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Research article

Mathematical analysis of a time delay predator-prey model for coconut pest management

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Abstract: In this work, a nonlinear delay differential equation model was proposed to investigate the interaction between healthy coconut leaves, infested leaves, rugose spiraling whiteflies, and the predator *Pseudomallada astur*. A discrete time delay was incorporated into the predator dynamics to account for the developmental period between prey consumption and predator recruitment. The positivity and boundedness of solutions were established, and the basic reproduction number was derived using the next-generation matrix approach. The existence of the pest-free and coexistence equilibria was investigated, and sufficient conditions for their local asymptotic stability were obtained. The influence of the developmental delay on system stability was analyzed, and conditions for the occurrence of delay-induced Hopf bifurcation were derived. In addition, normalized sensitivity analysis identified the parameters that most strongly influence pest invasion, while numerical simulations illustrated the effects of the maximum infestation rate, whitefly reproduction rate, predation rate, and predator developmental delay on the system dynamics. The results demonstrated that enhancing predator efficiency and reducing infestation and reproduction rates effectively suppress whitefly outbreaks. The proposed model provides useful mathematical insights for designing effective biological pest-management strategies for coconut plantations.

Keywords: rugose spiraling whitefly; delay differential equation; *Pseudomallada astur*; stability analysis; developmental delay

1. Introduction

India is one among the world's leading coconut-producing countries, where a large group of farmers depend on coconut cultivation for their income and livelihood. Since 2016, coconut trees in India have been affected by the exotic pest known as the rugose spiraling whitefly (RSW) [1]. It mostly infests broad-leaved hosts like palm trees where it lives naturally. In terms of behavior and general appearance,

the pest resembles the invasive whirling whitefly *Aleurodicus dispersus*, which was introduced into India in the middle of the 1990s. It damages the host plant by extracting water and important nutrients from the leaves. It also produces honeydew, which covers the lower leaf surface and promotes the growth of sooty mold. Although sooty mold is not a plant disease, its accumulation on the leaf surface may reduce photosynthetic activity [2]. In the southern districts of Tamil Nadu, which were badly affected by these pests, field surveys identified nine species of predators and one species of nymphal aphelinid parasitoid. Among the natural enemies of RSW, the aphelinid parasitoid *Encarsia guadeloupae* has been reported to contribute to reducing the pest population [3, 4].

Mathematical modeling has become an indispensable tool for understanding ecological interactions and designing efficient pest-management strategies. In agricultural systems, such models are essential for understanding pest outbreaks and devising effective pest-management strategies that reduce crop loss and dependence on chemical pesticides. Most crop–pest–predator models are analyzed using ordinary differential equations and include biological control, harvesting, or stage structure [5, 6]. These models provide substantial insights; however, they generally represent biological interactions without explicitly accounting for developmental and management-related time lags.

In real-world ecological systems, several biological and management processes are associated with inherent time lags. Delays arising from disease-management practices, such as rouging, have been incorporated into mathematical models of plant diseases [7]. Similarly, time delays associated with pest detection, farmer awareness, and the implementation of pest-control strategies have been considered in pest-management models [8]. Developmental delays, including vector developmental delays and the development of natural enemies, can substantially influence pest and plant–disease dynamics [9–11]. Delayed implementation of pest-control measures and biological control strategies has further been investigated in pest-management systems [12]. These studies demonstrate that incorporating temporal delays can substantially influence the dynamics of ecological systems, including their stability properties, oscillatory behavior, and bifurcation phenomena. In biological control systems, the developmental period of a predator is particularly important because prey consumption does not immediately result in the recruitment of new predator individuals. Field studies have identified several natural enemies of the RSW, with the predator *Pseudomallada astur* showing considerable potential for biological control [13–15]. Therefore, incorporating the population of *Pseudomallada astur* and its developmental delay provides a biologically motivated framework for investigating the temporal effects of predator recruitment on RSW dynamics.

Motivated by these findings, mathematical models have been developed to investigate rugose spiraling whitefly (RSW) infestation and its control in coconut ecosystems, including the effects of pest dynamics and awareness on pest management [16, 17]. Further studies have incorporated biological control through predators and parasites, and have investigated analytical approaches for pest-control models [18, 19]. More recently, a delay differential equation model was developed to account for temporal effects associated with whitefly infestation, and its solutions were studied using the homotopy perturbation method [20]. These models, however, do not describe the coupled interaction among healthy and infested coconut leaves, the whitefly population, and the predator *Pseudomallada astur*, with a delay in predator recruitment, and they do not analyze how such a delay changes the stability of the system through a Hopf bifurcation. The present study addresses this gap. The leaf population is divided into healthy and infested classes, and the transfer of healthy leaves to the infested class is described by a saturating Holling type-II function. Predation of the RSW by *P. astur* was described

by a Holling type-III functional response, which was supported by experimental feeding studies of this predator [14]. A discrete developmental delay was introduced in the predator recruitment term to represent the developmental period between prey consumption and the recruitment of new predators, and a constant release rate represented augmentative biological control. Thus, the contribution of the present study is not the use of a time delay or Holling functional responses individually, but their integration with the coconut leaf–RSW–predator interaction in a unified framework, together with a qualitative analysis of how the developmental delay affects the equilibria, stability, sensitivity, and delay-induced oscillations of the system.

Motivated by these considerations, the proposed model is analyzed through positivity and boundedness of solutions, equilibrium analysis, derivation of the basic reproduction number, local stability analysis, delay-induced Hopf bifurcation analysis, sensitivity analysis, and numerical simulations. Particular attention is given to the influence of the predator developmental delay on the stability and dynamical behavior of the coexistence equilibrium. The results provide mathematical insights into the role of the developmental period of the predator in the biological control of the RSW in coconut ecosystems.

The workflow of the paper is as follows. In Section 2, a mathematical model of the system is proposed. The model considers healthy coconut leaves, infested coconut leaves, the whitefly population, and the population of *Pseudomallada astur*, with a discrete time delay introduced to represent the developmental period associated with predator recruitment. Section 3 discusses the positive invariance and boundedness of the solution. Further, the basic reproduction number of the pest is derived, and the existence and stability of the equilibrium points are presented. Section 4 provides the sensitivity analysis of the parameters. Section 5 presents numerical simulations illustrating the analytical conditions obtained in the previous sections. Section 6 presents the overall conclusion and future research directions of the proposed model.

2. Mathematical model

In [19], a nonlinear ordinary differential equation model was proposed to investigate the biological control of whitefly populations in the presence of natural enemies. However, rugose spiraling whitefly infestation adversely affects coconut leaves through sap feeding, honeydew secretion, and the subsequent development of sooty mold, which collectively reduce the photosynthetic efficiency and physiological activity of the leaves. To represent this biological process more realistically, the coconut leaf population is divided into healthy and infested classes.

Motivated by these observations, we formulate a delay differential equation model describing the interactions among healthy coconut leaves $H(t)$, infested coconut leaves $I(t)$, the whitefly population $W(t)$, and the larval population of the predator *Pseudomallada astur*, denoted by $P(t)$. Since the primary objective of this study is to investigate the overall population dynamics rather than the dynamics of individual life stages, the whitefly compartment represents the aggregated biologically active whitefly population responsible for leaf infestation and population persistence.

The infestation of healthy coconut leaves by rugose spiraling whitefly (RSW) is studied with the Michaelis–Menten-type (or Holling type-II) functional response [6,12]. As the RSW population increases, the infestation of healthy coconut leaves also increases, but the rate of infestation eventually approaches a saturation level. This reflects the limited availability of healthy leaf resources and suitable

feeding and oviposition sites for the RSW. The Holling type-II functional response is characterized by a decelerating rate and is commonly represented by a rectangular hyperbola, known as Holling's disc equation, which describes saturation arising from limited availability or processing capacity.

The predator–prey interaction is represented by a Holling type-III functional response, which describes relatively low predation at low prey densities, followed by an increase in predation efficiency at intermediate prey densities and eventual saturation at high prey densities due to the predator's limited handling and consumption capacity. Experimental studies on the feeding behavior of *Pseudomallada astur* against the rugose spiraling whitefly have demonstrated a type-III functional response for all larval instars, thereby supporting its use in the present model [14].

A discrete time delay τ is incorporated into the predator equation to represent the developmental period between prey consumption and the recruitment of new predatory larvae. Both delayed quantities, $W(t-\tau)$ and $P(t-\tau)$, are evaluated at the earlier time because predator recruitment at time t is assumed to depend on the predation interaction that occurred at time $t-\tau$ [11]. Here, $W(t-\tau)$ represents the whitefly population available as prey, while $P(t-\tau)$ denotes the predator population participating in predation at that time. Thus, the delayed recruitment term accounts for the contribution of this past interaction to the predator population at the current time. In the present formulation, mortality during the developmental delay is neglected and is not modeled separately.

The parameters used in the model have the following biological meanings:

- r denotes the intrinsic growth rate of healthy coconut leaves, while k represents the carrying capacity of the coconut leaf population;
- α denotes the maximum infestation rate of healthy leaves by the whitefly population, while m is the half-saturation constant associated with the saturating infestation function;
- δ represents the removal rate of infested leaves due to natural senescence or management practices;
- β denotes the recruitment rate of the whitefly population supported by infested leaves, while γ represents the natural mortality rate of the whitefly population;
- a denotes the maximum predation rate of the larval stage of *Pseudomallada astur* on the whitefly population, while b is the half-saturation constant associated with the Holling type-III functional response;
- d denotes the conversion parameter relating prey consumption to predator recruitment. The discrete time delay τ represents the developmental period between predation and the recruitment of new predatory larvae;
- σ denotes the natural mortality rate of the predator, whereas ε represents the constant release rate of predators into the coconut ecosystem.

In view of the above assumptions, the proposed delay differential equation model is governed by the following system:

$$\begin{aligned}
\frac{dH}{dt} &= rH\left(1-\frac{H+I}{k}\right)-\frac{\alpha HW}{m+H},\\
\frac{dI}{dt} &= \frac{\alpha HW}{m+H}-\delta I,\\
\frac{dW}{dt} &= \beta I-\frac{aW^2P}{b^2+W^2}-\gamma W,\\
\frac{dP}{dt} &= \frac{adW^2(t-\tau)P(t-\tau)}{b^2+W^2(t-\tau)}-\sigma P+\varepsilon.
\end{aligned} \tag{2.1}$$

System (2.1) is considered together with the initial conditions

$$H(\nu)>0, \qquad I(\nu)>0, \qquad W(\nu)>0, \qquad P(\nu)>0, \qquad \nu\in[-\tau,0]. \tag{2.2}$$

Let $C=C([-\tau,0],\mathbb{R}^4)$ denote the Banach space of continuous functions

$$\Psi:[-\tau,0]\to\mathbb{R}^4$$

equipped with the norm

$$\|\Psi\|=\sup_{-\tau\le\nu\le0}\{|\Psi_1(\nu)|,|\Psi_2(\nu)|,|\Psi_3(\nu)|,|\Psi_4(\nu)|\}.$$

Define the positive cone

$$C_+=\{\Psi\in C:\Psi_i(\nu)\ge0,\ \nu\in[-\tau,0],\ i=1,2,3,4\}.$$

The initial conditions of system (2.1) are specified by

$$H(\nu)=\Psi_1(\nu), \qquad I(\nu)=\Psi_2(\nu), \qquad W(\nu)=\Psi_3(\nu), \qquad P(\nu)=\Psi_4(\nu), \qquad \nu\in[-\tau,0].$$

Thus, the initial function is

$$\Psi=(\Psi_1,\Psi_2,\Psi_3,\Psi_4)\in C_+.$$

For biological relevance, the initial functions satisfy

$$\Psi_i(\nu)\ge0, \qquad \nu\in[-\tau,0], \qquad \Psi_i(0)>0, \qquad i=1,2,3,4.$$

System (2.1) can be expressed in the abstract form

$$\frac{dX}{dt}=F(X),$$

where

$$X(\nu)=(H(\nu),I(\nu),W(\nu),P(\nu))^T\in C_+$$

and

$$F=(F_1,F_2,F_3,F_4)^T,$$

where F_i, $i=1,2,3,4$, denotes the right-hand sides of system (2.1). By the fundamental theorem of functional differential equations [21], there exists a unique solution

$$(H(t),I(t),W(t),P(t))$$

to system (2.1) satisfying the initial conditions (2.2).

3. Analytical study of the system dynamics

3.1. Positivity and boundedness

Biologically, positivity ensures that all population densities remain non-negative over time, thereby preserving their biological significance. Although population models are generally expected to admit non-negative solutions, the presence of a time delay may affect this property. Therefore, it is essential to establish that the solutions of system (1) remain non-negative for all $t > 0$. To establish this property, we follow the approach of Bodnar [22] and Yang et al. (1996) [23], which leads to the following theorem.

Theorem 1. *All solutions of system (2.1) with initial conditions (2.2) remain non-negative for all $t \geq 0$.*

Proof. Using Lemma 2 in [23] and Theorem 1.1 in [22], we show that the solution of system (2.1) exists in the region $\mathbb{R}^4$ and remains non-negative for all $t \geq 0$. It is straightforward to verify that, for any initial function $X(\upsilon) \in \mathbb{R}^4_+$, if one of the state variables reaches zero while the others remain non-negative, then the corresponding component of the vector field satisfies

$$F_i(X)\big|_{x_i=0,\, X\in\mathbb{R}^4_+} \geq 0, \qquad i = 1,2,3,4,$$

where

$$x_1(t) = H(t), \qquad x_2(t) = I(t), \qquad x_3(t) = W(t), \qquad x_4(t) = P(t).$$

Therefore, the vector field points inward on each boundary hyperplane of the positive cone. By Lemma 2 in [23] together with the existence and uniqueness theorem for functional differential equations [22], every solution $X(t) = X(t, X(\upsilon))$ with the initial function $X(\upsilon) \in C_+$ satisfies

$$X(t) \in \mathbb{R}^4_+, \qquad \forall t \geq 0.$$

Hence, all solutions of system (2.1) remain non-negative for all $t \geq 0$, and consequently, the positive cone $\mathbb{R}^4_+$ is positively invariant. □

Theorem 2. *Suppose that all parameters of system (2.1) are positive and that $\sigma > ad$. Then every solution of system (2.1), with non-negative initial functions on $[-\tau, 0]$, remains non-negative and uniformly bounded for all $t \geq 0$.*

Proof. Let $N(t) = H(t) + I(t)$. From system (2.1),

$$\begin{aligned} \frac{dN}{dt} &= \frac{dH}{dt} + \frac{dI}{dt} \\ &= rH\left(1 - \frac{H+I}{k}\right) - \delta I \\ &= rH\left(1 - \frac{N}{k}\right) - \delta I. \end{aligned} \tag{3.1}$$

Since $I(t) \geq 0$, it follows that

$$\frac{dN}{dt} \leq rH\left(1 - \frac{N}{k}\right). \tag{3.2}$$

Whenever $N(t) \geq k$, we have

$$1 - \frac{N(t)}{k} \leq 0, \tag{3.3}$$

and hence

$$\frac{dN}{dt} \leq 0. \tag{3.4}$$

Therefore,

$$N(t) \leq \max\{N(0), k\} := M_1, \qquad t \geq 0. \tag{3.5}$$

Consequently,

$$0 \leq H(t) \leq M_1, \qquad 0 \leq I(t) \leq M_1, \qquad t \geq 0. \tag{3.6}$$

Next, from the equation for the whitefly population,

$$\frac{dW}{dt} = \beta I - \frac{aW^2P}{b^2 + W^2} - \gamma W, \tag{3.7}$$

and since

$$\frac{aW^2P}{b^2 + W^2} \geq 0 \tag{3.8}$$

and $I(t) \leq M_1$, we obtain

$$\frac{dW}{dt} \leq \beta M_1 - \gamma W. \tag{3.9}$$

By comparison with the solution of

$$Y'(t) = \beta M_1 - \gamma Y(t), \tag{3.10}$$

we obtain

$$W(t) \leq W(0)e^{-\gamma t} + \frac{\beta M_1}{\gamma}\left(1 - e^{-\gamma t}\right). \tag{3.11}$$

Hence,

$$0 \leq W(t) \leq \max\left\{W(0), \frac{\beta M_1}{\gamma}\right\} := M_2, \qquad t \geq 0. \tag{3.12}$$

It remains to establish the boundedness of the predator population $P(t)$. From the fourth equation of system (2.1),

$$\frac{dP}{dt} = \frac{ad\,W^2(t-\tau)P(t-\tau)}{b^2 + W^2(t-\tau)} - \sigma P + \varepsilon. \tag{3.13}$$

Since

$$0 \leq \frac{W^2(t-\tau)}{b^2 + W^2(t-\tau)} \leq 1, \tag{3.14}$$

we obtain the linear delay differential inequality

$$\frac{dP}{dt} \leq ad\,P(t-\tau) - \sigma P(t) + \varepsilon. \tag{1}$$

Under the assumption $\sigma > ad$, we define

$$M_3 = \max\left\{ \sup_{-\tau \le s \le 0} P(s), \frac{\varepsilon}{\sigma - ad} \right\}. \tag{3.15}$$

We claim that $P(t) \le M_3$ for all $t \ge 0$.
Indeed, suppose that $P(t)$ exceeds M_3 for some $t > 0$. Let $t_1 > 0$ be the first time at which $P(t)$ reaches the level M_3 from below. Then

$$P(t_1) = M_3, \qquad P(t) \le M_3 \quad \text{for } t \le t_1, \tag{3.16}$$

and in particular,

$$P(t_1 - \tau) \le M_3. \tag{3.17}$$

Evaluating inequality (1) at $t = t_1$, we obtain

$$\begin{aligned} \left.\frac{dP}{dt}\right|_{t=t_1} &\le ad\, P(t_1 - \tau) - \sigma P(t_1) + \varepsilon \\ &\le adM_3 - \sigma M_3 + \varepsilon \\ &= -(\sigma - ad)M_3 + \varepsilon. \end{aligned} \tag{3.18}$$

By the definition of M_3, we have

$$M_3 \ge \frac{\varepsilon}{\sigma - ad}, \tag{3.19}$$

so that

$$-(\sigma - ad)M_3 + \varepsilon \le 0. \tag{3.20}$$

Thus,

$$\left.\frac{dP}{dt}\right|_{t=t_1} \le 0, \tag{3.21}$$

which implies that $P(t)$ cannot cross the level M_3 from below. This contradiction proves that

$$0 \le P(t) \le M_3, \qquad t \ge 0. \tag{3.22}$$

Therefore, $H(t)$, $I(t)$, $W(t)$, and $P(t)$ are uniformly bounded for all $t \ge 0$. Hence, every solution of system (2.1) remains uniformly bounded. □

3.2. Existence of equilibria

We consider the following equilibrium points:

(i) The pest-free equilibrium

$$E_1 = \left(k, 0, 0, \frac{\varepsilon}{\sigma}\right),$$

(ii) A coexistence equilibrium

$$E_2 = (H^*, I^*, W^*, P^*),$$

where

$$W^* = \frac{r\delta\,(m+H^*)(k-H^*)}{\alpha(\delta k + rH^*)},$$

$$I^* = \frac{\alpha H^* W^*}{\delta(m+H^*)},$$

$$P^* = \frac{\varepsilon\left(b^2+W^{*2}\right)}{\sigma\,(b^2+W^{*2}) - adW^{*2}},$$

with H^* being a positive root of the polynomial

$$\mathscr{H}(H) = B_5H^5 + B_4H^4 + B_3H^3 + B_2H^2 + B_1H + B_0 = 0, \tag{3.23}$$

where

$$\begin{aligned}
B_5 &= a\alpha\beta d\delta^2r^2 - ad\delta^3\gamma r^2 - \alpha\beta\delta^2r^2\sigma + \delta^3\gamma r^2\sigma,\\
B_4 &= -2a\alpha\beta d\delta^2kr^2 + 2a\alpha\beta d\delta^2mr^2 + 2ad\delta^3\gamma kr^2 - 3ad\delta^3\gamma mr^2\\
&\quad + 2\alpha\beta\delta^2kr^2\sigma - 2\alpha\beta\delta^2mr^2\sigma - 2\delta^3\gamma kr^2\sigma + 3\delta^3\gamma mr^2\sigma - a\alpha\delta^2\varepsilon r^2,\\
B_3 &= a\alpha\beta d\delta^2k^2r^2 - 4a\alpha\beta d\delta^2kmr^2 + a\alpha\beta d\delta^2m^2r^2 - ad\delta^3\gamma k^2r^2\\
&\quad + 6ad\delta^3\gamma kmr^2 - 3ad\delta^3\gamma m^2r^2 - \alpha^3b^2\beta r^2\sigma + \alpha^2b^2\delta\gamma r^2\sigma\\
&\quad - \alpha\beta\delta^2k^2r^2\sigma + 4\alpha\beta\delta^2kmr^2\sigma - \alpha\beta\delta^2m^2r^2\sigma + \delta^3\gamma k^2r^2\sigma\\
&\quad - 6\delta^3\gamma kmr^2\sigma + 3\delta^3\gamma m^2r^2\sigma - a\alpha\delta^3\varepsilon kr + a\alpha\delta^2\varepsilon kr^2\\
&\quad - 2a\alpha\delta^2\varepsilon mr^2,\\
B_2 &= 2a\alpha\beta d\delta^2k^2mr^2 - 2a\alpha\beta d\delta^2km^2r^2 - 3ad\delta^3\gamma k^2mr^2 + 6ad\delta^3\gamma km^2r^2\\
&\quad - ad\delta^3\gamma m^3r^2 - 2\alpha^3b^2\beta\delta kr\sigma + 2\alpha^2b^2\delta^2\gamma kr\sigma + \alpha^2b^2\delta\gamma mr^2\sigma\\
&\quad - 2\alpha\beta\delta^2k^2mr^2\sigma + 2\alpha\beta\delta^2km^2r^2\sigma + 3\delta^3\gamma k^2mr^2\sigma - 6\delta^3\gamma km^2r^2\sigma\\
&\quad + \delta^3\gamma m^3r^2\sigma + a\alpha\delta^3\varepsilon k^2r - 2a\alpha\delta^3\varepsilon kmr + 2a\alpha\delta^2\varepsilon kmr^2\\
&\quad - a\alpha\delta^2\varepsilon m^2r^2,\\
B_1 &= a\alpha\beta d\delta^2k^2m^2r^2 - 3ad\delta^3\gamma k^2m^2r^2 + 2ad\delta^3\gamma km^3r^2\\
&\quad - \alpha^3b^2\beta\delta^2k^2\sigma + \alpha^2b^2\delta^3\gamma k^2\sigma + 2\alpha^2b^2\delta^2\gamma kmr\sigma\\
&\quad - \alpha\beta\delta^2k^2m^2r^2\sigma + 3\delta^3\gamma k^2m^2r^2\sigma - 2\delta^3\gamma km^3r^2\sigma\\
&\quad + 2a\alpha\delta^3\varepsilon k^2mr - a\alpha\delta^3\varepsilon km^2r + a\alpha\delta^2\varepsilon km^2r^2,\\
B_0 &= -ad\delta^3\gamma k^2m^3r^2 + \alpha^2b^2\delta^3\gamma k^2m\sigma + \delta^3\gamma k^2m^3r^2\sigma + a\alpha\delta^3\varepsilon k^2m^2r.
\end{aligned} \tag{3.24}$$

The leading and constant coefficients in (3.2) can be written as

$$\begin{aligned}
B_5 &= a\alpha\beta d\delta^2r^2 - ad\delta^3\gamma r^2 - \alpha\beta\delta^2r^2\sigma + \delta^3\gamma r^2\sigma\\
&= \delta^2r^2(ad-\sigma)(\alpha\beta - \delta\gamma)
\end{aligned} \tag{3.25}$$

and

$$\begin{aligned}
B_0 &= -ad\delta^3\gamma k^2m^3r^2 + \alpha^2b^2\delta^3\gamma k^2m\sigma + \delta^3\gamma k^2m^3r^2\sigma + a\alpha\delta^3\varepsilon k^2m^2r\\
&= \delta^3k^2m\left[\gamma m^2r^2(\sigma - ad) + \alpha^2b^2\gamma\sigma + a\alpha\varepsilon mr\right].
\end{aligned} \tag{3.26}$$

Assume that all model parameters are positive and

$$\sigma > ad, \qquad \alpha\beta > \delta\gamma. \tag{3.27}$$

Then

$$B_0 > 0 \qquad \text{and} \qquad B_5 < 0,$$

and consequently,

$$B_0 B_5 < 0. \tag{3.28}$$

Since

$$\mathscr{H}(0) = B_0 > 0$$

and

$$\lim_{H\to+\infty} \mathscr{H}(H) = -\infty,$$

the intermediate value theorem guarantees the existence of at least one positive real root $H^* > 0$ of $\mathscr{H}(H) = 0$. Furthermore, under the condition $\sigma > ad$, the denominator of P^* is strictly positive. In fact,

$$\begin{aligned} \sigma\left[b^2 + (W^*)^2\right] - ad(W^*)^2 &= \sigma b^2 + (\sigma - ad)(W^*)^2 \\ &> 0. \end{aligned} \tag{3.29}$$

Hence, $P^* > 0$.

For the coexistence equilibrium to be biologically feasible, we also require $W^* > 0$. Since

$$W^* = \frac{r\delta(m+H^*)(k-H^*)}{\alpha(\delta k + rH^*)}, \tag{3.30}$$

and all parameters are positive, we have

$$W^* > 0 \quad \Longleftrightarrow \quad 0 < H^* < k.$$

Consequently, a sufficient condition for the existence of a root in the biologically relevant interval $(0,k)$ is

$$\mathscr{H}(0)\mathscr{H}(k) < 0, \tag{3.31}$$

which follows directly from the intermediate value theorem. Since $\mathscr{H}(0) = B_0 > 0$ under condition (3.27), condition (3.31) is equivalently written as

$$\mathscr{H}(k) < 0.$$

Thus, under

$$\sigma > ad, \qquad \alpha\beta > \delta\gamma, \qquad \mathscr{H}(k) < 0, \tag{3.32}$$

there exists at least one $H^* \in (0,k)$, and the corresponding equilibrium

$$E_2 = (H^*, I^*, W^*, P^*)$$

is biologically admissible.

3.3. Basic reproduction number

The basic reproduction number of the system is denoted by $\mathscr{R}_0$. The next-generation matrix method is employed to derive the basic reproduction number.

Let

$$Z = (I, W)^T. \tag{3.33}$$

Then the infected subsystem can be written as

$$Z' = F(Z) - V(Z), \tag{3.34}$$

where

$$F(Z) = \begin{bmatrix} \dfrac{\alpha HW}{m+H} \\ 0 \end{bmatrix} \tag{3.35}$$

and

$$V(Z) = \begin{bmatrix} \delta I \\ -\beta I + \dfrac{aW^2P}{b^2+W^2} + \gamma W \end{bmatrix}. \tag{3.36}$$

The pest-free equilibrium of system (2.1) is

$$E_1 = \left(k, 0, 0, \frac{\varepsilon}{\sigma}\right). \tag{3.37}$$

The Jacobian matrices of $F(Z)$ and $V(Z)$ at E_1 are

$$DF(E_1) = \begin{bmatrix} 0 & \dfrac{\alpha k}{m+k} \\ 0 & 0 \end{bmatrix} \tag{3.38}$$

and

$$DV(E_1) = \begin{bmatrix} \delta & 0 \\ -\beta & \gamma \end{bmatrix}. \tag{3.39}$$

Further,

$$(DV(E_1))^{-1} = \frac{1}{\delta\gamma} \begin{bmatrix} \gamma & 0 \\ \beta & \delta \end{bmatrix}. \tag{3.40}$$

Hence,

$$DF(E_1)(DV(E_1))^{-1} = \begin{bmatrix} \dfrac{\alpha\beta k}{(m+k)\delta\gamma} & \dfrac{\alpha k}{(m+k)\gamma} \\ 0 & 0 \end{bmatrix}. \tag{3.41}$$

The eigenvalues of the next-generation matrix are

$$\lambda_1 = \frac{\alpha\beta k}{(m+k)\delta\gamma}, \qquad \lambda_2 = 0. \tag{3.42}$$

Therefore, the spectral radius of $DF(E_1)(DV(E_1))^{-1}$ gives the basic reproduction number as

$$\mathscr{R}_0 = \frac{\alpha\beta k}{(m+k)\delta\gamma}. \tag{3.43}$$

3.4. Stability analysis of the system without a delay

In this subsection, we investigate the local stability of the equilibria of system (2.1) in the absence of a delay, i.e., $\tau = 0$. Let

$$E = (H, I, W, P)$$

be any equilibrium point of the system.

The Jacobian matrix of system (2.1) evaluated at E is given by

$$J(E) = \begin{bmatrix} r\left(1 - \dfrac{2H+I}{k}\right) - \dfrac{\alpha m W}{(m+H)^2} & -\dfrac{rH}{k} & -\dfrac{\alpha H}{m+H} & 0 \\ \dfrac{\alpha m W}{(m+H)^2} & -\delta & \dfrac{\alpha H}{m+H} & 0 \\ 0 & \beta & -\dfrac{2ab^2WP}{(b^2+W^2)^2} - \gamma & -\dfrac{aW^2}{b^2+W^2} \\ 0 & 0 & \dfrac{2adb^2WP}{(b^2+W^2)^2} & \dfrac{adW^2}{b^2+W^2} - \sigma \end{bmatrix}. \tag{3.44}$$

Theorem 3. *The pest-free equilibrium E_1 of system (2.1) is locally asymptotically stable if $\mathscr{R}_0 < 1$ and unstable if $\mathscr{R}_0 > 1$, where*

$$\mathscr{R}_0 = \frac{\alpha\beta k}{(m+k)\delta\gamma}.$$

Proof. Linearizing system (2.1) about E_1 yields the Jacobian matrix

$$J|_{E_1} = \begin{bmatrix} -r & -r & -\dfrac{\alpha k}{m+k} & 0 \\ 0 & -\delta & \dfrac{\alpha k}{m+k} & 0 \\ 0 & \beta & -\gamma & 0 \\ 0 & 0 & 0 & -\sigma \end{bmatrix}. \tag{3.45}$$

The characteristic equation associated with (3.45) is

$$(\lambda + r)(\lambda + \sigma)\left(\lambda^2 + (\delta+\gamma)\lambda + \delta\gamma - \frac{\alpha\beta k}{m+k}\right) = 0.$$

The eigenvalues $-r$ and $-\sigma$ are always negative. The remaining two eigenvalues have negative real parts if and only if

$$\delta\gamma - \frac{\alpha\beta k}{m+k} > 0,$$

which is equivalent to

$$\mathscr{R}_0 = \frac{\alpha\beta k}{(m+k)\delta\gamma} < 1.$$

Hence, the pest-free equilibrium E_1 is locally asymptotically stable whenever $\mathscr{R}_0 < 1$ and unstable whenever $\mathscr{R}_0 > 1$.

□

Theorem 4. *The coexistence equilibrium E_2 of system (2.1) is locally asymptotically stable if the following Routh–Hurwitz conditions are satisfied:*

$$q_1 > 0, \qquad q_2 > 0, \qquad q_3 > 0, \qquad q_4 > 0,$$

$$q_1 q_2 - q_3 > 0,$$

and

$$q_1 q_2 q_3 - q_3^2 - q_1^2 q_4 > 0,$$

where q_1, q_2, q_3, and q_4 are the coefficients of the characteristic polynomial

$$\lambda^4 + q_1\lambda^3 + q_2\lambda^2 + q_3\lambda + q_4 = 0,$$

given explicitly by

$$\begin{aligned}
q_1 &= -r\left(1 - \frac{2H^* + I^*}{k}\right) + \frac{\alpha m W^*}{(m+H^*)^2} + \delta + \frac{2ab^2 W^* P^*}{(b^2+W^{*2})^2} + \gamma - \frac{adW^{*2}}{b^2+W^{*2}} + \sigma,\\
q_2 &= \left[r\left(1 - \frac{2H^* + I^*}{k}\right) - \frac{\alpha m W^*}{(m+H^*)^2}\right]\left[-\delta - \frac{2ab^2 W^* P^*}{(b^2+W^{*2})^2} - \gamma + \frac{adW^{*2}}{b^2+W^{*2}} - \sigma\right]\\
&\quad + \frac{\alpha r H^* m W^*}{k(m+H^*)^2} - \delta\left[-\frac{2ab^2 W^* P^*}{(b^2+W^{*2})^2} - \gamma + \frac{adW^{*2}}{b^2+W^{*2}} - \sigma\right] - \frac{\alpha\beta H^*}{m+H^*}\\
&\quad + \left[-\frac{2ab^2 W^* P^*}{(b^2+W^{*2})^2} - \gamma\right]\left[\frac{adW^{*2}}{b^2+W^{*2}} - \sigma\right] + \frac{2a^2 db^2 W^{*3} P^*}{(b^2+W^{*2})^3},\\
q_3 &= -\left[r\left(1 - \frac{2H^* + I^*}{k}\right) - \frac{\alpha m W^*}{(m+H^*)^2}\right]\left[\delta\left(\frac{2ab^2 W^* P^*}{(b^2+W^{*2})^2} + \gamma\right) - \delta\left(\frac{adW^{*2}}{b^2+W^{*2}} - \sigma\right) - \frac{\alpha\beta H^*}{m+H^*}\right.\\
&\qquad \left. + \left(-\frac{2ab^2 W^* P^*}{(b^2+W^{*2})^2} - \gamma\right)\left(\frac{adW^{*2}}{b^2+W^{*2}} - \sigma\right) + \frac{2a^2 db^2 W^{*3} P^*}{(b^2+W^{*2})^3}\right]\\
&\quad - \frac{\alpha r H^* m W^*}{k(m+H^*)^2}\left[-\frac{2ab^2 W^* P^*}{(b^2+W^{*2})^2} - \gamma + \frac{adW^{*2}}{b^2+W^{*2}} - \sigma\right] + \frac{2a^2 db^2 \delta W^{*3} P^*}{(b^2+W^{*2})^3}\\
&\quad + \left[\frac{adW^{*2}}{b^2+W^{*2}} - \sigma\right]\left[\delta\left(-\frac{2ab^2 W^* P^*}{(b^2+W^{*2})^2} - \gamma\right) + \frac{\alpha\beta H^*}{m+H^*}\right],
\end{aligned}$$

$$
\begin{aligned}
q_4 = &\left[r\left(1-\frac{2H^*+I^*}{k}\right)-\frac{\alpha m W^*}{(m+H^*)^2}\right]\left[(-\delta)\left(-\frac{2ab^2W^*P^*}{(b^2+W^{*2})^2}-\gamma\right)\left(\frac{adW^{*2}}{b^2+W^{*2}}-\sigma\right)\right.\\
&\left.-(-\delta)\left(-\frac{aW^{*2}}{b^2+W^{*2}}\right)\left(\frac{2adb^2W^*P^*}{(b^2+W^{*2})^2}\right)-\frac{\alpha H^*}{m+H^*}\beta\left(\frac{adW^{*2}}{b^2+W^{*2}}-\sigma\right)\right]\\
&+\left(-\frac{rH^*}{k}\right)\left(\frac{\alpha m W^*}{(m+H^*)^2}\right)\left[-\left(-\frac{2ab^2W^*P^*}{(b^2+W^{*2})^2}-\gamma\right)\left(\frac{adW^{*2}}{b^2+W^{*2}}-\sigma\right)\right.\\
&\left.+\left(-\frac{aW^{*2}}{b^2+W^{*2}}\right)\left(\frac{2adb^2W^*P^*}{(b^2+W^{*2})^2}\right)\right]-\frac{\alpha^2 mH^*W^*\beta}{(m+H^*)^3}\left(\frac{adW^{*2}}{b^2+W^{*2}}-\sigma\right).
\end{aligned}
\tag{3.46}
$$

Proof. Evaluating the Jacobian matrix of system (2.1) at the coexistence equilibrium E_2 gives

$$
J(E_2)=\begin{bmatrix}
r\left(1-\dfrac{2H^*+I^*}{k}\right)-\dfrac{\alpha m W^*}{(m+H^*)^2} & -\dfrac{rH^*}{k} & -\dfrac{\alpha H^*}{m+H^*} & 0\\
\dfrac{\alpha m W^*}{(m+H^*)^2} & -\delta & \dfrac{\alpha H^*}{m+H^*} & 0\\
0 & \beta & -\dfrac{2ab^2W^*P^*}{(b^2+W^{*2})^2}-\gamma & -\dfrac{aW^{*2}}{b^2+W^{*2}}\\
0 & 0 & \dfrac{2adb^2W^*P^*}{(b^2+W^{*2})^2} & \dfrac{adW^{*2}}{b^2+W^{*2}}-\sigma
\end{bmatrix}.
$$

The corresponding characteristic equation is

$$
\lambda^4+q_1\lambda^3+q_2\lambda^2+q_3\lambda+q_4=0, \tag{3.47}
$$

where the coefficients q_1, q_2, q_3, and q_4 are given in the statement of the theorem.

According to the Routh–Hurwitz criterion, all roots of (3.47) have negative real parts if and only if the following conditions are satisfied:

$$
q_1>0, \qquad q_2>0, \qquad q_3>0, \qquad q_4>0,
$$

$$
q_1q_2-q_3>0,
$$

and

$$
q_1q_2q_3-q_3^2-q_1^2q_4>0.
$$

Therefore, when these conditions hold, all eigenvalues of $J(E_2)$ have negative real parts. Consequently, the coexistence equilibrium E_2 of system (2.1) is locally asymptotically stable. This completes the proof. □

3.5. Hopf-bifurcation analysis

To investigate the possible occurrence of Hopf bifurcations in the non-delayed system at the coexistence equilibrium E_2, we consider α as the bifurcation parameter while keeping the other parameters fixed. Define the continuously differentiable function $\Psi:(0,\infty)\to\mathbb{R}$ by

$$
\Psi(\alpha):=q_1(\alpha)q_2(\alpha)q_3(\alpha)-q_3^2(\alpha)-q_4(\alpha)q_1^2(\alpha). \tag{3.48}
$$

The characteristic equation of the non-delayed system at the coexistence equilibrium E_2 is

$$\lambda^4 + q_1\lambda^3 + q_2\lambda^2 + q_3\lambda + q_4 = 0. \tag{3.49}$$

The assumptions for Hopf bifurcation to occur require that the spectrum of Eq. (3.49) is such that there exists $\alpha^* \in (0,\infty)$, at which a pair of complex conjugate characteristic roots $\lambda(\alpha^*)$ and $\overline{\lambda}(\alpha^*)$ satisfy

$$\mathrm{Re}\,\lambda(\alpha^*) = 0, \qquad \mathrm{Im}\,\lambda(\alpha^*) = \omega_0 > 0, \tag{3.50}$$

where ω_0 denotes the Hopf angular frequency at the critical parameter value $\alpha = \alpha^*$, corresponding to the purely imaginary characteristic roots $\lambda = \pm i\omega_0$. Furthermore, the transversality condition

$$\left.\frac{d\,\mathrm{Re}\,\lambda(\alpha)}{d\alpha}\right|_{\alpha=\alpha^*} \neq 0 \tag{3.51}$$

must be satisfied, and all other characteristic roots must have negative real parts.

Theorem 5. *Let $q_i(\alpha)$, $i = 1,\ldots,4$, be continuously differentiable in a neighborhood of $\alpha^* > 0$. Suppose that*

$$q_1(\alpha^*) > 0, \qquad q_3(\alpha^*) > 0, \qquad q_4(\alpha^*) > 0,$$

and

$$\Psi(\alpha^*) = q_1(\alpha^*)q_2(\alpha^*)q_3(\alpha^*) - q_3^2(\alpha^*) - q_1^2(\alpha^*)q_4(\alpha^*) = 0.$$

Then the characteristic equation associated with the coexistence equilibrium E_2 has a simple pair of purely imaginary eigenvalues

$$\lambda_{1,2} = \pm i\omega_0, \qquad \omega_0 = \sqrt{\frac{q_3(\alpha^*)}{q_1(\alpha^*)}},$$

and the remaining two eigenvalues have negative real parts. If, in addition,

$$(q_4' - q_2'\omega_0^2)(q_3 - 3q_1\omega_0^2) + 2\omega_0^2(q_3' - q_1'\omega_0^2)(q_2 - 2\omega_0^2) \neq 0, \tag{3.52}$$

where all quantities are evaluated at $\alpha = \alpha^*$, then the transversality condition holds. Hence, the coexistence equilibrium E_2 undergoes a Hopf bifurcation at $\alpha = \alpha^*$.

Proof. By the condition $\Psi(\alpha^*) = 0$, the characteristic equation

$$\lambda^4 + q_1\lambda^3 + q_2\lambda^2 + q_3\lambda + q_4 = 0 \tag{3.53}$$

can be written as

$$\left(\lambda^2 + \frac{q_3}{q_1}\right)\left(\lambda^2 + q_1\lambda + \frac{q_1 q_4}{q_3}\right) = 0, \tag{3.54}$$

where all coefficients are evaluated at $\alpha = \alpha^*$.

If the four roots are λ_i, $i = 1,2,3,4$, with the pair of purely imaginary roots at $\alpha = \alpha^*$ given by $\lambda_1 = i\omega_0$ and $\lambda_2 = -i\omega_0$, then

$$\begin{aligned} \lambda_3 + \lambda_4 &= -q_1, \\ \omega_0^2 + \lambda_3\lambda_4 &= q_2, \\ \omega_0^2(\lambda_3 + \lambda_4) &= -q_3, \\ \omega_0^2\lambda_3\lambda_4 &= q_4. \end{aligned} \tag{3.55}$$

Hence,

$$\omega_0 = \sqrt{\frac{q_3}{q_1}} > 0. \tag{3.56}$$

Let the remaining roots be λ_3 and λ_4. From the factorization, λ_3 and λ_4 are the roots of

$$\lambda^2 + q_1\lambda + \frac{q_1 q_4}{q_3} = 0.$$

If λ_3 and λ_4 are complex conjugates, then

$$2\,\mathrm{Re}(\lambda_3) = 2\,\mathrm{Re}(\lambda_4) = \lambda_3 + \lambda_4 = -q_1 < 0.$$

Hence both roots have negative real parts. If they are real, their sum is negative and their product is

$$\lambda_3\lambda_4 = \frac{q_1 q_4}{q_3} = \frac{q_4}{\omega_0^2} > 0.$$

Since their product is positive, λ_3 and λ_4 have the same sign; since their sum is negative, both roots are negative. Thus, the remaining eigenvalues have negative real parts.

To establish the existence of a Hopf bifurcation at $\alpha = \alpha^*$, we verify the transversality condition. Let the characteristic equation of the linearized non-delayed system at the coexistence equilibrium E_2 be

$$\rho(\lambda, \alpha) = \lambda^4 + q_1(\alpha)\lambda^3 + q_2(\alpha)\lambda^2 + q_3(\alpha)\lambda + q_4(\alpha) = 0. \tag{3.57}$$

Suppose that, at $\alpha = \alpha^*$, the characteristic equation has the pair of purely imaginary roots

$$\lambda_{1,2} = \pm i\omega_0, \qquad \omega_0 > 0.$$

Thus,

$$\rho(i\omega_0, \alpha^*) = 0.$$

Differentiating the identity $\rho(\lambda(\alpha), \alpha) = 0$ with respect to α, we obtain

$$\frac{\partial \rho}{\partial \lambda}\frac{d\lambda}{d\alpha} + \frac{\partial \rho}{\partial \alpha} = 0.$$

Consequently,

$$\frac{d\lambda}{d\alpha} = -\frac{\partial \rho/\partial \alpha}{\partial \rho/\partial \lambda}, \tag{3.58}$$

provided that $\rho_\lambda \neq 0$.

Here,

$$\frac{\partial \rho}{\partial \alpha} = q_1'\lambda^3 + q_2'\lambda^2 + q_3'\lambda + q_4',$$
$$\frac{\partial \rho}{\partial \lambda} = 4\lambda^3 + 3q_1\lambda^2 + 2q_2\lambda + q_3,$$

where the prime denotes differentiation with respect to α.

Evaluating these expressions at $\lambda = i\omega_0$ and $\alpha = \alpha^*$ gives

$$\begin{aligned}\frac{\partial \rho}{\partial \alpha} &= q_4' - q_2'\omega_0^2 + i\omega_0(q_3' - q_1'\omega_0^2),\\ \frac{\partial \rho}{\partial \lambda} &= q_3 - 3q_1\omega_0^2 + 2i\omega_0(q_2 - 2\omega_0^2).\end{aligned} \tag{3.59}$$

All coefficients and their derivatives are evaluated at $\alpha = \alpha^*$.

For simplicity, define

$$\begin{aligned}A &= q_4' - q_2'\omega_0^2,\\ B &= \omega_0(q_3' - q_1'\omega_0^2),\\ C &= q_3 - 3q_1\omega_0^2,\\ D &= 2\omega_0(q_2 - 2\omega_0^2).\end{aligned}$$

Substituting these expressions into (3.58), we obtain

$$\left.\frac{d\lambda}{d\alpha}\right|_{\alpha=\alpha^*} = -\frac{A+iB}{C+iD}.$$

Multiplying the numerator and denominator by the complex conjugate $C - iD$, we get

$$\left.\frac{d\lambda}{d\alpha}\right|_{\alpha=\alpha^*} = -\frac{(AC+BD)+i(BC-AD)}{C^2+D^2}.$$

Therefore, the derivative of the real part of the eigenvalue is

$$\left.\frac{d\,\mathrm{Re}\,\lambda}{d\alpha}\right|_{\alpha=\alpha^*} = -\frac{AC+BD}{C^2+D^2}. \tag{3.60}$$

Hence, provided that $i\omega_0$ is a simple root of the characteristic equation, the transversality condition is satisfied if

$$AC + BD \neq 0. \tag{3.61}$$

Equivalently, in terms of the coefficients of the fourth-degree characteristic equation, this condition is

$$\begin{aligned}&(q_4' - q_2'\omega_0^2)(q_3 - 3q_1\omega_0^2)\\ &\quad + 2\omega_0^2(q_3' - q_1'\omega_0^2)(q_2 - 2\omega_0^2) \neq 0.\end{aligned} \tag{3.62}$$

Furthermore, the simplicity of the imaginary root requires

$$(q_3 - 3q_1\omega_0^2)^2 + 4\omega_0^2(q_2 - 2\omega_0^2)^2 > 0. \tag{3.63}$$

Indeed, condition (3.63) is equivalent to $\rho_\lambda(i\omega_0, \alpha^*) \neq 0$.

Therefore, if conditions (3.62) and (3.63) hold, the real part of the eigenvalue crosses the imaginary axis with nonzero speed as the parameter α passes through α^*. This verifies the transversality condition and consequently a Hopf bifurcation occurs at $\alpha = \alpha^*$. □

3.6. Stability analysis of the system with a delay

In this subsection, we investigate the effect of the discrete time delay τ on the local stability of the equilibria of system (2.1).

Let

$$E = (H, I, W, P)$$

be an equilibrium of system (2.1). Linearizing the delayed system about E, we obtain the variational equation

$$\frac{dX(t)}{dt} = FX(t) + GX(t-\tau), \tag{3.64}$$

where

$$X(t) = \big(H(t), I(t), W(t), P(t)\big)^T,$$

and F, G are the corresponding 4×4 matrices.

The Jacobian matrix is

$$F = \begin{bmatrix} r\left(1 - \dfrac{2H^* + I^*}{k}\right) - \dfrac{\alpha m W^*}{(m+H^*)^2} & -\dfrac{rH^*}{k} & -\dfrac{\alpha H^*}{m+H^*} & 0 \\ \dfrac{\alpha m W^*}{(m+H^*)^2} & -\delta & \dfrac{\alpha H^*}{m+H^*} & 0 \\ 0 & \beta & -\gamma & -\dfrac{aW^{*2}}{b^2+W^{*2}} \\ 0 & 0 & 0 & -\sigma \end{bmatrix}.$$

The delayed Jacobian matrix corresponding to the predator equation is

$$G = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \dfrac{2adb^2 W^* P^*}{(b^2+W^{*2})^2} & \dfrac{adW^{*2}}{b^2+W^{*2}} \end{bmatrix}.$$

For system (3.64), the characteristic equation is

$$\left|\lambda I - F - e^{-\lambda\tau} G\right| = 0. \tag{3.65}$$

Hence,

$$\lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_3\lambda + a_4 + e^{-\lambda\tau}\left(b_1\lambda^3 + b_2\lambda^2 + b_3\lambda + b_4\right) = 0, \tag{3.66}$$

where

$$
\begin{aligned}
a_1 &= -r\left(1-\frac{2H^*+I^*}{k}\right)+\frac{\alpha mW^*}{(m+H^*)^2}+\delta+\gamma+\sigma,\\
a_2 &= \left[r\left(1-\frac{2H^*+I^*}{k}\right)-\frac{\alpha mW^*}{(m+H^*)^2}\right](-\delta-\gamma-\sigma)+(-\delta)(-\gamma-\sigma)+(-\gamma)(-\sigma)\\
&\quad -\left[\frac{\alpha mW^*}{(m+H^*)^2}\right]\left(-\frac{rH^*}{k}\right)-\beta\left(\frac{\alpha H^*}{m+H^*}\right),\\
a_3 &= \left[r\left(1-\frac{2H^*+I^*}{k}\right)-\frac{\alpha mW^*}{(m+H^*)^2}\right][\delta\gamma+\delta\sigma+\gamma\sigma]+\left[r\left(1-\frac{2H^*+I^*}{k}\right)-\frac{\alpha mW^*}{(m+H^*)^2}\right]\left[\frac{\alpha H^*}{m+H^*}\beta\right]\\
&\quad +(-\sigma)\left[-\delta(-\gamma)+\frac{\alpha mW^*}{(m+H^*)^2}\left(-\frac{rH^*}{k}\right)+\beta\left(\frac{\alpha H^*}{m+H^*}\right)\right]+\frac{\alpha mW^*}{(m+H^*)^2}\left(-\frac{rH^*}{k}\right)(-\gamma)\\
&\quad -\frac{\alpha mW^*}{(m+H^*)^2}\beta\left(-\frac{\alpha H^*}{m+H^*}\right),\\
a_4 &= \left[r\left(1-\frac{2H^*+I^*}{k}\right)-\frac{\alpha mW^*}{(m+H^*)^2}\right](-\sigma)\left[(-\delta)(-\gamma)-\beta\left(\frac{\alpha H^*}{m+H^*}\right)\right]\\
&\quad +(-\sigma)\left[-\frac{\alpha mW^*}{(m+H^*)^2}\left(-\frac{rH^*}{k}\right)(-\gamma)+\frac{\alpha mW^*}{(m+H^*)^2}\beta\left(-\frac{\alpha H^*}{m+H^*}\right)\right],\\
b_1 &= -\frac{adW^{*2}}{b^2+W^{*2}},\\
b_2 &= \left[\frac{adW^{*2}}{b^2+W^{*2}}\right]\left[r\left(1-\frac{2H^*+I^*}{k}\right)-\frac{\alpha mW^*}{(m+H^*)^2}-\delta-\gamma\right]-\left[-\frac{aW^{*2}}{b^2+W^{*2}}\right]\left[\frac{2adb^2W^*P^*}{(b^2+W^{*2})^2}\right],\\
b_3 &= \left[\frac{adW^{*2}}{b^2+W^{*2}}\right]\left[-\left(r\left(1-\frac{2H^*+I^*}{k}\right)-\frac{\alpha mW^*}{(m+H^*)^2}\right)(-\gamma)-(-\delta)(-\gamma)+\beta\left(\frac{\alpha H^*}{m+H^*}\right)+\frac{\alpha mW^*}{(m+H^*)^2}\left(-\frac{rH^*}{k}\right)\right.\\
&\quad \left.-\left(r\left(1-\frac{2H^*+I^*}{k}\right)-\frac{\alpha mW^*}{(m+H^*)^2}\right)(-\delta)\right]+\left[-\frac{aW^{*2}}{b^2+W^{*2}}\right]\left[\frac{2adb^2W^*P^*}{(b^2+W^{*2})^2}\right]\left[r\left(1-\frac{2H^*+I^*}{k}\right)-\frac{\alpha mW^*}{(m+H^*)^2}-\delta\right],\\
b_4 &= \left(-\frac{aW^{*2}}{b^2+W^{*2}}\right)\left(\frac{2adb^2W^*P^*}{(b^2+W^{*2})^2}\right)\left[-\left(r\left(1-\frac{2H^*+I^*}{k}\right)-\frac{\alpha mW^*}{(m+H^*)^2}\right)(-\delta)+\frac{\alpha mW^*}{(m+H^*)^2}\left(-\frac{rH^*}{k}\right)\right]\\
&\quad +\left(\frac{adW^{*2}}{b^2+W^{*2}}\right)\left[-\beta\left(\frac{\alpha H^*}{m+H^*}\right)\left(r\left(1-\frac{2H^*+I^*}{k}\right)-\frac{\alpha mW^*}{(m+H^*)^2}\right)+\frac{\alpha mW^*}{(m+H^*)^2}\beta\left(-\frac{\alpha H^*}{m+H^*}\right)\right.\\
&\quad \left.-\frac{\alpha mW^*}{(m+H^*)^2}\left(-\frac{rH^*}{k}\right)(-\gamma)+\left(r\left(1-\frac{2H^*+I^*}{k}\right)-\frac{\alpha mW^*}{(m+H^*)^2}\right)(-\delta)(-\gamma)\right].
\end{aligned}
\tag{3.67}
$$

Theorem 6. *The pest-free equilibrium E_1 of system (2.1) is locally asymptotically stable for all $\tau \geq 0$ if*

$$\mathscr{R}_0 = \frac{\alpha\beta k}{(m+k)\delta\gamma} < 1,$$

and unstable if $\mathscr{R}_0 > 1$.

Proof. At the pest-free equilibrium,

$$E_1 = \left(k, 0, 0, \frac{\varepsilon}{\sigma}\right),$$

and the delayed matrix satisfies

$$G(E_1) = 0.$$

Hence the characteristic equation reduces to

$$(\lambda + r)(\lambda + \sigma)\left[\lambda^2 + (\delta + \gamma)\lambda + \delta\gamma - \frac{\alpha\beta k}{m+k}\right] = 0.$$

Clearly, the eigenvalues $-r$ and $-\sigma$ are negative. The remaining quadratic has roots with negative real parts whenever

$$\delta\gamma - \frac{\alpha\beta k}{m+k} > 0,$$

which is equivalent to

$$\mathscr{R}_0 = \frac{\alpha\beta k}{(m+k)\delta\gamma} < 1.$$

Therefore the pest-free equilibrium is locally asymptotically stable for every $\tau \geq 0$ whenever $\mathscr{R}_0 < 1$. □

3.7. Hopf-bifurcation analysis with a delay

The characteristic equation (3.66) is a transcendental equation in λ. It is well-known that the endemic equilibrium E^* is locally asymptotically stable if all the roots of the corresponding characteristic equation have negative real parts. A loss of stability may occur when a pair of characteristic roots crosses the imaginary axis. Since the characteristic equation is transcendental in the presence of the delay τ, it generally possesses infinitely many complex roots, making the direct determination of the signs of their real parts difficult.

Therefore, we first consider the stability of the endemic equilibrium in the absence of a delay and subsequently investigate how the introduction of the delay affects stability.

Case I: When $\tau = 0$.

In the absence of a delay, $\tau = 0$, the characteristic equation (3.66) reduces to

$$\lambda^4 + (a_1 + b_1)\lambda^3 + (a_2 + b_2)\lambda^2 + (a_3 + b_3)\lambda + (a_4 + b_4) = 0. \tag{3.68}$$

For convenience, define

$$\begin{aligned} \sigma_1 &= a_1 + b_1, \\ \sigma_2 &= a_2 + b_2, \\ \sigma_3 &= a_3 + b_3, \\ \sigma_4 &= a_4 + b_4. \end{aligned} \tag{3.69}$$

Then (3.68) can be written as

$$\lambda^4 + \sigma_1\lambda^3 + \sigma_2\lambda^2 + \sigma_3\lambda + \sigma_4 = 0. \tag{3.70}$$

By the Routh–Hurwitz criterion, the endemic equilibrium E^* is locally asymptotically stable in the absence of a delay if

$$\sigma_1 > 0, \qquad \sigma_2 > 0, \qquad \sigma_3 > 0, \qquad \sigma_4 > 0, \qquad \sigma_1\sigma_2 - \sigma_3 > 0, \tag{3.71}$$

and

$$(\sigma_1\sigma_2 - \sigma_3)\sigma_3 - \sigma_1^2\sigma_4 > 0. \tag{3.72}$$

Hence, throughout the subsequent delay analysis, we assume that the Routh–Hurwitz conditions (3.71)–(3.72) are satisfied.

Case II: When $\tau > 0$.

For $\tau > 0$, the characteristic equation (3.66) is transcendental and consequently has infinitely many characteristic roots. To investigate the stability of E^*, we therefore examine the possibility of characteristic roots crossing the imaginary axis as the delay parameter τ varies. Following the framework established in [24, 25], we analyze the distribution of zeros of the transcendental function and determine conditions under which all characteristic roots have negative real parts.

A necessary condition for a change in stability is the existence of purely imaginary characteristic roots. Let

$$\lambda = i\omega, \qquad \omega > 0. \tag{3.73}$$

Substituting (3.73) into the characteristic equation (3.66) and separating the real and imaginary parts yields

$$\omega^4 - a_2\omega^2 + a_4 = (\omega^2 b_2 - b_4)\cos(\omega\tau) - (b_3\omega - b_1\omega^3)\sin(\omega\tau) \tag{3.74}$$

and

$$a_1\omega^3 - a_3\omega = (\omega^2 b_2 - b_4)\sin(\omega\tau) + (b_3\omega - b_1\omega^3)\cos(\omega\tau). \tag{3.75}$$

Squaring and adding (3.74) and (3.75) eliminates the trigonometric terms and gives

$$\omega^8 + \alpha_1\omega^6 + \alpha_2\omega^4 + \alpha_3\omega^2 + \alpha_4 = 0. \tag{3.76}$$

Introducing

$$l = \omega^2, \tag{3.77}$$

equation (3.76) becomes

$$l^4 + \alpha_1 l^3 + \alpha_2 l^2 + \alpha_3 l + \alpha_4 = 0. \tag{3.78}$$

The coefficients in (3.78) are given by

$$\begin{aligned} \alpha_1 &= a_1^2 - 2a_2 - b_1^2, \\ \alpha_2 &= a_2^2 + 2a_4 - 2a_1a_3 - b_2^2 + 2b_1b_3, \\ \alpha_3 &= a_3^2 - 2a_2a_4 + 2b_2b_4 - b_3^2, \\ \alpha_4 &= a_4^2 - b_4^2. \end{aligned} \tag{3.79}$$

Lemma 1. *If $\alpha_4 < 0$, then equation* (3.78) *has at least one positive root.*

Proof. Define

$$H(l) = l^4 + \alpha_1 l^3 + \alpha_2 l^2 + \alpha_3 l + \alpha_4. \tag{3.80}$$

From (3.80),

$$H(0) = \alpha_4 < 0, \tag{3.81}$$

while

$$\lim_{l\to\infty} H(l) = \infty. \tag{3.82}$$

Therefore, by the intermediate value theorem, there exists at least one $l_0 \in (0,\infty)$ such that

$$H(l_0) = 0. \tag{3.83}$$

Hence, equation (3.78) has at least one positive root. □

The derivative of $H(l)$ is

$$H'(l) = 4l^3 + 3\alpha_1 l^2 + 2\alpha_2 l + \alpha_3. \tag{3.84}$$

To determine the stationary points of $H(l)$, set $H'(l) = 0$ and introduce the transformation

$$z = l + \frac{\alpha_1}{4}. \tag{3.85}$$

Then (3.84) reduces to the depressed cubic

$$z^3 + \Lambda_1 z + \Lambda_2 = 0, \tag{3.86}$$

where

$$\begin{aligned} \Lambda_1 &= \frac{\alpha_2}{2} - \frac{3\alpha_1^2}{16}, \\ \Lambda_2 &= \frac{\alpha_1^3}{32} - \frac{\alpha_1\alpha_2}{8} + \frac{\alpha_3}{4}. \end{aligned} \tag{3.87}$$

Define the discriminant

$$\Delta = \frac{\Lambda_2^2}{4} + \frac{\Lambda_1^3}{27}, \qquad \gamma = \frac{-1+\sqrt{3}\,i}{2}. \tag{3.88}$$

Let

$$\begin{aligned} u &= \sqrt[3]{-\frac{\Lambda_2}{2} + \sqrt{\Delta}}, \\ v &= \sqrt[3]{-\frac{\Lambda_2}{2} - \sqrt{\Delta}}. \end{aligned} \tag{3.89}$$

Then the three roots of (3.86) are

$$\begin{aligned} z_1 &= u + v, \\ z_2 &= \gamma u + \gamma^2 v, \\ z_3 &= \gamma^2 u + \gamma v. \end{aligned} \tag{3.90}$$

Consequently, the corresponding stationary points of $H(l)$ are

$$l_i = z_i - \frac{\alpha_1}{4}, \qquad i = 1,2,3. \tag{3.91}$$

Lemma 2. Suppose that $\alpha_4 < 0$:

(i) If $\Delta \geq 0$, then (3.78) has positive roots if and only if $l_1 > 0$ and $H(l_1) < 0$;
(ii) If $\Delta < 0$, then (3.78) has positive roots if and only if there exists at least one $l^* \in \{l_1, l_2, l_3\}$ such that $l^* > 0$ and $H(l^*) \leq 0$.

Proof. (i) If $\Delta \geq 0$, then the depressed cubic (3.86) has a unique real root z_1, and hence $H'(l) = 0$ has a unique real root l_1. Since $H(l)$ is a differentiable function and

$$\lim_{l\to\infty} H(l) = \infty,$$

we have that l_1 is the unique stationary point of $H(l)$ and is the minimum point of $H(l)$. The sufficient condition follows directly. We prove the necessary condition only.

Here, either $l_1 \leq 0$ or $l_1 > 0$ and $H(l_1) > 0$. Since $\alpha_4 < 0$, we have

$$H(0) = \alpha_4 < 0.$$

If $l_1 \leq 0$, then $H(l)$ is increasing for $l > 0$ and therefore has a positive root. If $l_1 > 0$ and $H(l_1) > 0$, then the minimum of $H(l)$ for $l > 0$ is positive, which contradicts $H(0) < 0$. Hence, $H(l)$ has a positive root if and only if $l_1 > 0$ and $H(l_1) < 0$.

(ii) If $\Delta < 0$, then the depressed cubic (3.86) has three roots z_1, z_2, z_3, and consequently, $H'(l) = 0$ has three stationary points l_1, l_2, l_3. The remaining part of the proof is similar to that of (i). Therefore, $H(l)$ has a positive root if and only if there exists at least one $l^* \in \{l_1, l_2, l_3\}$ such that $l^* > 0$ and $H(l^*) \leq 0$.

Now, suppose that (3.78) has positive roots and denote them by l_k^*, $k = 1, 2, 3, 4$. Then

$$\omega_k = \sqrt{l_k^*}. \tag{3.92}$$

From (3.74) and (3.75), we obtain the value of τ as

$$\begin{aligned} \tau_k^n = \frac{1}{\omega_k} \cos^{-1} \left[\frac{(\omega_k^2 b_2 - b_4)(\omega_k^4 - a_2\omega_k^2 + a_4) + (b_3\omega_k - b_1\omega_k^3)(a_1\omega_k^3 - a_3\omega_k)}{(\omega_k^2 b_2 - b_4)^2 + (b_3\omega_k - b_1\omega_k^3)^2} \right] \\ + \frac{2\pi n}{\omega_k}, \qquad k = 1, 2, 3, 4 \quad \textit{and} \quad n = 0, 1, 2, \ldots. \end{aligned} \tag{3.93}$$

Thus, $\pm i\omega_k$ is a pair of purely imaginary roots of (3.66).

Let

$$\tau^* = \tau_{k_0}^{n_0} = \min_{\substack{n\geq 0 \\ k\geq 1}} \{\tau_k^n\}, \qquad \omega_0 = \omega_{k_0}, \qquad l_0^* = l_{k_0}^*. \tag{3.94}$$

□

Lemma 3. *Suppose that the Routh–Hurwitz conditions* (3.71)–(3.72) *hold, i.e., the non-delayed system is stable. Now,*

(i) *if one of the following holds:*
   (a) $\alpha_4 < 0$,
   (b) $\alpha_4 \geq 0$, $\Delta \geq 0$, $l_1 > 0$, and $H(l_1) \leq 0$,

(c) $\alpha_4 \geq 0$, $\Delta < 0$, and there exists an $l^* \in \{l_1, l_2, l_3\}$ such that $l^* > 0$ and $H(l^*) \leq 0$,

then all roots of the characteristic equation (3.66) have negative real parts when $\tau \in [0, \tau^*)$.

(ii) *if the conditions (a)–(c) of (i) are not satisfied, then all roots of the characteristic equation* (3.66) *have negative real parts for all* $\tau \geq 0$.

From Lemmas 2 and 3, we know that if the conditions (a)–(c) are not satisfied, then equation (3.66) has no roots with a zero real part for all $\tau \geq 0$. If one of (a), (b), or (c) holds, when $\tau \neq \tau^*$, equation (3.66) has no roots with a zero real part and τ^* is the minimum value of τ so that equation (3.66) has purely imaginary roots.

Lemma 4. *Suppose* $H'(l_0^*) \neq 0$. *If* $\tau = \tau^*$, *then* $\pm i\omega_0$ *is a pair of purely imaginary roots of* (3.66). *Moreover, if the conditions of Lemma 3(i) are satisfied, then the sign of*

$$\left.\frac{d(\mathrm{Re}\,\lambda)}{d\tau}\right|_{\tau=\tau^*}$$

is consistent with that of $H'(l_0^*)$.

Proof. Differentiating (3.66) with respect to τ, we get

$$\frac{d\tau}{d\lambda} = \frac{\left(4\lambda^3 + 3a_1\lambda^2 + 2a_2\lambda + a_3\right)e^{\lambda\tau}}{b_1\lambda^4 + b_2\lambda^3 + b_3\lambda^2 + b_4\lambda} + \frac{3b_1\lambda^2 + 2b_2\lambda + b_3}{b_1\lambda^4 + b_2\lambda^3 + b_3\lambda^2 + b_4\lambda} - \frac{\tau}{\lambda}. \tag{3.95}$$

Now, using the relations (3.74) and (3.75), one can obtain

$$\mathrm{Sgn}\left[\left.\frac{d(\mathrm{Re}\,\lambda)}{d\tau}\right|_{\tau=\tau^*}\right] = \mathrm{Sgn}\left[\mathrm{Re}\left(\left.\left(\frac{d\lambda}{d\tau}\right)^{-1}\right|_{\lambda=i\omega_0}\right)\right], \tag{3.96}$$

$$= \mathrm{Sgn}\left[\frac{4\omega_0^6 + 3\alpha_1\omega_0^4 + 2\alpha_2\omega_0^2 + \alpha_3}{(b_4 - b_2\omega_0^2)^2 + (b_3\omega_0 - b_1\omega_0^3)^2}\right]. \tag{3.97}$$

Now, since

$$(b_4 - b_2\omega_0^2)^2 + (b_3\omega_0 - b_1\omega_0^3)^2 \neq 0, \tag{3.98}$$

we can say that

$$\mathrm{Sgn}\left[\left.\frac{d(\mathrm{Re}\,\lambda)}{d\tau}\right|_{\tau=\tau^*}\right] = \mathrm{Sgn}\left[4\omega_0^6 + 3\alpha_1\omega_0^4 + 2\alpha_2\omega_0^2 + \alpha_3\right]. \tag{3.99}$$

Since $l_0^* = \omega_0^2$, we have

$$4\omega_0^6 + 3\alpha_1\omega_0^4 + 2\alpha_2\omega_0^2 + \alpha_3 = H'(l_0^*). \tag{3.100}$$

Therefore,

$$\mathrm{Sgn}\left[\left.\frac{d(\mathrm{Re}\,\lambda)}{d\tau}\right|_{\tau=\tau^*}\right] = \mathrm{Sgn}\left[H'(l_0^*)\right]. \tag{3.101}$$

If

$$H'(l_0^*) > 0, \tag{3.102}$$

then we can say that

$$\mathrm{Sgn}\left[\frac{d(\mathrm{Re}\,\lambda)}{d\tau}\bigg|_{\tau=\tau^*}\right] > 0, \tag{3.103}$$

i.e., if $H'(l_0^*) > 0$, the pair of characteristic roots crosses the imaginary axis from the left half-plane to the right half-plane as τ increases through τ^*.

Thus, from Lemmas 2, 3, and 4, we have the following theorem:

Theorem 7. *If the conditions (a)–(c) of Lemma 3 are not satisfied, then* (3.66) *has no roots with a zero real part for all $\tau \geq 0$, and the system remains locally asymptotically stable for all $\tau \geq 0$. If one of the conditions (a), (b), or (c) holds, then all roots of* (3.66) *have negative real parts for $\tau \in [0, \tau^*)$, where τ^* is the first critical delay. At $\tau = \tau^*$,* (3.66) *has a pair of purely imaginary roots $\lambda = \pm i\omega_0$. If*

$$\frac{d(\mathrm{Re}\,\lambda)}{d\tau}\bigg|_{\tau=\tau^*} > 0,$$

then this pair of characteristic roots crosses the imaginary axis from the left half-plane to the right half-plane as τ increases through τ^*. Consequently, the system loses local asymptotic stability at $\tau = \tau^*$ and undergoes a Hopf bifurcation.

4. Sensitivity analysis

In this section, we analyze the sensitivity of the basic reproduction number $\mathscr{R}_0$ with respect to the key model parameters in order to identify the factors that most strongly influence pest invasion and persistence. For system (2.1), the basic reproduction number is given by

$$\mathscr{R}_0 = \frac{\alpha\beta k}{(m+k)\delta\gamma}. \tag{4.1}$$

Here, α denotes the maximum infestation rate, m is the half-saturation constant of the Holling type-II incidence function, β is the whitefly reproduction rate, k is the carrying capacity of healthy leaves, δ is the removal rate of infested leaves, and γ is the natural mortality rate of the whitefly population.

4.1. Normalized sensitivity indices

The normalized sensitivity index of $\mathscr{R}_0$ with respect to a parameter p is defined by

$$\mathsf{E}_p = \frac{p}{\mathscr{R}_0}\frac{\partial \mathscr{R}_0}{\partial p}. \tag{4.2}$$

Using this definition, the partial derivatives of $\mathscr{R}_0$ are

$$\frac{\partial \mathscr{R}_0}{\partial \alpha} = \frac{\beta k}{(m+k)\delta\gamma}, \qquad \frac{\partial \mathscr{R}_0}{\partial \beta} = \frac{\alpha k}{(m+k)\delta\gamma}, \tag{4.3}$$

$$\frac{\partial \mathscr{R}_0}{\partial k} = \frac{\alpha\beta m}{(m+k)^2\delta\gamma}, \qquad \frac{\partial \mathscr{R}_0}{\partial m} = -\frac{\alpha\beta k}{(m+k)^2\delta\gamma}, \tag{4.4}$$

$$\frac{\partial \mathscr{R}_0}{\partial \delta} = -\frac{\alpha\beta k}{(m+k)\delta^2\gamma}, \qquad \frac{\partial \mathscr{R}_0}{\partial \gamma} = -\frac{\alpha\beta k}{(m+k)\delta\gamma^2}. \tag{4.5}$$

Consequently, the normalized sensitivity indices are

$$\mathsf{E}_\alpha = 1, \qquad \mathsf{E}_\beta = 1, \qquad \mathsf{E}_k = \frac{m}{m+k}, \qquad \mathsf{E}_m = -\frac{m}{m+k}, \qquad \mathsf{E}_\delta = -1, \qquad \mathsf{E}_\gamma = -1. \tag{4.6}$$

As shown in Figure 1, the sensitivity indices illustrate the relative influence of the model parameters on the basic reproduction number $\mathscr{R}_0$. The sensitivity analysis indicates that $\mathscr{R}_0$ increases proportionally with the maximum infestation rate α and the whitefly reproduction rate β, since both have normalized sensitivity indices equal to one. Similarly, an increase in the carrying capacity k increases $\mathscr{R}_0$, although its influence is moderated by the factor $\frac{m}{m+k}$ because of the saturation effect of the Holling type-II incidence function. Conversely, $\mathscr{R}_0$ decreases with increasing values of the half-saturation constant m, the removal rate of infested leaves δ, and the natural mortality rate of whiteflies γ. The effect of m is quantified by the normalized sensitivity index $-\frac{m}{m+k}$, indicating that an increase in the half-saturation constant reduces the effective infestation rate and consequently suppresses pest establishment. Therefore, effective pest-management strategies should aim to reduce the infestation rate and whitefly reproduction while increasing the removal of infested leaves and the mortality of whiteflies. In addition, increasing the half-saturation constant can further weaken the infestation process and reduce the invasion potential of the whitefly population.

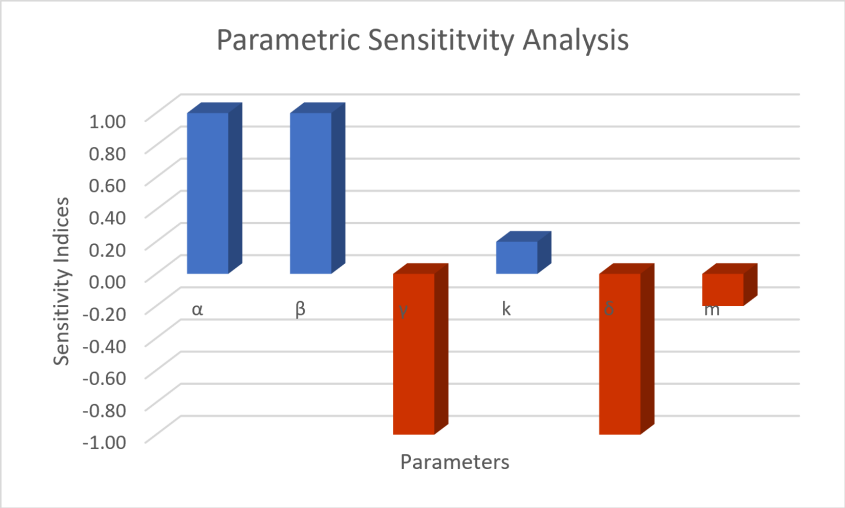


Figure 1. Graphical representation of sensitivity indices corresponding to parameters influencing $\mathscr{R}_0$.

5. Numerical simulation

A few representative numerical simulations are presented in this section to illustrate the dynamical behavior of the proposed delayed model and to examine the influence of several key biological parameters on the population dynamics. The initial history function is assumed to be constant over the interval $[-\tau,0]$, with $(H(\theta),I(\theta),W(\theta),P(\theta)) = (2,1,1,10)$, $\theta \in [-\tau,0]$, representing the initial

populations of healthy coconut leaves, infested leaves, rugose spiraling whiteflies, and predators, respectively. The parameter values used in the numerical simulations are selected with reference to the literature-derived ranges reported in Table 1. The baseline parameter values are $r = 0.20$, $k = 40$, $\alpha = 0.10$, $m = 10$, $\delta = 0.02$, $\beta = 0.02$, $a = 0.019$, $b = 10$, $\gamma = 0.01$, $d = 0.58$, $\sigma = 0.03$, $\varepsilon = 1$, and $\tau = 9$. Among these, the values of r, α, and ε are selected outside the literature-based ranges to illustrate the qualitative dynamical behavior of the system and to satisfy the analytical conditions considered in the stability and Hopf bifurcation analysis. These parameter values are used for illustrative numerical analysis and are not intended to represent a single field-calibrated parameter set. All numerical simulations are carried out using the MATLAB *dde23* solver for delay differential equations. For the additional bifurcation diagrams, the parameters α, a, and β are varied individually while the remaining parameters are fixed at their baseline values, and the corresponding long-term dynamics are obtained from direct numerical parameter sweeps using *dde23*. The critical parameter values reported in the bifurcation diagrams are identified from the resulting changes in the long-term dynamical behavior.

Table 1. Literature Based parameter ranges used as reference values for the model parameters [16, 18–20].

Parameter	Description	Literature-based Range	Units
r	Growth rate of healthy coconut leaves	$0.0027 - 0.004$	day^{-1}
k	Carrying capacity of coconut leaves	$30 - 40$	leaves per palm
α	Maximum infestation rate	$0.00001 - 0.0005$	day^{-1}
m	Half-saturation constant of the Holling type-II incidence	10	–
δ	Removal rate of infested leaves	$0.02 - 0.04$	day^{-1}
β	Whitefly reproduction rate from infested leaves	$0.02 - 0.04$	day^{-1}
a	Predation rate of whitefly by predator	0.019	$\text{predator}^{-1}\ \text{day}^{-1}$
b	Half-saturation constant of Holling type-III functional response	10	–
γ	Natural death rate of whitefly	$0.001 - 0.025$	day^{-1}
d	Predator conversion efficiency	$0.58 - 0.78$	day^{-1}
σ	Predator death rate	$0.02 - 0.04$	day^{-1}
ε	Constant predator release rate	0.10	$\text{predators day}^{-1}$
τ	Predator developmental delay	$8 - 15$	days

Figure 2a illustrates the time evolution of the system corresponding to the pest-free equilibrium ($\mathscr{R}_0 < 1$) obtained for $\alpha = 0.0001$. It is observed that the infested leaf population and rugose spiraling whitefly population gradually decline to zero, while the healthy leaf population approaches a stable positive level. In contrast, Figure 2b shows the endemic equilibrium ($\mathscr{R}_0 > 1$) corresponding to $\alpha = 0.1$, where all populations persist and converge to positive steady-state values.

Figure 3(a) illustrates the influence of the maximum infestation rate α on the population dynamics. As α increases, the healthy coconut leaf population decreases due to the enhanced infestation pressure, while the populations of infested leaves and whiteflies increase and exhibit more pronounced oscillatory behavior. The predator population also increases slightly because the greater whitefly abundance provides additional food resources for predator development. The corresponding bifurcation diagrams in Figure 3(b) reveal the occurrence of a Hopf bifurcation at the critical value

$\alpha_c = 0.135$. For $\alpha < 0.135$, the coexistence equilibrium is locally asymptotically stable, whereas for $\alpha > 0.135$, it loses stability and stable periodic oscillations appear.

Figure 4(a) shows the effect of the predation rate a on the system dynamics. Increasing the predation rate suppresses both the infested leaf and whitefly populations, whereas the healthy leaf population increases as pest pressure is reduced. The predator population approaches a stable positive level, indicating that enhanced predation improves the efficiency of biological control. The bifurcation diagrams in Figure 4(b) indicate that the predation rate also governs the qualitative dynamics of the system. A Hopf bifurcation occurs at the critical value $a_c = 0.013$. Increasing the predation rate beyond this threshold suppresses oscillations and restores the stability of the coexistence equilibrium.

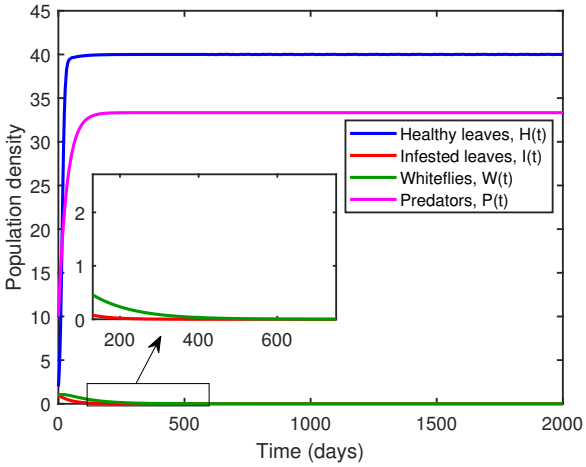
Figure 5(a) depicts the effect of varying the whitefly reproduction rate β. Larger values of β promote rapid growth of the whitefly population, resulting in a higher level of infestation and a corresponding decline in the healthy leaf population. The predator population also increases because the greater prey density supports predator persistence. Figure 5(b) shows that a Hopf bifurcation occurs at the critical whitefly reproduction rate $\beta_c = 0.028$. Beyond this threshold, the coexistence equilibrium becomes unstable and sustained oscillations emerge.

Stability regions of the coexistence equilibrium E_2 are presented in Figure 6. The figure illustrates the stability region of E_2 in the (β, a) parameter plane. The green region represents the combinations of the whitefly reproduction rate β and predation rate a for which the coexistence equilibrium E_2 is locally asymptotically stable, whereas the yellow region corresponds to an unstable coexistence equilibrium. The boundary separating the stable and unstable regions indicates the stability transition, where the equilibrium loses stability as the parameter combinations are varied. In particular, increasing the predation rate a generally enlarges the stable region, demonstrating the stabilizing effect of stronger predation on the pest–predator system. Thus, the figure identifies the parameter combinations that promote either stable coexistence or instability of the endemic equilibrium.

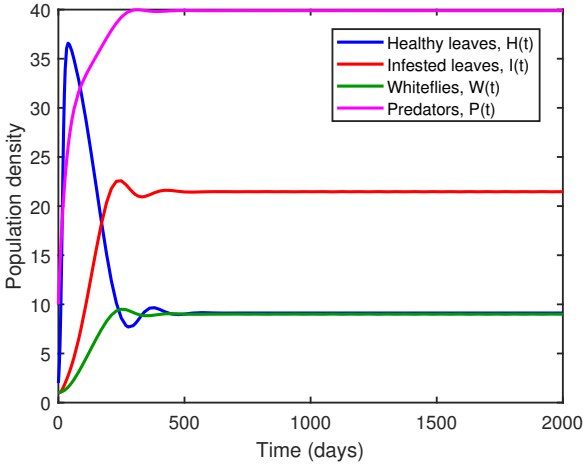
Figure 7 presents the time evolution of the system for different values of the developmental delay τ using the parameter values of Table 1. Increasing the delay prolongs the time required for predator recruitment, leading to a slightly larger increase in the whitefly and infested leaf populations. The healthy leaf and predator populations remain bounded and eventually converge to positive steady-state values. Hence, for the parameter values of Table 1, the coexistence equilibrium remains stable over the range of delays considered, and the delay affects mainly the transient dynamics.

The delay can, however, change the stability of the coexistence equilibrium when predation is stronger. Figure 7 illustrates this case for $a = 0.30$, $\beta = 0.10$, and $\sigma = 0.20$, with the remaining parameters as in Table 1. These values are chosen only to illustrate the delay-induced Hopf bifurcation. For these values, the equilibrium is stable in the absence of delay, and the critical delay obtained from the characteristic equation is $\tau^* \approx 13.15$. Figure 7(a) shows that for $\tau = 5$ and $\tau = 10$, which are below τ^*, all populations converge to the coexistence equilibrium, whereas for $\tau = 14$, which is above τ^*, the solutions settle into sustained periodic oscillations. The amplitude of these oscillations increases as τ moves further beyond τ^*. The bifurcation diagrams in Figure 7(b) confirm this transition: a single branch is observed for $\tau < \tau^*$, while the maximum and minimum values of the periodic solutions separate at the Hopf bifurcation point HB and the gap widens with τ. The numerical value of τ^* therefore agrees with the analytical Hopf condition. This shows that a long developmental delay can destabilize the predator–pest interaction even when strong predation stabilizes it for short delays.

Overall, the numerical simulations demonstrate that increasing the infestation rate and whitefly



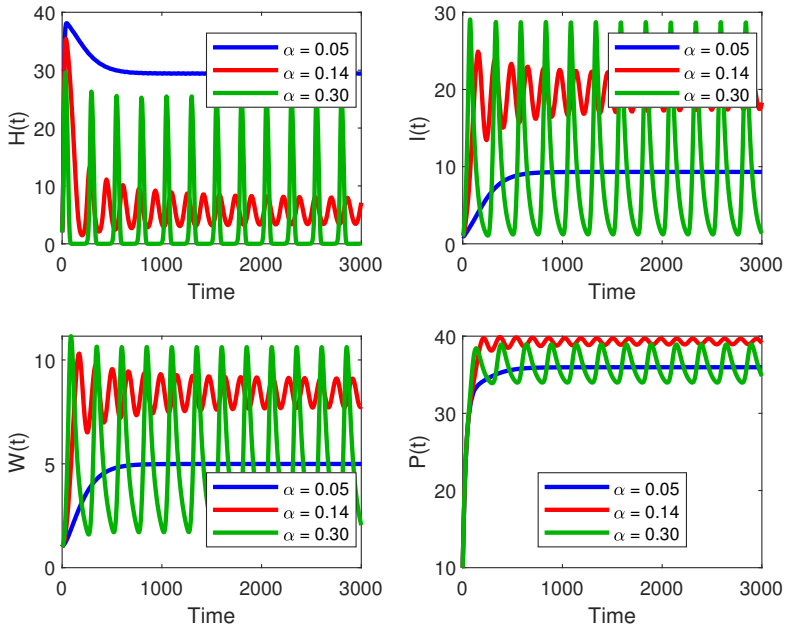
(a)



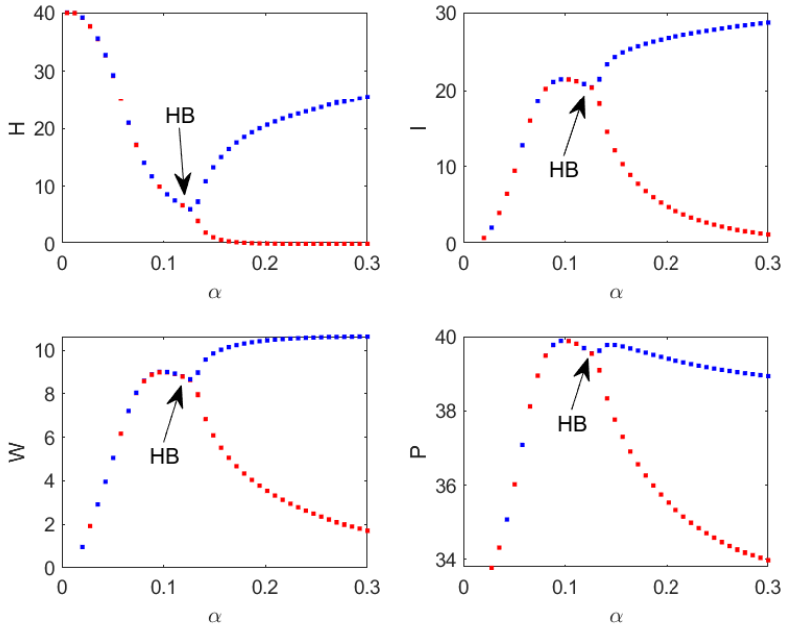
(b)

Figure 2. (a) Time evolution of the system dynamics corresponding to the pest-free equilibrium ($\mathscr{R}_0 < 1$) for $\alpha = 0.0001$. (b) Time evolution of the system dynamics corresponding to the endemic equilibrium ($\mathscr{R}_0 > 1$) for $\alpha = 0.1$. Other parameter values are as in Table 1.

reproduction rate promotes pest persistence and reduces the healthy coconut leaf population. In contrast, increasing the predation rate enhances biological control by suppressing the whitefly population and promoting the recovery of healthy leaves. The bifurcation analysis further confirms that the parameters α, β, and a play a crucial role in determining the stability of the coexistence equilibrium. For the parameter ranges considered in the numerical simulations, the developmental delay has only a transient effect when predation is weak, whereas for stronger predation, it induces a Hopf bifurcation at a critical delay τ^*, beyond which the coexistence equilibrium loses stability and periodic oscillations appear.

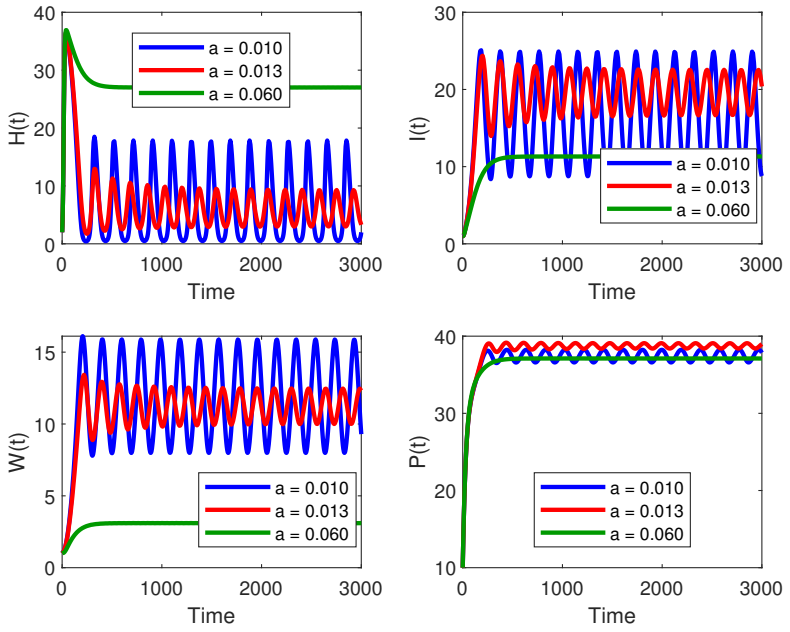


(a)

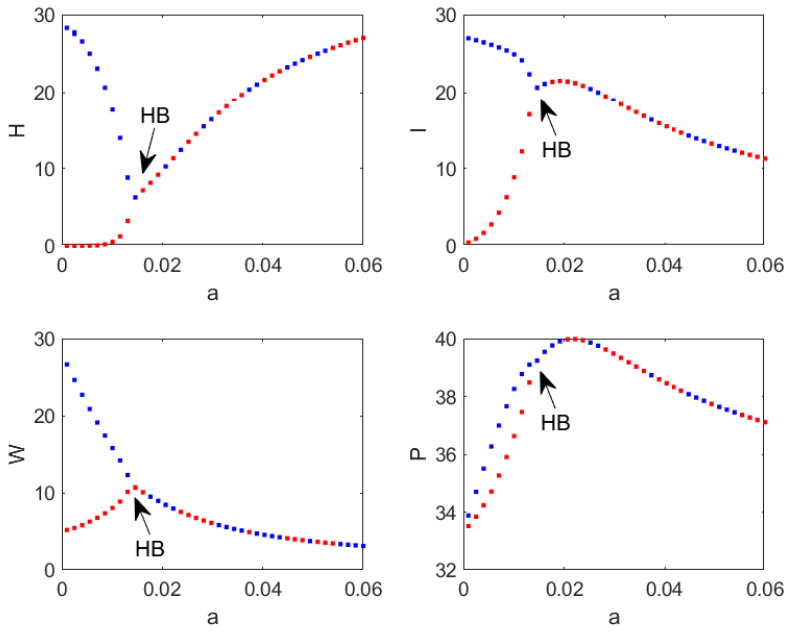


(b)

Figure 3. Effect of the maximum infestation rate α on the system dynamics. (a) Time evolution of the healthy coconut leaves $H(t)$, infested leaves $I(t)$, whitefly population $W(t)$, and predator population $P(t)$ for different values of α. (b) Bifurcation diagrams of H, I, W, and P with respect to α, where HB denotes the Hopf bifurcation point. We plot the maximum (blue dotted line) and minimum (red dotted line) values of the periodic solutions. Other parameter values are as given in Table 1.

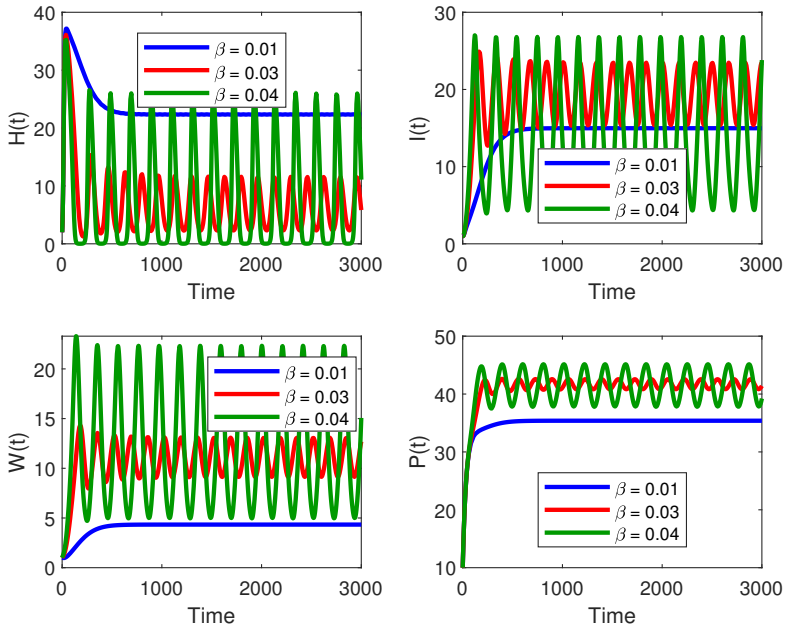


(a)

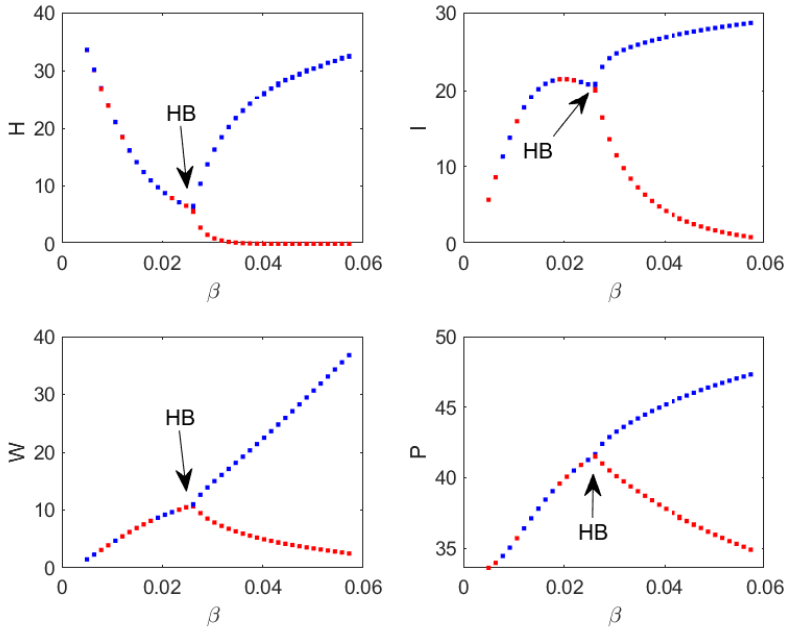


(b)

Figure 4. Effect of the predation rate a on the system dynamics. (a) Time evolution of the healthy coconut leaves $H(t)$, infested leaves $I(t)$, whitefly population $W(t)$, and predator population $P(t)$ for different values of a. (b) Bifurcation diagrams of H, I, W, and P with respect to the predation rate a, where HB denotes the Hopf bifurcation point. We plot the maximum (blue dotted line) and minimum (red dotted line) values of the periodic solutions. Other parameter values are as given in Table 1.

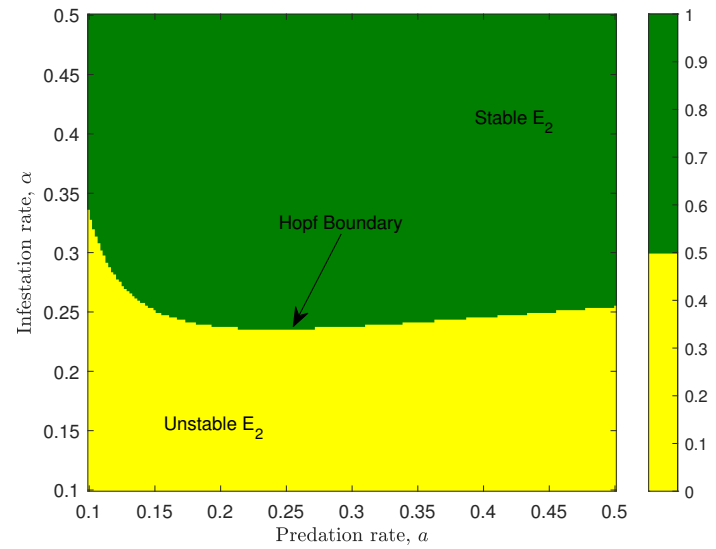


(a)

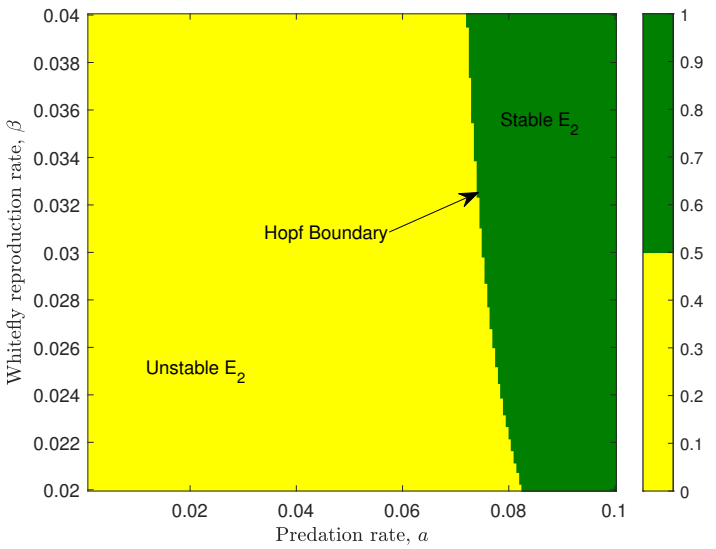


(b)

Figure 5. Effect of the whitefly reproduction rate β on the system dynamics. (a) Time evolution of the healthy coconut leaves $H(t)$, infested leaves $I(t)$, whitefly population $W(t)$, and predator population $P(t)$ for different values of β. (b) Bifurcation diagrams of H, I, W, and P with respect to the whitefly reproduction rate β, where HB denotes the Hopf bifurcation point. We plot the maximum (blue dotted line) and minimum (red dotted line) values of the periodic solutions. Other parameter values are as given in Table 1.

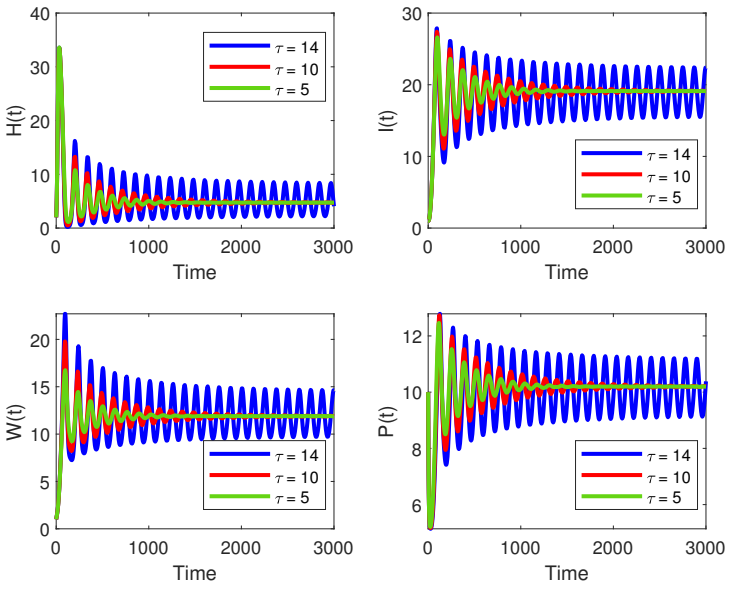


(a)

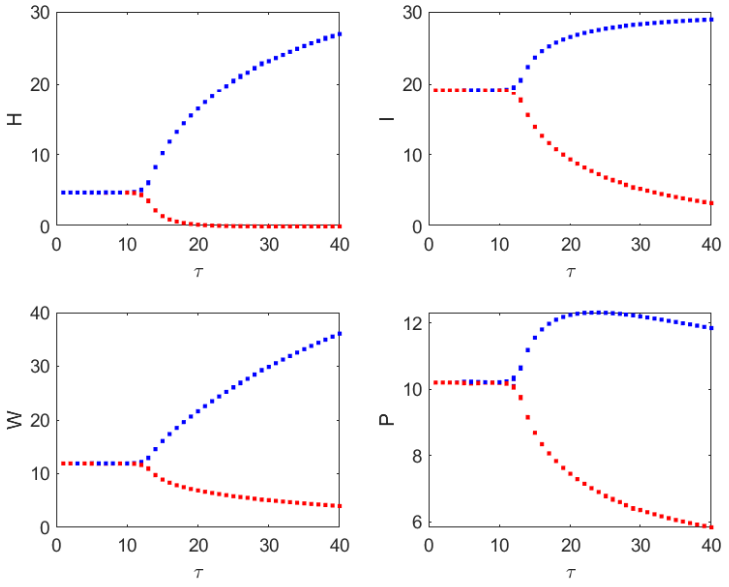


(b)

Figure 6. Stability region of the coexistence equilibrium E_2 in the β–a parameter plane. The color code represents the maximum real part of the eigenvalues λ of the Jacobian matrix at E_2. The green region indicates that E_2 is locally asymptotically stable, whereas the yellow region indicates that E_2 is unstable. The boundary separating the stable and unstable regions corresponds to the Hopf bifurcation boundary. Other parameter values are as given in Table 1.



(a)



(b)

Figure 7. Effect of the developmental delay τ on the system dynamics. (a) Time evolution of the healthy coconut leaves $H(t)$, infested leaves $I(t)$, whitefly population $W(t)$, and predator population $P(t)$ for $\tau = 5$, 10, and 14. The coexistence equilibrium is stable for $\tau < \tau^*$, whereas sustained oscillations appear for $\tau > \tau^*$. (b) Bifurcation diagrams of H, I, W, and P with respect to τ, where HB denotes the Hopf bifurcation point at $\tau^* \approx 13.15$. The maximum (blue dotted line) and minimum (red dotted line) values of the periodic solutions are plotted after the transient. The parameter values are $a = 0.30$, $\beta = 0.10$, and $\sigma = 0.20$, with the remaining parameters as given in Table 1.

6. Conclusions

This study developed a delay differential equation model to investigate the dynamics of rugose spiraling whitefly infestation in coconut plantations and its biological control using the natural predator *Pseudomallada astur*. The model incorporates a Holling type-III functional response together with a discrete developmental delay to account for the time required for predator development. The analytical results establish the positivity and boundedness of solutions and characterize the relevant equilibrium states. The basic reproduction number $\mathscr{R}_0$ is derived and shown to determine the local stability of the pest-free equilibrium. In particular, the pest-free equilibrium is locally asymptotically stable when $\mathscr{R}_0 < 1$, indicating the possibility of pest eradication, whereas it becomes unstable when $\mathscr{R}_0 > 1$, allowing persistence of the whitefly population.

The numerical simulations support the theoretical findings and illustrate the effects of key biological parameters on the system dynamics. An increase in the whitefly reproduction and infestation rates promotes pest persistence and reduces the healthy coconut leaf population, whereas an increase in the predation rate enhances biological control and facilitates the recovery of healthy leaves. Furthermore, for the parameter ranges considered in the numerical simulations, the predator developmental delay primarily affects the transient dynamics while the system maintains long-term coexistence. These results highlight the importance of incorporating predator developmental characteristics when evaluating biological control strategies for rugose spiraling whitefly management.

The present model provides a useful framework for understanding the interactions among coconut plants, rugose spiraling whiteflies, and their natural predator. However, several extensions can be considered to increase its biological realism and practical applicability. Future studies could incorporate seasonal variations in environmental conditions, spatial dispersal of whiteflies and predators, and stochastic environmental fluctuations. The effects of additional biocontrol agents and their interactions with *Pseudomallada astur* could also be investigated. Furthermore, optimal control approaches could be developed to identify effective and economically feasible pest-management strategies. Finally, calibration and validation of the model using experimental and field data would provide more reliable parameter estimates and help assess the predictive performance of the model. Such extensions may contribute to the development of more comprehensive and effective strategies for managing rugose spiraling whitefly infestations in coconut plantations.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that there are no conflicts of interest.

Data availability

The datasets used for this research are mentioned within this article.

Author contributions

E. Jenitta: Investigation, Methodology, Software, Writing—original draft, Visualization. **R. Senthamarai:** Conceptualization, Methodology, Validation, Resources, Writing—review and editing, Supervision.

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