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Research article

On the statistical description of large populations

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Abstract: A wide variety of works exists on the dynamics of large populations, ranging from simple heuristic modeling to those based on advanced computer-supported methods. However, their interconnections remain mostly vague, which, in particular, limits the effectiveness of computer methods in this domain. In this work, we propose and justify the following concept. Typically, the description of the population dynamics is based on the sole use of low-order correlations. As we demonstrate here, in important cases, where the local population structure is shaped by strong interactions, higher-order correlations become essential. To verify when one can or cannot rely on studying low-order correlations only, we suggest to explicitly use probability measures as micro-states. Among such states may be those whose adequate characterization is based on their low-order correlation functions. In particular, this is the case for sub-Poissonian states where the large n asymptotics of the probability of finding n particles in a given vessel is similar to that for non-interacting entities, which can completely be described by the density of the particles. To illustrate this concept, a general individual-based model of an infinite population of interacting entities is analyzed. The evolution of this model preserves the sub-Poissonian states, which allows one to describe it through the correlation functions of such states for which a chain of evolution equations is obtained. The corresponding kinetic equation is derived, numerically solved, and analyzed.

Keywords: population of migrants; Fokker-Planck equation; Poisson state; occupation probability; kinetic equation

1. Introduction

There exists a broad spectrum of publications which study the dynamics of large populations at various levels, ranging from advanced phenomenology [1–4] to more mathematically sophisticated works [5–10] (kinetic approach), [11, 12] (measure-valued stochastic processes), [13–16] (statistical dynamics).

Before 1970s, ecological modelers mainly employed macroscopic phenomenological (rather heuristic) models: Verhulst, Lotka-Volterra, SIS, SEIR, etc., described by simple equations, see [2], which resemble phenomenological thermodynamic models of physical substances like gases, liquids, etc. In such models, similar to their physical prototypes, the main (often the sole) quantitative characteristic is the population density, or a similar aggregate parameter. However, from a statistical physics perspective, this type of modeling provides only a grainy (macroscopic) sketch of the picture that can be seen at the microscopic level by using individual-based models [17, 18]. In such models, the entities which constitute the system are explicitly taken into consideration by assigning them individual traits, such as spatial location, type, etc. Then interactions in the system—that form its local structure—are described as functions of the traits. Therefore, analogously to the case of physical substances, a comprehensive description of the dynamics of a large population should be carried out at various scales—primarily the macro- and micro scale. This also means that the population itself should be considered as infinite if one strives to clearly distinguish between the global and the local aspects of its description. This seems to be true even if the population consists only of thousands—not billions—of members. In statistical physics, the necessity of taking an infinite volume (in fact, infinite particle) limit to describe phase transitions (interaction-induced collective phenomena) was accepted after a heated discussion followed by voting; see [19]. Since that time, it has become clear that the individual-based mathematical theory of a statistically large complex system should employ a probabilistic/statistical apparatus, even if the elementary evolutionary acts are deterministic.

Generally speaking, our aim in this work is to clarify the interconnections between the macro- and microscopic descriptions of the evolution of large populations based on the experience elaborated in the non-equilibrium statistical mechanics of infinite systems of interacting physical particles and some recent developments in this direction. This seems to be even more important in view of the wide variety of powerful computational methods (e.g., [8]) which can efficiently be employed here. In statistical physics, the conceptual fundamentals of this framework were laid by Bogoliubov, who proposed to describe micro-states of large systems of physical particles as probability measures on appropriate phase spaces, characterized by their correlation functions; see [20] for a contemporary presentation of Bogoliubov's approach. In recent years, this conception has been developed in the theory of large populations; see [16,18] and also [21–24], as well as the works quoted therein. The key feature of these and similar works consists in describing the evolution of population states as that of the corresponding correlation functions. Among them are the first and second-order correlation functions, which describe the population density and the binary spatial correlations, respectively. Here, similarly to Bogoliubov's theory, the evolution of the whole collection of correlation functions is usually governed by an infinite chain of coupled equations, which turns their simultaneous solving into a hard task. Approximate solutions of this chain are usually obtained by its decoupling with the help of a certain *ansatz*, which yields closed evolution equations for the densities alone or jointly with the binary spatial correlations; see [14, 15, 23]. An important advantage of this procedure is that here one can apply appropriate numerical methods to get a deeper insight into the properties of their solutions; see [8, 23] or [25, 26]. However, in these and similar works, it is tacitly assumed (believed) that the description obtained in this way reliably approximates the evolution of states of the considered system. One of the key aims of this article is to bring to the attention—primarily of the ecological modelers community—that this is not always the case, and that the use of low order correlation functions only ought to be preceded by a detailed analysis of the evolution of states of the underlying individual-based model.

Among the states of a large population, one can distinguish the Poissonian states, in which population members dwell in the habitat independently of each other. Such states are completely characterized by their first-order correlation functions. To distinguish between the cases where one can or cannot restrict the theory to studying only the low-order correlation functions, we propose to use the notion of a sub-Poissonian state. In some sense, such states can sufficiently be characterized by their correlation functions of low orders. At the same time, the set of such states is large enough as it includes states of thermal equilibrium of interacting physical particles that undergo a phase transition; see [27, 28]. Thus, to base the description solely on the low-order correlation functions, one should be sure that the evolution preserves the set of states that can be fully characterized by such functions. If this is the case, then the evolution of states can approximately be described either by decoupling the second-order correlation function into the product of two densities (mean-field approximation), or going beyond mean-field by decoupling, e.g., the third-order function, and getting a coupled system of equations for the first and second correlation functions. Such mean-field-like approximations can be mathematically justified by passing to a macro-scale with the help of a scaling procedure; see [22] and the literature quoted therein.

As mentioned above, we intend to reconsider the steps that lead from the microscopic dynamics to its description with the help of a kind of kinetic equation. To this end, we study the model introduced in [24], which is a generalization of the known Bolker-Pacala model [1]. In this model, the motionless population members (e.g., plants) dwell in a continuous habitat. The dynamic acts amount to appearance (from outside and from the existing population) and disappearance–also due to local competition. All of these acts are random. The arrival and departure rates depend on the environment and on the existing population.

The article's structure is as follows. In Section 2, we begin by specifying the very notion of a state of a possibly infinite population, where we introduce an essential notion: a sub-Poissonian state, for which the large n asymptotic of the occupation probabilities should be at most as in a Poisson state. In subsection 2.3, we provide an example of the state in which the occupation probabilities have a *heavy tail*, and thus the high-order correlations essentially determine its properties. In Section 3, we present our microscopic model and state that its dynamics preserves the set of sub-Poisonian states. Consequently, the evolution of states of the considered model can reasonably be approximated by kinetic-like equations, cf. [24, Section 3.4]. In subsection 3.4, we present the results of the numerical study of such equations. Finally, in Section 4 we discuss the main aspects of our study and make some comparisons, mostly with the approach of [14].

2. States of infinite populations

In this section, we briefly present mathematical tools for describing infinite populations. We denote the set of positive integers $1, 2, \dots$ by $\mathbf{N}$, whereas $\mathbf{R}^d$, $d \geq 1$, will stand for the usual Euclidean space of vectors $x = (x^1, \dots x^d)$ with real components x^j. It will serve us as the habitat of the considered systems. A "vessel", Λ, is a bounded closed subset of the habitat. It will always be involved when discussing local properties. For an appropriate function f, we write

$$\int f(x)dx \;=\; \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} f(x^1, \dots, x^d)dx^1 \cdots dx^d, \tag{2.1}$$

$$\int_\Lambda f(x)dx \;\; = \;\; \int f(x)\mathbf{1}_\Lambda(x)dx,$$

where

$$\mathbf{1}_\Lambda(x) = \begin{cases} 1, & \text{if } \; x \in \Lambda; \\ 0, & \text{otherwise.} \end{cases} \tag{2.2}$$

Sometimes, by $\int$ we also denote other integrals, the meaning of which becomes clear from the context.

2.1. The states

The population that we are going to study consists of point entities–particles–distributed over the habitat $\mathbf{R}^d$. Their evolution amounts to appearance (immigration and from the existing population) and disappearance (emigration or death), both affected by the existing population and the environment. From the very beginning, we allow the population to be infinite. At the same time, we will assume that it is locally finite, i.e., each vessel Λ can contain only finitely many particles. We assume that the state of each particle is fully characterized by its position $x_i \in \mathbf{R}^d$. For simplicity, we do not assume any spatial motion in our model. This advantage proves decisive for constructing solutions of the corresponding evolution equations. Therefore, pure states of the system are infinite collections of positions $x_i \in \mathbf{R}^d$, which we will denote $\mathbf{x}$. By writing $x \in \mathbf{x}$, we mean that this x is a member of the collection, i.e., $x = x_i$ for some i. Keeping this in mind, by writing $\sum_{x\in\mathbf{x}} f(x)$, we will mean $\sum_i f(x_i)$, where the sum may be infinite. However, this latter fact does not cause harm if the function f has bounded support, i.e., vanishes outside a certain vessel Λ. Note that we allow more than one particle to have the same location, i.e., some of the x_i's in a given $\mathbf{x}$ may coincide. For $x \in \mathbf{x}$, by writing $\mathbf{x} \setminus x$, we refer to the collection obtained from $\mathbf{x}$ by removing x. Similarly, by writing $\mathbf{x} \cup x$, we refer to the collection obtained from $\mathbf{x}$ by adding the mentioned x. The mathematical theory of spaces consisting of such collections is well-developed; see, e.g., [29].

Let $F(\mathbf{x})$ be a real-valued function, which will be an observable or test function in our description. Following Bogoliubov, we will consider the next class of such functions. Let Θ denote the set of all continuous functions $\theta(x)$ such that $\theta(x) = 0$ whenever x is taken outside a θ-specific vessel Λ. Then for $\theta \in \Theta$, we set

$$F^\theta(\mathbf{x}) = \prod_{x\in\mathbf{x}}(1 + \theta(x)) = \exp\left(\sum_{x\in\mathbf{x}} \ln(1 + \theta(x))\right). \tag{2.3}$$

If $\theta(x) \in (-1, 0]$, then $0 < F^\theta(\mathbf{x}) \leq 1$ for all $\mathbf{x}$. We extend (2.3) also to the case where the collection $\mathbf{x}$ is empty. This collection is denoted $\mathbf{0}$, for which

$$F^\theta(\mathbf{0}) = \prod_{x\in\mathbf{0}}(1 + \theta(x)) = 1. \tag{2.4}$$

As mentioned above, we consider possibly infinite populations. In particular, this means that one cannot expect that the probability laws that describe the population states have densities. Such states, denoted by μ, are real-valued linear maps $F \mapsto \mu(F)$ such that: (a) $\mu(F) \geq 0$ whenever $F(\mathbf{x}) \geq 0$ for all $\mathbf{x}$; (b) $\mu(F) = C$ whenever $F(\mathbf{x}) \equiv C$. In terms of Lebesgue's integrals, they are $\mu(F) = \int F d\mu$, where the integrals are taken over the set of all $\mathbf{x}$, and $d\mu$ is the corresponding probability measure defined

on this set. In the case of finite populations contained in a given vessel Λ and states characterized by densities, this amounts to the following

$$\mu(F) = F^{(0)} f_\mu^{(0)} + \sum_{n=1}^{\infty} \frac{1}{n!} \int_\Lambda \cdots \int_\Lambda F^{(n)}(x_1, \dots, x_n) f_\Lambda^{(n)}(x_1, \dots, x_n) dx_1 \cdots dx_n,$$

which is pretty similar to the grand canonical average, cf. [20]. Here $F^{(0)} = F(\mathbf{0})$ is just a real number, whereas $f_\mu^{(0)}$ is the probability that the system consists of zero particles, i.e., is empty. Bogoliubov's brilliant idea consists in using the following representation, cf. (2.4),

$$\mu(F^\theta) = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \int \cdots \int \varrho_\mu^{(n)}(x_1, \dots, x_n) \theta(x_1) \cdots \theta(x_n) dx_1 \cdots dx_n, \tag{2.5}$$

which holds for all $\theta \in \Theta$ and those states μ that have correlations functions $\varrho_\mu^{(n)}$ of all orders $n = 0, 1, \dots$, such that the sum and the integrals in (2.5) converge. Fortunately, the collection of such states is big enough to develop a comprehensive theory of this kind, in which the map $\theta \mapsto \mu(F^\theta)$ is called Bogoliubov's functional. More on this subject can be found in [18, 20, 21] and [22, 24, 27, 30], and in the publications quoted therein.

The first correlation function $\varrho_\mu^{(1)}$ is the particle density in state μ. Correspondingly, $\int_\Lambda \varrho_\mu^{(1)}(x) dx$ is the expected number of particles (population members) contained in the vessel Λ. Likewise,

$$c_\mu(n, \Lambda) = \int_\Lambda \cdots \int_\Lambda \varrho_\mu^{(n)}(x_1, \dots, x_n) dx_1 \cdots dx_n, \qquad n \geq 1, \tag{2.6}$$

is the expected number of n-clusters contained in Λ. Mostly, in the kinetic approach works, one deals only with $\varrho_\mu^{(1)}$ and $\varrho_\mu^{(2)}$. At the same time, an important characteristic of μ can be the large n behavior of $c_\mu(n, \Lambda)$. In particular, the appearance of *clustering* (clumping, cf. [11]) can be detected by studying this behavior.

2.2. Sub-Poissonian states

The correlation functions of all orders are symmetric with respect to the permutations of its arguments. That is, for each $n \geq 2$,

$$\varrho_\mu^{(n)}(x_1, \dots, x_n) = \varrho_\mu^{(n)}(x_{\sigma(1)}, \dots, x_{\sigma(n)}), \tag{2.7}$$

which holds for all $\sigma \in S_n$, where the latter is the symmetric group. In particular, this means $\varrho_\mu^{(2)}(x_1, x_2) = \varrho_\mu^{(2)}(x_2, x_1)$.

Among the states characterized by correlation functions are Poisson states π_ϱ, for which

$$\varrho_{\pi_\varrho}^{(n)}(x_1, \dots, x_n) = \varrho(x_1) \cdots \varrho(x_n), \qquad n \geq 1, \tag{2.8}$$

where $\varrho(x) \geq 0$ is the density that completely characterizes the state π_ϱ. Then, by (2.8) and (2.5), we get

$$\pi_\varrho(F^\theta) = \exp\left(\int \theta(x) \varrho(x) dx \right). \tag{2.9}$$

The homogeneous Poisson state $\pi_\varkappa$ corresponds to $\varrho(x) \equiv \varkappa > 0$. In this state, the particles are independently distributed over the space with a constant density. Keeping these facts in mind, we introduce the following notion. A state μ is called *sub-Poissonian* if its Bogoliubov functional $\mu(F^\theta)$ can be written as in (2.5) and its correlation functions $\varrho_\mu^{(k)}$ satisfy the following estimates

$$0 \leq \varrho^{(k)}(x_1, \dots, x_k) \leq \varkappa^k, \tag{2.10}$$

holding for some $\varkappa > 0$ and all k. This inequality is called *Ruelle's bound*, see [27, 28]. The least such $\varkappa$ will be called the *type* of this μ. For sub-Poissonian states, the expected number of n-clusters in Λ (2.6) satisfies

$$c_\mu(n, \Lambda) \leq (\varkappa|\Lambda|)^n, \qquad n \geq 1. \tag{2.11}$$

Hence, dense clusters are not more expected in such states than they are in the Poisson state $\pi_\varkappa$ with the corresponding $\varkappa$.

Along with the expected numbers $c_\mu(n, \Lambda)$, one may use *occupation probabilities*, which are defined in the following way. For a vessel Λ and a state μ, the probability $p_\mu(n, \Lambda)$ of having n population members in Λ in state μ is defined as follows. First, we take $\theta(x) = \zeta \mathbf{1}_\Lambda(x)$ with real ζ; see (2.2). Then, we set

$$\phi_{\mu,\Lambda}(\zeta) = \mu(F^{\zeta \mathbf{1}_\Lambda}). \tag{2.12}$$

By (2.3), it follows that

$$F^{\zeta \mathbf{1}_\Lambda}(\mathbf{x}) = \prod_{x \in \mathbf{x}} (1 + \zeta \mathbf{1}_\Lambda(x)) = (1 + \zeta)^{N_\Lambda(\mathbf{x})},$$

where $N_\Lambda(\mathbf{x})$ is the number of particles in the collection $\mathbf{x}$ contained in Λ. Then, the probabilities $p_\mu(n, \Lambda)$, $n \geq 0$, are to be calculated by taking derivatives of the corresponding generating function

$$\phi_{\mu,\Lambda}(\zeta) = \mu\left((1 + \zeta)^{N_\Lambda}\right) = \sum_{n=0}^{\infty} (1 + \zeta)^n p_\mu(n, \Lambda),$$

which yields

$$p_\mu(n, \Lambda) = \phi_{\mu,\Lambda}^{(n)}(-1)/n!, \tag{2.13}$$

where $\phi_{\mu,\Lambda}^{(n)}(-1)$ stands for the usual derivatives of ϕ with respect to ζ at $\zeta = -1$. Note that by (2.9) it follows that $\phi_{\pi_\varrho,\Lambda}(\zeta) = \exp(\zeta \langle \varrho \rangle_\Lambda)$. By (2.5), we obtain

$$\begin{aligned} \phi_{\mu,\Lambda}^{(n)}(\zeta) &= \sum_{m=0}^{\infty} \frac{\zeta^m}{m!} \int_\Lambda \cdots \int_\Lambda \varrho_\mu^{(n+m)}(x_1, \dots, x_{n+m}) dx_1 \cdots dx_{n+m} \\ &= \sum_{m=n}^{\infty} \frac{m!}{(m-n)!} (1 + \zeta)^{m-n} p_\mu(m, \Lambda). \end{aligned} \tag{2.14}$$

By the latter equality in (2.14), one may see that $\phi_{\mu,\Lambda}^{(n)}$ is an increasing function of ζ on $[-1, 0]$, which by the first equality yields

$$\phi_{\mu,\Lambda}^{(n)}(-1) \leq \phi_{\mu,\Lambda}^{(n)}(0) = \int_\Lambda \cdots \int_\Lambda \varrho_\mu^{(n)}(x_1, \dots, x_n) dx_1 \cdots dx_n \leq (\varkappa|\Lambda|)^n. \tag{2.15}$$

Then, by (2.13), it follows that

$$p_\mu(n,\Lambda) \le (\varkappa|\Lambda|)^n/n!, \tag{2.16}$$

which holds for any sub-Poissonian state μ, where $\varkappa > 0$ is its type. Note that (2.16) differs from (3.5) by a constant factor only, and hence the large n asymptotic of the occupation probabilities for sub-Poissonian states is subordinated by the corresponding Poissonian asymptotic. If the occupation probabilities fail to satisfy (3.3), then one may say that they have a *heavy tail*; cf. [31]. Here, it is worthwhile to recall that the states of thermal equilibrium of a particle system with super-stable interactions are sub-Poissonian; see [27].

Sub-Poissonian probability distributions are employed in various disciplines, ranging from genetics [32] to quantum optics [33]. Therein, a random variable, say X, is called sub-Poissonian if its Fano factor

$$F = [\langle X^2\rangle - \langle X\rangle^2]/\langle X\rangle$$

satisfies $F < 1$. It is evidently a different property from that corresponding to (2.11), as it does not take higher-order moments into account. Indeed, let us consider X, for which $P(X = k) = (1-p)p^{k-1}$, $k \ge 1$, $p \in (0,1)$. Then, the corresponding analog of the generating function (2.2) is

$$\phi(\zeta) = \sum_{k=1}^{\infty}(1+\zeta)^k(1-p)p^{k-1} = \frac{(1+\zeta)(1-p)}{1-p(1+\zeta)}, \tag{2.17}$$

which yields

$$\phi^{(n)}(0) = n!\frac{p^{n-1}}{(1-p)^n}, \tag{2.18}$$

in contrast to our estimates in (2.11) and (2.15). At the same time, this X is sub-Poissonian in the sense that Fano's factor $F = p/(1-p)$ is smaller than 1 for $p < 1/2$. This example once again points to the importance of taking higher moments into account.

2.3. Heavy tails

In the example considered right above, the probability distribution $p_k = (1-p)p^{k-1}$ decays with k exponentially, while in the Poisson case, it decays roughly as k^{-k}. This fact finds its reflection in the large n asymptotic of $\phi^{(n)}(0)$; see (2.18). Returning to population statistics, one may say that the occupation probabilities (2.15) manifest a heavy tail property if the large n decay of $p_\mu(n,\Lambda)$ is essentially slower than that for a Poisson state. As a result, the large n behavior of the expected values of n-particle clusters $c_\mu(n,\Lambda)$ is essentially different from that for the sub-Poissonian states; see (2.6) and (2.11). This can be interpreted as a kind of *clustering* occurring in such states, for which the account of higher-order correlations is crucial. In this part of the article, we provide an example of the state where this effect takes place.

The integral $\int \theta(x)\varrho(x)dx$ in (2.9) can be interpreted as the average of θ taken with respect to the measure $\varrho(x)dx$. Let us consider a generalization of this average, in which it is substituted by

$$\langle\theta\rangle = \sum_{y\in\mathbf{y}}\theta(y),$$

where $\mathbf{y}$ is a certain infinite collection of points y_i, fixed by far. Note that this is equivalent to taking ϱ in the form $\varrho(x) = \sum_{y\in\mathbf{y}}\delta(x-y)$, where δ is Dirac's delta-function. That is, the corresponding state

by no means is sub-Poissonian as its first correlation function is a distribution – not a usual real-valued function, which definitely does not obey the Ruelle bound (2.10). Let $\Pi_{\mathbf{y}}$ be the corresponding state, i.e., the one for which

$$\Pi_{\mathbf{y}}(F^{\theta}) = \exp\left(\sum_{y\in\mathbf{y}} \theta(y)\right) = \prod_{y\in\mathbf{y}} e^{\theta(y)} = F^{\upsilon}(\mathbf{y}), \qquad 1 + \upsilon(y) = e^{\theta(y)}.$$

Now, assume that $\mathbf{y}$ is distributed over $\mathbf{R}^d$ according to the homogeneous Poisson law $\pi_{\varkappa}$. Then, the compound law—Cox' cluster measure η—can be identified by its values $\eta(F^{\theta})$ given in the form

$$\eta(F^{\theta}) = \int \Pi_{\mathbf{y}}(F^{\theta})\pi_{\varkappa}(d\mathbf{y}) = \int F^{\upsilon}(\mathbf{y})\pi_{\varkappa}(d\mathbf{y}) = \pi_{\varkappa}(F^{\upsilon})$$
$$= \exp\left(\varkappa \int \upsilon(x)dx\right) = \exp\left(\varkappa \int [e^{\theta(x)} - 1]dx\right).$$

Now, we take $\theta(x) = \zeta \mathbf{1}_{\Lambda}(x)$, see (2.2) and (2.12), and obtain

$$\phi_{\eta,\Lambda}(\zeta) = \exp\left(\varkappa \int [e^{\zeta \mathbf{1}_{\Lambda}(x)} - 1]dx\right) = \exp\left(\varkappa|\Lambda|[e^{\zeta} - 1]\right),$$

where $|\Lambda|$ is the volume of the vessel Λ. According to (2.13), to obtain $p_{\eta}(n, \Lambda)$, we have to calculate the corresponding derivatives of $\phi_{\eta,\Lambda}$. To this end, we find the first derivative

$$\phi'_{\eta,\Lambda}(\zeta) = \varkappa|\Lambda|e^{\zeta}\phi_{\eta,\Lambda}(\zeta),$$

by which we obtain

$$\phi^{(n)}_{\eta,\Lambda}(-1) = \varkappa|\Lambda|e^{-1}\sum_{l=0}^{n-1} \phi^{(l)}_{\eta,\Lambda}(-1).$$

The latter recursion can be solved in the form

$$\phi^{(n)}_{\eta,\Lambda}(-1) = a_{\Lambda}T_n(b_{\Lambda}),$$
$$a_{\Lambda} = \exp\left(-\varkappa|\Lambda|(1 - e^{-1})\right), \qquad b_{\Lambda} = \varkappa|\Lambda|e^{-1},$$

where T_n is Touchard's polynomial, cf. [34], attributed also to Grunert and studied by Ranamujan; see the historical notes in [35] and [36]. At the same time, the first correlation function (i.e., the density) corresponding to η is

$$\varrho_{\eta}(x) = \phi'_{\eta,\Lambda}(0)/|\Lambda| = \varkappa.$$

That is, both $\pi_{\varkappa}$ and η have the same densities. However, the asymptotic properties of the corresponding occupation probabilities are essentially different. To see this, assume that $b_{\Lambda} \geq 1$, which can be obtained by taking the vessel big enough. Then,

$$T_n(b_{\Lambda}) \geq b_{\Lambda}T_n(1) = b_{\Lambda}B_n,$$

where B_n, $n \in \mathbf{N}$, are Bell's numbers; see [36] and [37]. Combining this with (3.8) and (2.13), we obtain

$$p_\eta(n,\Lambda) \geq a_\Lambda b_\Lambda B_n/n! \geq \frac{a_\Lambda b_\Lambda}{\sqrt{2\pi}(\ln n)^n}, \tag{2.19}$$

with the latter estimate holding for big n. This can be compared with, cf. (3.5),

$$p_{\pi_\varkappa}(n,\Lambda) = (\varkappa|\Lambda|)^n e^{-\varkappa|\Lambda|}/n! \leq \frac{a_\Lambda b_\Lambda}{\sqrt{2\pi n}}\left(\frac{[\varkappa|\Lambda|]^2}{n}\right)^n.$$

Thus, the occupation probabilities (2.19) decay much slower than those for the Poisson states with the same density.

3. The population of migrants

3.1. The model

Here, we present the model introduced and studied in [24]. It describes an infinite population of motionless point entities (e.g., plants) dwelling in $\mathbf{R}^d$, $d \geq 1$. Each population member is located at some $x \in \mathbf{R}^d$. The (infinite) collection of the locations of all population members is denoted by $\mathbf{x}$. It is a locally finite subset of the habitat $\mathbf{R}^d$, referred to as a *configuration*. The local finiteness means that the intersection of $\mathbf{x}$ with every vessel Λ is a finite set. The mathematical theory of such configurations, considered as counting measures, can be found in [29].

The population members appear in and disappear from the habitat at random. The appearance can occur from outside and from the existing population. In the latter case, the newcomers are considered as progenies of the existing members. Disappearance has also two components: intrinsic death and death due to local competition. As every evolution model of this kind, it is defined by its generator, also called the Kolmogorov operator, which acts on test functions, e.g., on the function given in (2.3). For the model in question, the generator reads

$$LF(\mathbf{x}) = L^+F(\mathbf{x}) + L^-F(\mathbf{x}) = \int (D^+F)(y,\mathbf{x})dy - \sum_{y\in\mathbf{x}}(D^-F)(y,\mathbf{x}), \tag{3.1}$$

where the integral is taken over the trait space $\mathbf{R}^d$; see (2.1). The parts L^+ and L^- correspond to the aforementioned appearance and disappearance of the population members, respectively. The gradients $D^\pm$ are defined by the following expressions:

$$\begin{aligned}(D^+F)(y,\mathbf{x}) &= E^+(y,\mathbf{x})\left[F(\mathbf{x}\cup y) - F(\mathbf{x})\right], \qquad y \in \mathbf{R}^d,\\ (D^-F)(y,\mathbf{x}) &= E^-(y,\mathbf{x}\setminus y)\left[F(\mathbf{x}) - F(\mathbf{x}\setminus y)\right], \qquad y \in \mathbf{x},\end{aligned}$$

where the rate functions $E^\pm$ read

$$E^\pm(y,\mathbf{x}) = b^\pm(y) + \sum_{x\in\mathbf{x}} a^\pm(y-x), \qquad y \in \mathbf{R}^d. \tag{3.2}$$

In (3.2), the kernel a^+ (resp. a^-) describes attraction (resp. repulsion) of the population members by the existing population, while $b^\pm$ describe the interaction with the environment. At the same time, b^-

can be interpreted as the space dependent intrinsic death factor, while b^+ describes the immigration intensity. The interactions corresponding to a^- and a^+ can also be interpreted as competition and dispersal (progeny production), respectively. Thus, $(D^+F)(y, \mathbf{x})$ describes the change of F by adding a new member located at y to the population $\mathbf{x}$, while $(D^-F)(y, \mathbf{x})$ corresponds to the change of F by removing y from the population. Now, the Markov evolution equation is presented in the following form:

$$\frac{\partial}{\partial t}F_t(\mathbf{x}) = (LF_t)(\mathbf{x}) = \int (D^+F_t)(y, \mathbf{x})dy - \sum_{y\in\mathbf{x}}(D^-F_t)(y, \mathbf{x}). \tag{3.3}$$

The model described by (3.1) with $b^+ \equiv 0$ is known as the Bolker-Pacala-Diekmann-Law (or just Bolker-Pacala) model, which was introduced in [1]; additionally see [3,4]. Its mathematical study was performed in [11, 12, 21]. Our present study is mostly based on the technique developed in [21].

In the sequel, we assume that the functions $a^\pm$ and $b^\pm$ are continuous and bounded, i.e.,

$$0 \le a^\pm(x) \le a^\pm_*, \qquad 0 \le b^\pm(x) \le b^\pm_*$$

Additionally, we assume that both $a^\pm$ are symmetric and integrable, that is, $a^\pm(x - y) = a^\pm(y - x)$ and

$$\int a^\pm(x)dx < \infty.$$

The relationship between the interaction intensities $a^\pm$ can be of the following types, cf. [21]:

(i) long competition: there exists $\omega > 0$ such that $a^-(x) \ge \omega a^+(x)$ for all x;
(ii) short competition: for each $\omega > 0$, there exists x such that $a^-(x) < \omega a^+(x)$.

In the case of long competition, the effect of repulsion from the population by the existing members prevails. In the case of long competition, a^- decays faster than a^+, and hence some of the attracted (by a^+) newcomers can be out of the competitive action of the existing population; see [24, Section 3.1]. Particular cases of the model specified by (2.3) were studied in the following works: (a) [21], case of $b^+(x) \equiv 0$; and (b) [22], case of $a^+(x) \equiv 0$. A detailed analysis of both these cases in the Bolker-Pacala model based on numerical solutions can be found in [23].

3.2. The statement

For the model considered here, the direct solution of the Kolmogorov equation (3.3) is rather impossible. Instead, one considers its weaker version that involves states μ, i.e., that describes their evolution $\mu_0 \to \mu_t$. This is the Fokker-Planck equation

$$\mu_t(F) = \mu_0(F) + \int_0^t \mu_s(LF)ds, \tag{3.4}$$

which has to be solved for a sufficiently big class of test functions F, for which the action of L is defined. As such functions, one can take F^θ defined in (2.3) and restrict (3.4) to measures that satisfy (2.5). To optimize the notations, we introduce variables $\mathbf{u}$ and functions $\varrho_\mu(\mathbf{u})$ as follows. As $\mathbf{u}$, we take the pair $\{n, (u_1, \dots, u_n)\}$, where $n \in \mathbf{N}$ and $u_i \in \mathbf{R}^d$. Then,

$$\varrho_\mu(\mathbf{u}) = \varrho^{(n)}_\mu(u_1, \dots, u_n) \text{ if } \mathbf{u} = \{n, (u_1, \dots, u_n)\}, \tag{3.5}$$

where $\varrho_\mu^{(n)}$ is the n-th order correlation function of μ, see (2.5), which is symmetric, see (2.7). We extend (3.5) also to the case $n = 0$, i.e., to $\mathbf{u} = \mathbf{0} := \{0, \varnothing\}$, by setting $\varrho_\mu(\mathbf{0}) = 1$. In the sequel, we call ϱ_μ the correlation function of μ and still use 'correlation function of order n' if we mean $\varrho_\mu^{(n)}$. In a similar way, for a collection of functions $g^{(n)}$, $n \geq 0$, which are symmetric as in (2.7), we define $g(\mathbf{u})$ by (3.5). In particular, we will deal with

$$e(\theta; \mathbf{u}) = \theta(u_1) \cdots \theta(u_n), \qquad e(\theta; \mathbf{0}) = 1,$$

with θ as in (2.5). Similarly as in the case of infinite collections $\mathbf{x}$, by writing $x \in \mathbf{u}$, we mean that $x = u_i$ for some i. Likewise, for $\mathbf{u} = \{n, (u_1, \dots, u_n)\}$, we write $\mathbf{u} \cup x = \{n+1, (u_1, \dots, u_n, x)\}$. For $x \in \mathbf{u}$, i.e., for $\mathbf{u} = \{n, (u_1, \dots, u_{n-1}, x)\}$, by writing $\mathbf{u} \setminus x$, we mean $\{n-1, (u_1, \dots, u_{n-1})\}$. With these notations, (2.5) can be rewritten in the form

$$\mu(F^\theta) = \int \varrho_\mu(\mathbf{u}) e(\theta; \mathbf{u}) d\mathbf{u}.$$

The main advantage of passing to correlation functions consists in the possibility to present $\mu(LF^\theta)$ in the form

$$\mu(LF^\theta) = \int (L_\varrho \varrho_\mu)(\mathbf{u}) e(\theta; \mathbf{u}) d\mathbf{u},$$

where the operator L_ϱ—which acts on correlation functions—can be calculated from the Kolmogorov operator L with the help of a certain technique. For L as in (3.3), it yields

$$\begin{aligned} (L_\varrho \varrho)(\mathbf{u}) &= \sum_{x \in \mathbf{u}} E^+(x, \mathbf{u} \setminus x) \varrho(\mathbf{u} \setminus x) + \int \sum_{x \in \mathbf{u}} a^+(x-y) \varrho(\mathbf{u} \setminus x \cup y) dy \\ &- \left(\sum_{x \in \mathbf{u}} E^-(x, \mathbf{u} \setminus x) \right) \varrho(\mathbf{u}) - \int \left(\sum_{x \in \mathbf{u}} a^-(x-y) \right) \varrho(\mathbf{u} \cup y) dy. \end{aligned} \tag{3.6}$$

Employing this operator, we may pass to the following initial value problem:

$$\frac{\partial}{\partial t} \varrho_t = L_\varrho \varrho_t, \qquad \varrho_t|_{t=0} = \varrho_{\mu_0}, \quad t > 0, \tag{3.7}$$

where ϱ_{μ_0} is the correlation function of μ_0 that appears in (3.4). By means of the methods developed in [21, 22], we prove the following statement.

Proposition 1. *[24, Theorem 3.9] Let μ_0 be a sub-Poissonian state and ϱ_{μ_0} be its correlation function. Then, the initial value problem in (3.7) has a unique (global in time) solution $t \mapsto \varrho_t$ that satisfies the Ruelle bound (2.10) with a t-dependent right-hand side and is such that, for each $t > 0$, ϱ_t is the correlation function of a unique sub-Poissonian state μ_t. Moreover, these states μ_t, $t \geq 0$ are such that the map $t \mapsto \mu_t$ solves the Fokker-Planck equation (3.4), and hence describes the evolution of our model, in the course of which no clustering occurs since the corresponding occupation probabilities obey (2.16).*

The proof of this statement can be done by means of the technique used in the proof of [21]. A more detailed presentation of the mathematical aspects of the model discussed here may be found in [24]. It is based on a nontrivial combination of various functional-analytic methods, and thus is definitely beyond the scope of the present work.

3.3. The evolution of correlation functions

At first glance, besides the conclusion concerning clustering, the result formulated in Proposition 1 has only existential character and provides not so much information on the evolution details. However, after further developing, it may yield much more. To see this, we begin by rewriting (3.7). According to (3.5), for $n \geq 2$, we have

$$\begin{aligned}(L_\varrho \varrho^{(n)})(x_1,\dots,x_n) &= \sum_{i=1}^{n}\left(b^+(x_i)+\sum_{j\neq i}a^+(x_i-x_j)\right)\varrho^{(n-1)}(x_1,\dots,x_{i-1},x_{i+1},\dots x_n)\\ &+ \sum_{i=1}^{n}\int a^+(x_i-y)\varrho^{(n)}(x_1,\dots,x_{i-1},y,x_{i+1},\dots x_n)dy\\ &- \left(\sum_{i=1}^{n}\left[b^-(x_i)+\sum_{j\neq i}a^-(x_i-x_j)\right]\right)\varrho^{(n)}(x_1,\dots,x_n)\\ &- \int\left(\sum_{i=1}^{n}a^-(x_i-y)\right)\varrho^{(n+1)}(x_1,\dots,x_n,y)dy. \end{aligned} \tag{3.8}$$

The expression for $L_\varrho\varrho^{(1)}$ can be obtained from that above by setting $\sum_{j\neq i}a^{\pm}(x_i-x_j)=0$, which is in accordance with the usual convention that $\sum_{x\in\mathbf{0}}=0$. Moreover, the latter yields that $L_\varrho\varrho^{(0)}=0$, which means that $\varrho_t^{(0)}=\varrho_0^{(0)}=1$ for all $t>0$. Now, we write

$$\frac{\partial}{\partial t}\varrho_t^{(n)}(x_1,\dots,x_n))=(L_\varrho\varrho_t)^{(n)}(x_1,\dots,x_n),$$

and apply (3.8) in the right-hand side of the latter. For $n=1$, this yields

$$\begin{aligned}\frac{\partial}{\partial t}\varrho_t^{(1)}(x) &= b^+(x)+\int a^+(x-y)\varrho^{(1)}(y)dy\\ &- b^{-1}(x)\varrho^{(1)}(x)-\int a^-(x-y)\varrho^{(2)}(x,y)dy.\end{aligned} \tag{3.9}$$

Likewise, we can obtain the other members of the chain. By (3.8), with $n=2$, it follows that

$$\begin{aligned}\frac{\partial}{\partial t}\varrho_t^{(2)}(x_1,x_2) &= b^+(x_1)\varrho^{(1)}(x_2)+b^+(x_2)\varrho_t^{(1)}(x_1)\\ &+ \int\left[a^+(x_1-y)\varrho_t^{(2)}(y,x_2)+a^+(x_2-y)\varrho_t^{(2)}(x_1,y)\right]dy\\ &- \left[b^-(x_1)+b^-(x_2)+2a^-(x_1-x_2)\right]\varrho^{(2)}(x_1,x_2)\\ &- \int\left[a^-(x_1-y)+a^-(x_2-y)\right]\varrho^{(3)}(x_1,x_2,y)dy.\end{aligned} \tag{3.10}$$

Similarly, by means of (3.8), one can write the next members of the chain. By Proposition 1, we know that the evolution of the considered population preserves the sub-Poissonicity of the system states,

which are "similar" to the Poissonian states characterized by their densities only. Thus, the ansatz $\varrho_t^{(2)}(x_1, x_2) \simeq \varrho_t^{(1)}(x_1)\varrho_t^{(1)}(x_2)$ seems reasonable. It allows one to split the chain and to turn (3.9) into a nonlinear equation, which corresponds to the mean-field approximation widely known in statistical physics. However, now one can go beyond this approximation by making the ansatz in (3.10) that allows for expressing $\varrho_t^{(3)}$ through $\varrho_t^{(2)}$ and $\varrho_t^{(1)}$. The most popular one is the Kirkwood superposition approximation [26], which was applied in similar situations in [23, 38]. If necessary, one can try to split (3.7) at higher levels.

3.4. The kinetic equation

In this subsection, we turn to the kinetic equation

$$\begin{aligned} \frac{\partial}{\partial t}\varrho_t(x) &= b^+(x) + \int a^+(x-y)\varrho_t(y)dy \\ &- \left[b^-(x) + \int a^-(x-y)\varrho_t(y)dy\right]\varrho_t(x), \end{aligned} \tag{3.11}$$

which is obtained from (3.9) by the ansatz $\varrho^{(2)}(x, y) \simeq \varrho^{(1)}(x)\varrho^{(1)}(y)$ and setting $\varrho^{(1)} = \varrho$. Our aim is to demonstrate what kind of information can be obtained at this level of description. Of course, the first thing that would have to be done is proving the existence and uniqueness of its solutions. Moreover, the very deduction of (3.11) from that in (3.7) can be done by means of a scaling technique. Assuming this is done, we proceed to study such solutions for various types of model parameters. At the beginning, we assume that the described system is spatially homogeneous, which might be its simplest possible version. In this case, $b^\pm$—and the initial density ϱ_0—are constant functions of x. Correspondingly, we will look for homogeneous solutions ϱ_t. This assumption used in (3.11) leads to the following nonlinear ordinary differential equation:

$$\dot{\varrho}_t = b + \alpha\varrho_t - a\varrho_t^2, \qquad \varrho_t|_{t=0} = \varrho_0 \geq 0. \tag{3.12}$$

Here, $\dot{\varrho}_t$ stands for the time derivative of ϱ_t and

$$b = b^+ \geq 0, \quad a = \int a^-(x)dx \geq 0, \quad \alpha = \int a^+(x)dx - b^-.$$

Note that the parameter α can be positive or negative, correspondingly to the outcome of the tradeoff between the immigration due to the attraction by the existing population and the intrinsic emigration. Fortunately, (3.12) is a standard differential Riccati-type equation, which can be explicitly solved. Then, its solution is presented in the form

$$\varrho_t = \lambda_+ \frac{\varrho_0 - \lambda_- - \delta(\lambda_+ - \varrho_0)e^{-\omega t}}{\varrho_0 - \lambda_- + (\lambda_+ - \varrho_0)e^{-\omega t}}, \tag{3.13}$$

where

$$\lambda_\pm = \frac{1}{2a}\left(\alpha \pm \sqrt{\alpha^2 + 4ab}\right), \qquad \omega = \sqrt{\alpha^2 + 4ab},$$

$$\delta = \frac{4ab}{(\alpha + \sqrt{\alpha^2 + 4ab})^2}.$$

The typical behavior of the solution (3.13) is plotted in Figure 1. It has one steady state $\varrho_t = \lambda_+$. Then, the solution either decays or grows to the level λ_+, depending on whether the initial condition exceeds it or not.

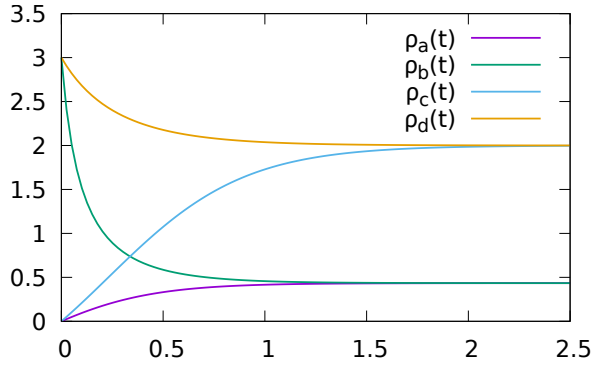

Figure 1. Solutions (3.13) of Eq (3.12) with different parameters and initial conditions. Value of ϱ_t against t. Solutions ρ_a and ρ_b illustrate the case with $a = 3$, $b = 1$, and $\alpha = -1$ with initial conditions $\varrho_0 = 0$ in the case of ρ_a and $\varrho_0 = 3$ in the case of ρ_b. Solutions ρ_c and ρ_d correspond to the case with $a = 1$, $b = 2$, and $\alpha = 1$. Initial conditions as in the first case. Visualization with gnuplot.

The study of the spatially homogeneous case can give a rather superficial insight into the system's evolution. Clearly, considering non-constant parameters $b^{\pm}$ and/or initial condition ρ_0 can yield a far richer spectrum of possibilities. To illustrate some of them, we consider the initial condition ρ_0, and the rates $b^{\pm}$ either as constants or periodic Gaussians $G_p(c, r, s)$; see (3.14). The kernels $a^{\pm}$ are taken as symmetrically shifted Gaussians $G_s(c, r, s)$; see (3.15).

The periodic Gaussian case is defined by the formula

$$G_p(x; c, r, s) = \sum_{n\in\mathbb{Z}} G(x - s + np; c, r), \tag{3.14}$$

where p is the period, and the symmetrically shifted Gaussian by

$$G_s(x; c, r, s) = \frac{1}{2}\left(G(x + s; c, r) + G(x - s; c, r)\right). \tag{3.15}$$

Gaussian $G(c, r)$ is defined as

$$G(x; c, r) = \frac{c}{r\sqrt{2\pi}} \exp\left(-\frac{x^2}{2r^2}\right).$$

In the numerical calculations presented below, the Gaussian tails were cut after reaching small enough values, and the infiniteness of the domain was simulated using the periodic boundary conditions.

In Figure 2, the role of the rates $b^{\pm}$ is illustrated. Even starting from a constant initial condition, for non-constant $b^{\pm}$, the system evolves towards a heterogeneous stationary state. In the domain where b^{+} is relatively stronger, the density ϱ_t attains higher values; it attains lower values in the domain where b^{-} is stronger. On the presented graphs, one can observe how the initially constant densities ($\varrho_0 = 1$) evolve towards irregular stationary states. The only difference between the presented cases is the range

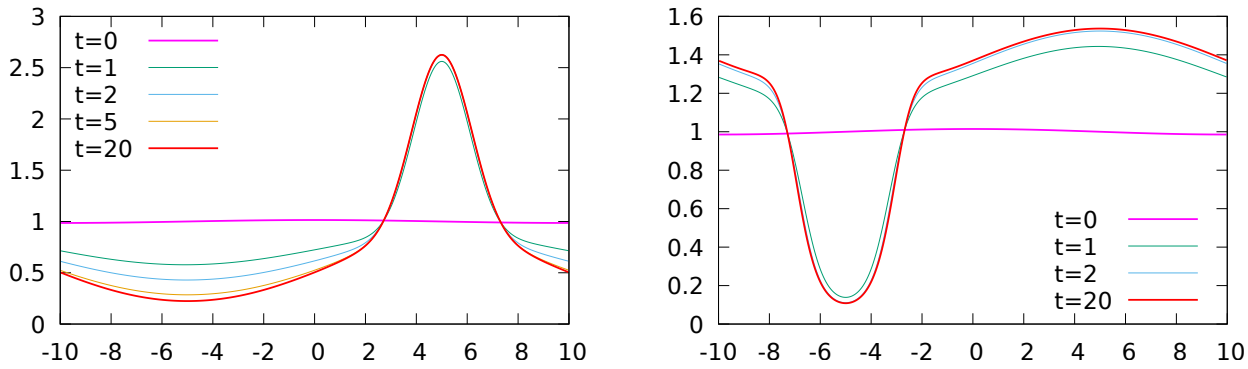

Figure 2. The role of parameters b^+ and b^- in kinetic equation (3.11). Value of $\varrho_t(x)$ versus x for selected moments of time t. Model parameters in the form of Gaussians: $a^\pm = G(1,1)$ on both graphs, $b^+ = G_p(10,1;5)$, $b^- = G_p(10,5;-5)$ on the left graph, $b^+ = G_p(10,5;5)$, $b^- = G_p(10,1;-5)$ on the right. Constant initial condition $\rho_0 \equiv 1$ in both cases. Visualization with gnuplot.

of the rates $b^\pm$. On the left side, the range of b^- is higher than that of b^+, and vice versa on the right side.

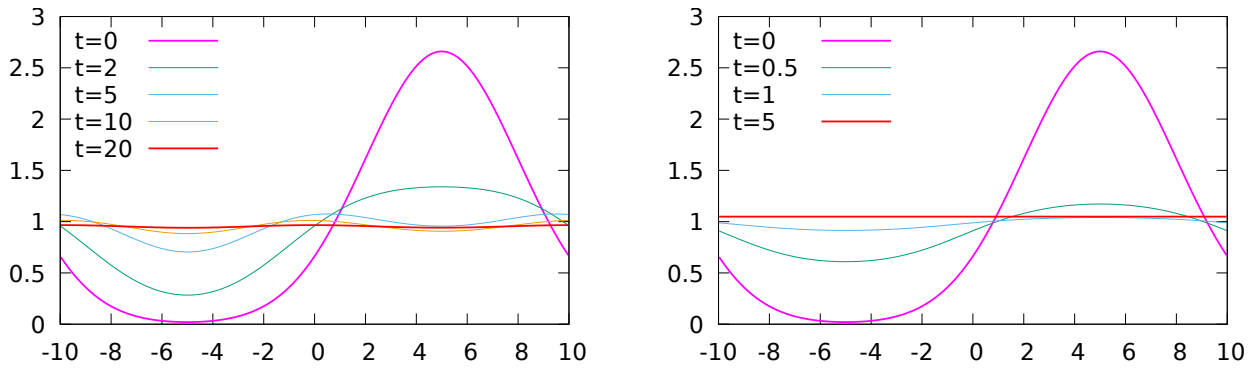

Figure 3. Short and long range of parameters a^+ and a^- in kinetic equation (3.11). Value of $\varrho_t(x)$ versus x for selected moments of time t. Model parameters taken as $a^+ = G(2,0.5)$, $a^- = G(2,5)$ on the left graph and $a^+ = G(2,5)$, $a^- = G(2,0.5)$ on the right, $b^\pm \equiv 0.1$ on both. Same initial condition $\rho_0 = G_p(20,3;5)$. Visualization with gnuplot.

In Figure 3, the interplay of the kernels $a^\pm$ is illustrated. In both cases, the initial condition is the same—periodic Gaussian, and the rates $b^\pm$ are constant. The magnitude of the kernels $a^\pm$ is comparable, but they differ in the range. It seems that the long-range a^- and short-range a^+ (left graph) may result in a much slower approach to the stationary state than in the opposite case (right graph).

Figures 4 and 5 illustrate how the shift of the kernels a^- can substantially change the long-time behavior of the system. In the cases presented in Figure 4, the only difference in the model parameters is the extent of the shift of a^-. In the first case, where the shift is smaller, the system evolves towards a homogeneous density, while in the second case, the bigger shift seems to result in evolving towards a

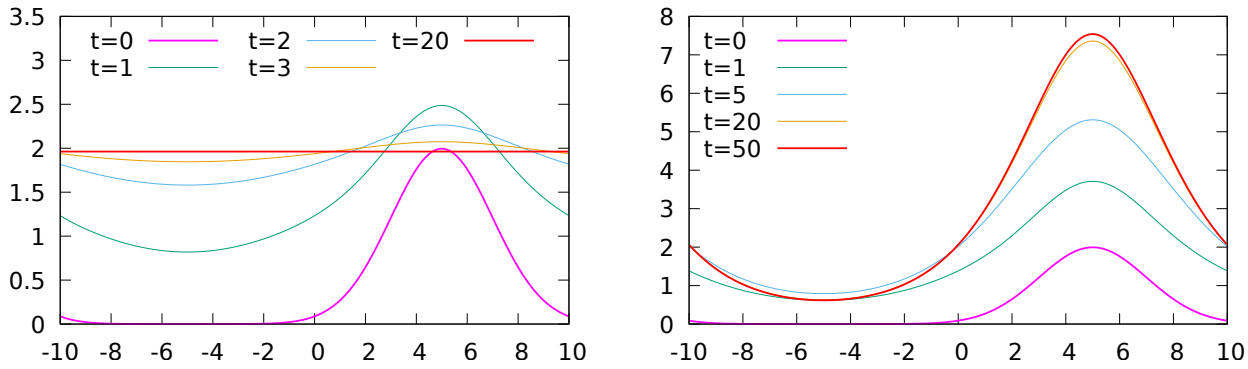

Figure 4. Shifted parameter a^- in kinetic equation (3.11). Value of $\varrho_t(x)$ versus x for selected moments of time t. Model parameters $a^+ = G(2, 4)$, $b^\pm \equiv 0.1$ on both graphs, a^- in the form of $G_s(1, 3, 5)$ on the left and $G_s(1, 3; 10)$ on the right. Same initial condition $\rho_0 = G_p(10, 2; 5)$. Visualization with gnuplot.

heterogeneous stationary state. When a significant shift of a^- is accompanied by a relatively strong a^+ with small range, the system may experience an unbounded growth. This is illustrated in Figure 5.

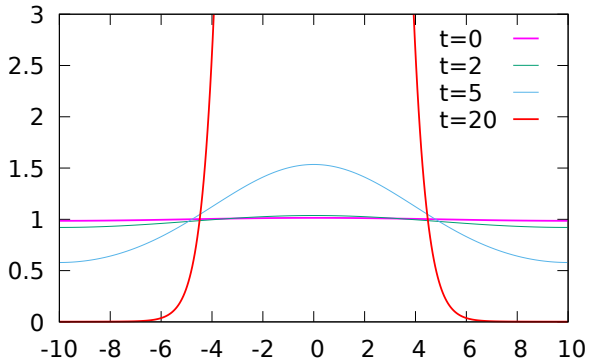

Figure 5. Unbounded growth with shifted parameter a^-. Value of $\varrho_t(x)$ versus x for selected moments of time t. Model parameters $a^+ = G(1, 1)$, $a^- = G_s(1, 2; 10)$ and $b^\pm \equiv 0.1$. Initial condition in the form of $G_p(20, 10)$. Visualization with gnuplot.

A more extensive numerical study of the model could reveal some other interesting phenomena that occur in such systems, though it is beyond the scope of this paper.

3.5. Summary

Now, let us emphasize the basic principles of the theory developed in this work.

- Similarly as in statistical physics, individual based modeling of large populations ought to be based on statistical/probabilistic methods and multi-scale analysis. In particular, following Bogoliubov's idea, population states are to be described by probability measures on appropriate phase spaces. Their evolution is obtained by solving model-dependent Fokker-Planck equations (FPE); see (3.4) for our model.

- For some models, the solutions of the Fokker-Planck equations may have the useful property of being sub-Poissonian, and thus characterized by correlation functions. To establish this, one transforms the FPE into an evolution equation for correlation functions, see (3.6), (3.7), solves the latter equation, and then proves that its solution is indeed the correlation function for a solution of the FPE. The low-order correlation functions provide such aggregated characteristics as particle density, spatial correlations, etc.
- Since multi-particle effects in sub-Poissonian states are inessential, their properties are mostly determined by the low-order correlation functions. Then, the solutions of the evolution equation for correlation functions can be approximated by solving their "decoupled" versions (i.e., kinetic-like equations) that neglect multi-particle effects. Noteworthy, such procedures also decouple the macroscopic (phenomenological) theory from its microscopic (individual-based) background. Of course, this might be admissible only if the proof mentioned in the preceding item has been performed, as well as for states where multi-particle effects are inessential. At the same time, decoupling opens the possibility to apply powerful methods of numerical analysis, e.g., like those developed in [23, 25, 38].

Here, it is worth noting that the full realization of the steps just outlined has been done for a number of models similar to that in (3.3); see [21, 24].

4. Comparison and discussion

4.1. Similar approaches

There exists a wide variety of works which deal with kinetic equations, or their "improved" versions based on the correlation functions of orders two or three that take spatial correlations into account. However, only a few of them appeal to micro-states [5, 7, 13–15, 17, 21]. The most typical and, at the same time, most comprehensive recent publication of this sort is [14], where the authors deal with individual based models defined by the corresponding Kolmogorov operators L, transformed afterwards into operators which act on correlation functions. In our case, these operators appear in the evolution equations (3.3) and (3.6), respectively. In [14], a general framework was elaborated for calculating the time evolution of quantities such as our density $\varrho_t^{(1)}(x)$ and the "truncated" correlation function $\varrho_t^{(2)}(x,y) - \varrho_t^{(1)}(x)\varrho_t^{(1)}(y) =: u_t(x-y)$. The key step of this approach (see [14]) consists in presenting them in the form of the asymptotic expansions

$$\varrho_t^{(1)}(x) = q_t(x) + \varepsilon p_t(x) + o(\varepsilon), \qquad u_t(x) = \varepsilon g_t(x) + o(\varepsilon),$$

where $\varepsilon > 0$ is a spatial scaling parameter such that taking the limit $\varepsilon \to 0$ corresponds to passing to a macro-scale. By tremendous calculations (see the supplementary part of [14]), the authors derive separate model-dependent differential evolution equations for each of q_t, p_t, g_t, see [14], intended to cover a huge range of models, and provide a kind of computational machine for constructing these equations for a concrete model and then solving them numerically and (if possible) analytically. This approach is evidently based on the assumption that microscopic states of systems "of an unlimited level of complexity" [14] can reasonably fully be specified by their low-order correlation functions; practically, by $\varrho^{(1)}$ and $\varrho^{(2)}$. The main message of this work is that this assumption should be based on a detailed analysis of the microscopic model, as it is done here. An example where this assumption

becomes problematic is provided by Cox' cluster state described in subsection 2.3, for which $\varrho^{(1)}$ is the same as for the corresponding homogeneous Poisson state and which, therefore, cannot be distinguished from the latter by using only low-order correlation functions. At the same time, Cox states are typical for systems of entities randomly distributed in a fluctuating environment, which is not so exotic in applications. In fact, instances of states with essential clustering—and hence with important high-order correlations—are well-known; see [18]. On the other hand, the very solvability of the chain equations as in (3.9) and (3.10) may be a problem: the existence of solutions $\varrho_t^{(n)}$ of all orders has been established only for some of the models mentioned in [14]; see [21, 24]. Moreover, the available techniques cover only two-particle interactions. And not for all of them, the solutions exist globally in time.

4.2. Discussion

The rich experience of modeling physical substances and processes shows how important it is to find a proper balance between the model complexity and the possibility of its study by rigorous mathematical tools. Clearly, complex systems cannot be modeled in an adequate way by simple models. On the other hand, for complex models, the mathematically rigorous results of their study are usually too general, and thus are not so expressive to the modelers. Here, the typical way of finding balance is to employ usually uncontrolled approximations, called "ansätze". The mean field approximation mentioned is among the most popular. Such approximations yield a more detailed description, whereas the possible weakness of their mathematical justification may be compensated by comparison with the corresponding experimental data. Fortunately for physicists, their experiments are incomparably more reproducible than those in other natural sciences. To their consolation, mathematicians involved in modeling physical systems are able to show (at least in some cases) that a given ansatz yields an exact result, but for another model, that is not purely individual-based and should be considered as a toy model—a convenient "caricature" of the initial model. As good examples may serve individual-based models with Curie-Weiss (nonlocal) interactions, providing a "mathematical justification" of the naive mean field approximation; see [19].

Since the experimental results that concern the majority of the complex systems studied in life sciences are much less reliable than those mentioned above, the accuracy of the theoretical modeling of such systems predetermines the correctness of their understanding. Hence, one can safely say that the adequate description of individual-based models "containing interactions of an unlimited level of complexity"—either by means of the technique of [14], or by any other machinery of this kind—is barely possible.

The principal novel aspects of the present study can be summarized as follows.

- The complete utilization of an individual-based model characterized by its Kolmogorov operator L, should be performed by constructing the evolution of states that solve the corresponding Fokker-Planck equation. Such solutions provide complete information concerning the evolution of the considered model. To facilitate this study, one can use a suitable class of states to which the possible solutions of the latter are confined. The most natural one is the class of sub-Poissonian states defined in subsection 2.2, the members of which are characterized by their correlation functions. In such states, clustering is not essential, and thus the spatial scaling procedures developed and used in [14] minimally distort the description. Significantly, the relevant states of systems of interacting physical particles are sub-Poissonian [27]. At the same

time, many of the states important in applications are not sub-Poissonian, and may not have correlation functions at all.

- One can transform the Fokker-Planck equation to an evolution equation, solutions of which may (but need not) be the correlation functions for some sub-Poissonian states. Such equations appear as [14] and (3.7) in our work. Then, one has to prove that their solutions are indeed correlation functions, which is missing in the framework of [14]. This step establishes a crucial link between the evolution of the initial individual-based model and its description in terms of $\varrho_t^{(n)}$.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare there is no conflict of interest.

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