(\mu + \beta_{rs})(1 + \epsilon) + \gamma(1 + \eta)].$$

As β_{rg} increases, the system is more likely to exhibit bistability, and the positive gang equilibrium generally becomes larger.

These results also clarify the role of Theorem 3.6. When $\beta_{rg} = 0$, the model has standard forward-threshold behavior: If $\mathcal{R}_0^{\text{simp}} < 1$, the gang-free equilibrium is globally asymptotically stable in the homogeneous-contact reduction. The bifurcation diagrams in this subsection show what changes when relapse is reintroduced. A sufficiently strong relapse pathway can create a positive gang equilibrium even below the threshold and can make elimination depend on initial conditions.

The practical implication is that exit programs are unlikely to be sufficient if reformed individuals remain strongly exposed to relapse pathways. Reducing β_{rg} is not merely a secondary intervention; it changes the qualitative dynamics by weakening or removing the backward-bifurcation mechanism.

4.5. Effect of return from reformed to susceptible status

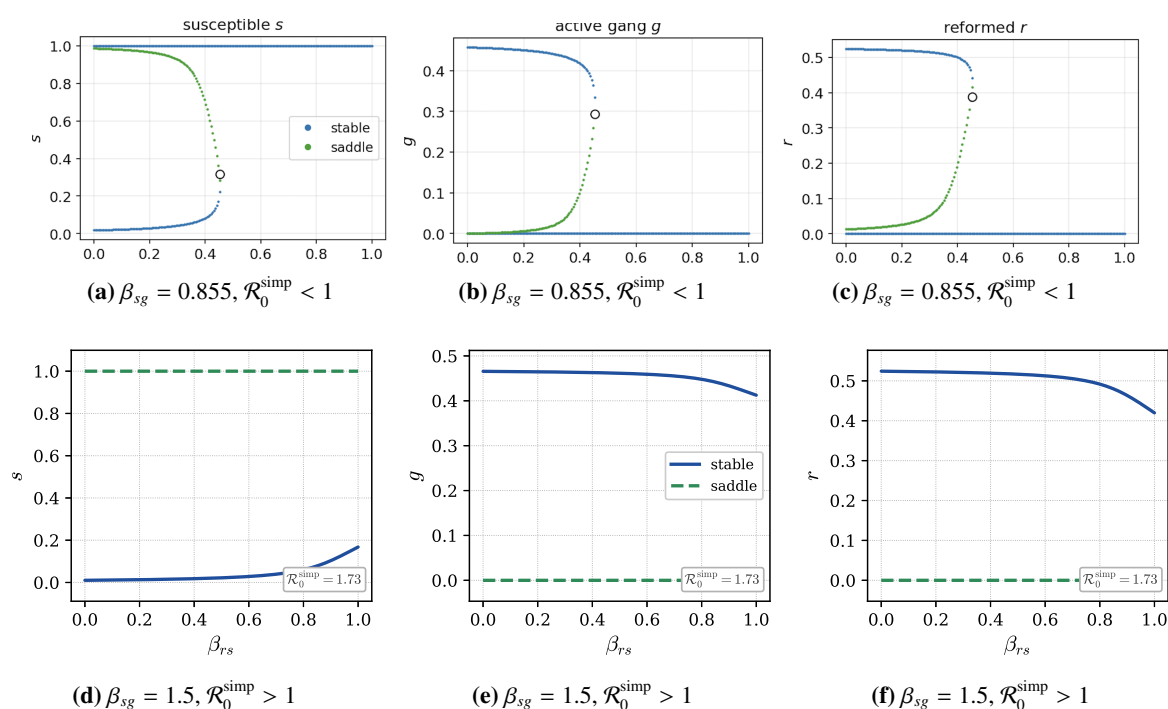


Figure 6. Bifurcation diagrams for the homogeneous-contact SGR system with β_{rs} as the bifurcation parameter. Parameter β_{rs} represents the transition from the reformed class to a susceptible or neutral status; under the decoupled, formulation it does not affect advocacy, which is governed by the independent effectiveness parameter η . Solid blue curves denote locally stable equilibria, and dashed green curves denote saddle equilibria; open circles mark saddle-node points. Fixed parameters are $\mu = 0.005$, $\gamma = 0.86$, $\beta_{rg} = 1.63$, $\eta = 0$, and $\epsilon = 0.95$. The fixed value of $\mathcal{R}_0^{\text{simp}}$ is annotated in each panel. Branches are obtained by solving the steady-state equations and classifying stability from the Jacobian eigenvalues, so the diagrams do not depend on initial conditions.

Figure 6 illustrates how the transition rate β_{rs} affects equilibrium structure. In the model, β_{rs} represents movement from the reformed class back to a susceptible or neutral state. This transition may correspond to reintegration, completion of a reform program, reduced engagement in active advocacy, or return to a general at-risk community environment.

The role of β_{rs} should be interpreted through the theoretical results. In the tractable case $\beta_{rs} = 0$, Theorem 3.3 shows that multiple positive gang equilibria can occur when relapse is sufficiently strong. Thus, even without return from R to S , relapse alone can support the equilibrium structure associated with backward bifurcation. When $\beta_{rs} > 0$, reformed individuals leave the reformed class for the susceptible or neutral class at rate β_{rs} , which drains the reformed reservoir that sustains relapse.

Figure 6 shows the consequence of this reintegration outflow. In the lower recruitment regime, increasing β_{rs} removes the positive gang equilibria and closes the bistable interval, because a smaller reformed class supplies fewer relapses into active gang involvement. In the higher recruitment regime, where $\mathcal{R}_0^{\text{simp}} > 1$, a single positive gang equilibrium persists and its level declines gradually as β_{rs} increases. Under the decoupled formulation, β_{rs} is a clean index of reintegration alone: advocacy is governed by the independent effectiveness parameter η through the term ηrs in the recruitment denominator, so reintegration and advocacy act as separate levers. This separation lets the durability of reformed status and the per-contact effectiveness of advocacy be calibrated independently, consistent with the model formulation in Section 2.5 and the first-order advocacy threshold of Corollary 3.5.

4.6. Effect of reformed-led advocacy

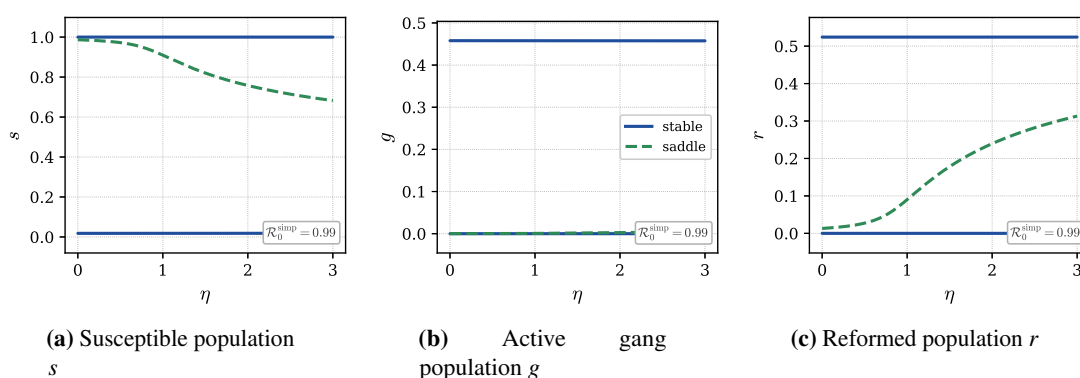


Figure 7. Bifurcation diagrams for the homogeneous-contact SGR system with reformed-led advocacy effectiveness η as the bifurcation parameter. Solid blue curves denote locally stable equilibria, and dashed green curves denote saddle equilibria; open circles mark saddle-node points. Fixed parameters are $\mu = 0.005$, $\gamma = 0.86$, $\beta_{sg} = 0.855$, $\beta_{rg} = 1.63$, $\beta_{rs} = 0.0001$, and $\epsilon = 0.95$. The fixed value of $\mathcal{R}_0^{\text{simp}}$ is annotated in each panel. Branches are obtained by solving the steady-state equations and classifying stability from the Jacobian eigenvalues, so the diagrams do not depend on initial conditions.

Figure 7 examines the effect of the advocacy-effectiveness parameter η . In the recruitment-reduction function, η scales the suppressive effect of the reformed–susceptible interaction term rc_{rs} . Larger values of η therefore represent stronger reformed-led mentoring, outreach, public testimony, or counternarrative influence per reformed–susceptible contact.

The effect of η is also visible in the backward-bifurcation condition from Theorem 3.4. Since η appears in

$$\gamma(1 + \eta),$$

larger η raises the relapse threshold required for backward bifurcation. By the threshold decomposition Eq (3.18) and the sensitivity $\partial\tilde{\beta}_{rg}^*/\partial\eta = 1$, advocacy effectiveness enters the threshold additively and at first order, so stronger advocacy raises the threshold one-for-one and can make relapse-driven bistability substantially more difficult to sustain.

Because advocacy effectiveness is decoupled from the reintegration rate β_{rs} , this lever is available independently of how durable reformed status is. The magnitude of advocacy’s effect on equilibrium gang prevalence within each branch still depends on the size of the reformed class and the rate of reformed–susceptible interaction, but its effect on the bifurcation threshold itself is first order in η .

4.7. Local sensitivity analysis

To identify which parameters most strongly influence long-term gang prevalence near the illustrative baseline, we perform a baseline-centered PRCC sensitivity analysis using Latin hypercube sampling and partial rank correlation coefficients [47, 48], following standard approaches for nonlinear dynamical models [34, 49]. We sample 1000 parameter sets, varying each parameter uniformly within $\pm 20\%$ of its illustrative baseline value; the advocacy-effectiveness parameter η , whose baseline is zero, is instead sampled uniformly over the advocacy-active range $[0, 1]$. For each parameter set, the long-term active gang fraction g^* is recorded at the high-prevalence equilibrium E_+ , computed directly as the largest- g stable equilibrium of the homogeneous-contact system; this is equivalent to the steady state reached from an initial condition in the high-gang basin, but it isolates the sensitivity of the established-gang equilibrium and makes the response independent of the initial condition.

The parameters included in the sensitivity analysis are the recruitment rate β_{sg} , reformation rate γ , relapse rate β_{rg} , return-to-susceptible rate β_{rs} , perceived-risk deterrence strength ϵ , and advocacy-effectiveness parameter η . A positive PRCC indicates that increasing the parameter is associated with an increase in the long-term active gang fraction, while a negative PRCC indicates that increasing the parameter is associated with a decrease in the long-term active gang fraction.

Because the model can exhibit bistability, the PRCC values characterize the sensitivity of the high-prevalence equilibrium E_+ , conditional on the system occupying the high-gang basin. They do not describe how parameters move the system between basins, which is a separate question governed by the bifurcation structure of Section 3. Across the sampled sets a positive gang equilibrium exists for every parameter combination, so g^* is well defined throughout. Because the sampling range is limited to $\pm 20\%$ of the baseline values (and to $[0, 1]$ for η), the results should be interpreted as local sensitivity

information around the chosen baseline, not as a global ranking over all possible parameter regimes. Under the decoupled formulation β_{rs} and the advocacy effectiveness, η enter the model through separate terms, so their PRCC values are interpreted as independent sensitivity indices.

The PRCC results are consistent with the theoretical analysis. The reformation rate γ and the relapse rate β_{rg} show the largest-magnitude associations with long-term gang prevalence, with opposite signs and comparable magnitude. Reformation is strongly suppressive because it moves individuals out of active gang involvement and appears in the denominator of $\mathcal{R}_0^{\text{simp}}$, whereas relapse is strongly enhancing because, by Theorem 3.4, sufficiently strong relapse is the mechanism that generates backward bifurcation and sustains the high-prevalence equilibrium. Recruitment parameter β_{sg} has a smaller positive effect: It raises $\mathcal{R}_0^{\text{simp}}$ and promotes invasion, but its influence on the established-gang equilibrium is limited because susceptibles are depleted at E_+ , so recruitment contributes comparatively little to sustaining an entrenched gang. The dominant relapse–reformation balance in the sensitivity ranking mirrors the mechanism responsible for the backward bifurcation; the full set of PRCC values with their 95% bootstrap confidence intervals is reported in Table 4.

Table 4. Partial rank correlation coefficients for the long-term active gang fraction g^* at the high-prevalence equilibrium E_+ , from $N = 1000$ Latin hypercube samples (five parameters within $\pm 20\%$ of the illustrative baseline; advocacy η sampled over $[0, 1]$; seed fixed for reproducibility). Confidence intervals are 95% bootstrap percentile intervals from 1000 resamples; a coefficient is flagged significant (*) when its interval excludes zero.

Parameter	PRCC	95% CI	Significant
Reformation γ	−0.977	[−0.981, −0.968]	*
Relapse β_{rg}	+0.976	[+0.968, +0.981]	*
Recruitment β_{sg}	+0.167	[+0.102, +0.228]	*
Deterrence ϵ	−0.059	[−0.126, +0.009]	
Advocacy η	+0.018	[−0.043, +0.076]	
Reintegration β_{rs}	−0.005	[−0.067, +0.059]	

Parameters ϵ , β_{rs} , and the advocacy effectiveness η act through distinct channels. Increasing ϵ suppresses recruitment and raises the relapse threshold in Eq (3.16). Increasing η strengthens the recruitment–reduction term ηrs and raises the bifurcation threshold at first order. The reintegration rate β_{rs} governs return from the reformed class to the susceptible/neutral class. At the high-prevalence equilibrium, these three parameters have small PRCC magnitudes whose 95% bootstrap confidence intervals include zero, so their marginal effect on established-gang prevalence is weak near the baseline. Their more consequential role is on the bifurcation threshold itself Eq (3.19), where advocacy effectiveness η in particular acts at first order; the PRCC on equilibrium prevalence and the analytical sensitivity of the threshold therefore answer complementary questions.

Under the decoupled formulation, β_{rs} governs only the reformed-to-susceptible outflow, and advocacy effectiveness is carried by the independent parameter η . The PRCC for β_{rs} therefore admits a clean mechanistic interpretation as the sensitivity of long-term gang prevalence to reformed-class outflow, and the sensitivity to advocacy effectiveness is read directly from the η PRCC.

Overall, the numerical results support three qualitative conclusions that link directly to the theoretical analysis. First, recruitment reduction is essential because β_{sg} determines the invasion threshold. Second, relapse prevention is critical because β_{rg} is the parameter that controls the backward bifurcation underlying the bistability. Third, reformation and reformed-led prevention can reduce long-term gang prevalence, but their effectiveness depends on maintaining a sufficiently large reformed class, reducing relapse, and keeping trajectories within the basin of attraction of the gang-free or low-gang state.

Table 5. Summary of qualitative predictions: Each modeled mechanism, its qualitative effect on long-term active gang prevalence, and the analytical or numerical result that supports the prediction. The mechanism-to-intervention column lists the principal intervention class associated with reducing each driver.

Mechanism (parameter)	Effect on long-term g^*	Mechanism-to-intervention mapping	Supporting result
Recruitment (β_{sg})	+ (increases)	recruitment-prevention programs; reduction of gang-exposure pathways for youth	$\mathcal{R}_0^{\text{full}}$ formula (3.2); Figure 3
Reformation (γ)	– (decreases)	exit programs, deradicalization support, vocational and educational pathways out of gang involvement	denominator of $\mathcal{R}_0^{\text{simp}}$; negative PRCC in Figure 8
Relapse (β_{rg})	+ (increases); drives the backward bifurcation underlying bistability	relapse-prevention services for former gang members; continued reintegration support	Theorem 3.4; Corollary 3.5; Figure 5
Perceived-risk deterrence (ϵ)	– (decreases positive equilibrium level; raises backward-bifurcation threshold)	public-safety visibility, awareness campaigns, legal-consequence messaging	(3.16); Figure 4
Return from reformed to neutral (β_{rs})	reintegration outflow (advocacy governed separately by η)	reintegration completion, program graduation, transition to neutral community status	Figure 6; PRCC caveat in Section 4.7
Reformed-led advocacy (η)	– (small structural decrease via recruitment reduction; first-order lever on the threshold, $\partial \tilde{\beta}_{rg}^* / \partial \eta = 1$; PRCC unresolved)	mentoring programs, public testimony by former gang members, credible counter-narrative campaigns	Figure 7; (3.16); Table 4

Table 5 summarizes the qualitative effect of each modeled mechanism on long-term active gang

prevalence, together with the analytical result and numerical figure that supports the predicted direction. The table is intended as a compact reference for readers who wish to map specific intervention strategies (recruitment-prevention messaging, perceived-risk deterrence, exit programs, relapse prevention, reintegration support, reformed-led mentoring) to the model components that govern them.

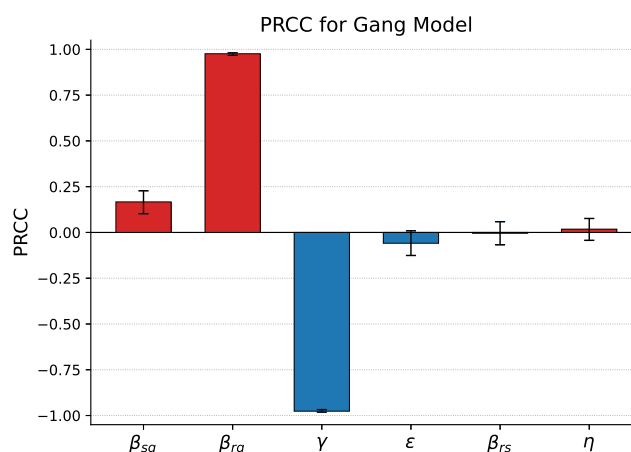


Figure 8. Partial rank correlation coefficients for the long-term active gang fraction g^* at the high-prevalence equilibrium E_+ , from Latin hypercube sampling of $N = 1000$ parameter sets (five parameters within $\pm 20\%$ of their illustrative baseline; advocacy effectiveness η over $[0, 1]$). Positive PRCC values indicate association with a larger long-term active gang fraction (red bars); negative values indicate a suppressive association (blue bars). Error bars are 95% bootstrap confidence intervals. Reformation γ and relapse β_{rg} have the largest-magnitude, opposite-sign associations; recruitment β_{sg} is a smaller positive driver; deterrence ϵ , reintegration β_{rs} , and advocacy η have confidence intervals overlapping zero at the equilibrium level.

The qualitative predictions in Table 5 are intended as model-derived expectations for an at-risk population, whose dynamics are well approximated by the present formulation; the applicability conditions are discussed in Section 5.1.

The negative direction listed for advocacy in Table 5 is the structural direction obtained from the one-parameter advocacy bifurcation (Figure 7) and the recruitment-reduction denominator, in which larger η lowers recruitment and, hence, g^* . This effect is small at the $\eta = 0$ baseline, which is consistent with the advocacy PRCC of +0.018 in Table 4, whose 95% confidence interval contains zero: the small positive PRCC is not evidence of a positive effect, and the two tables report complementary quantities (a structural sign versus a sampled monotone association) rather than a contradiction.

5. Conclusions: Relapse-driven bistability and implications for gang intervention

In this paper, we developed and analyzed a nonlinear SGR model for gang dynamics in an at-risk population, with susceptible individuals S , active gang members G , and reformed individuals R . The model incorporates recruitment into gang involvement, exit into the reformed class, relapse from R to G , return from R to a susceptible or neutral class, perceived-risk deterrence, and reformed-led advocacy. The weighted-contact formulation yields the gang-free equilibrium

$$\mathcal{E}_0 = (1, 0, 0)$$

and the invasion threshold

$$\mathcal{R}_0^{\text{full}} = \frac{a_{sg}\beta_{sg}}{a_{ss}(\mu + \gamma)}.$$

We show that $\mathcal{E}_0$ is locally asymptotically stable when $\mathcal{R}_0^{\text{full}} < 1$, unstable when $\mathcal{R}_0^{\text{full}} > 1$, and that the active gang class is uniformly persistent when $\mathcal{R}_0^{\text{full}} > 1$ under the assumptions of Theorem 3.2. A conservative sufficient condition for global convergence to $\mathcal{E}_0$ is also obtained.

The central mathematical finding is that relapse can generate a backward bifurcation and, for the illustrative parameter regime, bistability. In the homogeneous-contact reduction, where

$$\mathcal{R}_0^{\text{simp}} = \frac{\beta_{sg}}{\mu + \gamma},$$

the explicit equilibrium analysis and center-manifold calculation show that a sufficiently large relapse rate β_{rg} can produce backward bifurcation at $\mathcal{R}_0^{\text{simp}} = 1$. Thus, for some parameter values with $\mathcal{R}_0^{\text{simp}} < 1$, the gang-free equilibrium and a positive gang equilibrium may coexist. This means that reducing the local invasion threshold below one may not be sufficient for elimination if the community is already in the basin of attraction of a positive gang state. Consequently, two communities with the same parameter values may approach different long-term outcomes depending on initial gang prevalence, reformed-class size, and basin membership.

We now summarize how the analysis answers the five motivating research questions. First, perceived-risk deterrence reduces gang recruitment by decreasing the effective transition from S to G . Its effect on long-term gang prevalence is therefore suppressive, but not purely linear: Increasing ϵ can lower positive gang equilibria and raise the relapse threshold required for backward bifurcation, yet it may not eliminate gang activity once the system lies in the basin of attraction of a persistent positive equilibrium. Thus, perceived-risk deterrence is most effective when applied early or combined with relapse prevention and exit support.

Second, reformed individuals influence the system through three distinct pathways. They represent successful exit from active gang involvement through the transition $G \rightarrow R$; they may reduce recruitment through reformed–susceptible interaction in the advocacy term ηrc_{rs} ; and they may relapse into active gang involvement through the transition $R \rightarrow G$. Hence, the reformed class can either suppress or sustain long-term gang prevalence depending on whether reformation and advocacy dominate relapse.

This dual role is central to the model: reformed individuals are not only an endpoint of intervention but also an active component of the feedback structure.

Third, relapse drives the backward bifurcation, producing a subthreshold positive gang equilibrium when it is strong enough, and hence the bistability seen numerically even below the local invasion threshold. The condition in Theorem 3.4 identifies β_{rg} as the key parameter controlling the bifurcation direction. When β_{rg} exceeds the relapse threshold, the bifurcation at $\mathcal{R}_0^{\text{simp}} = 1$ is backward; when the inequality is reversed, the bifurcation is forward. In the no-relapse case $\beta_{rg} = 0$, the model returns to a standard forward-threshold structure, confirming that relapse is the mechanism responsible for the nonstandard threshold behavior. The dimensionless recasting in Corollary 3.5 further decomposes the threshold into a structural baseline of unity plus two perturbative drag terms (with limiting-case structure documented in Section 3.5). For the illustrative parameter set, the structural baseline dominates by nearly two orders of magnitude (a factor of about 90), reducing the bifurcation condition to the approximate requirement $\beta_{rg} > \mu + \gamma$. A worked instance at $\mathcal{R}_0^{\text{simp}} = 0.5$ (Section 4.1) exhibits the gang-free and high-gang attractors separated by a low-prevalence saddle with substantial reformed-class reservoir.

Fourth, the homogeneous-contact reduction (the reciprocity-consistent proportional-mixing case of the weighted-contact formulation, obtained when all contact-preference weights are equal) reveals the relapse-driven bifurcation mechanism in a mathematically tractable setting. Proportional mixing is the contact structure consistent with the reciprocity balance at every population state; because requiring that balance to hold at all states forces proportional mixing, extending the relapse-driven backward bifurcation to genuinely heterogeneous, balance-consistent mixing requires a state-dependent or relaxed-reciprocity construction and remains a substantive open problem.

Fifth, the analytical and numerical results identify recruitment β_{sg} , relapse β_{rg} , and reformation γ as the parameters that most directly shape gang dynamics, acting at different stages. Recruitment sets the invasion threshold and governs whether a gang can establish; relapse strengthens the nonlinear feedback responsible for backward bifurcation and bistability, and sustains an already-established gang; and reformation suppresses gang prevalence by moving individuals from G to R . The baseline-centered PRCC analysis (conducted on the homogeneous-contact reduction, sampled within $\pm 20\%$ of an illustrative baseline) is consistent with these conclusions: relapse β_{rg} (positive) and reformation γ (negative) show the largest-magnitude, opposite-sign associations with long-term gang prevalence, while recruitment β_{sg} is a smaller positive driver. The effects of ϵ , β_{rs} , and the advocacy effectiveness η on long-term prevalence at the high-gang equilibrium are small, with bootstrap confidence intervals overlapping zero, because they act through nonlinear recruitment reduction and reformed-susceptible interaction rather than through the dominant relapse-reformation balance; under the decoupled formulation the β_{rs} PRCC reflects reformed-class outflow alone, while advocacy effectiveness is governed by the independent parameter η , as discussed in Section 4.7. The analytical sensitivity of the backward-bifurcation threshold itself (Eq (3.19)) provides a complementary ranking: the threshold is most sensitive in absolute terms to changes in $\tilde{\mu}$, $\tilde{\beta}_{rs}$, and the advocacy effectiveness η (with $\partial \tilde{\beta}_{rg}^* / \partial \eta = 1$) at the illustrative point. These rankings should therefore be interpreted locally within the sampled parameter regime rather than as a global ordering across all possible bifurcation regimes.

Drawing together the analytical and numerical results yields three quantitative features of the illustrative

parameter set. First, the dimensional relapse rate must exceed the natural decay rate $\mu + \gamma$ of the active gang class for backward bifurcation to arise (Corollary 3.5 together with the decomposition in Eq (3.18)); reducing β_{rg} below this level rules out bistability altogether. Second, when the system is in the backward-bifurcation regime, the bistable wedge extends across approximately $0.0876 < \mathcal{R}_0^{\text{simp}} < 1$ for the illustrative parameter set (Section 4.1); threshold-lowering alone therefore corresponds to reducing recruitment by approximately 91% below the standard invasion threshold of unity before convergence to the gang-free equilibrium is asymptotically guaranteed for all initial conditions. Third, within the bistable interior, the topology of the saddle separator (low active-gang prevalence with a substantial reformed-class population) implies that interventions targeting active gang prevalence alone are insufficient; what moves the system across the separatrix is preventing relapse and supporting durable reintegration, rather than simply reducing the number of reformed individuals. These threshold values are illustrative for the chosen parameter set, not prescriptive recommendations; the analytical structure (Corollary 3.5, the decomposition, and the partial derivatives in Eq (3.19)) provides the recipe for recomputing them at community-specific parameter values derived from longitudinal data.

Several limitations should be noted. The backward-bifurcation theorem is derived for the proportional-mixing case; because the reciprocity constraint (2.5), imposed at all population states, forces proportional mixing, extending the bifurcation analysis to genuinely heterogeneous balance-consistent mixing requires a state-dependent or relaxed-reciprocity contact model and is a substantive open problem rather than a routine extension. The model is deterministic and does not include stochastic effects, age structure, spatial movement, gang territory, school or neighborhood networks, social-media exposure, or explicit peer-network structure. The numerical values are illustrative and are intended to demonstrate possible nonlinear mechanisms rather than calibrated rates for a specific community.

A modeling choice worth noting concerns the advocacy term ηrc_{rs} : The dimensionless effectiveness η is treated as independent of the reintegration rate β_{rs} , so advocacy effectiveness and reformed-status durability are separate parameters with distinct social interpretations. This decoupling lets advocacy effectiveness be calibrated against intervention data on community-based programs independently of the reintegration rate inferred from desistance data, and it makes advocacy a first-order lever on the backward-bifurcation threshold ($\partial \tilde{\beta}_{rg}^* / \partial \eta = 1$; Corollary 3.5). The advocacy term remains phenomenological: It represents recruitment suppression generated through reformed–susceptible contact rather than a mechanistic model of specific programs, and calibrating η to particular interventions is left to future work informed by program-evaluation data.

5.1. Applicability domain

The qualitative predictions summarized in Table 5 apply to the class of social settings in which the present SGR model is a reasonable mechanistic abstraction. The model is appropriate for at-risk adolescent and young-adult populations characterized by: (i) A pproximately balanced demographic turnover, so that the constant reference population N_0 is a reasonable closure over the time horizon of interest; (ii) social-transition processes (recruitment, exit, relapse, reintegration) that occur on timescales of months to a few years, faster than the demographic turnover; (iii) gang involvement that is shaped predominantly by peer-contact dynamics rather than by hierarchical or strategic organizational structure;

and (iv) the absence of major external shocks during the modeled period.

Conversely, the model is *not* appropriate, without substantial modification, for the following settings. Organized criminal enterprises with internal hierarchy, recruitment quotas, and strategic decision-making are not well represented by a frequency-dependent contact model; predator-prey or game-theoretic formulations of the kind developed by Sooknanan and coauthors [30] are more suitable. Transient territorial conflicts between specific gangs, in which spatial movement and territory control dominate, are better captured by reaction-diffusion or agent-based approaches [27, 31]. Populations subject to major external shocks during the modeled period (mass incarceration events, large-scale deportation, sudden program cancellations, or violent suppression campaigns) introduce non-stationary transitions that the deterministic constant-coefficient formulation does not capture. Finally, populations in which a substantial fraction of recruitment occurs through online or social-media exposure rather than direct peer contact may require explicit network or media-exposure terms beyond this mass-action formulation.

Within the applicability domain, the model is best interpreted as a mechanism-revealing framework rather than a predictive tool calibrated to a specific community. Quantitative point predictions require parameter estimation against community-specific longitudinal data, which is beyond the scope of the present paper.

5.2. *Directions for future empirical work*

The model is uncalibrated, and its predictions are qualitative and structural rather than quantitative. We do not claim falsifiable predictions. Moreover, testing whether relapse-driven bistability operates in a given setting would require an explicit observation model linking the model states and parameters to measured quantities, including community-level recruitment, exit, and relapse transitions, advocacy exposure, and comparable initial conditions, together with a specified statistical test. Such an observation model is not developed in this paper, so neither bistability nor the local sensitivity ordering derived above should be treated as directly testable without it. The structure of the model may nonetheless help motivate future data collection. Longitudinal studies such as the Rochester Youth Development Study [16] and multi-site program evaluations such as the Gang Resistance Education and Training (G.R.E.A.T.) evaluation [50] indicate the kind of individual-trajectory and matched-community data that such work would need; we identify them only as motivation for empirical follow-up, ideally in collaboration with the criminological-research community.

In summary, relapse-driven backward bifurcation and bistability can make gang reduction more difficult than a simple threshold condition predicts. Effective intervention should therefore not focus only on reducing recruitment. The model suggests that successful long-term reduction requires coordinated strategies that reduce recruitment, strengthen exit pathways, prevent relapse, sustain reentry support, and support credible reformed-led prevention so that community trajectories remain within, or are shifted toward, the basin of attraction of the gang-free or low-gang state.

Use of AI tools declaration

The authors declare they have used Artificial Intelligence (AI) tools in the creation of this article.

AI tools used: Claude (Anthropic).

How were the AI tools used? For editorial review and prose suggestions on text drafted by the authors. The tools were not used to generate scientific content, mathematical derivations, numerical results, or figures.

Where in the article is the information located? Language and presentation throughout the manuscript. All scientific content, mathematical analysis, numerical computation, and final wording are the authors' responsibility; the authors reviewed and edited all output and take full responsibility for the content of this publication.

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Conflict of interest

Yun Kang is an editorial board member for Mathematical Biosciences and Engineering and was not involved in the editorial review or the decision to publish this article. All authors declare that there are no competing interests.

Data and code availability

No new empirical dataset was analyzed in this study; the numerical results are illustrative and are produced from the mathematical model defined in Eq (2.1) and the parameter values reported in Tables 1 and 2. The introduction cites national gang statistics and literature-informed parameter ranges for context, but no such data were fit. The code used to generate every bifurcation diagram and the PRCC sensitivity analysis is openly available in a companion reproducibility repository at <https://github.com/machyman/kwofie2026relapse>, allowing independent verification and re-running of every figure and sensitivity result reported in this paper. The repository provides two independent implementations: annotated Python notebooks that run in a browser without local installation, and a MATLAB implementation that requires no toolboxes. The two agree to within numerical round-off on the

equilibria and on the reported PRCC values. The repository documents the equilibrium solver, the branch computation, and the sensitivity pipeline, and includes an automated test suite. A companion summary of the model and its results is available at <https://machyman.github.io/kwofie2026relapse/>.

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Appendix

A. Proofs of Theorems

A.1. Proof of Theorem 3.1

Proof. On the boundary $s = 0$,

$$\dot{s} = \mu + \beta_{rs}rc_{rs} \geq 0.$$

On the boundary $g = 0$, both $c_{sg} = 0$ and $c_{rg} = 0$, since their numerators are proportional to g . Hence

$$\dot{g} = 0.$$

On the boundary $r = 0$,

$$\dot{r} = \gamma g \geq 0.$$

Thus, the vector field points inward or is tangent on each coordinate boundary of the nonnegative orthant.

The total population satisfies

$$\frac{d}{dt}(s + g + r) = \mu - \mu(s + g + r).$$

Therefore, if $s(0) + g(0) + r(0) = 1$, then $s(t) + g(t) + r(t) = 1$ for all $t \geq 0$. Hence $\mathcal{D}$ is positively invariant. $\square$

A.2. Proof of Theorem 3.2

Proof. Part (a). The Jacobian matrix of (2.10)–(2.12) evaluated at $\mathcal{E}_0 = (1, 0, 0)$ is

$$J(\mathcal{E}_0) = \begin{pmatrix} -\mu & -\frac{a_{sg}\beta_{sg}}{a_{ss}} & \beta_{rs} \\ 0 & \frac{a_{sg}\beta_{sg}}{a_{ss}} - (\mu + \gamma) & 0 \\ 0 & \gamma & -(\mu + \beta_{rs}) \end{pmatrix}.$$

The eigenvalues are

$$-\mu, \quad \frac{a_{sg}\beta_{sg}}{a_{ss}} - (\mu + \gamma), \quad -(\mu + \beta_{rs}).$$

The middle eigenvalue is

$$(\mu + \gamma)(\mathcal{R}_0^{\text{full}} - 1).$$

Here, $-\mu$ is the eigenvalue transverse to the invariant simplex, arising from extending the equations off $\mathcal{D}$, while $\frac{a_{sg}\beta_{sg}}{a_{ss}} - (\mu + \gamma)$ and $-(\mu + \beta_{rs})$ are the eigenvalues of the reduced system on $\mathcal{D}$ that govern stability within the physical phase space. All eigenvalues have negative real parts if $\mathcal{R}_0^{\text{full}} < 1$, while the reduced eigenvalue $(\mu + \gamma)(\mathcal{R}_0^{\text{full}} - 1)$ is positive if $\mathcal{R}_0^{\text{full}} > 1$. This shows part (a).

Part (b). We establish uniform persistence using the average Lyapunov function method of Hutson [51]. The framework requires (i) a compact positively invariant state space X ; (ii) a closed, positively invariant boundary set $X_0 \subset X$ whose maximal compact invariant set is the gang-free equilibrium $\mathcal{E}_0$, to which all boundary trajectories converge; and (iii) a non-negative continuous function P that vanishes on the boundary, is strictly positive in the interior, and whose logarithmic derivative $\dot{P}/P$ has positive time-average along the boundary invariant set. We verify each of these conditions in turn.

Let

$$X = \mathcal{D}, \quad X_0 = \{(s, g, r) \in \mathcal{D} : g = 0\}, \quad X_+ = \{(s, g, r) \in \mathcal{D} : g > 0\}.$$

By Theorem 3.1, X is compact and positively invariant, verifying (i). The boundary set X_0 is also positively invariant, because when $g = 0$, $c_{sg} = 0$, $c_{rg} = 0$, and hence $\dot{g} = 0$; this is the first half of (ii).

On X_0 , the boundary dynamics satisfy

$$\dot{r} = -\beta_{rs}rc_{rs} - \mu r \leq -\mu r.$$

Thus, $r(t) \rightarrow 0$. Since $s + g + r = 1$ and $g = 0$ on X_0 , it follows that $s(t) \rightarrow 1$. Therefore, every boundary trajectory approaches $\mathcal{E}_0 = (1, 0, 0)$, and $\mathcal{E}_0$ is the only invariant set in X_0 ; this completes (ii).

Define the average Lyapunov function

$$P(s, g, r) = g.$$

Then $P > 0$ on X_+ , $P = 0$ on X_0 , and P is continuous (in fact, linear) on X . For $g > 0$,

$$\frac{\dot{P}}{P} = \frac{\dot{g}}{g} = \frac{\beta_{sg}sc_{sg}}{g(1 + \eta rc_{rs} + \epsilon g)} + \frac{\beta_{rg}rc_{rg}}{g} - (\gamma + \mu).$$

Define

$$\Psi(s, g, r) = \frac{\beta_{sg} s c_{sg}}{g(1 + \eta r c_{rs} + \epsilon g)} + \frac{\beta_{rg} r c_{rg}}{g} - (\gamma + \mu).$$

Because c_{sg} and c_{rg} are proportional to g (the factor g in the numerator cancels the g in the denominator of $\dot{P}/P$), Ψ extends continuously to the boundary $g = 0$. At $\mathcal{E}_0$,

$$\Psi(\mathcal{E}_0) = \frac{a_{sg} \beta_{sg}}{a_{ss}} - (\gamma + \mu) = (\gamma + \mu)(\mathcal{R}_0^{\text{full}} - 1).$$

If $\mathcal{R}_0^{\text{full}} > 1$, then $\Psi(\mathcal{E}_0) > 0$. Because every boundary trajectory approaches $\mathcal{E}_0$ and Ψ is continuous, the time average of Ψ along any orbit in X_0 converges to $\Psi(\mathcal{E}_0) > 0$. Since $\mathcal{E}_0$ is the only invariant set on X_0 , the time average of Ψ along the boundary invariant set is positive; this verifies (iii). Therefore, by Hutson's average Lyapunov persistence theorem [51], X_0 is a uniform repeller for X_+ . Hence, there exists $\delta > 0$, independent of the initial condition (provided $g(0) > 0$), such that every solution satisfies

$$\liminf_{t \rightarrow \infty} g(t) \geq \delta.$$

This shows uniform persistence. Operationally, uniform persistence means that gang prevalence cannot asymptotically approach zero along trajectories starting from any positive gang fraction, no matter how small; the lower bound δ is uniform across all such initial conditions.

Part (c). We construct a single scalar Lyapunov-type bound on $g(t)$ using the simplest possible candidate $L(g) = g$; the strategy is to majorize $\dot{g}$ by a linear function of g whose coefficient is negative under condition (3.4). Since $\mathcal{D}$ is positively invariant, $0 \leq s, g, r \leq 1$ and $s + g + r = 1$. The recruitment-reduction denominator satisfies

$$1 + \eta r c_{rs} + \epsilon g \geq 1,$$

so dropping the denominator only enlarges the upper bound on $\dot{g}$. Using the definitions of c_{sg} and c_{rg} ,

$$c_{sg} = \frac{a_{sg} g}{a_{ss} s + a_{sg} g + a_{sr} r} \leq \frac{a_{sg}}{\min\{a_{ss}, a_{sg}, a_{sr}\}} g,$$

and

$$c_{rg} = \frac{a_{rg} g}{a_{rs} s + a_{rg} g + a_{rr} r} \leq \frac{a_{rg}}{\min\{a_{rs}, a_{rg}, a_{rr}\}} g.$$

Each of these bounds is obtained by replacing the contact-environment denominator with its smallest possible value over the feasible simplex, which is the conservative choice that maximizes c_{ij}/g . Thus,

$$\begin{aligned} \dot{L} = \dot{g} &= \frac{\beta_{sg} s c_{sg}}{1 + \eta r c_{rs} + \epsilon g} + \beta_{rg} r c_{rg} - (\gamma + \mu) g \\ &\leq \left[\frac{\beta_{sg} a_{sg}}{\min\{a_{ss}, a_{sg}, a_{sr}\}} s + \frac{\beta_{rg} a_{rg}}{\min\{a_{rs}, a_{rg}, a_{rr}\}} r - (\gamma + \mu) \right] g. \end{aligned}$$

Define

$$A = \frac{\beta_{sg} a_{sg}}{\min\{a_{ss}, a_{sg}, a_{sr}\}}, \quad B = \frac{\beta_{rg} a_{rg}}{\min\{a_{rs}, a_{rg}, a_{rr}\}}.$$

Quantities A and B are upper bounds for the per-capita recruitment and relapse rates with the maximum possible contact-preference adjustment, taken over the feasible state space. Since $s + r \leq 1$,

$$As + Br \leq \max\{A, B\}(s + r) \leq \max\{A, B\}.$$

This is the key step that converts the two-parameter bound into a single conservative scalar bound: The worst case is achieved when the remaining population mass is concentrated in whichever class has the larger feedback coefficient. Therefore,

$$\dot{L} \leq (\max\{A, B\} - (\gamma + \mu))g.$$

Condition (3.4) is precisely

$$\max\{A, B\} < \gamma + \mu,$$

so there exists $k > 0$ such that

$$\dot{L} \leq -kg.$$

By Gronwall's inequality, $g(t) \leq g(0) \exp(-kt)$, so $g(t) \rightarrow 0$ exponentially.

From the r -equation,

$$\dot{r} = \gamma g - \beta_{rg}rc_{rg} - \beta_{rs}rc_{rs} - \mu r \leq \gamma g - \mu r.$$

Since $g(t) \rightarrow 0$, comparison implies $r(t) \rightarrow 0$. Finally, $s + g + r = 1$ gives $s(t) \rightarrow 1$. Hence,

$$(s(t), g(t), r(t)) \rightarrow (1, 0, 0) = \mathcal{E}_0.$$

This shows part (c). □

A.3. Proof of Theorem 3.3

Proof. When $\beta_{rs} = 0$, the equilibrium equation for r reduces to (3.13). Since

$$\mu\epsilon + \beta_{sg} < \beta_{rg},$$

the denominator

$$\beta_{rg}(\mu\epsilon + \beta_{sg} - \beta_{rg})$$

in a_1 and a_2 is negative.

If $\mathcal{R}_0^{\text{simp}} > 1$, then

$$1 - \mathcal{R}_0^{\text{simp}} < 0.$$

From Eq (3.15), we obtain $a_2 < 0$. A quadratic with positive leading coefficient and negative constant term has exactly one positive root and one negative root. This shows part (a), subject to feasibility of the associated s^* and g^* .

If $\mathcal{R}_0^{\text{simp}} < 1$, then

$$1 - \mathcal{R}_0^{\text{simp}} > 0,$$

and hence $a_2 > 0$. For $r^2 + a_1 r + a_2 = 0$, the product of the roots is $a_2 > 0$, and the sum of the roots is $-a_1$. Thus, both roots are positive precisely when the roots are real and their sum is positive. This requires

$$a_1 < 0, \quad a_1^2 - 4a_2 > 0.$$

The two positive roots are

$$r_{1,2}^* = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2}}{2}.$$

They yield feasible positive gang equilibria only if the associated g_i^* and s_i^* satisfy the stated feasibility conditions. This shows part (b).

If the discriminant is negative, no real roots exist. If real roots exist but fail the positivity or feasibility conditions, then no feasible positive gang equilibrium is obtained from Eq (3.13). This shows part (c). $\square$

A.4. Center-manifold criterion used in Theorem 3.4

Theorem A.1 (Center-manifold bifurcation criterion [46]). *Consider the parameter-dependent system*

$$\frac{dx}{dt} = f(x, \phi), \quad f : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^n, \quad f \in C^2, \quad (\text{A.1})$$

where $x = 0$ is an equilibrium for all sufficiently small ϕ . Assume that:

(A1) *The Jacobian matrix*

$$M = D_x f(0, 0)$$

has a simple zero eigenvalue, and all other eigenvalues have negative real parts.

(A2) *The matrix M has a right eigenvector w and a left eigenvector u associated with the zero eigenvalue, normalized so that $u \cdot w = 1$, if desired.*

Let f_k denote the k -th component of f , and define

$$a = \sum_{k,i,j=1}^n u_k w_i w_j \frac{\partial^2 f_k}{\partial x_i \partial x_j}(0, 0),$$

and

$$b = \sum_{k,i=1}^n u_k w_i \frac{\partial^2 f_k}{\partial x_i \partial \phi}(0, 0).$$

Then the local dynamics near $x = 0$ are determined by the signs of a and b . In particular, if

$$a > 0 \quad \text{and} \quad b > 0,$$

then the system exhibits a backward bifurcation at $\phi = 0$. If

$$a < 0 \quad \text{and} \quad b > 0,$$

then the bifurcation is forward.

Remark. In Theorem 3.4, the bifurcation parameter is

$$\phi = \beta_{sg} - \beta_{sg}^*, \quad \beta_{sg}^* = \mu + \gamma.$$

Thus, $\phi = 0$ is equivalent to

$$\mathcal{R}_0^{\text{simp}} = 1.$$

The equilibrium $x = 0$ in the reduced (g, r) -system corresponds to the gang-free equilibrium $(s, g, r) = (1, 0, 0)$ in the original normalized model.

A.5. Proof of Theorem 3.4

Proof. We use the center-manifold criterion of Castillo-Chavez and Song [46] (Theorem A.1). On the invariant plane $s = 1 - g - r$, let

$$x_1 = g, \quad x_2 = r.$$

The homogeneous-contact system becomes

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2, \beta_{sg}) = \frac{\beta_{sg}(1 - x_1 - x_2)x_1}{1 + \eta x_2(1 - x_1 - x_2) + \epsilon x_1} + \beta_{rg}x_1x_2 - (\gamma + \mu)x_1, \\ \dot{x}_2 &= f_2(x_1, x_2, \beta_{sg}) = \gamma x_1 - \beta_{rg}x_1x_2 - \mu x_2 - \beta_{rs}(1 - x_1 - x_2)x_2. \end{aligned}$$

The bifurcation point is

$$\beta_{sg} = \beta_{sg}^* = \mu + \gamma,$$

equivalent to $\mathcal{R}_0^{\text{simp}} = 1$. At $(x_1, x_2) = (0, 0)$ and $\beta_{sg} = \beta_{sg}^*$, the Jacobian is

$$J = \begin{pmatrix} 0 & 0 \\ \gamma & -(\mu + \beta_{rs}) \end{pmatrix}.$$

The eigenvalues are 0 and $-(\mu + \beta_{rs}) < 0$. A right eigenvector and left eigenvector associated with the zero eigenvalue are

$$w = \left(\frac{\mu + \beta_{rs}}{\gamma}, 1 \right)^T, \quad u = (1, 0).$$

We do not normalize $u \cdot w = 1$, since multiplying u or w by a positive constant does not change the signs of the center-manifold coefficients used to determine bifurcation direction.

The coefficient b is

$$b = \sum_{k,i=1}^2 u_k w_i \frac{\partial^2 f_k}{\partial x_i \partial \beta_{sg}} \Big|_{(0,0,\beta_{sg}^*)}.$$

Since $u = (1, 0)$, only f_1 contributes. Direct calculation gives

$$b = \frac{\mu + \beta_{rs}}{\gamma} > 0.$$

The coefficient a is

$$a = \sum_{k,i,j=1}^2 u_k w_i w_j \frac{\partial^2 f_k}{\partial x_i \partial x_j} \Big|_{(0,0,\beta_{sg}^*)}.$$

If only f_1 contributes, then direct calculation gives

$$\begin{aligned} \frac{\partial^2 f_1}{\partial x_1^2} \Big|_{(0,0,\beta_{sg}^*)} &= -2\beta_{sg}^*(1 + \epsilon), \\ \frac{\partial^2 f_1}{\partial x_1 \partial x_2} \Big|_{(0,0,\beta_{sg}^*)} &= \beta_{rg} - \beta_{sg}^*(1 + \eta), \end{aligned}$$

and

$$\frac{\partial^2 f_1}{\partial x_2^2} \Big|_{(0,0,\beta_{sg}^*)} = 0.$$

Therefore,

$$\begin{aligned} a &= \left(\frac{\mu + \beta_{rs}}{\gamma} \right)^2 \left[-2\beta_{sg}^*(1 + \epsilon) \right] + 2 \left(\frac{\mu + \beta_{rs}}{\gamma} \right) \left[\beta_{rg} - \beta_{sg}^*(1 + \eta) \right] \\ &= \frac{2(\mu + \beta_{rs})}{\gamma^2} \left[\gamma\beta_{rg} - \beta_{sg}^* ((\mu + \beta_{rs})(1 + \epsilon) + \gamma(1 + \eta)) \right]. \end{aligned}$$

We record the final closed forms of the two center-manifold coefficients explicitly:

$$b = \frac{\mu + \beta_{rs}}{\gamma}, \tag{A.2}$$

$$a = \frac{2(\mu + \beta_{rs})}{\gamma^2} \left[\gamma\beta_{rg} - \beta_{sg}^* ((\mu + \beta_{rs})(1 + \epsilon) + \gamma(1 + \eta)) \right]. \tag{A.3}$$

Since $\mu > 0$, $\beta_{rs} \geq 0$, and $\gamma > 0$, (A.2) gives $b > 0$ unconditionally. The sign of a is determined by the sign of the bracketed expression in (A.3), since the prefactor $2(\mu + \beta_{rs})/\gamma^2$ is strictly positive. Specifically,

$$\begin{aligned} a > 0 &\iff \gamma\beta_{rg} > \beta_{sg}^* ((\mu + \beta_{rs})(1 + \epsilon) + \gamma(1 + \eta)) \\ &\iff \beta_{rg} > \frac{\beta_{sg}^*}{\gamma} [(\mu + \beta_{rs})(1 + \epsilon) + \gamma(1 + \eta)], \end{aligned}$$

which is precisely the backward-bifurcation condition Eq (3.16) stated in Theorem 3.4. Conversely, $a < 0$ corresponds to the reversed inequality and yields a forward bifurcation. The degenerate case $a = 0$ (equality in Eq (3.16)) is not covered by the second-order normal form used here; in that case, the bifurcation direction is determined by higher-order terms.

Since $b > 0$, the Castillo-Chavez and Song criterion implies that the bifurcation is backward if $a > 0$, confirming the result. $\square$

A.6. Proof of Theorem 3.6

Proof. Assume $\beta_{rg} = 0$. Then the g -equation of the homogeneous-contact system becomes

$$\dot{g} = \frac{\beta_{sg}sg}{1 + \eta rs + \epsilon g} - (\gamma + \mu)g.$$

Since $0 \leq s \leq 1$ and

$$1 + \eta rs + \epsilon g \geq 1,$$

we obtain

$$\dot{g} \leq (\beta_{sg} - (\gamma + \mu))g.$$

If $\mathcal{R}_0^{\text{simp}} < 1$, then $\beta_{sg} < \gamma + \mu$, and hence there exists $k > 0$, such that

$$\dot{g} \leq -kg.$$

Therefore, $g(t) \rightarrow 0$.

The r -equation satisfies

$$\dot{r} = \gamma g - \mu r - \beta_{rs}rs \leq \gamma g - \mu r.$$

Since $g(t) \rightarrow 0$, comparison implies $r(t) \rightarrow 0$. Finally, $s + g + r = 1$ gives $s(t) \rightarrow 1$. Therefore, $\mathcal{E}_0$ is globally asymptotically stable in $\mathcal{D}$ when $\mathcal{R}_0^{\text{simp}} < 1$.

If $\mathcal{R}_0^{\text{simp}} > 1$, local instability of $\mathcal{E}_0$ follows from the local stability result above. Setting $\beta_{rg} = 0$ in the center-manifold coefficient a gives

$$a = -\frac{2(\mu + \beta_{rs})\beta_{sg}^*}{\gamma^2} [(\mu + \beta_{rs})(1 + \epsilon) + \gamma(1 + \eta)] < 0,$$

while $b > 0$. Hence, by the Castillo-Chavez and Song criterion, the bifurcation at $\mathcal{R}_0^{\text{simp}} = 1$ is forward. Therefore, a positive gang equilibrium emerges locally from the gang-free equilibrium for $\mathcal{R}_0^{\text{simp}} > 1$ sufficiently close to one. $\square$



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