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Research article

Stability of nonlinear stochastic systems under event-triggered impulsive control with distributed-delay impulses

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Abstract: This paper investigated the stability of nonlinear stochastic systems with distributed-delay impulses within the framework of event-triggered impulsive control (ETIC). A continuous event-triggered mechanism (ETM) with a fixed waiting time and a periodic ETM with a fixed sampling period were proposed, effectively eliminating the occurrence of Zeno behavior. By employing the Lyapunov method and mathematical induction, a set of sufficient conditions was established to ensure the p-th moment uniform stability (p-US) and p-th moment exponential stability (p-ES) of the considered system. Furthermore, the theoretical results were applied to a class of nonlinear stochastic systems. Utilizing the linear matrix inequality (LMI) approach, a joint design of the ETM and impulsive control gains was achieved. Finally, numerical examples were provided to demonstrate the effectiveness and feasibility of the proposed theoretical results.

Keywords: nonlinear stochastic system; event-triggered impulsive control; distributed-delay impulses; stability

1. Introduction

As a typical class of hybrid systems, impulsive systems integrate continuous-time dynamic with discrete-time state jumps, and can effectively model abrupt dynamic changes observed in phenomena such as spike pulse [1, 2], epidemic transmission [3], and network communication [4]. From a control perspective, impulse systems offer significant advantages in engineering applications due to their simple structure, flexible control schemes, and the fact that control inputs are applied only at discrete moments, thus avoiding the high energy consumption and system burden associated with continuous control. In particular, impulsive control (IC) strategies have been widely applied in fields such as aerospace engineering [5], biomedicine [6], and ecology [7, 8]. However, most existing studies are based on a pre-scheduled sequence of impulse times, i.e., time-triggered impulsive control strategies. Although this mechanism is easy to implement, it often overlooks whether the system's current state truly requires control input, which can lead to resource waste in practical applications.

To overcome the limitations of traditional time-triggered mechanisms, event-triggered control (ETC) has been introduced into the design of control systems in recent years. This strategy determines whether to apply control based on whether the system state satisfies a predefined triggering condition, thereby significantly reducing unnecessary control updates and resource consumption. It has attracted widespread attention across various fields [9–12]. Recently, researchers have further developed the event-triggered impulsive control (ETIC) strategy, which integrates the advantages of both ETC and IC. In this approach, control impulses are applied only when the system state deviates from the desired behavior and a predefined triggering condition is satisfied. This strategy not only preserves the simplicity and instantaneous intervention characteristics inherent to IC, but also enhances the adaptability of the control scheme and improves resource utilization efficiency. As a result, it has garnered significant interest from researchers and has been widely applied in numerous areas [13–17]. For instance, [13] investigated the stabilization problem of nonlinear stochastic systems using an ETIC scheme, while [14] proposed both continuous and dynamic ETIC methods to analyze input-to-state stability of nonlinear systems.

It is worth noting that most of the aforementioned studies did not take time delays into consideration. However, time delays are often unavoidable in practical systems, particularly during communication transmission and information sampling processes. For example, [18] discussed the impulse stabilization problem of nonlinear systems and proposed two event-triggered algorithms to update impulsive control signals with actuation delays. Li et al. [19] investigated the ETIC approach to analyze the input-to-state stability of nonlinear systems with delays. Peng et al. [20] analyzed the stability of stochastic time-delay systems using the ETIC approach. However, both [19] and [20] did not consider the existence of delays within impulses. In recent years, some progress has been made in research on ETIC strategies with delayed impulses [21–24]. For instance, [22] probed into how event-triggered delayed impulsive control (ETDIC) ensures finite-time stability in time-varying nonlinear systems. Shi et al. [23] investigated the stability of inertial delayed neural networks under dynamic ETIC. However, most of these studies focus on discrete delays. In contrast, distributed delays, as another important form of delay, describe the accumulated influence of past states over a time interval through an integral form and are widely present in various fields [25–28]. For example, Liu et al. [25] studied the exponential stability of nonlinear delayed systems under distributed-delay-dependent impulsive control. Liu et al. [26] explored Lyapunov stability of general nonlinear systems subjected to distributed-delay impulses by ETIC, and applied the theoretical findings to the synchronization of complex neural networks.

In addition, most of the abovementioned studies are based on deterministic systems. In the presence of stochastic effects, whether the existing ETIC methods are still applicable requires further investigation and verification. To this end, some researchers have conducted related studies, see [13, 29–32]. For example, [13] proposed feasible schemes for the stability of nonlinear stochastic systems under the framework of ETIC, but did not consider the existence of delays within the impulses. In [29], two types of event-triggered mechanisms (ETMs) were designed based on the Lyapunov functional method to study the stabilization of stochastic functional differential systems, and the Zeno behavior was effectively avoided. In [31], the p-th moment exponential stability of stochastic delayed systems was investigated under ETDIC, where the impulse times were generated based on historical state information. However, most of these studies only consider discrete delays, and relatively little attention has been paid to distributed delays.

Inspired by the discussions above, this paper investigates the p-th moment uniform stability (p-US) and p-th moment exponential stability (p-ES) of nonlinear stochastic systems with distributed-delay impulses under the ETIC, and applies the theoretical results to a class of nonlinear stochastic systems. This paper makes the following primary contributions:

(i) Compared with the existing literature [13,15,20,22,24], this paper thoroughly considers the effect of distributed-delay impulses and explicitly establishes the relationship among triggering parameters, impulse intensity, and distributed delays.

(ii) Two types of ETMs are designed: continuous and periodic. For each mechanism, analytical frameworks are established to guarantee p-US and p-ES of the system, respectively. In contrast to existing studies [17, 20, 24, 27], which typically consider only a single triggering strategy, this paper derives sufficient stability conditions for both mechanisms under a unified theoretical framework, thereby enhancing the adaptability of the proposed approach.

(iii) A time regularization parameter is introduced in the continuous ETM to strictly ensure a minimum inter-event time (MIET), effectively eliminating Zeno behavior. Additionally, an explicitly computable upper bound for the sampling period is provided for the proposed periodic ETM.

(iv) Compared with [13] and [33], the theoretical results in this paper are extended to nonlinear stochastic systems. Leveraging linear matrix inequality (LMI) techniques, a codesign method is proposed for determining both the control gain and triggering parameters.

The remaining sections of this paper are organized as follows. Section 2 introduces some necessary preliminaries. In Section 3, several sufficient conditions are provided to ensure the stability of nonlinear stochastic systems. Section 4 demonstrates the application of the theoretical results. In Section 5, two illustrative examples verify the obtained results. Finally, Section 6 summarizes the conclusions of this paper and suggests possible future research directions.

2. Problem formulation and preliminaries

Notations: N, N^+, $\mathbb{R}_+$, $\mathbb{R}^m$, $\mathbb{R}^{m\times n}$ denote the set of natural numbers, positive integers, nonnegative real numbers, and the m-dimensional and $m\times n$-dimensional Euclidean spaces equipped with the Euclidean norm $\|\cdot\|$. $\mathcal{I}$ denotes the identity matrix of suitable dimension. L^T and L^{-1} represent the transpose and the inverse of matrix L, respectively. A matrix O is symmetric and positive (or negative) definite if $O > 0$ ($O < 0$). $x \vee y$ ($x \wedge y$) corresponds to $\max\{x,y\}$ ($\min\{x,y\}$). $\mathbb{E}$ denotes the expectation operator with respect to the probability measure $\mathcal{P}$. $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathcal{P})$ is a complete probability space equipped with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$. $\mathcal{PC}([-\varrho,0];\mathbb{R}^m) = \left\{\hbar : [-\varrho,0] \to \mathbb{R}^m \text{with norm } \|\hbar\|_\varrho, \text{ where } \|\hbar\|_\varrho = \sup_{\nu\in[-\varrho,0]}|\hbar(\nu)|\right\}$, $\mathcal{PC}^b_{\mathcal{F}_0}([-\varrho,0];\mathbb{R}^m)$ represents the space of all bounded $\mathcal{F}_0$-measurable, $\mathcal{PC}([-\varrho,0];\mathbb{R}^m)$-valued random variables. $\mathcal{K} = \{\beta(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+ \mid \beta(\cdot)$ is continuous and strictly increasing, $\beta(0) = 0\}$. $\mathcal{K}_\infty = \{\beta(\cdot) \in \mathcal{K} \mid \beta(\cdot)$ is unbounded$\}$. $trace(\cdot)$ denotes the trace of a matrix.

Consider the following nonlinear stochastic system with distributed-delay impulses:

$$\begin{cases} dz(t) = \Gamma(t, z(t))dt + \Upsilon(t, z(t))d\omega(t), \ t \neq t_s, \\ z(t) = \mathcal{D}_s(z_{\tau_s}(t)), \ t = t_s, \\ z_{t_0} = \hbar, \end{cases} \tag{2.1}$$

where $z_{\tau_s}(t) = \int_{-\tau_s}^{0} z(t+r)dr$, τ_s is the set of distributed delays in impulses and satisfies $\tau_s \leq \tau,\ s \in N^+$. $z_{t_0}(\nu) = z(t_0+\nu)$, $\nu \in [-\varrho, 0]$. $\hbar \in \mathcal{PC}^b_{\mathcal{F}_0}([-\varrho, 0]\,; \mathbb{R}^m)$ is initial value. $\Gamma : \mathbb{R}_+ \times \mathbb{R}^m \to \mathbb{R}^m$, $\Upsilon : \mathbb{R}_+ \times \mathbb{R}^m \to \mathbb{R}^{m\times n}$, $\omega(t)$ is an n-dimensional Brownain motionan defined on $(\Omega, \mathcal{F}, \mathcal{P})$. $\mathcal{D}_s : \mathbb{R}^m \to \mathbb{R}^m$ denotes the control input for impulse instant t_s. $\{t_s,\ s \in N^+\}$ is a sequence of impulse times determined by a well-designed ETM.

If the functions Γ, Υ, and $\mathcal{D}_s$ satisfy both the Lipschitz condition and the linear growth condition, then system (2.1) admits a unique solution. Moreover, if $\Gamma(t, 0) \equiv 0$, $\Upsilon(t, 0) \equiv 0$, $\mathcal{D}_s \equiv 0$ for $s \in N^+$, then the solution is trivially given by $z(t) \equiv 0$.

Definition 1. *[13] Let $z(t) = z(t, t_0, \hbar)$ be the solution of system (2.1) with initial state $\hbar$. The trivial solution of system (2.1) is said to be:*
$(A1)$ p-th moment stable: $\forall \epsilon > 0, \exists\, \delta > 0$ such that $\mathbb{E}\,\|\hbar\|^p < \delta$, implies $\mathbb{E}\,\|z(t)\|^p < \epsilon$, $t \geq t_0$.
$(A2)$ p-US: it is p-th moment stable and δ in $(A1)$ is unrelated to $\hbar$.
$(A3)$ p-ES: $\exists$ positive constants ∇ and σ such that $\mathbb{E}\,\|z(t)\| \leq \nabla e^{-\sigma(t-t_0)}\mathbb{E}\,\|\hbar\|^p$, $t \geq t_0$.
For special case $p = 2$, it is considered to be mean-square exponentially stable.

Definition 2. *[34] A function $\mathcal{H}(t, z)$: $\mathbb{R}_+ \times \mathbb{R}^m \to \mathbb{R}_+$ is said to belong to the class χ_0, if:*
$(B1)$ On each sets of $[t_s, t_{s+1}) \times \mathbb{R}^m,\ s \in N$, $\mathcal{H}(t, z)$ is continuously once differentiable in t and twice differentiable in z;
$(B2)$ $\mathcal{H}(t, z)$ is continuous and $\lim_{(t,\ \alpha)\to(t^-_{s+1},\ \beta)} \mathcal{H}(t,\ \alpha) = \mathcal{H}(t^-_{s+1},\ \beta)$ exists.

For $\mathcal{H}(t, z) \in \chi_0$, we define the following operator $\mathcal{LH}$ for system (2.1):

$$\mathcal{LH}(t, z(t)) = \frac{\partial\mathcal{H}(t, z(t))}{\partial t} + \frac{\partial\mathcal{H}(t, z(t))}{\partial z}\Gamma(t, z(t)) + \frac{1}{2}trace\left[\Upsilon^T(t, z(t))\frac{\partial^2\mathcal{H}(t, z(t))}{\partial z^2}\Upsilon(t, z(t))\right].$$

Assumption 1. *Given a function $\mathcal{H}(t, z(t)) \in \chi_0$ and positive constants κ_1, κ_2, μ, ς_s, $s \in N^+$ such that*
$(C1)$ $\kappa_1\,\|z(t)\|^p \leq \mathcal{H}(t, z(t)) \leq \kappa_2\,\|z(t)\|^p$, $\forall z(t) \in \mathbb{R}^m$;
$(C2)$ $\mathcal{LH}(t, z(t)) \leq \mu\mathcal{H}(t, z(t))$, $t \neq t_s$;
$(C3)$ $\mathcal{H}(t, \mathcal{D}_s(z_{\tau_s}(t))) \leq e^{-\varsigma_s}\int_{-\tau_s}^{0}\mathcal{H}(z(t+r))dr$, $t = t_s$.

Remark 1. *In Assumption 1, condition $(C1)$ is indispensable for the analysis of p-ES. If we are only concerned with the p-US of system (2.1), then condition $(C1)$ may be replaced by $\beta_1(\|z\|^p) \leq \mathcal{H}(t, z(t)) \leq \beta_2(\|z\|^p)$, where β_1,$\beta_2 \in \mathcal{K}_\infty$, with β_1 is a convex function and β_2 is a concave function. During the continuous evolution phase, condition $(C2)$ regulates the exponential increase of the Lyapunov function, thereby reflecting the rate of energy change in system (2.1). Condition $(C3)$ can be regarded as an impulse generator, indicating that the occurrence of impulses is determined by the system's behavior over a past time interval.*

3. Main results

Sufficient conditions for p-US and p-ES of system (2.1) are established in this section through ETIC. Two distinct ETMs are proposed: 1) continuous ETM; 2) periodic ETM.

3.1. Continuous ETM

To eliminate the possibility of Zeno behavior, the following continuous ETM with temporal regularization is introduced:

$$t_{s+1} = inf\left\{t \geq t_s + \aleph : \mathcal{H}(t, z(t)) > e^{\theta_{s+1}}\mathcal{H}(t_s, z(t_s))\right\}, \tag{3.1}$$

where $\aleph > 0$ serves as a temporal regularization parameter that enforces a minimum inter-event time, thereby providing rigorous exclusion of Zeno behavior in ETM (3.1). And θ_s is an event-triggered parameter and satisfies $\theta = \inf\{\theta_s, s \in N^+\} > 0$.

Theorem 1. *Given that Assumption 1 holds and provided that the parameters of ETM* (3.1) *fulfill*

$$\tau < \aleph \leq \frac{\theta}{\mu}, \tag{3.2}$$

$$\sum_{s=1}^{k}(\theta_s - \varsigma_s + \ln\tau_s) + \theta_{k+1} \leq \mathcal{G}, \tag{3.3}$$

where $\mathcal{G}$ is a positive constant and $k \in N^+$ is the index of the kth event trigger, then system (2.1) is p-US.

Proof. Let $z(t) = z(t, t_0, \hbar)$ denote the solution to system (2.1) with initial condition $\hbar$; for convenience, we define $\mathcal{H}(t) = \mathcal{H}(t, z(t))$. In accordance with ETM (3.1), system (2.1) behavior can be categorized into two distinct cases based on triggering frequency: events occurring an infinite number of times and a finite number of times.

Case I: Events happen an infinite number of times. For $t \in [t_0, t_0 + \aleph)$, from conditions (C2) and (3.2), we have

$$\mathbb{E}\mathcal{H}(t) \leq \mathbb{E}\left[e^{\int_{t_0}^{t}\mu du}\mathcal{H}(t_0)\right] \leq e^{\theta_1}\mathbb{E}\mathcal{H}(t_0), \tag{3.4}$$

for $t \in [t_0 + \aleph, t_1)$, applying ETM (3.1) yields

$$\mathbb{E}\mathcal{H}(t) \leq e^{\theta_1}\mathcal{H}(t_0). \tag{3.5}$$

The below inequality is proved to hold by using mathematical induction.

$$\mathbb{E}\mathcal{H}(t) \leq e^{\sum_{j=1}^{s}(\theta_j - \varsigma_j + \ln\tau_j)}\mathbb{E}\mathcal{H}(t_0),\ t = t_s,\ s \in N^+, \tag{3.6}$$

$$\mathbb{E}\mathcal{H}(t) \leq e^{\sum_{j=1}^{s}(\theta_j - \varsigma_j + \ln\tau_j) + \theta_{s+1}}\mathbb{E}\mathcal{H}(t_0),\ t \in (t_s, t_{s+1}),\ s \in N^+. \tag{3.7}$$

At the first trigger moment t_1, from (C3), (3.4), and (3.5), it can be deduced that

$$\mathbb{E}\mathcal{H}(t_1) \leq \mathbb{E}\left[e^{-\varsigma_1}\int_{-\tau_1}^{0}\mathcal{H}(t_1 + r)dr\right] \leq e^{\theta_1 - \varsigma_1 + \ln\tau_1}\mathcal{H}(t_0), \tag{3.8}$$

for $t \in (t_1, t_1 + \aleph)$, according to conditions (C2), (3.2), and (3.8), it is possible to obtain

$$\mathbb{E}\mathcal{H}(t) \leq \mathbb{E}\left[e^{\int_{t_1}^{t}\mu du}\mathcal{H}(t_1)\right] \leq e^{\theta_2}\mathbb{E}\mathcal{H}(t_1) \leq e^{\theta_1 - \varsigma_1 + \ln\tau_1 + \theta_2}\mathbb{E}\mathcal{H}(t_0),$$

for $t \in [t_1 + \aleph, t_2)$, by ETM (3.1), it follows that

$$\mathbb{E}\mathcal{H}(t) \leq e^{\theta_2}\mathcal{H}(t_1) \leq e^{\theta_1-\varsigma_1+\ln\tau_1+\theta_2}\mathbb{E}\mathcal{H}(t_0).$$

Assume that inequalities (3.6) and (3.7) hold for $t \in [t_s, t_{s+1})$. Then, for $t = t_{s+1}$, one has

$$\mathbb{E}\mathcal{H}(t_{s+1}) \leq \mathbb{E}\left[e^{-\varsigma_{s+1}}\int_{-\tau_{s+1}}^{0}\mathcal{H}(t_{s+1}+r)dr\right] \leq e^{\theta_{s+1}-\varsigma_{s+1}+\ln\tau_{s+1}}\mathbb{E}\mathcal{H}(t_s) \leq e^{\sum_{j=1}^{s+1}(\theta_j-\varsigma_j+\ln\tau_j)}\mathbb{E}\mathcal{H}(t_0). \tag{3.9}$$

For $t \in (t_{s+1}, t_{s+1} + \aleph)$, following the derivation for $t \in (t_1, t_1 + \aleph)$, there is

$$\mathbb{E}\mathcal{H}(t) \leq \mathbb{E}\left[e^{\int_{t_{s+1}}^{t}\mu du}\mathcal{H}(t_{s+1})\right] \leq e^{\theta_{s+2}}\mathbb{E}\mathcal{H}(t_{s+1}) \leq e^{\sum_{j=1}^{s+1}(\theta_j-\varsigma_j+\ln\tau_j)+\theta_{s+2}}\mathbb{E}\mathcal{H}(t_0),$$

and for $t \in [t_{s+1} + \aleph, t_{s+2})$, from (3.1),

$$\mathbb{E}\mathcal{H}(t) \leq e^{\theta_{s+2}}\mathbb{E}\mathcal{H}(t_{s+1}) \leq e^{\sum_{j=1}^{s+1}(\theta_j-\varsigma_j+\ln\tau_j)+\theta_{s+2}}\mathbb{E}\mathcal{H}(t_0).$$

As a result, for $t \in [t_s, t_{s+1})$, both (3.6) and (3.7) hold. Then, combining conditions (C1) and (3.3), one can get

$$\mathbb{E}\mathcal{H}(t) \leq e^{\mathcal{G}}\mathbb{E}\mathcal{H}(t_0),$$

which yields that

$$\mathbb{E}\,\|z(t)\|^p \leq \frac{\kappa_2}{\kappa_1}e^{\mathcal{G}}\mathbb{E}\,\|\hbar\|^p\,,\ \forall t \geq t_0.$$

For $\forall\epsilon > 0$, select $\delta = \delta(\epsilon)$ such that $\delta \leq \frac{\kappa_1}{\kappa_2}e^{-\mathcal{G}}\epsilon$, then

$$\mathbb{E}\,\|\hbar\|^p \leq \delta \ \Rightarrow\ \mathbb{E}\,\|z(t)\|^p \leq \epsilon,\ t \geq t_0.$$

Case II: Events happen a limited number of times. Analogously to the argument for Case I, for all $t \in [t_s, t_{s+1})\,,\ s \in N,\ s < M$, the following inequality holds:

$$\mathbb{E}\mathcal{H}(t) \leq e^{\sum_{j=1}^{s}(\theta_j-\varsigma_j+\ln\tau_j)+\theta_{s+1}}\mathbb{E}\mathcal{H}(t_0). \tag{3.10}$$

Thus, for $t \in (t_M, \infty)$, no event occurs. Then, from ETM (3.1) as well as (3.10), it can be derived that

$$\mathbb{E}\mathcal{H}(t) \leq e^{\theta_{M+1}}\mathbb{E}\mathcal{H}(t_M) \leq e^{\theta_{M+1}-\varsigma_M}\int_{-\tau_M}^{0}\mathbb{E}\mathcal{H}(t_M+r)dr \leq e^{\theta_{M+1}-\varsigma_M+\theta_M+\ln\tau_M}\mathbb{E}\mathcal{H}(t_M) \leq e^{\sum_{j=1}^{M}(\theta_j-\varsigma_j+\ln\tau_j)+\theta_{M+1}}\mathbb{E}\mathcal{H}(t_0).$$

Again, we get from (3.3) that

$$\mathbb{E}\mathcal{H}(t) \leq e^{\mathcal{G}}\mathbb{E}\mathcal{H}(t_0),\ t \geq t_0.$$

For arbitrary $\epsilon > 0$, choose $\delta = \delta(\epsilon)$ such that $\delta \leq \frac{\kappa_1}{\kappa_2}e^{-\mathcal{G}}\epsilon$, then

$$\mathbb{E}\,\|\hbar\|^p \leq \delta \ \Rightarrow\ \mathbb{E}\,\|z(t)\|^p \leq \epsilon,\ t \geq t_0.$$

Therefore, system (2.1) is p-US under ETM (3.1).

Remark 2. *In order to validate the effectiveness of ETM, it is necessary to first ensure the exclusion of Zeno behavior. Typically, this is achieved by demonstrating the existence of a strictly positive MIET between any two successive impulse moments. However, due to the influence of stochastic noise, traditional methods developed for deterministic systems cannot be directly applied [14, 24, 27]. Therefore, we will estimate the MIET in terms of mathematical expectation [35]. In the proposed ETM* (3.1)*, a waiting time parameter* $\aleph > 0$ *is employed to guarantee that the time interval between any two triggering instants satisfies* $t_{s+1} - t_s \geq \aleph$*, which effectively eliminates the possibility of Zeno behavior. Notably, under ETM* (3.1)*, within the interval* $(t_s, t_s + \aleph)$*, neither system state monitoring nor triggering condition evaluation is required. Compared with existing approaches in the literature [24, 27], this design significantly reduces communication and computation burdens.*

Remark 3. *Theorem 1 provides two sufficient conditions to ensure p-US for system* (2.1) *under the continuous ETM* (3.1)*, given that Assumption 1 holds. Condition* (3.2) *imposes a constraint on the waiting time* $\aleph$*, indicating that its value is directly related to the continuous dynamics parameter* μ *and the triggering parameter* θ*, and also guarantees the exclusion of Zeno behavior. Condition* (3.3) *characterizes the relationship among the triggering parameter, impulse strength, and time delay, providing a key foundation for the stability analysis.*

Under Theorem 1, based on ETM (3.1), events may occur finitely many times. In this case, the asymptotic behavior of the solution is generally not guaranteed, so the following ETM with forced impulse interval Ξ_s is introduced:

$$\begin{aligned} \bar{t}_{s+1} &= \inf\left\{t \geq t_s + \aleph : \mathcal{H}(t, z(t)) > e^{\theta_{s+1}}\mathcal{H}(t_s, z(t_s))\right\}, \\ t_{s+1} &= \min\left\{\bar{t}_{s+1}, t_s + \Xi_s\right\}, \end{aligned} \tag{3.11}$$

where Ξ_s, and $s \in N$ is the sth forced impulse interval and satisfies $\aleph < \Xi_s < \Xi$. The introduction of Ξ_s effectively prevents the event from being triggered only a finite number of times, thereby ensuring the asymptotic behavior of the solution. If no event is triggered within the forced impulse interval Ξ_s, an impulsive control action will be enforced at $t_{s+1} = t_s + \Xi_s$.

Theorem 2. *Under Assumption 1, if ETM* (3.11) *satisfies condition* (3.2) *and there exist positive constants* ζ *and* ι *such that*

$$\sum_{s=1}^{k}(\theta_s - \varsigma_s + \ln \tau_s) \leq \zeta - k\iota, \ k \in N^+, \tag{3.12}$$

then, system (2.1) *achieves p-ES under ETM* (3.11)*, with a convergence rate* $\frac{\iota}{\Xi}$.

Proof. From ETM (3.11), it follows that the triggering events will occur infinitely. Through analogous methodology to Case I of Theorem 1, it can be obtained that

$$\mathbb{E}\mathcal{H}(t) \leq e^{\sum_{j=1}^{s}(\theta_j - \varsigma_j + \ln \tau_j)}\mathbb{E}\mathcal{H}(t_0), \text{ for } t = t_s, \ s \in N^+. \tag{3.13}$$

Substituting (3.12) into (3.13) yields

$$\mathbb{E}\mathcal{H}(t_s) \leq e^{\zeta - s\iota}\mathbb{E}\mathcal{H}(t_0), \ k \in N^+. \tag{3.14}$$

By ETM (3.11), we get that $t_{s+1} - t_s \leq \Xi, \ s \in N$, which means

$$t_s - t_0 \leq s\Xi. \tag{3.15}$$

By further combining conditions (C2), (3.14), and (3.15), it can be concluded that for any $t \in [t_s, t_{s+1})$, $s \in N$,

$$\mathbb{E}\mathcal{H}(t) \le e^{\mu(t-t_s)}\mathbb{E}\mathcal{H}(t_s) \le e^{\mu(t-t_s)}e^{\zeta - s\iota}\mathbb{E}\mathcal{H}(t_0) \le e^{\mu(t-t_s)}e^{\zeta-\frac{\iota}{\Xi}(t_s-t_0)}\mathbb{E}\mathcal{H}(t_0) \le e^{\mu\Xi+\zeta+\iota}e^{-\frac{\iota}{\Xi}(t-t_0)}\mathbb{E}\mathcal{H}(t_0).$$

Utilizing condition (C1) leads to

$$\mathbb{E}\,\|z\|^p \le \Delta e^{-\frac{\iota}{\Xi}(t-t_0)}\mathbb{E}\,\|\hbar\|^p\,,\ t \ge t_0,$$

where $\Delta = \frac{\kappa_2}{\kappa_1}e^{\mu\Xi+\zeta+\iota}$.

Consequently, system (2.1) is p-ES based on ETM (3.11).

Remark 4. *The forced impulse interval Ξ_s introduced in Theorem 2 not only ensures the asymptotic stability of system* (2.1)*, but also, in conjunction with the waiting time, $\aleph$ effectively eliminates the occurrence of Zeno behavior. Furthermore, condition* (3.12) *establishes a precise relationship among the triggering parameter θ_s, the impulse intensity ς_s, and time delay τ_s. This represents a more stringent constraint compared to condition* (3.2) *in Theorem 1.*

Corollary 1. *Consider system* (2.1) *under Assumption 1, and assume the ETM* (3.11) *parameters fulfill condition* (3.2) *and $\theta_s + \ln \tau_s < \varsigma_s$. Under ETM* (3.11)*, system* (2.1) *achieves p-ES with convergence rate* $\min_{s\in N^+}\{\varsigma_s - \theta_s - \ln\tau_s\}/\Xi$.

3.2. Periodic ETM

Design the following periodic ETM:

$$t_{s+1} = t_s + \min_{m\in N^+}\left\{m\eth : \mathcal{H}(t_s + m\eth, z(t_s + m\eth)) > e^{\theta_{s+1}}\mathcal{H}(t_s, z(t_s))\right\}, \tag{3.16}$$

where $\eth$ represents the sampling period, and θ_s is the triggering parameter and satisfies $\theta = \inf\{\theta_s, s \in N^+\} > 0$. The impulse time sequence $\{t_s,\ s \in N^+\}$ is generated by ETM (3.16).

Theorem 3. *Under Assumption 1, if the parameters of ETM* (3.16) *satisfy*

$$\tau < \eth \le \frac{\theta}{\mu}, \tag{3.17}$$

$$\sum_{s=1}^{k}(2\theta_s - \varsigma_s + \ln\tau_s) + 2\theta_{k+1} \le \mathcal{G}, \tag{3.18}$$

then, system (2.1) *exhibits p-US based on ETM* (3.16)*, where $\mathcal{G} > 0$.*

Proof. ETM (3.16) suggests that events may be triggered a limited or unlimited number of times. Accordingly, we consider several cases:

Case I: An infinite number of events are triggered. We divide the time interval t into three separate cases for analysis:

case i: for

$$\widetilde{m}_s = 0,\ \amalg_{s,0} = (t_s, t_{s+1})\,;$$

case ii: for

$$\widetilde{m}_s = 1, \begin{cases} Ш_{s,0} = (t_s, t_s + ð), \\ Ш_{s,1} = [t_s + ð, t_{s+1}); \end{cases}$$

case iii: for

$$\widetilde{m}_s \geq 2, \begin{cases} Ш_{s,0} = (t_s, t_s + ð), \\ Ш_{s,m} = [t_s + mð, t_s + (m+1)ð), \\ Ш_{s,\widetilde{m}_s} = [t_s + \widetilde{m}_s ð, t_{s+1}), \end{cases}$$

where we define $\widetilde{m}_s = \max\{m \in N : t_s + mð < t_{s+1}\}$, and when $\widetilde{m}_s \geq 2$, $0 < m < \widetilde{m}_s,\ m \in N^+$.

In the subsequent proof, we focus exclusively on case iii. Clearly, the conclusion of case iii also holds for the other two cases, ensuring that the overall result remains unaffected. Therefore, for $t \in [t_s, t_{s+1})$, we proceed with the analysis of case iii as follows.

case iii: for $t \in Ш_{0,0}$, by (C2) and (3.17), one has

$$\mathbb{E}\mathcal{H}(t) \leq \mathbb{E}\left[e^{\int_{t_0}^{t} \mu du}\mathcal{H}(t_0)\right] \leq e^{\theta_1}\mathbb{E}\mathcal{H}(t_0). \tag{3.19}$$

For $t \in Ш_{0,m}$, from (C2), (3.16), and (3.17), it can be deduced that

$$\mathbb{E}\mathcal{H}(t) \leq \mathbb{E}\left[e^{\int_{t_0+mð}^{t} \mu du}\mathcal{H}(t_0 + mð)\right] \leq e^{\theta_1}\mathbb{E}\mathcal{H}(t_0 + mð) \leq e^{2\theta_1}\mathbb{E}\mathcal{H}(t_0). \tag{3.20}$$

For $t \in Ш_{0,\widetilde{m}_0}$,

$$\mathbb{E}\mathcal{H}(t) \leq \mathbb{E}\left[e^{\int_{t_0+\widetilde{m}_0ð}^{t} \mu du}\mathcal{H}(t_0 + \widetilde{m}_0ð)\right] \leq e^{\theta_1}\mathbb{E}\mathcal{H}(t_0 + \widetilde{m}_0ð) \leq e^{2\mu_1}\mathbb{E}\mathcal{H}(t_0). \tag{3.21}$$

In fact, not only does (3.19) hold for $t \in Ш_{0,0}$ in case i, but (3.19) and (3.20) also hold for $t \in Ш_{0,0}$ and $t \in Ш_{0,m}$ in case ii. Likewise, the same reasoning holds for each time interval $[t_s, t_{s+1})$.

We shall prove the following inequality by means of mathematical induction:

$$\mathbb{E}\mathcal{H}(t) \leq e^{\sum_{j=1}^{s}(2\theta_j - \varsigma_j + \ln\tau_j)}\mathbb{E}\mathcal{H}(t_0), \text{ for } t = t_s, s \in N^+, \tag{3.22}$$

$$\mathbb{E}\mathcal{H}(t) \leq e^{\sum_{j=1}^{s}(2\theta_j - \varsigma_j + \ln\tau_j) + \theta_{s+1}}\mathbb{E}\mathcal{H}(t_0), \text{ for } t \in Ш_{s,0},\ s \in N^+, \tag{3.23}$$

$$\mathbb{E}\mathcal{H}(t) \leq e^{\sum_{j=1}^{s}(2\theta_j - \varsigma_j + \ln\tau_j) + \theta_{s+1}}\mathbb{E}\mathcal{H}(t_0), \text{ for } t \in Ш_{s,m} \text{ and } t \in Ш_{s,\widetilde{m}_s}, s \in N^+. \tag{3.24}$$

For $t = t_1$, from condition (C3) and similar reasoning in (3.8), it follows that

$$\mathbb{E}\mathcal{H}(t_1) \leq \mathbb{E}\left[e^{-\varsigma_1}\int_{-\tau_1}^{0}\mathcal{H}(t_1 + r)dr\right] \leq e^{2\theta_1 - \varsigma_1 + \ln\tau_1}\mathcal{H}(t_0),$$

and $t \in Ш_{1,0}$, from (C2) and (3.17), we get

$$\mathbb{E}\mathcal{H}(t) \leq \mathbb{E}\left[e^{\int_{t_1}^{t} \mu du}\mathcal{H}(t_1)\right] \leq e^{\theta_2}\mathbb{E}\mathcal{H}(t_1) \leq e^{2\theta_1 - \varsigma_1 + \ln\tau_1 + \theta_2}\mathbb{E}\mathcal{H}(t_0), \tag{3.25}$$

for $t \in Ш_{1,m}$, according to (C2), (3.17), and (3.16), we have

$$\mathbb{E}\mathcal{H}(t) \leq \mathbb{E}\left[e^{\int_{t_1+mð}^{t} \mu du}\mathcal{H}(t_1 + mð)\right] \leq e^{\theta_2}\mathbb{E}\mathcal{H}(t_1 + mð) \leq e^{2\theta_1 - \varsigma_1 + \ln\tau_1 + 2\theta_2}\mathbb{E}\mathcal{H}(t_0), \tag{3.26}$$

and for $t \in \amalg_{1,\widetilde{m}_1}$, it follows from (C2), (3.17), and (3.16) that

$$\mathbb{E}\mathcal{H}(t) \le e^{\theta_2}\mathbb{E}\mathcal{H}(t_1 + \widetilde{m}_1ð) \le e^{2\theta_1 - \varsigma_1 + \ln\tau_1 + 2\theta_2}\mathbb{E}\mathcal{H}(t_0). \tag{3.27}$$

Hypotheses (3.22)–(3.24) hold on $t \in [t_s, t_{s+1})$. Then, for $t = t_{s+1}$, it follows from (3.9) that

$$\mathbb{E}\mathcal{H}(t_{s+1}) \le \mathbb{E}\left[e^{-\varsigma_{s+1}}\int_{-\tau_{s+1}}^{0}\mathcal{H}(t_{s+1}+r)dr\right] \le e^{2\theta_{s+1}-\varsigma_{s+1}+\ln\tau_{s+1}}\mathbb{E}\mathcal{H}(t_s) \le e^{\sum_{j=1}^{s+1}(2\theta_j - \varsigma_j + \ln\tau_j)}\mathbb{E}\mathcal{H}(t_0).$$

Following a similar derivation process to that used for (3.25)–(3.27), we establish the following inequality:
for $t \in \amalg_{s+1,0}$,

$$\mathbb{E}\mathcal{H}(t) \le \mathbb{E}\left[e^{\int_{t_{s+1}}^{t}\mu du}\mathcal{H}(t_{s+1})\right] \le e^{\sum_{j=1}^{s+1}(2\theta_j - \varsigma_j + \ln\tau_j) + \theta_{s+2}}\mathbb{E}\mathcal{H}(t_0);$$

for $t \in \amalg_{s+1,m}$,

$$\mathbb{E}\mathcal{H}(t) \le \mathbb{E}\left[e^{\int_{t_{s+1}+mð}^{t}\mu du}\mathcal{H}(t_{s+1}+mð)\right] \le e^{\sum_{j=1}^{s+1}(2\theta_j - \varsigma_j + \ln\tau_j) + 2\theta_{s+2}}\mathbb{E}\mathcal{H}(t_0);$$

for $t \in \amalg_{s+1,\widetilde{m}_{s+1}}$,

$$\mathbb{E}\mathcal{H}(t) \le \mathbb{E}\left[e^{\int_{t_{s+1}+\widetilde{m}_{s+1}ð}^{t}\mu du}\mathcal{H}(t_{s+1}+\widetilde{m}_{s+1}ð)\right] \le e^{\sum_{j=1}^{s+1}(2\theta_j - \varsigma_j + \ln\tau_j) + 2\theta_{s+2}}\mathbb{E}\mathcal{H}(t_0).$$

Thus, for any $t \in [t_s, t_{s+1})$, (3.22)–(3.24) hold. Again, based on (3.18) and (C1), we can have

$$\mathbb{E}\left\|z(t)\right\|^p \le \frac{\kappa_2}{\kappa_1}e^{\mathcal{G}}\mathbb{E}\left\|\hbar\right\|^p, \ \forall t \ge t_0.$$

In the same way, for $\forall\epsilon > 0$, choose $\delta = \delta(\epsilon)$ such that $\delta \le \frac{\kappa_1}{\kappa_2}e^{-\mathcal{G}}\epsilon$, then

$$\mathbb{E}\left\|\hbar\right\|^p \le \delta \ \Rightarrow \ \mathbb{E}\left\|z(t)\right\|^p \le \epsilon, \ t \ge t_0.$$

Case II: A limited number of events are triggered. Considering this scenario and following a proof procedure analogous to Case II in Theorem 1, we can also realize that system (2.1) is p-US in accordance with ETM (3.16).

Consequently, the p-US property holds for system (2.1) under ETM (3.16) in either case.

To guarantee that system (2.1) is p-ES, we consider a periodic ETM of the following form:

$$\begin{aligned} \bar{t}_{s+1} &= t_s + \min_{m\in N^+}\left\{mð : \mathcal{H}(t_s + mð) > e^{\theta_{s+1}}\mathcal{H}(t_s)\right\}, \\ t_{s+1} &= \min\left\{\bar{t}_{s+1}, t_s + \widetilde{m}ð\right\}, \end{aligned} \tag{3.28}$$

where ð and θ_s, $s \in N^+$ are similar to the parameters given in ETM (3.16), constant $\widetilde{m}$ is known in advance, and $\widetilde{m}ð$ stands for forced impulse interval.

Theorem 4. *Under Assumption 1, if ETM* (3.28) *satisfies* (3.17) *and there exist positive constants ζ and ι such that*

$$\sum_{s=1}^{k}(2\theta_s - \varsigma_s + \ln\tau_s) \le \zeta - k\iota, \ k \in N^+, \tag{3.29}$$

then, system (2.1) *is p-th ES under ETM* (3.28) *and has convergence rate* $\frac{\iota}{\widetilde{m}ð}$.

Proof. According to ETM (3.28), it is known that the event is triggered an infinite number of times. Similar to the derivation of Case I of Theorem 3, we can have

$$\mathbb{E}\mathcal{H}(t) \leq e^{\sum_{j=1}^{s}(2\theta_j-\varsigma_j+\ln\tau_j)}\mathbb{E}\mathcal{H}(t_0), \text{ for } t = t_s, \forall s \in N^+. \tag{3.30}$$

Combining this with (3.29) gives

$$\mathbb{E}\mathcal{H}(t_s) \leq e^{\zeta - s\iota}\mathbb{E}\mathcal{H}(t_0), \ k \in N^+.$$

Then, from ETM (3.28) we can see that $t_{s+1} - t_s \leq \widetilde{m}\eth$, and, further, we can deduce that

$$t_s - t_0 \leq s\widetilde{m}\eth. \tag{3.31}$$

From (C2), (3.30), and (3.31), one has that

$$\mathbb{E}\mathcal{H}(t) \leq e^{\mu(t-t_s)}e^{\zeta-s\iota}\mathbb{E}\mathcal{H}(t_0) \leq e^{\mu(t-t_s)}e^{\zeta-\frac{\iota}{\widetilde{m}\eth}(t_s-t_0)}\mathbb{E}\mathcal{H}(t_0) \leq e^{\mu\widetilde{m}\eth+\zeta+\iota}e^{-\frac{\iota}{\widetilde{m}\eth}(t-t_0)}\mathbb{E}\mathcal{H}(t_0) \tag{3.32}$$

holds on all $t \in [t_s, t_{s+1})$. Utilizing condition (C1) leads to

$$\mathbb{E}\|z\|^p \leq \Delta e^{-\frac{\iota}{\widetilde{m}\eth}(t-t_0)}\mathbb{E}\|\hbar\|^p, \ t \geq t_0,$$

where $\Delta = \frac{\kappa_2}{\kappa_1}e^{\mu\widetilde{m}\eth+\zeta+\iota}$.

Therefore, system (2.1) is p-ES under ETM (3.28) and has convergence rate $\frac{\iota}{\widetilde{m}\eth}$.

Remark 5. *In existing works [17, 24, 27], most ETIC strategies are designed utilizing continuous ETM, which necessitate continuous monitoring of the system state. This can lead to increased communication burden and higher control energy consumption. In contrast, Theorems 3 and 4 propose ETIC strategies based on periodic ETM, which inherently avoid continuous monitoring and guarantee a MIET, thereby excluding Zeno behavior. However, such mechanisms disregard state information within the sampling intervals.*

Corollary 2. *Under assumption 1 holds, if the parameters of ETM* (3.28) *satisfy* (3.17) *as well as* $2\theta_s + \ln\tau_s < \varsigma_s$, *then system* (2.1) *is p-ES and converges exponentially with rate* $\min_{s\in N^+}\{\varsigma_s - 2\theta_s - \ln\tau_s\}/\widetilde{m}\eth$ *under ETM* (3.28).

4. Application

In what follows, we apply the established theoretical results to stochastic nonlinear systems governed by distributed-delay impulses effects.

$$\begin{cases} dz(t) = [\mathcal{A}z(t) + \mathcal{B}\Gamma(z(t))]\,dt + \Upsilon(t, z(t))d\omega(t), \ t \neq t_s, \\ z(t) = (\mathcal{I} + \mathcal{J})\int_{t-\tau_s}^{t} z(u)du, \ t = t_s, \end{cases} \tag{4.1}$$

where $\Gamma(\cdot) : \mathbb{R}^m \to \mathbb{R}^m$ is a Lipschitz function and has a Lipschitz matrix $\beth$. $\mathcal{A}$ and $\mathcal{B}$ is an $m \times m$ real matrix. τ_s is time delay in impulses, $\{t_s, \ s \in N^+\}$ is the impulse time sequence determined by the designed ETM, $\mathcal{J}$ is the impulsive control gain matrix, and $\mathcal{I}$ is the identity matrix.

Assumption 2. Υ *is locally Lipschitz continuous and there exists a matrix* Λ *of appropriate dimensions such that for any* η,

$$trace\left\{\Upsilon^T(t,\eta)\Upsilon(t,\eta)\right\} \leq \eta^T\Lambda^T\Lambda\eta.$$

Lemma 1. *For arbitrary given real matrices* Ω_1, Ω_2, P, *and a scalar* σ *with* $P > 0$, $\sigma > 0$, *the following inequality holds*

$$\Omega_1^T\Omega_2 + \Omega_2^T\Omega_1 \leq \sigma\Omega_1^T P\Omega_1 + \sigma^{-1}\Omega_2^T P^{-1}\Omega_2.$$

4.1. Continuous ETM

Theorem 5. *Under Assumption 2, given a matrix Λ, if there exist positive constants θ, ς, α, q, τ_s, μ satisfying $\tau_s < \alpha \le e^{\varsigma-\theta} \wedge \frac{\theta}{\mu}$, and $m \times m$ matrix $Q > 0$, $m \times m$ diagonal matrix $Z > 0$, and $m \times m$ real matrix S, such that $Q \le q\mathcal{I}$ as well as the following conditions hold:*

$$\begin{pmatrix} \mathcal{A}^T Q + Q\mathcal{A} + \beth Z \beth + q\Lambda^T\Lambda - \mu Q & Q\mathcal{B} \\ * & -Z \end{pmatrix} < 0, \tag{4.2}$$

$$\begin{pmatrix} \frac{1}{\alpha} e^{-\varsigma} Q & Q + S \\ * & -Q \end{pmatrix} < 0, \tag{4.3}$$

then system (4.1) *is mean-square exponentially stable under the control gain $\mathcal{J} = Q^{-1}S^T$ and ETM:*

$$\begin{cases} \bar{t}_{s+1} = \inf\left\{t \ge t_s + \aleph : z^T(t)Qz(t) > e^{\theta} z^T(t_s)Qz(t_s)\right\}, \\ t_{s+1} = \min\{\bar{t}_{s+1}, t_s + \Xi_s\}, \end{cases} \tag{4.4}$$

where $\aleph \le \frac{\theta}{\mu}$, $\aleph < \Xi_s < \Xi$.

Proof. Consider the Lyapunov function $\mathcal{H}(t) = \mathcal{H}(t, z(t)) = z^T(t)Qz(t)$. From (4.1), Assumption 2, Lemma 1, and (4.2), we obtain that

$$\begin{aligned} \mathcal{L}\mathcal{H}(t) &= 2z^T(t)Q\left[\mathcal{A}z(t) + \mathcal{B}\Gamma(z(t))\right] + trace\left[\Upsilon^T(t, z(t))Q\Upsilon(t, z(t)\right] \\ &\le z^T(t)(Q\mathcal{A} + \mathcal{A}^T Q)z(t) + z^T(t)Q\mathcal{B}Z^{-1}\mathcal{B}^T Qz(t) + \Gamma^T(z(t))Z\Gamma(z(t)) + qz^T(t)\Lambda^T\Lambda z(t) \\ &\le z^T(Q\mathcal{A} + \mathcal{A}^T Q + Q\mathcal{B}Z^{-1}\mathcal{B}^T Q + \beth Z \beth + q\Lambda^T\Lambda)z(t) \\ &\le \mu z^T(t)Qz(t) \\ &= \mu\mathcal{H}(t). \end{aligned}$$

When $t = t_s,\ s \in N^+$, from (4.3), we know that

$$\begin{aligned} \mathcal{H}(t_s) &= z^T(t_s)Qz(t_s) \\ &= \left[(\mathcal{I} + \mathcal{J}) \int_{t_s-\tau_s}^{t_s} z(u)du\right]^T Q \left[(\mathcal{I} + \mathcal{J}) \int_{t_s-\tau_s}^{t_s} z(u)du\right] \\ &\le \frac{1}{\alpha} e^{-\varsigma} \left[\int_{t_s-\tau_s}^{t_s} z(u)du\right]^T Q \left[\int_{t_s-\tau_s}^{t_s} z(u)du\right] \\ &\le e^{-\varsigma} \int_{t_s-\tau_s}^{t_s} z^T(u)Qz(u)du \\ &= e^{-\varsigma} \int_{t_s-\tau_s}^{t_s} \mathcal{H}(u)du. \end{aligned}$$

Therefore, it is easy to verify that all the conditions of Theorem 2 are satisfied. This also shows that system (4.1) is mean-square exponentially stable under the control gains $\mathcal{J} = Q^{-1}S^T$ and ETM (4.4).

4.2. Periodic ETM

Theorem 6. *Under Assumption 2, given a matrix Λ, if there exist positive constants θ, ς, α, q, τ_s, μ satisfying $\tau_s < \alpha \le e^{\varsigma-2\theta} \wedge \frac{\theta}{\mu}$, and $m \times m$ matrix $Q > 0$, $m \times m$ diagonal matrix $Z > 0$, and $m \times m$ real matrix S, such that $Q \le q\mathcal{I}$ as well as* (4.2) *and* (4.3) *hold, then system* (4.1) *is mean-square exponentially stable under the control gain $\mathcal{J} = Q^{-1}S^T$ and ETM:*

$$\begin{cases} \bar{t}_{s+1} = t_s + \min_{m \in N^+} \left\{ m\eth : z^T(t_s + m\eth)Qz(t_s + m\eth) > e^{\theta} z^T(t_s)Qz(t_s) \right\}, \\ t_{s+1} = \min \left\{ \bar{t}_{s+1}, t_s + \widetilde{m}\eth \right\}, \end{cases} \tag{4.5}$$

where $\eth \le \frac{\theta}{\mu}$ and $\widetilde{m}$ is a positive constant.

Proof. The process of changing the proof is similar to Theorem 5, so it is omitted.

Remark 6. *Assumption 2 and Lemma 1 play a crucial role in the stability analysis of nonlinear stochastic systems. Specifically, Assumption 2 imposes a local Lipschitz condition on the noise term Υ. This assumption not only aligns with the actual dependence between noise and system states in practical systems, but also guarantees the existence and uniqueness of solutions to the stochastic differential equations. Moreover, Assumption 2 is consistent with those adopted in existing studies [29, 31], and has been widely employed and verified to demonstrate good applicability. Therefore, this assumption is both reasonable and necessary in the context of our work. Lemma 1 is employed to estimate the nonlinear cross terms, thereby facilitating the transformation of the stability analysis into an LMI problem. This lemma has been widely adopted in existing studies [20, 24, 27, 31]. The combined application of Lemma 1 and Assumption 2 provides rigorous theoretical foundations for deriving the system's stability conditions.*

Remark 7. *This paper proposes an ETIC strategy with distributed-delay impulses. Compared with other control approaches, such as: [36] containment control for fractional-order multi-agent systems, [37] logic-based switching gain control for uncertain nonlinear impulsive systems, and [38] geometric maneuver control for underactuated vertical takeoff and landing (VTOL) vehicles. The core idea of the proposed control strategy is to trigger control signal updates only when the system state satisfies certain conditions, thereby significantly reducing communication and computational resource consumption. Moreover, compared with existing ETDIC strategies [21, 22, 31], which mostly focus on discrete delays, studies specifically targeting distributed-delay impulsive controllers are relatively limited. Although [27] investigates an ETIC with distributed delays, it is limited to the stability analysis of deterministic systems and does not take into account the effect of stochastic disturbances. This paper proposes a novel ETIC method incorporating distributed delays for the stability analysis of nonlinear stochastic systems, thereby extending its applicability within stochastic dynamic frameworks.*

5. Numerical examples

Two numerical examples are provided in this section to verify the effectiveness of the developed ETMs.

Example 1. Consider the following impulsive stochastic system:

$$\begin{cases} \mathrm{d}z(t) = \cos(t)z(t)\mathrm{d}t + \mathcal{E}z(t)\mathrm{d}\omega(t), \ t \neq t_s, \\ z(t) = e^{-\varsigma} \int_{-\tau}^{0} z(t+r)\mathrm{d}r, \ t = t_s, \\ z(t_0) = 1, \end{cases} \tag{5.1}$$

where $\{t_s, \ s \in N^+\}$ denotes the impulse time sequence determined by the designed ETM and $\mathcal{E}$ represents the noise intensity. For any $\mathcal{E} > 0$, system (5.1) is unstable in the absence of impulsive control input. In this example, we set $\mathcal{E} = 0.3$ for the subsequent numerical simulations.

To begin, select the Lyapunov function $\mathcal{H}(t, z(t)) = \|z(t)\|^2$; it can be deduced that $\mathcal{LH}(t, z(t)) = (2\cos(t) + \mathcal{E}^2)\|z(t)\|^2 \leq (2 + \mathcal{E}^2)\mathcal{H}(t, z(t))$. Based on condition (3.2), when the influence of random noise is relatively small, a longer waiting time can be adopted. Setting $\theta = 1.1$, the continuous ETM can then be constructed as follows:

$$\begin{cases} \bar{t}_{s+1} = \inf\left\{t \geq t_s + \aleph : \|z(t)\|^2 > e^{1.1}\|z(t_s)\|^2\right\}, \\ t_{s+1} = \min\{\bar{t}_{s+1}, t_s + \Xi_s\}. \end{cases} \tag{5.2}$$

Based on Theorem 2, we select the parameters $\varsigma = 0.85$, $\aleph = 0.5$, $\tau = 0.4$, and $\Xi_s = \Xi = 10$. Simulation results demonstrate that system (5.1) is mean-square exponentially stable under ETM (5.2) (see Figure 1: blue curve). When the forced impulse interval Ξ is not taken into account, the expression of ETM (5.2) can be simplified as follows:

$$t_{s+1} = \inf\left\{t \geq t_s + \aleph : \|z(t)\|^2 > e^{1.1}\|z(t_s)\|^2\right\}. \tag{5.3}$$

From the red curve in Figure 1, it can be seen that system (5.1) is not mean-square exponentially stable under ETM (5.3). However, according to Theorem 1, it can be concluded that system (5.1) is mean-square uniformly stable.

Next, by choosing the same $\theta = 1.1$, the following periodic ETM can be designed:

$$\begin{cases} \bar{t}_{s+1} = t_s + \min_{m \in N^+}\left\{m\eth : \|z(t_s + m\eth)\|^2 > e^{1.1}\|z(t_s)\|^2\right\}, \\ t_{s+1} = \min\{\bar{t}_{s+1}, t_s + \widetilde{m}\eth\}. \end{cases} \tag{5.4}$$

According to Theorem 4, we choose the parameters $\varsigma = 1.4$, $\eth = 0.5$, and $\widetilde{m} = 10$, namely, forced impulse interval $\widetilde{m}\eth = 5$. Simulation results indicate that system (5.1) is mean-square exponentially stable under ETM (5.4), as shown by the blue curve in Figure 2. Under the same parameter settings as ETM (5.4), at $t_s + \widetilde{m}\eth$, if the verification of the $(s+1)$th event occurrence is omitted, ETM (5.4) can be simplified as follows:

$$t_{s+1} = t_s + \min_{m \in N^+}\left\{m\eth : \|z(t_s + m\eth)\|^2 > e^{1.1}\|z(t_s)\|^2\right\}. \tag{5.5}$$

Through simulation, it is clear that system (5.1) is not mean-square exponentially stable under ETM (5.5) (see Figure 2: red curve). It follows from Theorem 3 that system (5.1) is mean-square uniformly stable under ETM (5.5).

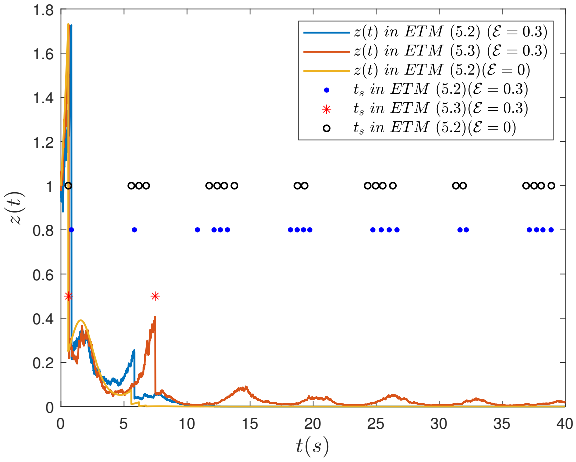

Figure 1. State trajectories of system (5.1) under ETM (5.2) and (5.3).

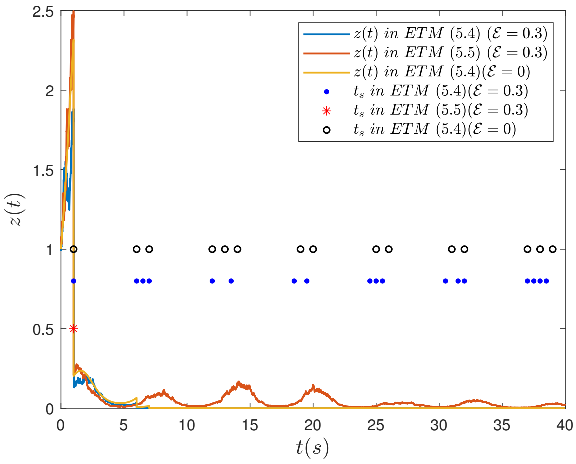

Figure 2. State trajectories of system (5.1) under ETM (5.4) and (5.5).

When $\mathcal{E} = 0$, system (5.1) reduces to a deterministic model, which has been extensively studied in the literature, including [18,24,27]. Simulation results indicate that under both continuous and periodic event-triggered mechanisms, system (5.1) exhibits stable convergence when $\mathcal{E} = 0$, as illustrated by the yellow curves in Figures 1 and 2. This behavior is similar to the stability results reported in the aforementioned studies. Moreover, a comparison between the yellow and blue trajectories in Figures 1 and 2 demonstrates that, even in the presence of stochastic disturbances, the system achieves mean-square exponential convergence under the proposed control strategies ((5.2) and (5.4)). This fully demonstrates that the proposed method in this paper has good adaptability and stability performance in the face of both deterministic and stochastic disturbances.

Example 2. System (4.1) is considered under the following conditions:

$$\mathcal{A} = \begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix}, \quad \mathcal{B} = \begin{pmatrix} 0.51 & 0.19 \\ 0.51 & -1 \end{pmatrix}, \quad \Lambda = \mathcal{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$\Gamma(z) = \tanh(z)$, $\Upsilon = \frac{\sqrt{2}}{2}\begin{bmatrix} z_1(t) & 0 \\ 0 & z_2(t) \end{bmatrix}$. It is evident that system (4.1) is unstable in the absence of impulsive control. Now we choose the parameters $\mu = 2$, $\alpha = 0.45$, $q = 0.1$, $\varsigma = 1.1$, and by solving the LIMs (4.2) and (4.3) via MATLAB, we obtain

$$Q = \begin{pmatrix} 8.7833 & -3.3766 \\ -3.3766 & 4.6548 \end{pmatrix}, \quad S = \begin{pmatrix} -5.5914 & 2.2841 \\ 2.2841 & -2.7919 \end{pmatrix}.$$

From $\mathcal{J} = Q^{-1}S^T$, the control gain matrix $\mathcal{J}$ can be calculated as

$$\mathcal{J} = \begin{pmatrix} -0.6212 & 0.0409 \\ 0.0401 & -0.5701 \end{pmatrix}.$$

Then, continuous ETM (4.4) can be expressed as follows:

$$\begin{cases} \bar{t}_{s+1} = \inf\left\{t \geq t_s + \aleph : z^T(t)Qz(t) > e^{0.95}z^T Qz(t)\right\}, \\ t_{s+1} = \min\{\bar{t}_{s+1}, t_s + 4.7\}, \end{cases} \tag{5.6}$$

where $\aleph = 0.47$, $\Xi = 4.7$. From Theorem 5, it can be obtained that system (4.1) is mean-square exponentially stable under ETM (5.6) (see Figure 3). In the following, we consider the periodic ETM (4.5) by setting the parameters θ to the same values as in (5.6), as well as $\widetilde{m}\eth = 10\eth = \Xi$. Then, periodic ETM (4.5) can be designed as follows:

$$\begin{cases} \bar{t}_{s+1} = t_s + \min_{m\in N^+}\left\{m\eth : z^T(t)Qz(t) > e^{0.95}z^T Qz(t)\right\}, \\ t_{s+1} = \min\{\bar{t}_{s+1}, t_s + 4.7\}. \end{cases} \tag{5.7}$$

Based on Theorem 6, the system (4.1) is shown to be mean-square exponentially stable by simulation (see Figure 4).

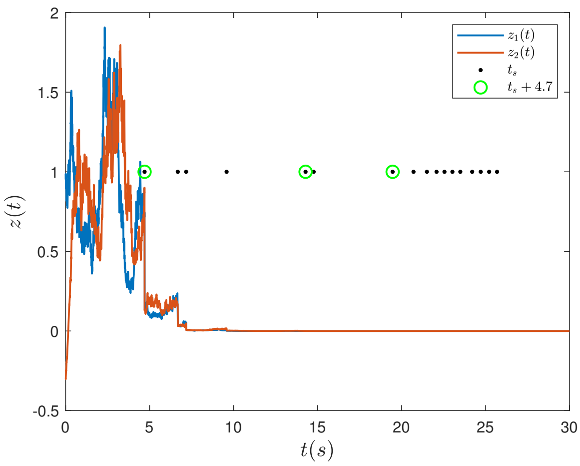

Figure 3. State trajectories of system (4.1) under ETM (5.6).

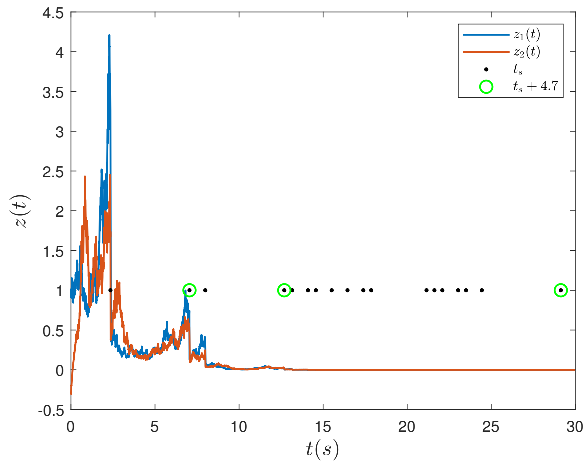

Figure 4. State trajectories of system (4.1) under ETM (5.7).

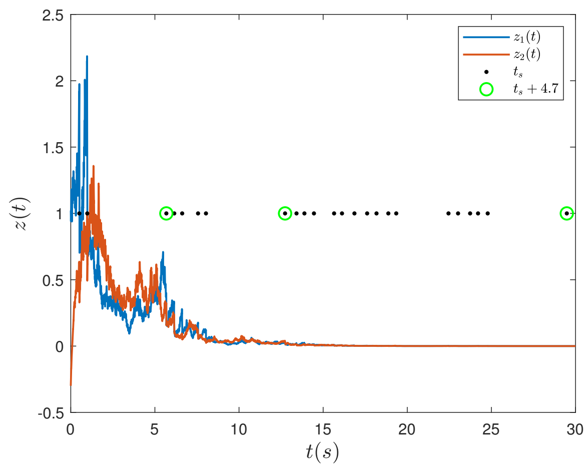

Figure 5. System state evolution under the continuous ETM strategy proposed in [13].

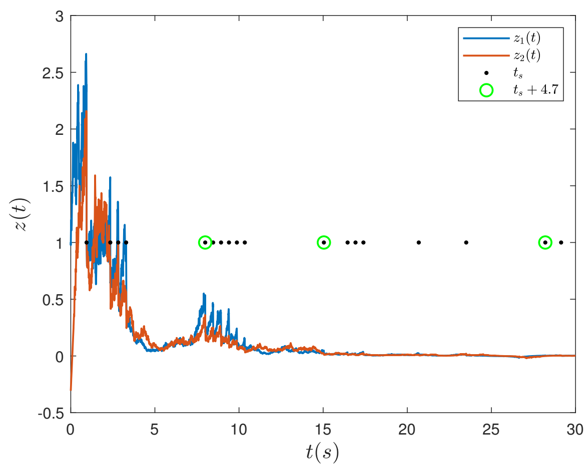

Figure 6. System state evolution under the periodic ETM strategy proposed in [13].

Further, we ignore the distributed delay term in the impulses of system (4.1), reducing it to the nonlinear stochastic system studied in [13]. Figures 5 and 6 respectively show the system state responses under the continuous ETM and periodic ETM proposed in [13], while Figures 3 and 4 correspond to the control results obtained by the proposed method under the system (4.1). The comparison reveals that, despite the increased complexity of the system considered in this paper, the proposed event-triggered control strategies achieve faster convergence and lower triggering frequency, demonstrating superior control performance and strong robustness.

6. Conclusions and outlook

Under the framework of ETIC, this paper investigates the p-US and p-ES of nonlinear stochastic systems, with full consideration of distributed delays at the impulse instants. By employing the Lyapunov method and impulsive control theory, and integrating a specifically designed ETM, a group of criteria sufficient for ensuring system stability is proposed. Furthermore, the theoretical results are applied to nonlinear stochastic systems, where the design of ETM parameters and impulsive control gains is carried out using the LMI approach, thereby enhancing the practical feasibility of the proposed method. Finally, numerical simulations are given to verify the effectiveness and feasibility of the proposed method. In future work, the proposed control strategy can be further applied to a broader class of time-delay systems, such as continuous-time systems with time delays or systems with discrete delays. It can also be extended to more complex scenarios, including fractional-order multi-agent systems, nonlinear triangular impulsive systems with unknown impulsive effects, and geometric maneuver control of underactuated VTOL vehicles.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflict of interest.

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