



Research article

Effects of human population and forestry trees on the hydrologic cycle: A modeling-based study

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Abstract: The hydrologic cycle is increasingly disrupted due to the rising human population and the associated decline in forest trees. The rationale of this work was to address the disruption in the hydrologic cycle, which is caused by the dual adverse effects of human population growth: reducing forestry trees and diminishing clouds' formation. The proposed model assumes that the density of forestry trees decreases due to harvesting activities to fulfill the resource demands of human population. Additionally, it posits that the transpiration from forestry trees contributes to an increased density of vapor clouds' formation, while population growth adversely impacts the natural formation rate of vapor clouds. The model was analyzed by employing qualitative analysis, demonstrating the feasibility and stability of equilibrium solutions. Furthermore, to capture the consequences of environmental fluctuations on the model's dynamics, the proposed deterministic model was extended to a stochastic framework. The analytical and numerical work sought to provide the directives for understanding and mitigating the adverse effects of human activities on the hydrologic cycle, promoting sustainable practices to restore ecological equilibrium. Results of the model analysis reveal that an increase in human population leads to a decline in both rainfall and forestry trees. However, reforestation with high-transpiration tree species can mitigate rainfall decline and restore balance to the hydrologic cycle. Moreover, the maximum density of forest trees is achieved when the utility of rain by the forest trees and the natural formation of vapor clouds are maximal. Also, the minimal anthropogenic hindrance in reducing the natural formation of vapor clouds, combined with the maximal efficiency of vapor clouds to naturally convert into raindrops, facilitates maximum rainfall.

Keywords: hydrologic cycle; stochastic model; transpiration; stability; interquartile range

1. Introduction

At the global scale, the hydrologic cycle is the endless recirculatory process of water movement within the atmosphere and across the Earth's surface. This cycle comprises three primary processes: evaporation, condensation, and precipitation. The Earth's hydrologic cycle is driven and sustained by various natural sources and processes, including oceans, lakes, ponds, forests, sunlight, and atmospheric dynamics. Sunlight facilitates the evaporation of water from major water bodies, such as oceans, lakes, and ponds, forming water vapor that ascends into the atmosphere. Additionally, forest, trees, plants, and agricultural crops contribute to atmospheric moisture through the transpiration process [1, 2]. When water vapor reaches to the atmosphere, it undergoes the process of condensation and aggregation to form vapor clouds. These vapor clouds convert into raindrops, which eventually precipitate back to the Earth's surface under the influence of gravity. This cyclical exchange of water between the Earth's surface and atmosphere, mediated by evaporation, transpiration, condensation, and precipitation, ensures the continuity of the global hydrologic cycle.

1.1. Forest trees and rainfall

Water is a vital resource for sustaining human life and underpins ecosystem's services crucial for maintaining ecosystems. Human population relies on the hydrologic cycle to fulfill their demand of fresh water, yet disrupts it through their unsustainable development activities. Based on an analysis of a global data set of large lakes and rivers, [3] concluded that transpiration recycles approximately $62,000 \pm 8000 \text{ Km}^3$ of water annually to the atmosphere. It is also observed that evapotranspiration via forest trees plays a major role in regulating fluxes of atmospheric moisture, but human-induced large-scale deforestation weakens regional moisture recycling and leads to rainfall suppression [4–6]. On a global scale, a massive deforestation has been done for the construction of buildings, roads, highways, agricultural land, etc., and the root cause of this is overpopulation, which demands space for living, more food production for survival, and rapid development [7]. Research studies show that, before the development of human civilization, the Earth's surface was covered by approximately 60 million square kilometers of forest. However, due to extensive deforestation and land-use changes, the forest cover has been reduced to approximately 40 million square kilometers [8]. As of 2012, approximately 20% of the Amazon rainforest had been transformed into agricultural areas, pasture lands, and other human-dominated landscapes [9–11]. Environmental studies show that several anthropogenic activities have significantly altered the ecological dynamics of forest ecosystems, which lead to disruption in forming vapor clouds and rainfall patterns. According to an analysis conducted by Smith et al. [12], the impact of forest loss between 2003 and 2017 on precipitation within a 200-km area was quantified and showed that a 1% increase in forest loss was associated with a reduction in monthly precipitation of $0.25 \pm 0.1 \text{ mm}$. A precipitationshed moisture tracking framework indicates that the forested regions in Rondonia (Brazilian Amazon) contribute to approximately 48% of the annual rainfall on average [13, 14].

1.2. Anthropogenic interference in the hydrologic cycle

The reduction in evapotranspiration due to deforestation, disturbance in the formation of vapor clouds due to less evaporation or a deteriorated condensation-nucleation process for forming raindrops are all associated with population growth and its developmental activities. The expansion in

settlements of growing human population has led to the loss of small water bodies, such as ponds and open water sources, which significantly diminishes the evaporation from these surfaces to the global water cycle [15]. It is estimated that over the last century, damaged ecosystems have lost approximately 37,000 Km³ of freshwater from their small hydrologic cycles, and in the last 20 years, deforestation and urbanization have caused an annual fresh water loss of 760 Km³ [16].

Alongside this, larger aquatic systems, including lakes and ponds, are increasingly affected by nutrient enrichment (eutrophication), promoting excessive growth of algae and macrophytes that form dense mats of pollution over the water surface [17]. This algal proliferation inhibits evaporation by reducing the exposed surface area of water. Additionally, urbanization and infrastructure expansion, such as the construction of residential complexes and roads/highways, and extensive coverage of land with impermeable materials like concrete and cement, have drastically reduced ground evaporation [18, 19].

Besides water pollution, anthropogenic activities also cause atmospheric pollution, which affects the condensation-nucleation of vapor clouds and the agglomeration of clouds to form raindrops. Thus, the research studies reflect the negative impacts of increased human activities on hydrologic processes and the natural balance of the Earth's water cycle [20].

1.3. Background

Some modeling researches have been carried out by the researchers to address the problem of deforestation caused by human population and industrialization [21, 22]. It is emphasized in these studies that the forestry resources may become very low because of large population pressure and its augmented industrialization, even if the population growth is partially dependent on forestry resources. In 2020, Lata and Misra [23] conducted a modeling study to assess the impact of forestry trees on precipitation patterns, concluding that tree species with a higher intrinsic growth rate contribute positively in enhancing rainfall. Based on the experimental and observational studies, it is noticed that the growing human population affects the dynamics of forest trees, rainfall, as well as the hydrologic cycle [24–26]. Further, some modeling studies to analyze the effects of human population and its augmented pollution on rainfall have been conducted, as discussed in [27, 28] and the references therein. The results show that factors, such as urbanization, industrialization, and atmospheric pollutants negatively impact the clouds' formation, and thus rainfall. To address the challenges associated with deforestation, conservation of forest resources [23], atmospheric pollution [29, 30], rainfall suppression [31], and artificial rainfall [32–34], various modeling studies have been undertaken aiming to improve precipitation patterns and alleviate environmental impacts, which can be studied in the cited studies and the references therein.

1.4. Objective

For the continuous process of the hydrologic cycle, transpiration and evaporation are the important factors. The forest determines the recycling of water by releasing the water vapor into the atmosphere via their stomata (microscopic pores located on leaves and other tree surfaces). In addition to transpiration, evaporation from open water bodies, soils, and other wet surfaces significantly contributes to atmospheric water vapor, leading to cloud formation, and thus raindrops. However, recent trends reveal a decline in wet surfaces, deforestation, depletion of open water sources, and

increased concentration of aerosols. These changes are driven by increasing human population and associated developmental activities, such as constructing impervious surfaces, expanding urban areas, industrial air and water pollution, etc.

The dynamics of rainfall as well as the hydrologic cycle, influenced by environmental fluctuations, can be effectively analyzed using nonlinear ordinary and stochastic differential models. Therefore, in this study, an integrated investigation of the effects of human population on forestry trees' density, cloud formation, and rainfall dynamics is undertaken. Thus, the present study develops and analyzes a novel mathematical model for the assessment of the hydrologic cycle under the combined influences of human population and forestry trees.

2. Formulation of the mathematical model

From the above literature review, we observe that forest trees and the human population play a significant role in the regulation of the water cycle. Therefore, to understand the dynamic relationship between rainfall, forest trees, and human population, we introduce a four-dimensional nonlinear mathematical model involving the density of forest trees, the density of human population, and the densities of vapor clouds and raindrops as the dynamic variables, representing them by $F(t)$, $N(t)$, $V(t)$, and $R(t)$ at any time t , respectively.

2.1. Basic assumptions of model formulation

We establish the base of our model on several key speculations, which are as follows:

- Forest trees grow logistically having intrinsic growth rate s_0 and carrying capacity s_0/s_1 . Here, s_1 represents an intra-specific competition coefficient, which encapsulates the effects of resource limitations, including sunlight, water, land, and nutrients on the forestry trees [35]. Furthermore, the growth of the human population is also modeled by a logistic growth equation characterized by an intrinsic growth rate r_0 and carrying capacity defined as r_0/r_1 . Here, r_1 represents an intra-specific competition coefficient, which arises due to the constraints imposed by limited food, water, and land resources [36].
- Moreover, it is observed that the increasing human population cuts down the forest trees for resource needs, like the expansion of cities and towns, construction of highways, housing, roads, and industries, agricultural purposes, logging for wood products and fuelwood, etc. [8, 9, 11]. Therefore, we assume that the per capita density of forest trees depletes at a rate $\alpha_1 N$ caused by harvesting for the survival of the human population. Consequently, the density of the human population increases proportionally to the depletion of the density of forest trees, represented by the term $\pi_1 \alpha_1 N F$, where π_1 is the proportionality constant and $\pi_1 \in (0, 1)$.
- Vapor clouds are naturally formed in the atmosphere at a constant rate Q ; however the reduction in the formation of vapor clouds is observed due to reduced evaporation. Observational studies show that the extinction of open water resources, such as ponds, lakes, rivers, etc., and eutrophied/polluted water sources, are the major causes for water evaporation reduction, which results in less formation of vapor clouds [15–17]. Along with this, the development of roads, industrialization, urbanization, etc., covers most of the land with concrete/cement in cities to

fulfill the demand of the increasing human population [18, 19]. This significantly reduces the evaporation of water from the ground. Also, the atmospheric pollutants, released due to the anthropogenic activities, deteriorate the condensation-nucleation process to form the clouds [28]. Therefore, we assume that the human population significantly reduces the natural formation rate of vapor clouds (Q) following a Holling type II functional response $\frac{\theta N}{m+N}$. Here, θ represents the reduction coefficient associated with the formation of vapor clouds due to human population, and m represents the half saturation constant.

- It is observed that water vapor condenses around atmospheric aerosols and creates microscopic cloud droplets. These droplets grow by colliding and merging with others within the cloud. When these cloud droplets become too heavy to float in the atmosphere, they fall on the land as rainfall [37–39]. So, we assume that the natural depletion of vapor clouds occurs at a rate δV because of their conversion into raindrops. Consequently, the raindrops are naturally formed in the atmosphere due to the conversion of vapor clouds into raindrops at a rate $\lambda \delta V$, and the natural depletion of rainfall occurs at a rate $\delta_1 R$. Here, λ is a proportionality constant, such that $\lambda \in (0, 1)$. It is observed that trees recharge atmospheric moisture, contributing to vapor clouds' formation by evapotranspiring [40–42]. We represent this contribution by the term νF , where ν is the formation rate of vapor clouds due to transpiration from forest trees.
- Studies depict that the growth of forest trees influences and is being influenced by rainfall. The forest trees absorb the rain water through their roots, stems, leaves, etc., as water is essential for photosynthesis, nutrient transport, and cellular functions. So, we assume that the density of raindrops depletes at a rate γRF , and the density of forest trees increases proportionally to this depletion rate. We represent this growth of forest trees by the term $\eta \gamma RF$. Here, η is a proportionality constant and $\eta \in (0, 1)$.

The above discussed assumptions are now systematically expressed in the form of nonlinear ordinary differential equations. The given equations depict the rate of change of our considered dynamical variables with respect to time and provide a mathematical representation of the interactions among various factors that influence the hydrological cycle:

$$\begin{aligned}
 \frac{dF}{dt} &= s_0 F - s_1 F^2 - \alpha_1 NF + \eta \gamma RF, \\
 \frac{dN}{dt} &= r_0 N - r_1 N^2 + \pi_1 \alpha_1 NF, \\
 \frac{dV}{dt} &= Q \left(1 - \frac{\theta N}{m + N} \right) - \delta V + \nu F, \\
 \frac{dR}{dt} &= \lambda \delta V - \delta_1 R - \gamma RF,
 \end{aligned} \tag{2.1}$$

with initial conditions $F(0) \geq 0$, $N(0) \geq 0$, $V(0) > 0$, and $R(0) > 0$.

2.2. Model parameters and schematic representation of the modeling phenomena

A visual representation of all the discussed assumptions is framed in a schematic diagram, Figure 1. The diagram represents the main hydrological factors, along with associated variables and parameters,

involved in modeling the system (2.1). Further, a table consisting of the description of all parameters with their numerical values and units is presented in Table 1.

Table 1. Numerical values of parameters with their description.

Parameter	Description	Numerical value	Unit
s_0	Intrinsic growth rate of forest trees	0.2	year ⁻¹
s_1	Depletion rate of forest trees due to intra-specific competition	2.0×10^{-5}	trees ⁻¹ year ⁻¹
α_1	Depletion rate of forest trees due to harvesting by the human population	1.0×10^{-8}	person ⁻¹ year ⁻¹
η	Growth of forest trees due to rainfall, $\eta \in (0, 1)$	0.0476	trees mm ⁻¹
γ	Depletion rate of raindrops due to its absorption by forest trees	2.7203×10^{-7}	trees ⁻¹ year ⁻¹
r_0	Intrinsic growth rate of the human population	0.08	year ⁻¹
r_1	Decay rate of the human population due to intra-specific competition	1.0×10^{-5}	person ⁻¹ year ⁻¹
π_1	A proportionality constant representing the growth of the human population due to harvesting of the forest for their survival	0.01	person trees ⁻¹
Q	Natural formation rate of vapor clouds	285	mm year ⁻¹
θ	A reducing factor in the natural formation of vapor clouds due to anthropogenic activities	0.3	-
m	Half saturation constant due to the human population	50	person
δ	Natural depletion rate of vapor clouds	0.1	year ⁻¹
ν	Formation rate of vapor clouds due to transpiration from forest trees	2.044×10^{-4}	mm trees ⁻¹ year ⁻¹
λ	Conversion of vapor clouds into raindrops, $\lambda \in (0, 1)$	0.8	-
δ_1	Natural depletion rate of raindrops	0.13	year ⁻¹

Now, to analyze the proposed model system (2.1), we apply the qualitative theory of differential equations as it provides insights into the long-term behavior of the nonlinear model system and extracts meaningful information about the dynamics of the system.

Lemma 1. *The bounded set*

$$\Omega = \left\{ (F, N, V, R) \in \mathbb{R}_+^4 : 0 \leq F + V + R \leq \frac{(s_0 + \nu)^2 + 4s_1 Q}{4s_1 \min\{\delta(1 - \lambda), \delta_1\}} =: \rho_m, \quad 0 \leq N \leq \frac{r_0 + \pi_1 \alpha_1 \rho_m}{r_1} \right\},$$

encloses the region of attraction for the model system (2.1) for all its solutions in the interior of the positive orthant. The proof of this lemma can be followed from [43, 44].

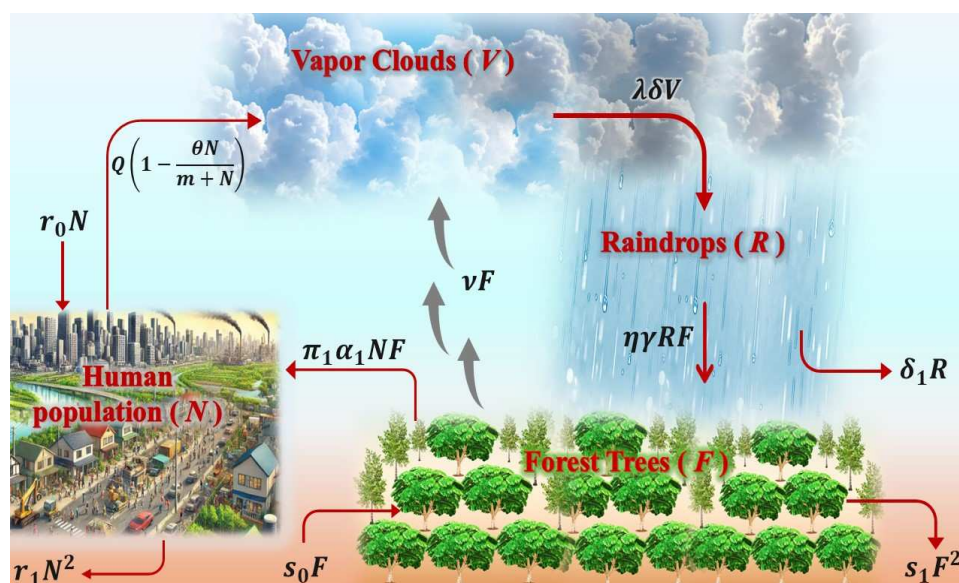


Figure 1. Schematic representation of the interplay between all dynamic variables and parameters involved in modeling the process of the hydrologic cycle as in model system (2.1).

3. Equilibrium solutions and their stability

To evaluate the equilibrium solutions of the model system (2.1), we equate the time derivative of all the dynamical variables to zero [45]. For this, we have the following set of algebraic equations:

$$F(s_0 - s_1 F - \alpha_1 N + \eta\gamma R) = 0, \quad (3.1)$$

$$N(r_0 - r_1 N + \pi_1 \alpha_1 F) = 0, \quad (3.2)$$

$$Q\left(1 - \frac{\theta N}{m + N}\right) - \delta V + vF = 0, \quad (3.3)$$

$$\lambda\delta V - \delta_1 R - \gamma RF = 0. \quad (3.4)$$

From the above set of algebraic equations, four feasible equilibrium solutions of the formulated model system (2.1) are obtained, which are as follows:

- i. At $F = 0$ and $N = 0$, we obtain a boundary equilibrium $E_0\left(0, 0, \frac{Q}{\delta}, \frac{\lambda Q}{\delta_1}\right)$, which is always feasible. From this equilibrium solution, the dynamics of vapor clouds and raindrops can be studied, while forestry trees and the human population act as bystanders in the hydrological process.

This represents the functioning of the hydrological cycle including natural formation of vapor clouds through evaporation, and the condensation-nucleation of clouds into raindrops, despite the participation of forest trees (which contribute in forming vapor clouds through the transpiration process) and the human population (affecting the natural formation rate of vapor clouds and density of forest trees) in the system.

- ii. At $F = 0$ and $N \neq 0$, we obtain an equilibrium, represented by E_1 as

$$E_1\left(0, \frac{r_0}{r_1}, \frac{Q}{\delta} \left(\frac{mr_1 + (1-\theta)r_0}{mr_1 + r_0}\right), \frac{\lambda Q}{\delta_1} \left(\frac{mr_1 + (1-\theta)r_0}{mr_1 + r_0}\right)\right), \text{ which is feasible without any condition. In analyzing this}$$

equilibrium solution, we consider the impact of the human population on the dynamics of vapor clouds and raindrops, whereas the impact of transpiration from forestry trees remains excluded on other dynamical variables.

This reflects the process of the hydrologic cycle, where the influence of forestry trees, such as their harvesting for human survival, and transpiration for regulating atmospheric moisture, remains uninvolved. However, other atmospheric processes, such as evaporation from open land water resources, condensation, and the nucleation of vapor clouds to form raindrops, persist to regulate the dynamics of the hydrologic cycle.

- iii. At $F \neq 0$ and $N = 0$, we obtain an equilibrium, represented by E_2 given as $E_2(\hat{F}, 0, \hat{V}, \hat{R})$, provided $\hat{F} > s_0/s_1$. By evaluating the Eqs (3.1), (3.3) and (3.4) at $N = 0$ and $F \neq 0$, we get a quadratic equation in dynamic variable F as $s_1\gamma F^2 - (s_0\gamma + \eta\gamma\lambda\nu - s_1\delta_1)F - (s_0\delta_1 + \eta\gamma\lambda Q) = 0$, which has one positive and one negative root. Let, the one positive root be $\hat{F}$, provided $\hat{F} > s_0/s_1$. Using this positive value of $\hat{F}$, we can easily get the positive value of the variables V and R as $\hat{V}$ and $\hat{R}$, respectively. Thus, we have an equilibrium E_2 as $E_2(\hat{F}, 0, \hat{V}, \hat{R})$, provided $\hat{F} > s_0/s_1$. This equilibrium represents a scenario where the forestry trees influence the dynamics of vapor clouds and raindrops, despite the participation of the human population.

In analyzing this equilibrium solution, we explore the hydrologic cycle without anthropogenic interference, such as urbanization and industrialization, tree harvesting, etc. This equilibrium captures the natural processes of vapor cloud formation through evaporation and transpiration. Additionally, the condensation and nucleation of vapor clouds lead to the formation of raindrops, which nourish forest trees and support their growth.

- iv. The interior equilibrium is obtained when $F \neq 0$ and $N \neq 0$, i.e., when the effect of all the dynamic variables is visible in the system. To attain this, from Eq (3.3), we have

$$V = \frac{1}{\delta} \left\{ Q \left(1 - \frac{\theta N}{m + N} \right) + \nu F \right\}. \quad (3.5)$$

Placing the value of V from Eq (3.5) in Eq (3.4), we get

$$R = \frac{\lambda}{(\delta_1 + \gamma F)} \left\{ Q \left(1 - \frac{\theta N}{m + N} \right) + \nu F \right\}. \quad (3.6)$$

Further, substituting the value of R from Eq (3.6) in Eq (3.1), we have

$$s_0 - s_1 F - \alpha_1 N + \frac{\eta\lambda\gamma}{(\delta_1 + \gamma F)} \left\{ Q \left(1 - \frac{\theta N}{m + N} \right) + \nu F \right\} = 0. \quad (3.7)$$

Also, from Eq (3.2), we have

$$r_0 - r_1 N + \pi_1 \alpha_1 F = 0. \quad (3.8)$$

Now, analysis of isocline (3.7) infers the following interpretations:

- (i) At $N = 0$, we obtain a quadratic equation in the variable F , having a positive and a negative root. The one positive root F^+ (say) exists provided $F^+ > s_0/s_1$.
(ii) At $F = 0$, we get a quadratic equation in the variable N , having a positive root and a negative

root. Let the one positive root be N^+ .

(iii) It may be noted that Eq (3.7) has three asymptotes. Two negative asymptotes, parallel to the N -axis and F -axis are obtained as $F = -\delta_1/\gamma$ and $N = -m$, respectively. Also, a slant asymptote is obtained as $s_1F + \alpha_1N = s_0 + \eta\lambda\nu$. It can be clearly seen that the slant asymptote has intercepts in the $N - F$ plane as $((s_0 + \eta\lambda\nu)/s_1, 0)$ and $(0, (s_0 + \eta\lambda\nu)/\alpha_1)$ (denoted by $(N_a, 0)$ and $(0, F_a)$) having slope $-\alpha_1/s_1$.

(iv) $dF/dN < 0$ in the first quadrant above the slant asymptote.

Proceeding further, analysis of isocline (3.8) manifests the equation of a straight line having intersections in the $N - F$ plane as $(r_0/r_1, 0)$ and $(0, -r_0/(\pi_1\alpha_1))$.

Thus, in the first quadrant of the $N - F$ plane, the point of intersection of both the isoclines (3.7) and (3.8) yields a positive value of variables N and F , denoted as (N^*, F^*) if $N^+ > r_0/r_1$, $F^+ > s_0/s_1$, $(s_0 + \eta\lambda\nu)/s_1 > r_0/r_1$, and $Q > \nu\delta_1/\gamma$. Furthermore, we attain the positive values of variables V and R as V^* (say) and R^* (say) by using the positive values N^* and F^* . Hence, we have the interior equilibrium, represented as $E^*(F^*, N^*, V^*, R^*)$ under the mentioned conditions. The graphical representation for the analysis of both the isoclines (3.7) (in pink) and (3.8) (in blue) is shown in Figure 2.

This equilibrium reflects a real-world scenario, where the process of the hydrological cycle is significantly impacted by both forestry trees and the human population. By analyzing this equilibrium, we explore the dynamics of the formation of vapor clouds and raindrops under the effect of developmental activities, such as timber harvesting, road construction, concreted/cemented floors, atmospheric and surface water pollution, etc.

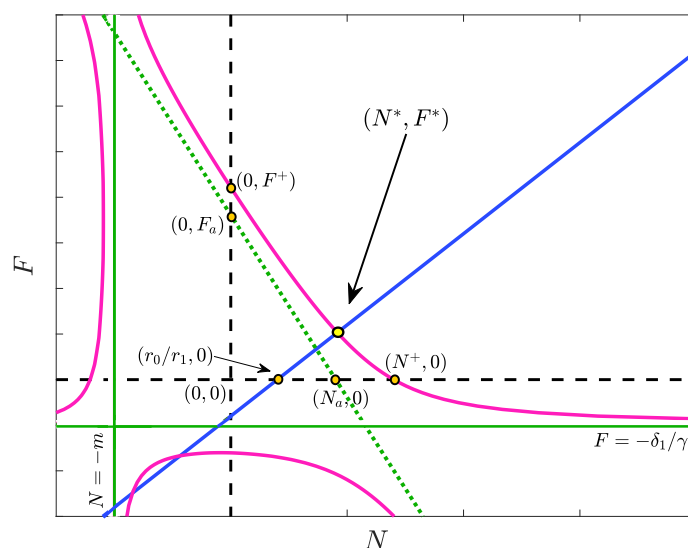


Figure 2. Plot showing the feasibility of (N^*, F^*) for model system (2.1).

Stability results: The interplay between forest trees and human activities affects the formation of vapor clouds and raindrops, and can stabilize or destabilize the dynamical system. Therefore, we check the stability of each equilibrium solution, which helps us to understand the long-term

sustainability of the system for the hydrological cycle. The stability analysis assesses how the formulated model system (2.1) behaves when subjected to perturbations near to its equilibrium solution. By analyzing the stability of equilibrium solutions, we can predict whether the system will return to equilibrium after a disturbance (indicating stability) or will deviate further away (indicating instability). Local stability analysis focuses on our system's behavior in a small neighborhood of an equilibrium solution. For analyzing the local stability of the equilibria E_0 , E_1 , and E_2 , we calculate the Jacobian matrix and determine the sign of its eigenvalues at the corresponding equilibrium solution to obtain the stability [46]. Furthermore, the dynamics for the local and global stability of the equilibrium solution E^* for formulated model system (2.1) is obtained using Lyapunov's direct method. In this regard, we provide the underlying result:

Result 1. For the proposed deterministic model system (2.1):

- i. Boundary equilibrium $E_0(0, 0, \frac{Q}{\delta}, \frac{\lambda Q}{\delta_1})$ is always unstable.
- ii. The equilibrium $E_1(0, \frac{r_0}{r_1}, \frac{Q}{\delta}(\frac{mr_1+(1-\theta)r_0}{mr_1+r_0}), \frac{\lambda Q}{\delta_1}(\frac{mr_1+(1-\theta)r_0}{mr_1+r_0}))$ is unstable if $s_0r_1 - \alpha_1r_0 > 0$.
- iii. The equilibrium $E_2(\hat{F}, 0, \hat{V}, \hat{R})$ is always unstable.
- iv. The interior equilibrium $E^*(F^*, N^*, V^*, R^*)$ exhibits local and global asymptotic stability if the conditions $\frac{3\eta\delta\lambda^2}{4(\delta_1+\gamma F^*)R^*} < \frac{4}{3} \min\left\{\frac{\delta s_1}{v^2}, \frac{r_1\delta(m+N^*)^4}{\pi_1\theta^2m^2Q^2}\right\}$ and $\frac{3\eta\delta\lambda^2}{4\delta_1R^*} < \frac{4}{3} \min\left\{\frac{\delta s_1}{v^2}, \frac{r_1\delta(m+N^*)^2}{\pi_1Q^2\theta^2}\right\}$ hold, respectively.

The following remarks address some analytical findings and their practical relevance in real-world applications:

Remark 1. The condition $Q > v\delta_1/\gamma$ obtained for the feasibility of the interior equilibrium can be written as $\frac{Q}{vF^*} > \frac{\delta_1R^*}{\gamma R^*F^*}$. This condition ecologically describes a balancing ratio to sustain rainfall and represents an interplay between vapor clouds, rainfall, and forest trees. It can be easily noted that for the feasibility of the interior equilibrium, there must exist the natural formation of vapor clouds. Also, the ratio of the natural formation rate of vapor clouds to the transpiration rate from forest trees should be greater than the ratio of the natural depletion rate of raindrops to the absorption rate of raindrops by the forest trees.

Remark 2. It is worth noting that $\frac{dN^*}{d\theta} < 0$, if $s_1F^* + \alpha_1N^* - (s_0 + \eta\lambda v) > 0$. Further, if $\frac{dN^*}{d\theta} < 0$, then: i. $\frac{dF^*}{d\theta} < 0$, ii. $\frac{dV^*}{d\theta} < 0$, provided $\frac{vF^*}{\theta} > \frac{Qm(N^*-r_0/r_1)}{(m+N^*)^2}$, iii. $\frac{dR^*}{d\theta} < 0$ provided $\frac{vF^*}{\theta} > \frac{Qm(N^*-r_0/r_1)}{(m+N^*)^2} + \frac{\gamma R^*F^*}{\lambda\theta}$. This indicates that if the mentioned conditions are satisfied, and the human population density exceeds its carrying capacity, then an increase in the reducing factor for the natural formation rate of vapor clouds due to human activities leads to a significant decline in the densities of both vapor clouds and raindrops. This signifies a potential real-world scenario where human activities, such as urbanization, industrialization, and deforestation, disrupt natural processes of forming vapor clouds in the atmosphere. This disruption leads to a significant reduction in the clouds' formation, and thus rainfall.

Remark 3. It may be noted that $\frac{dV^*}{dv} > 0$, if $s_1 F^* + \alpha_1 N^* - (s_0 + \eta \lambda v) > 0$, and $\frac{v F^*}{\theta} < \frac{Qm(N^* - r_0/r_1)}{(m+N^*)^2}$. Further, $\frac{dR^*}{dv} > 0$, if $\frac{v F^*}{\theta} < \frac{Qm(N^* - r_0/r_1)}{(m+N^*)^2} + \frac{\gamma R^* F^*}{\lambda \theta}$. Thus, under the specified conditions, an increase in the transpiration rate from forest trees elevates atmospheric water vapor levels, thereby enhancing the density of vapor clouds. This, in turn, increases the density of raindrops. This significance also endorses the role of transpiration in the hydrologic cycle, where trees release water vapor into the atmosphere. The released moisture contributes significantly to clouds' formation, which condenses under favorable atmospheric conditions. The resultant increase in cloud density amplifies precipitation processes, such as droplet coalescence and collision, ultimately influencing the regional rainfall patterns.

4. Stochastic model: incorporating environmental fluctuations into deterministic model system (2.1)

This section presents the stochastic version of the model system (2.1) to get a more flexible and realistic representation of the processes involved in the modeling. Environmental fluctuations, such as seasonal variations, droughts, floods, etc., significantly impact our ecological system and may provide more realistic results when considered in the modeling phenomenon. To evaluate the influence of these fluctuations, we convert the deterministic model system (2.1) into a stochastic system by introducing an environmental noise term into each equation (for details, see [47–49]). In this regard, we consider that the environmental variations deviate the intrinsic growth rate of forest trees (s_0), intrinsic growth rate of human population (r_0), natural depletion rate of vapor clouds (δ), and natural depletion rate of raindrops (δ_1). Thus, to incorporate the randomness in the proposed deterministic model system (2.1), we consider the following perturbations:

$s_0 \rightarrow s_0 + \sigma_1 \xi_1(t)$, $r_0 \rightarrow r_0 + \sigma_2 \xi_2(t)$, $\delta \rightarrow \delta - \sigma_3 \xi_3(t)$, and $\delta_1 \rightarrow \delta_1 - \sigma_4 \xi_4(t)$. Here, $\xi_i(t)$, $i = 1, 2, 3, 4$, are white noise terms that are mutually independent. Thus, we obtain the following system:

$$\begin{aligned}\frac{dF}{dt} &= (s_0 + \sigma_1 \xi_1(t))F(t) - s_1 F^2(t) - \alpha_1 N(t)F(t) + \eta \gamma R(t)F(t), \\ \frac{dN}{dt} &= (r_0 + \sigma_2 \xi_2(t))N(t) - r_1 N^2(t) + \pi_1 \alpha_1 N(t)F(t), \\ \frac{dV}{dt} &= Q \left(1 - \frac{\theta N(t)}{m + N(t)} \right) - (\delta - \sigma_3 \xi_3(t))V(t) + v F(t), \\ \frac{dR}{dt} &= \lambda \delta V(t) - (\delta_1 - \sigma_4 \xi_4(t))R(t) - \gamma R(t)F(t).\end{aligned}$$

Here, $\xi_i(t)$, $i = 1, 2, 3, 4$, are described as $\langle \xi_i(t) \rangle, \langle \xi_2(t) \rangle, \langle \xi_3(t) \rangle, \langle \xi_4(t) \rangle = 0$ and $\langle \xi_i(t) \xi_j(t_1) \rangle = \delta_{ij} \delta(t - t_1)$. Also, δ_{ij} represents the Kronecker delta, while $\delta(\cdot)$ is the Dirac- δ function. Also, $\sigma_i^2 > 0$, $i = 1, 2, 3, 4$, denotes the intensities of environmental fluctuations. Hence, we derive the following stochastic model system:

$$\begin{aligned}dF &= \left(s_0 F(t) - s_1 F^2(t) - \alpha_1 N(t)F(t) + \eta \gamma R(t)F(t) \right) dt + \sigma_1 F(t) dB_1(t), \\ dN &= \left(r_0 N(t) - r_1 N^2(t) + \pi_1 \alpha_1 N(t)F(t) \right) dt + \sigma_2 F(t) dB_2(t), \\ dV &= \left(Q \left(1 - \frac{\theta N(t)}{m + N(t)} \right) - \delta V(t) + v F(t) \right) dt + \sigma_3 V(t) dB_3(t),\end{aligned}$$

$$dR = (\lambda \delta V(t) - \delta_1 R(t) - \gamma R(t) F(t)) dt + \sigma_4 R(t) dB_4(t). \quad (4.1)$$

Here, B_i 's, $i = 1, 2, 3, 4$, represent the one-dimensional independent standard Brownian motion. Also, $dB_i = \xi_i(t)dt$, $i = 1, 2, 3, 4$, denote the relation between Brownian motion and the white noise terms [50].

5. Preliminaries

Assume that $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $\{\mathcal{F}_t\}_{t \geq 0}$ is a complete probability space and satisfies the usual conditions [51]. Further, it is also considered that $\text{Int}(\mathbb{R}_+^n) = \{y \in \mathbb{R}^n : y_i > 0, i = 1, 2, \dots, n\}$.

Let us consider the n -dimensional stochastic differential equation (for details, see [48])

$$dy(t) = f(y(t), t) dt + g(y(t), t) dB(t) \text{ for } t \geq t_0, \quad (5.1)$$

with $y(t_0) = y_0 \in \mathbb{R}^n$. Here, $B(t)$ signifies an l -dimensional Brownian motion defined over a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Furthermore, let $Z(y, t)$ be a function of class $C^{2,1}$ and defined on $\mathbb{R}^l \times [t_0, \infty]$. L is the differential operator associated with Eq (5.1) and is defined as

$$L = \frac{\partial}{\partial t} + \sum_{i=1}^n f_i(y, t) \frac{\partial}{\partial y_i} + \frac{1}{2} \sum_{i,j=1}^n [g^T(y, t) g(y, t)]_{ij} \frac{\partial^2}{\partial y_i \partial y_j}.$$

If L acts on a function Z , then $LZ = Z_t(y, t) + Z_y(y, t)f(y, t) + \frac{1}{2} \text{trace} [g^T(y, t) Z_{yy}(y, t) g(y, t)]$.

$$\text{Here, } Z_t = \frac{\partial Z}{\partial t}, Z_y = \left(\frac{\partial Z}{\partial y_1}, \frac{\partial Z}{\partial y_2}, \dots, \frac{\partial Z}{\partial y_n} \right), Z_{yy} = \left(\frac{\partial^2 Z}{\partial y_i \partial y_j} \right)_{n \times n}.$$

Applying Itô's formula, we have $dZ(y(t), t) = LZ(y(t), t) dt + Z_y(y(t), t) g(y(t), t) dB(t)$. Further, the criteria for the stationary distribution is given in Lemma 2. Proceeding further, we assume a homogeneous Markov process $\mathcal{Y}(t)$, defined in E^n (n -dimensional Euclidean space), which is given as:

$$d\mathcal{Y}(t) = e(\mathcal{Y}(t)) dt + \sum_{r=1}^l g_r(\mathcal{Y}) dB_r(t).$$

The diffusion matrix is

$$A(y) = (a_{ij}(y)), \quad a_{ij} = \sum_{r=1}^l g_r^i(y) g_r^j(y).$$

Here, we consider that with a regular boundary Γ , the bounded domain $D \subset E^n$ exists having the underlying properties:

H_1 : The lowest eigenvalue of the diffusion matrix $A(y)$ is bounded away from zero in the domain D and some neighborhood thereof.

H_2 : If $y \in E^n \setminus D$, the mean time τ at which a path initiating from y reaching the set D is finite and $\sup_{y \in M} E_y \tau < \infty$ for any compact subset $M \subset E^n$.

Lemma 2. *The Markov process $\mathcal{Y}(t)$ has a stationary distribution $\theta(\cdot)$, if H_1 and H_2 are satisfied. Let $h(\cdot)$ represent an integrable function with respect to the measure θ and then*

$$P_y \left\{ \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T h(\mathcal{Y}(t)) dt = \int_{E^n} h(y) \theta(dy) \right\} = 1, \quad \forall y \in E^n.$$

5.1. Existence and uniqueness of a global positive solution

Here, by using a change of variables, the existence of a unique positive solution of model system (4.1) is shown. Furthermore, by employing Lyapunov's method [52–54], we demonstrate that this solution is global.

Lemma 3. *The model system (4.1) possesses a unique positive local solution $(F(t), N(t), V(t), R(t))$ with initial values $(F(0), N(0), V(0), R(0)) \in \text{Int}(\mathbb{R}_+^4)$ for $t \in [0, \tau_e)$ almost surely, where τ_e denotes the explosion time.*

Proof. To establish the proof of the lemma, we consider $v_1(t) = \log F(t)$, $v_2(t) = \log N(t)$, $v_3(t) = \log V(t)$, $v_4(t) = \log R(t)$. Here, for any time $t \geq 0$, $v_1(0) = \log F(0)$, $v_2(0) = \log N(0)$, $v_3(0) = \log V(0)$, and $v_4(0) = \log R(0)$. Now, we derive the underlying system by employing Itô's formula:

$$\begin{aligned} dv_1(t) &= \left[s_0 - \frac{\sigma_1^2}{2} - s_1 e^{v_1(t)} - \alpha_1 e^{v_2(t)} \right] dt + \sigma_1 dB_1(t), \\ dv_2(t) &= \left[r_0 - \frac{\sigma_2^2}{2} - r_1 e^{v_2(t)} + \pi_1 \alpha_1 e^{v_1(t)} \right] dt + \sigma_2 dB_2(t), \\ dv_3(t) &= \left[\frac{Q_d}{e^{v_3(t)}} \left(1 - \frac{\theta e^{v_2(t)}}{m + e^{v_2(t)}} \right) - \delta - \frac{\sigma_3^2}{2} + \frac{\nu e^{v_1(t)}}{e^{v_3(t)}} \right] dt + \sigma_3 dB_3(t), \\ dv_4(t) &= \left[\frac{\lambda \delta e^{v_3(t)}}{e^{v_4(t)}} - \delta_1 - \frac{\sigma_4^2}{2} - \gamma e^{v_4(t)} \right] dt + \sigma_4 dB_4(t). \end{aligned}$$

It may be noted that since all the coefficients of system (4.1) fulfill the local Lipschitz condition, therefore, a unique local solution $v_1(t), v_2(t), v_3(t), v_4(t)$ exists on $[0, \tau_e)$. Therefore, for initial values $F(0) > 0, N(0) > 0, V(0) > 0, R(0) > 0$, positive local solutions $F(t) = e^{v_1(t)}, N(t) = e^{v_2(t)}, V(t) = e^{v_3(t)}, R(t) = e^{v_4(t)}$ of model system (4.1) exist. $\square$

Theorem 1. *The model system (4.1) possesses a unique positive solution $(F(t), N(t), V(t), R(t))$ for $t \geq 0$ corresponding to any initial value $(F(0), N(0), V(0), R(0)) \in \text{Int}(\mathbb{R}_+^4)$. Furthermore, this solution remains within $\text{Int}(\mathbb{R}_+^4)$ with probability 1.*

Proof. To establish the proof, let us choose a non-negative large number $\varrho_0 > 0$, so that $F(0), N(0), V(0)$ and $R(0)$ belong to $\left[\frac{1}{\varrho_0}, \varrho_0\right]$. Now, the sequence of stopping times for each integer $\varrho \geq \varrho_0$ can be defined as follows:

$$\tau_\varrho = \inf \left\{ t \in [0, \tau_e) : (F(t), N(t), V(t), R(t)) \notin \left(\frac{1}{\varrho}, \varrho \right) \right\}.$$

Here, $\inf \emptyset = \infty$, while $\emptyset$ accounts for the empty set. Apparently, as $\varrho \rightarrow \infty$, τ_ϱ is increasing. Furthermore, let $\tau_\infty = \lim_{\varrho \rightarrow \infty} \tau_\varrho$ and then $\tau_\infty \leq \tau_e$ almost surely [23, 27]. To establish $\tau_e = \infty$, it suffices to prove that $\tau_\infty = \infty$ almost surely. For the sake of contradiction, suppose that this assertion is false. So, a constant $T > 0$ exists, and $\epsilon \in (0, 1)$ such that $\mathbb{P}\{\tau_\infty \leq T\} > \epsilon$. Thus, it follows that an integer $\varrho_1 \geq \varrho_0$ exists such that

$$\mathbb{P}\{\tau_\varrho \leq T\} \geq \epsilon, \quad \forall \varrho \geq \varrho_1. \quad (5.2)$$

Now, consider a C^2 - function $X : \text{Int}(\mathbb{R}_+^4) \rightarrow \text{Int}(\mathbb{R}_+)$ as:

$$X(F, N, V, R) = (F + 1 - \log F) + (N + 1 - \log N) + (V + 1 - \log V) + (R + 1 - \log R).$$

Here, the non-negativity of this function is evident by the expression $v + 1 - \log v \geq 0$ for $v > 0$. Thus, employing Itô's formula, the derived expression is

$$\begin{aligned} dX(F, N, V, R) &= \left[\left(1 - \frac{1}{F}\right) \left(s_0 F - s_1 F^2 - \alpha_1 N F + \eta \gamma R F\right) + \left(1 - \frac{1}{N}\right) \left(r_0 N - r_1 N^2 + \pi_1 \alpha_1 N F\right) \right. \\ &\quad + \left(1 - \frac{1}{V}\right) \left(Q \left(1 - \frac{\theta N}{m + N}\right) - \delta V + \nu F\right) + \left(1 - \frac{1}{R}\right) \left(\lambda \delta V - \delta_1 R - \gamma R F\right) + \frac{1}{2} \sum_{i=1}^4 \sigma_i^2 \Bigg] dt \\ &\quad + \sigma_1 (F - 1) dB_1(t) + \sigma_2 (N - 1) dB_2(t) + \sigma_3 (V - 1) dB_3(t) + \sigma_4 (R - 1) dB_4(t), \\ &= \left[\left(s_0 F - s_1 F^2 - \alpha_1 N F + \eta \gamma R F - s_0 + s_1 F + \alpha_1 N - \eta \gamma R\right) \right. \\ &\quad + \left(r_0 N - r_1 N^2 + \pi_1 \alpha_1 N F - r_0 + r_1 N - \pi_1 \alpha_1 F\right) \\ &\quad + \left(Q \left(1 - \frac{\theta N}{m + N}\right) - \delta V + \nu F - \frac{Q}{V} \left(1 - \frac{\theta N}{m + N}\right) + \delta - \frac{\nu F}{V}\right) \\ &\quad + \left(\lambda \delta V - \delta_1 R - \gamma R F - \frac{\lambda \delta V}{R} + \delta_1 + \gamma F\right) + \frac{1}{2} \sum_{i=1}^4 \sigma_i^2 \Bigg] dt \\ &\quad + \sigma_1 (F - 1) dB_1(t) + \sigma_2 (N - 1) dB_2(t) + \sigma_3 (V - 1) dB_3(t) + \sigma_4 (R - 1) dB_4(t). \end{aligned}$$

Therefore,

$$\begin{aligned} dX(F, N, V, R) &\leq [C_1 - (s_0 + s_1 + \nu + \gamma)F - (r_0 + r_1 + \alpha_1)N - \delta(1 - \lambda)V - (\delta_1 + \eta \gamma)R] dt \\ &\quad + \sigma_1 (F - 1) dB_1(t) + \sigma_2 (N - 1) dB_2(t) + \sigma_3 (V - 1) dB_3(t) + \sigma_4 (R - 1) dB_4(t). \end{aligned}$$

$$\begin{aligned} \text{Here, } C_1 &= s_0 \left(1 + \frac{s_0}{s_1} + \frac{s_1}{s_0}\right) + \frac{(\nu + \gamma)^2}{s_1} + 2 \left(1 + \frac{s_0}{s_1}\right) (\nu + \gamma) + r_0 \left(1 + \frac{r_0}{r_1} + \frac{r_1}{r_0}\right) + \frac{\alpha_1^2}{r_1} \\ &\quad + 2 \left(1 + \frac{r_0}{r_1}\right) \alpha_1 + Q + \delta + \delta_1 + \frac{1}{2} \sum_{i=1}^4 \sigma_i^2. \end{aligned}$$

Hence, we can write

$$\begin{aligned} dX(F, N, V, R) &\leq C_1 dt + \sigma_1 (F - 1) dB_1(t) + \sigma_2 (N - 1) dB_2(t) + \sigma_3 (V - 1) dB_3(t) \\ &\quad + \sigma_4 (R - 1) dB_4(t). \end{aligned} \tag{5.3}$$

If $t_1 \leq T$, then we integrate inequality (5.3) between 0 and $\tau_\varrho \wedge t_1$, and get the following after taking the expectation:

$$\begin{aligned} \int_0^{\tau_\varrho \wedge t_1} dX(F, N, V, R) &\leq C_1 \int_0^{\tau_\varrho \wedge t_1} 1 dt + \sigma_1 \int_0^{\tau_\varrho \wedge t_1} (F - 1) dB_1(t) + \sigma_2 \int_0^{\tau_\varrho \wedge t_1} (N - 1) dB_2(t) \\ &\quad + \sigma_3 \int_0^{\tau_\varrho \wedge t_1} (V - 1) dB_3(t) + \sigma_4 \int_0^{\tau_\varrho \wedge t_1} (R - 1) dB_4(t). \end{aligned}$$

$$E \left[X \left(F(\tau_\varrho \wedge t_1), N(\tau_\varrho \wedge t_1), V(\tau_\varrho \wedge t_1), R(\tau_\varrho \wedge t_1) \right) \right] \leq E \left[X(F(0), N(0), V(0), R(0)) \right] + C_1 E(\tau_\varrho \wedge t_1).$$

$$\text{So, } E \left[X \left(F(\tau_\varrho \wedge t_1), N(\tau_\varrho \wedge t_1), V(\tau_\varrho \wedge t_1), R(\tau_\varrho \wedge t_1) \right) \right] \leq X(F(0), N(0), V(0), R(0)) + C_1 T. \quad (5.4)$$

Let $\Omega_\varrho = \{\omega : \tau_\varrho \leq T\} \forall \varrho \geq \varrho_1$, and thus from (5.2), $\mathbb{P}(\Omega_\varrho) \geq \epsilon$. Therefore, $\forall \omega \in \Omega_\varrho$, there is at least one of $F(\tau_\varrho, \omega), N(\tau_\varrho, \omega), V(\tau_\varrho, \omega), R(\tau_\varrho, \omega)$ equal either ϱ or $\frac{1}{\varrho}$. So, $X(F(\tau_\varrho, \omega), N(\tau_\varrho, \omega), V(\tau_\varrho, \omega), R(\tau_\varrho, \omega))$ is no less than either

$$\varrho + 1 - \log \varrho \text{ or } \frac{1}{\varrho} + 1 - \log \frac{1}{\varrho} = \frac{1}{\varrho} + 1 + \log \varrho.$$

Therefore, $X(F(\tau_\varrho, \omega), N(\tau_\varrho, \omega), V(\tau_\varrho, \omega), R(\tau_\varrho, \omega)) \geq (\varrho + 1 - \log \varrho) \wedge \left(\frac{1}{\varrho} + 1 + \log \varrho \right)$.

Further, from inequalities (5.4), we get

$$\begin{aligned} X(F(0), N(0), V(0), R(0)) + C_1 T &\geq E \left[1_{\Omega_\varrho}(\omega) X \left(F(\tau_\varrho, \omega), N(\tau_\varrho, \omega), V(\tau_\varrho, \omega), R(\tau_\varrho, \omega) \right) \right] \\ &\geq \epsilon \left[(\varrho + 1 - \log \varrho) \wedge \left(\frac{1}{\varrho} + 1 + \log \varrho \right) \right], \end{aligned}$$

where 1_{Ω_ϱ} denotes the indicator function of Ω_ϱ . Now, as $\varrho \rightarrow \infty$, we obtain

$$\infty > X(F(0), N(0), V(0), R(0)) + C_1 T = \infty.$$

This leads to the contradiction. Hence, it must be the case that $\tau_\infty = \infty$ almost surely. This result shows that $F(t), N(t), V(t)$, and $R(t)$ does not explode within finite time with probability 1. $\square$

5.2. Boundedness

To show the ultimate boundedness of model system (4.1) in mean, we have the following theorem.

Theorem 2. *Model system (4.1) is stochastically ultimately bounded.*

Proof. To establish the proof of this theorem, consider

$$H(F(t), N(t), V(t), R(t)) = F(t) + N(t) + V(t) + R(t).$$

Employing Itô's formula,

$$dH(t) = LHdt + \sigma_1 F dB_1(t) + \sigma_2 N dB_2(t) + \sigma_3 V dB_3(t) + \sigma_4 R dB_4(t).$$

Here,

$$\begin{aligned} LH &= s_0 F - s_1 F^2 - \alpha_1 NF + \eta \gamma RF + r_0 N - r_1 N^2 + \pi_1 \alpha_1 NF + Q \left(1 - \frac{\theta N}{m + N} \right) - \delta V + \nu F + \lambda \delta V - \delta_1 R - \gamma RF \\ &\leq 2(s_0 + \nu)F - s_1 F^2 - \alpha_1(1 - \pi_1)NF - \gamma(1 - \eta)RF + 2r_0 N - r_1 N^2 + Q - (s_0 + \nu)F - r_0 N - \delta(1 - \lambda)V - \delta_1 R, \end{aligned}$$

$$\leq \left(Q + \frac{(s_0 + \nu)^2}{s_1} + \frac{r_0^2}{r_1} \right) - (s_0 + \nu)F - r_0N - \delta(1 - \lambda)V - \delta_1R.$$

Therefore,

$$\begin{aligned} dH \leq & \left(Q + \frac{(s_0 + \nu)^2}{s_1} + \frac{r_0^2}{r_1} \right) - (s_0 + \nu)F - r_0N - \delta(1 - \lambda)V - \delta_1R \\ & + \sigma_1 F dB_1(t) + \sigma_2 N dB_2(t) + \sigma_3 V dB_3(t) + \sigma_4 R dB_4(t). \end{aligned}$$

Furthermore, by employing the generalized Itô formula and taking the expectation, the following expression is obtained:

$$\begin{aligned} e^{\beta t} E \left[H((F(t), N(t), V(t), R(t)), t) \right] &= E \left[H(F(0), N(0), V(0), R(0)) \right] + E \int_0^t e^{\beta s} \left(\beta H((F(s), N(s), V(s), R(s)), s) \right. \\ &\quad \left. + LH((F(s), N(s), V(s), R(s)), s) \right) ds \\ &\leq H(F(0), N(0), V(0), R(0)) + E \int_0^t e^{\beta s} \left(\left(Q + \frac{(s_0 + \nu)^2}{s_1} + \frac{r_0^2}{r_1} \right) \right. \\ &\quad \left. - (s_0 + \nu - \beta)F(s) - (r_0 - \beta)N(s) - (\delta(1 - \lambda) - \beta)V(s) - (\delta_1 - \beta)R(s) \right) ds, \end{aligned}$$

where the parameter β is a positive constant and chosen as $\beta = \min\{(s_0 + \nu), r_0, \delta(1 - \lambda), \delta_1\}$. Then,

$$e^{\beta t} E \left[H((F(t), N(t), V(t), R(t)), t) \right] = H(F(0), N(0), V(0), R(0)) + \left(Q + \frac{(s_0 + \nu)^2}{s_1} + \frac{r_0^2}{r_1} \right) \int_0^t e^{\beta s} ds.$$

This implies that

$$\limsup_{t \rightarrow \infty} E \left[H((F(t), N(t), V(t), R(t)), t) \right] \leq \frac{(s_1 r_1 Q + r_1 (s_0 + \nu)^2 + s_1 r_0^2)}{s_1 r_1 \beta}.$$

Hence, the proof is complete. $\square$

5.3. Asymptotic behavior of stochastic model system (4.1)

This subsection presents the investigation about the asymptotic behavior of the stochastic model system (4.1) around the interior equilibrium point (E^*) of the deterministic model (2.1), referring to how it behaves over the long term as randomness continues to influence it.

Theorem 3. *If the underlying inequalities are satisfied,*

$$4\sigma_3^2 < \delta, \quad \sigma_4^2 < \delta_1, \quad \frac{\delta \nu \lambda^2}{(\delta_1 - \sigma_4^2)R^*} < \min \left\{ \frac{s_1 \delta}{\nu^2}, \frac{r_1 \delta (m + N^*)^2}{\pi_1 \theta^2 Q^2} \right\}, \quad (5.5)$$

then the solution of stochastic model system (4.1) initialized at any point $(F(0), N(0), V(0), R(0)) \in \text{Int}(\mathbb{R}_+^4)$ possesses the following property:

$$\limsup_{t \rightarrow \infty} \frac{1}{t} E \int_0^t \left((F(s) - F^*)^2 + (N(s) - N^*)^2 + (V(s) - V^*)^2 + (R(s) - R^*)^2 \right) ds \leq \frac{\mathfrak{D}_\sigma}{\mathfrak{D}}, \quad (5.6)$$

where the values of $\mathfrak{D}$ and $\mathfrak{D}_\sigma$ are provided within the proof.

Proof. To establish the proof of the above expressed Theorem 3, let $x = (F, N, V, R)^T$ and the C^2 -function $X : \text{Int}(\mathbb{R}_+^4) \rightarrow \text{Int}(\mathbb{R}_+)$. Then

$$X(x) = \left(F - F^* - F^* \log \frac{F}{F^*}\right) + \mathfrak{C}_1 \left(N - N^* - N^* \log \frac{N}{N^*}\right) + \frac{\mathfrak{C}_2}{2}(V - V^*)^2 + \frac{\mathfrak{C}_3}{2}(R - R^*)^2,$$

where $\mathfrak{C}_1, \mathfrak{C}_2, \mathfrak{C}_3 > 0$ and will be chosen appropriately. Now, by employing Itô's formula,

$$dX(x) = LXdt + \left(1 - \frac{F^*}{F}\right)\sigma_1 F dB_1(t) + \mathfrak{C}_1 \left(1 - \frac{N^*}{N}\right)\sigma_2 N dB_2(t) + \mathfrak{C}_2(V - V^*)\sigma_3 V dB_3(t) + \mathfrak{C}_3(R - R^*)\sigma_4 R dB_4(t).$$

Using model system (2.1) and doing simple calculations, we have

$$\begin{aligned} LX(x) = & -s_1(F - F^*)^2 - \mathfrak{C}_1 r_1(N - N^*)^2 - \mathfrak{C}_2 \delta(V - V^*)^2 - \mathfrak{C}_3(\delta_1 + \gamma F)(R - R^*)^2 \\ & - \alpha_1(N - N^*)(F - F^*) + \eta\gamma(R - R^*)(F - F^*) + \mathfrak{C}_1 \pi_1 \alpha_1(N - N^*)(F - F^*) \\ & - \mathfrak{C}_2 \frac{\theta m Q}{(m + N)(m + N^*)}(N - N^*)(V - V^*) + \mathfrak{C}_2 \nu(V - V^*)(F - F^*) \\ & + \mathfrak{C}_3 \lambda \delta(V - V^*)(R - R^*) - \mathfrak{C}_3 \gamma R^*(F - F^*)(R - R^*) + \frac{1}{2}\sigma_1^2 F^* + \mathfrak{C}_1 \frac{1}{2}\sigma_2^2 N^* \\ & + \mathfrak{C}_2 \frac{1}{2}\sigma_3^2 V^2 + \mathfrak{C}_3 \frac{1}{2}\sigma_4^2 R^2. \end{aligned}$$

Choosing the constants $\mathfrak{C}_1$ as $\mathfrak{C}_1 = 1/\pi_1$ and $\mathfrak{C}_3$ as $\mathfrak{C}_3 = \eta/R^*$, the above expression reduces to

$$\begin{aligned} LX(x) = & -s_1(F - F^*)^2 - \frac{r_1}{\pi_1}(N - N^*)^2 - \mathfrak{C}_2 \delta(V - V^*)^2 - \frac{\eta(\delta_1 + \gamma F)}{R^*}(R - R^*)^2 \\ & - \mathfrak{C}_2 \frac{\theta m Q}{(m + N)(m + N^*)}(N - N^*)(V - V^*) + \mathfrak{C}_2 \nu(V - V^*)(F - F^*) \\ & + \frac{\eta \lambda \delta}{R^*}(V - V^*)(R - R^*) + \frac{1}{2}\sigma_1^2 F^* + \frac{1}{2\pi_1}\sigma_2^2 N^* + \mathfrak{C}_2 \frac{1}{2}\sigma_3^2 V^2 + \frac{\eta}{2R^*}\sigma_4^2 R^2. \end{aligned}$$

Further, following simple algebraic manipulations as done in [23, 27], we obtain

$$\begin{aligned} LX(x) \leq & \left(s_1 - \mathfrak{C}_2 \frac{\nu^2}{\delta}\right)(F - F^*)^2 - \left(\frac{r_1}{\pi_1} - \mathfrak{C}_2 \frac{\theta^2 m^2 Q^2}{\delta(m + N)^2(m + N^*)^2}\right)(N - N^*)^2 \\ & - \mathfrak{C}_2 \left(\frac{1}{4}\delta - \sigma_3^2\right)(V - V^*)^2 - \left(\frac{\eta(\delta_1 + \gamma F)}{R^*} - \frac{\delta \eta^2 \lambda^2}{\mathfrak{C}_2 R^{*2} - \frac{\eta}{R^*}\sigma_4^2}\right)(R - R^*)^2 \\ & + \frac{1}{2}\sigma_1^2 F^* + \frac{1}{2\pi_1}\sigma_2^2 N^* + \mathfrak{C}_2 \sigma_3^2 V^{*2} + \frac{\eta}{R^*}\sigma_4^2 R^{*2}. \end{aligned}$$

Hence, we get

$$LX(x) \leq -\mathfrak{D}_1(F - F^*)^2 - \mathfrak{D}_2(N - N^*)^2 - \mathfrak{D}_3(V - V^*)^2 - \mathfrak{D}_4(R - R^*)^2 + \mathfrak{D}_\sigma,$$

where

$$\mathfrak{D}_1 = \left(s_1 - \mathfrak{C}_2 \frac{\nu^2}{\delta}\right), \quad \mathfrak{D}_2 = \left(\frac{r_1}{\pi_1} - \mathfrak{C}_2 \frac{\theta^2 Q^2}{\delta(m + N^*)^2}\right), \quad \mathfrak{D}_3 = \frac{\mathfrak{C}_2}{4}(\delta - 4\sigma_3^2),$$

$$\mathfrak{D}_4 = \frac{\eta}{R^*} \left(\delta_1 - \frac{\delta\eta\lambda^2}{\mathfrak{C}_2 R^*} - \sigma_4^2 \right), \quad \mathfrak{D}_\sigma = \frac{1}{2}\sigma_1^2 F^* + \frac{1}{2\pi_1}\sigma_2^2 N^* + \mathfrak{C}_2\sigma_3^2 V^{*2} + \eta\sigma_4^2 R^*.$$

Here, it is worth noting that $\mathfrak{D}_1, \mathfrak{D}_2, \mathfrak{D}_3$, and $\mathfrak{D}_4$ are all positive if underlying inequalities are satisfied:

$$\mathfrak{C}_2 < \frac{s_1\delta}{\nu^2}, \quad \mathfrak{C}_2 < \left(\frac{r_1\delta(m+N^*)^2}{\pi_1\theta^2 Q^2} \right), \quad (\delta - 4\sigma_3^2) > 0, \quad \mathfrak{C}_2 > \frac{\eta\delta\lambda^2}{(\delta_1 - \sigma_4^2)R^*}. \quad (5.7)$$

Now, from the above-mentioned inequalities (5.7), we can choose the positive value of the constant $\mathfrak{C}_2$ as

$$\frac{\eta\delta\lambda^2}{(\delta_1 - \sigma_4^2)R^*} < \mathfrak{C}_2 < \min \left\{ \frac{\delta s_1}{\nu^2}, \frac{r_1\delta(m+N^*)^2}{\pi_1 Q^2 \theta^2} \right\}.$$

Thus, $\mathfrak{D}_1, \mathfrak{D}_2, \mathfrak{D}_3$, and $\mathfrak{D}_4$ are positive if

$$4\sigma_3^2 < \delta, \quad \sigma_4^2 < \delta_1, \quad \frac{\delta\nu\lambda^2}{(\delta_1 - \sigma_4^2)R^*} < \min \left\{ \frac{s_1\delta}{\nu^2}, \frac{r_1\delta(m+N^*)^2}{\pi_1\theta^2 Q^2} \right\}.$$

Further, we have

$$dX(x) = LXdt + (F - F^*)\sigma_1 dB_1(t) + \frac{1}{\pi_1}(N - N^*)\sigma_2 dB_2(t) + \mathfrak{C}_2\sigma_3 V(V - V^*)dB_3(t) + \frac{\eta}{R^*}\sigma_4 R(R - R^*)dB_4(t). \quad (5.8)$$

Integrating (5.8) over 0 to t , and subsequently taking the expectation, leads us to the expression

$$E \left[\int_0^t \left\{ K_1(F(s) - F^*)^2 + K_2(N(s) - N^*)^2 + K_3(V(s) - V^*)^2 + K_4(R(s) - R^*)^2 \right\} ds \right] \leq X(x_0) + \mathfrak{D}_\sigma t. \quad (5.9)$$

Now, both sides of (5.9) are multiplied by $\frac{1}{t}$, and taking the limit $t \rightarrow \infty$ yields

$$\limsup_{t \rightarrow \infty} \frac{1}{t} E \left[\int_0^t \left\{ (F(s) - F^*)^2 + (N(s) - N^*)^2 + (V(s) - V^*)^2 + (R(s) - R^*)^2 \right\} ds \right] \leq \frac{\mathfrak{D}_\sigma}{\mathfrak{D}}, \quad (5.10)$$

where $\mathfrak{D} = \min\{\mathfrak{D}_1, \mathfrak{D}_2, \mathfrak{D}_3, \mathfrak{D}_4\}$. □

Remark 4. The condition for global asymptotic behavior also suggests that for the system to stabilize, the intensity of random fluctuations must be within certain limits. If the randomness is too large, it may prevent the system to stabilize around equilibrium E^* . The condition (5.5) clearly depicts that the intensities of environmental fluctuations in the natural depletion rate of vapor clouds and raindrops, denoted by σ_3 and σ_4 , respectively, must be less than their depletion rates δ and δ_1 , respectively. Thus, in a stochastic framework, the system will eventually approach the equilibrium state, but the path to that equilibrium may not be straightforward (as we obtained in the deterministic model system) due to the random fluctuations or noise.

Remark 5. It can be observed that the solution of our stochastic model described by Eq (4.1) exhibits fluctuations around the equilibrium point E^* of the deterministic system (2.1). These fluctuations are the consequences of the stochastic nature of the model, arising due to the presence of noise terms. Additionally, it is noted that the amplitude of these fluctuations decreases as the intensity of the noise terms diminishes, indicating that the trajectories of the stochastic system approach the equilibrium point E^* as the magnitude of the perturbations becomes small. This behavior suggests that, under conditions of sufficiently small stochastic perturbations, the trajectories of the model system (4.1) remain near the equilibrium E^* of deterministic model (2.1).

Theorem 4. *Contingent upon the fulfillment of the conditions specified in Eq (5.5), the stochastic model system (4.1) possesses a unique stationary distribution $\theta(\cdot)$, characterized by an ergodic property.*

Proof. To substantiate the proof of the above-stated theorem, the diffusion coefficient $g(x)$, associated with the stochastic model described in Eq (4.1), is expressed as follows:

$$g(x) = \begin{pmatrix} \sigma_1 F & 0 & 0 & 0 \\ 0 & \sigma_2 N & 0 & 0 \\ 0 & 0 & \sigma_3 V & 0 \\ 0 & 0 & 0 & \sigma_4 R \end{pmatrix}, \quad x = (F, N, V, R)^T \in \text{Int}(\mathbb{R}_+^4). \quad (5.11)$$

It is readily apparent that the rank of $g(x)$ is 4 and the positive definite diffusion matrix in $\text{Int}(\mathbb{R}_+^4)$ is

$$A(x) = g(x) g(x)^T = \begin{pmatrix} \sigma_1^2 F^2 & 0 & 0 & 0 \\ 0 & \sigma_2^2 N^2 & 0 & 0 \\ 0 & 0 & \sigma_3^2 V^2 & 0 \\ 0 & 0 & 0 & \sigma_4^2 R^2 \end{pmatrix}. \quad (5.12)$$

Hence, $g(x)$ is continuous with respect to x . Therefore, in any compact set $M \subset \text{Int}(\mathbb{R}_+^4)$, $A(x)$ is uniformly elliptic. To complete the proof, it suffices to construct a Lyapunov function $X(t)$ along with a compact set $M \subset \text{Int}(\mathbb{R}_+^4)$ such that $LX(x) \leq -d$, $d > 0$, and $x \in \text{Int}(\mathbb{R}_+^4) \setminus M$. Furthermore, the Eq (5.7) provides the Lyapunov function $X : \text{Int}(\mathbb{R}_+^4) \rightarrow \text{Int}(\mathbb{R}_+)$, and the expression for $LX(x)$ derived from the proof of Theorem 3 is extracted as

$$LX(x) \leq -\mathfrak{D}_1(F - F^*)^2 - \mathfrak{D}_2(N - N^*)^2 - \mathfrak{D}_3(V - V^*)^2 - \mathfrak{D}_4(R - R^*)^2 + \mathfrak{D}_\sigma.$$

Thus,

$$\mathfrak{D}_1(F - F^*)^2 + \mathfrak{D}_2(N - N^*)^2 + \mathfrak{D}_3(V - V^*)^2 + \mathfrak{D}_4(R - R^*)^2 = \mathfrak{D}_\sigma,$$

entirely contained within $\text{Int}(\mathbb{R}_+^4)$. Consequently, there exists a positive constant d and a compact set $M \subset \text{Int}(\mathbb{R}_+^4)$, such that for any $x \in \text{Int}(\mathbb{R}_+^4) \setminus M$, the following holds:

$$\mathfrak{D}_1(F - F^*)^2 + \mathfrak{D}_2(N - N^*)^2 + \mathfrak{D}_3(V - V^*)^2 + \mathfrak{D}_4(R - R^*)^2 \geq \mathfrak{D}_\sigma + d.$$

Therefore, for any $x \in \text{Int}(\mathbb{R}_+^4) \setminus M$, $LX(x) \leq -d < 0$. Hence the proof is complete. $\square$

Now, we proceed to perform the numerical simulation for the assessment of deterministic and stochastic dynamics of our system. The system involves high nonlinearity, so we follow a numerical approach to visualize and interpret the system's behavior, and specifically, the dynamics of rainfall and vapor clouds under the influence of forestry trees and the human population.

6. Numerical results and discussion

This section presents a numerical simulation for the formulated deterministic and stochastic model systems, given in (2.1) and (4.1), respectively, to explore the system's dynamics, which imparts deeper insights into the modeling phenomena and corroborates the obtained analytical results. To perform the numerical simulation, we consider a set of parameters, on the basis of the modeling phenomenon and assumptions with the help of the data set provided in [23]. We take the same parameter set as given in Table 1 throughout the numerical simulation (except where the changes are explicitly mentioned).

6.1. Validation and discussion of deterministic model system (2.1)

For the aforementioned parameter set, given in Table 1, the obtained equilibria and their stability are as follows:

- i. The components of boundary equilibrium E_0 are $(0, 0, 2850 \text{ mm}, \approx 1754 \text{ mm})$, and the corresponding eigenvalues are $-0.13, -0.10, 0.20, 0.08$. Thus, E_0 is unstable.
- ii. The components of equilibrium E_1 are $(0, 8000 \text{ persons}, \approx 2000 \text{ mm}, \approx 1231 \text{ mm})$, and the corresponding eigenvalues are $-0.13, -0.10, -0.08, 0.19999$. Thus, E_1 is unstable.
- iii. The components of equilibrium E_2 are $(\approx 10001 \text{ trees}, 0, \approx 2870 \text{ mm}, \approx 1730 \text{ mm})$, and the corresponding eigenvalues are $-0.20, -0.09, -0.1327, 0.08$. Thus, E_2 is unstable.
- iv. The components of interior equilibrium $E^*(F^*, N^*, V^*, R^*)$ are

$$F^* \approx 9997 \text{ trees}, \quad N^* \approx 8000 \text{ persons}, \quad V^* \approx 2021 \text{ mm}, \quad R^* \approx 1218 \text{ mm}.$$

The eigenvalues of the Jacobian matrix at the interior equilibrium E^* are computed as $-0.1999, -0.09999, -0.1327, -0.8000$, which are negative, confirming the local asymptotic stability for equilibrium E^* of deterministic model (2.1). The conditions for the local and global asymptotic stability of equilibrium E^* provided in Result 1 iv. are also satisfied. Moreover, Figure 3(a),(b) portray the nonlinear stability behavior of equilibrium E^* in $F - N - V$ and $F - N - R$ spaces, respectively. Here, the solution trajectories of deterministic model (2.1) starting from distinct initial points in $F - N - V$ and $F - N - R$ spaces approach the equilibrium point E^* , which manifests the global asymptotic stability of $E^*(F^*, N^*, V^*, R^*)$ in these spaces, respectively.

Proceeding further, bar plots are presented to elucidate the variations in the densities of vapor clouds and raindrops corresponding to the transpiration rate of forestry trees (ν) in Figure 4(a),(b), respectively. Here, the transpiration rate $\nu \in [0.0002, 0.003]$ and varies as $0.0002 : 0.0001 : 0.003$, i.e., ν increases from 0.0002 to 0.003 with a step size of 0.0001. These figures depict that as the parameter ν increases, the densities of vapor clouds ($V(t)$) and raindrops ($R(t)$) also increase. This physically signifies that the increasing transpiration from forestry trees increases the amount of water vapor released into the atmospheric environment, enhancing the moisture available for the formation of clouds. As more water vapor accumulates in the atmosphere, it condenses into droplets more efficiently, increasing the density of vapor clouds. This, in turn, accelerates the coalescence of droplets, leading to more raindrop formation and rainfall.

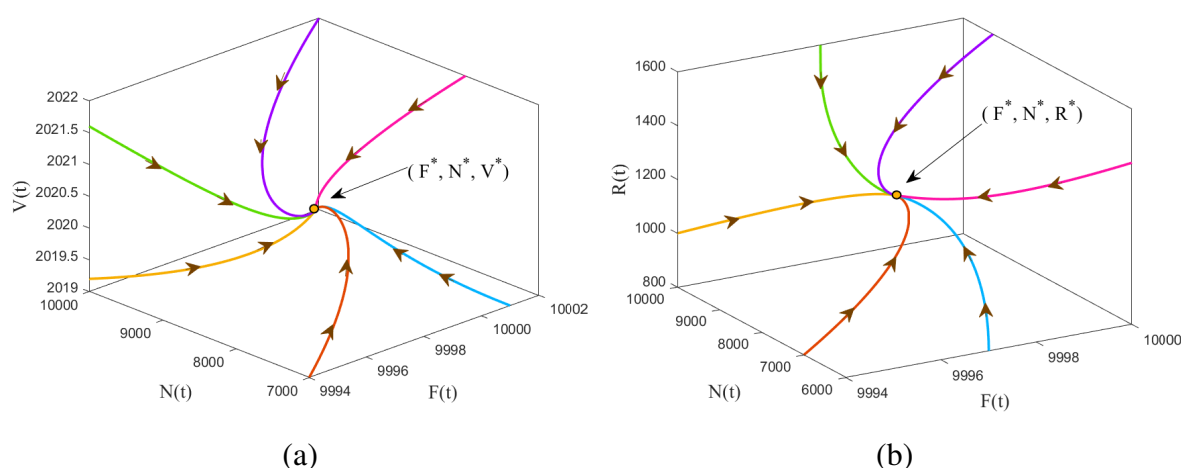


Figure 3. Plot showing the nonlinear stability behavior of interior equilibrium E^* of deterministic model system (2.1) in $F - N - V$ and $F - N - R$ spaces.

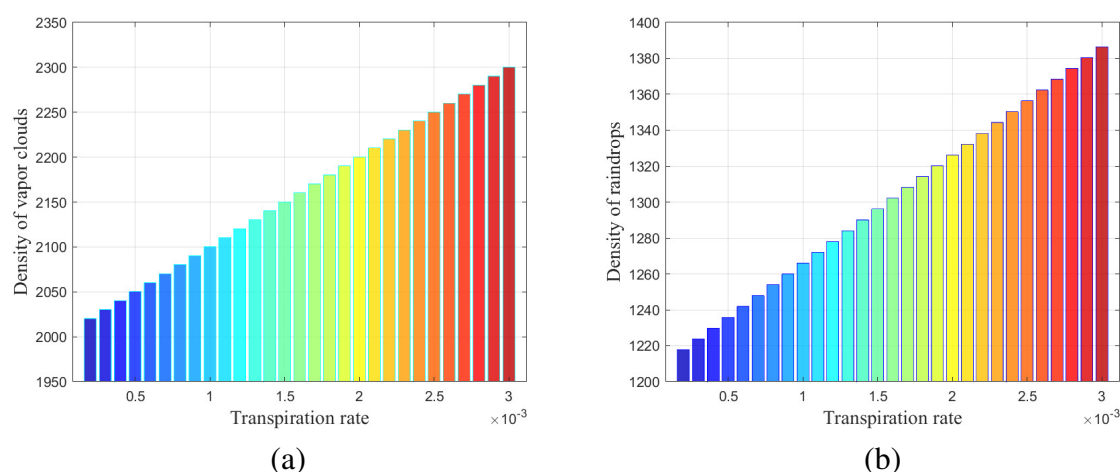


Figure 4. Bar plots of densities of (a) vapor clouds, and (b) raindrops corresponding to the transpiration rate from forest trees for $\nu \in [0.0002, 0.003]$ for deterministic model system (2.1).

Additionally, the variation plots of densities of the forest trees and vapor clouds with respect to time, corresponding to the reducing factor for the natural formation rate of vapor clouds due to anthropogenic activities, are displayed in Figure 5(a),(b), respectively. Here, the reducing factor $\theta \in [0.3, 0.7]$ and varies as $0.3 : 0.001 : 0.7$, i.e., θ increases from 0.3 to 0.7 with a step size of 0.001. These figures depict that when θ (as shown in the color bar) increases, the densities of forest trees and vapor clouds both decrease significantly. Here, it is worth noting that anthropogenic activities, such as urbanization, concrete/cemented ground, extinction of open water sources, industrial emissions, etc., alter the physical and chemical properties of vapor clouds, leading to a reduction in their density. This disruption in cloud dynamics adversely affects local microclimates by decreasing precipitation. Consequently, the forest ecosystem experiences reduced moisture availability and environmental stress, resulting in significant declines in trees' density. This establishes a feedback loop, where

diminished forest cover further suppresses vapor cloud formation, exacerbating climate change impacts.

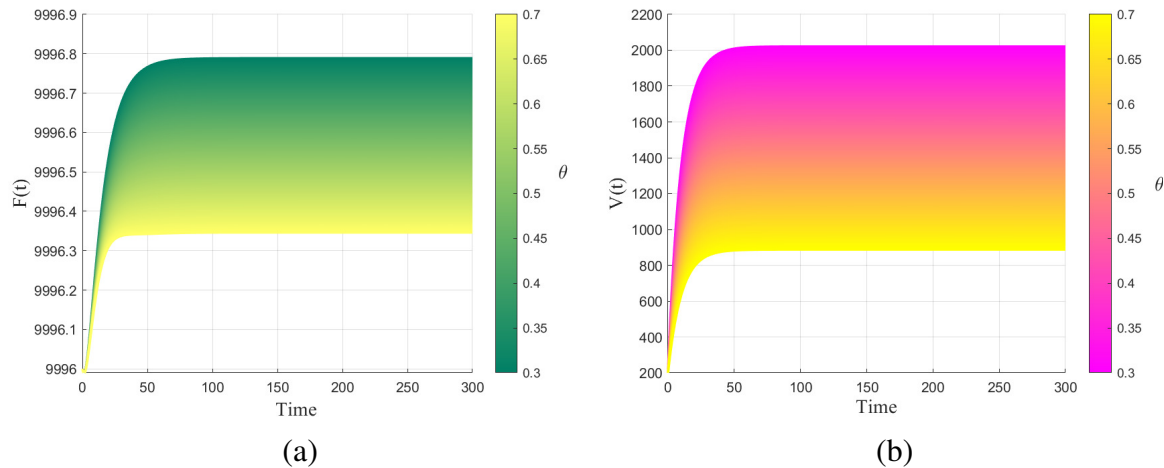


Figure 5. Variation plots of densities of (a) forestry trees, and (b) vapor clouds corresponding to the reducing factor due to anthropogenic activities for $\theta \in [0.3, 0.7]$ for deterministic model system (2.1).

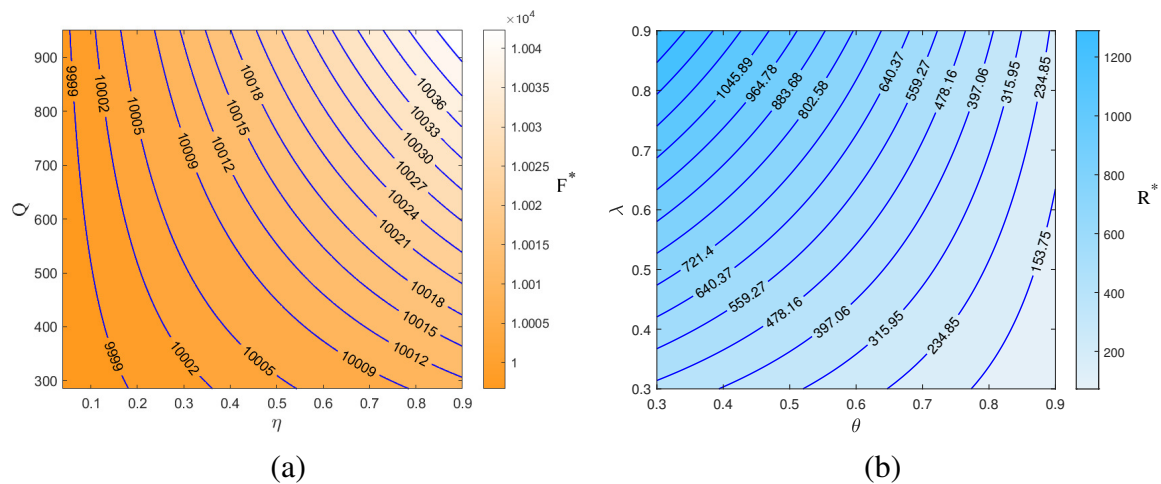


Figure 6. Contour plots of equilibrium values of the density of (a) forestry trees corresponding to the parameters η and Q , and (b) raindrops corresponding to the parameters θ and λ for deterministic model system (2.1).

Further, to explore the combined impact of different parameters on equilibrium values of the density of forest trees and raindrops, contour plots are shown in Figure 6(a),(b), respectively. The contour plot Figure 6(a) illustrates the combined effect of the growth rate of forest trees due to rainfall (η) and the natural formation rate of vapor clouds (Q) on the equilibrium density of forest trees (F^*). Here, $\eta \in [0.04, 0.9]$ and varies as $0.04 : 0.01 : 0.9$, i.e., η increases from 0.04 to 0.9 with a step size of 0.01, and $Q \in [285, 900]$. The minimum value of F^* is observed at the minimum value of both η and Q . For a fixed value of Q , an increase in η results in a corresponding increase in F^* .

Likewise, for a fixed value of η , increasing Q also leads to an increase in F^* . The maximum value of F^* is achieved when both η and Q reach their maximum values. This relationship underscores the interconnectedness between forest trees and rainfall processes. A higher η indicates the better utilization of rain by the forest trees for the soil moisture, nutrient uptake, and photosynthesis, leading to increased growth of forest trees and their density. Also, a higher Q indicates more natural formation of vapor clouds, ensuring the reduced drought conditions or consistent rainfall. Thus, these parameters mutually amplify each other's impact and contribute to supporting a healthy forest ecosystem. Further, the contour plot Figure 6(b) illustrates the combined effect of a reducing factor in the natural formation of vapor clouds due to anthropogenic activities (θ) and the conversion of vapor clouds into raindrops (λ) on the equilibrium density of raindrops (R^*). Here, θ and λ both belongs to $[0.3, 0.9]$ and vary as $0.3 : 0.01 : 0.9$, i.e., θ and λ increase from 0.3 to 0.9 with a step size of 0.01. The maximum value of R^* is observed at the minimum value of θ and the maximum value of λ . For a fixed value of θ , an increase in λ results in a corresponding increase in R^* . Whereas, for a fixed value of λ , increasing θ leads to a decrease in R^* . The minimum value of R^* is obtained when θ is maximum and λ is minimum. This underscores the importance of minimizing anthropogenic interference and optimizing the conversion of vapor clouds into rainfall to achieve optimal precipitation. Human activities such as air and water pollution, deforestation, urbanization, and the creation of impervious surfaces disrupt the natural processes of cloud nucleation and formation, leading to reduced rainfall. Additionally, anthropogenic interference results in fewer clouds forming, and those that do form are less efficiently converted into rain. This ultimately leads to diminished precipitation, disrupting water availability and contributing to drought-like conditions. In this case, optimizing vapor cloud to raindrop conversion can enhance rainfall, but it is most effective when natural cloud formation is less hindered by human development activities.

6.2. Validation and discussion of the stochastic model system (4.1)

For analyzing the stochastic model system (4.1) numerically, we apply Milstein's method [50] and discretize the equations of model (4.1) as follows:

$$\begin{aligned} F_{i+1} - F_i &= \left(s_0 F_i - s_1 F_i^2 - \alpha_1 N_i F_i + \eta \gamma R_i F_i \right) \Delta t + \sigma_1 F_i \sqrt{\Delta t} \zeta_i + \frac{\sigma_1^2}{2} F_i (\zeta_i^2 - 1) \Delta t, \\ N_{i+1} - N_i &= \left(r_0 N_i - r_1 N_i^2 + \pi_1 \alpha_1 N_i F_i \right) \Delta t + \sigma_2 N_i \sqrt{\Delta t} \zeta_i + \frac{\sigma_2^2}{2} N_i (\zeta_i^2 - 1) \Delta t, \\ V_{i+1} - V_i &= \left(Q \left(1 - \frac{\theta N_i}{m + N_i} \right) - \delta V_i + \nu F_i \right) \Delta t + \sigma_3 V_i \sqrt{\Delta t} \zeta_i + \frac{\sigma_3^2}{2} V_i (\zeta_i^2 - 1) \Delta t, \\ R_{i+1} - R_i &= \left(\lambda \delta V_i - \delta_1 R_i - \gamma R_i F_i \right) \Delta t + \sigma_4 R_i \sqrt{\Delta t} \zeta_i + \frac{\sigma_4^2}{2} R_i (\zeta_i^2 - 1) \Delta t. \end{aligned}$$

Here, ζ_i , $i = 1, 2, 3, \dots, n$, are independent Gaussian random variables drawn from a normal distribution $N(0, 1)$. They approximate the Wiener increments as $\Delta B_i = \zeta_i \sqrt{\Delta t}$. Thus, taking this into consideration, we simulate the model system (4.1) at the parameter set given in Table 1. To encapsulate how environmental randomness interacts with the deterministic dynamics of the model system (2.1), we set the intensities of environmental fluctuations as $\sigma_1 = 0.002$, $\sigma_2 = 0.001$, $\sigma_3 = 0.002$, $\sigma_4 = 0.003$. Here, it is worth noting that the condition outlined

in Eq (5.5) is fulfilled at these parameters, which manifests that the stochastic model system (4.1) possesses a unique stationary distribution.

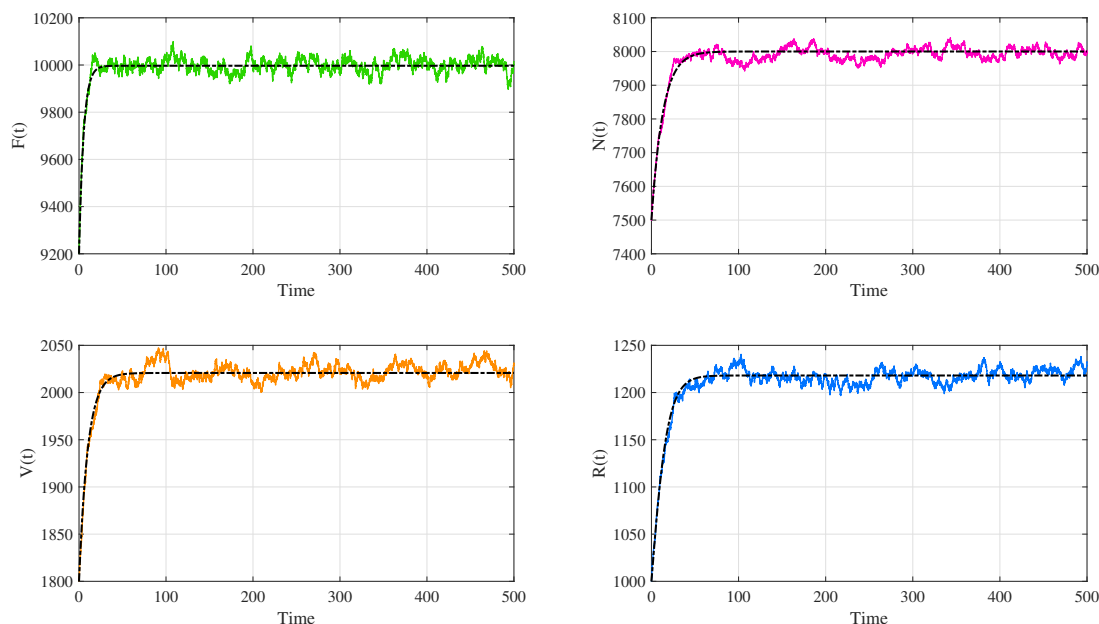


Figure 7. Solution trajectories of stochastic model system (4.1) with $\sigma_1 = 0.002, \sigma_2 = 0.001, \sigma_3 = 0.002, \sigma_4 = 0.003$, and deterministic model system (2.1).

Moving further, solution trajectories of stochastic model system (4.1) with $\sigma_1 = 0.002, \sigma_2 = 0.001, \sigma_3 = 0.002, \sigma_4 = 0.003$ along with deterministic model system (2.1) are plotted in Figure 7. From this figure, it is apparent that under the low intensities of noise terms, the solution of stochastic model system (4.1) exhibits small fluctuations, resulting in minor oscillations with small amplitudes around the solution trajectories of deterministic model (2.1) for all the dynamic variables. Whereas, by increasing the intensities of the noise terms to $\sigma_1 = 0.05, \sigma_2 = 0.04, \sigma_3 = 0.05, \sigma_4 = 0.06$, the solution of stochastic system (4.1) exhibits slightly larger fluctuations, resulting in oscillations with large amplitudes around the solution trajectories of deterministic model (2.1) for all the dynamic variables, as shown in Figure 8.

Proceeding further, to understand how stochastic processes behave in the long run, balancing the effects of environmental randomness and system dynamics, we present the stationary distribution graph. For this, we showcase Figure 9 by setting the intensities of the noise terms at $\sigma_1 = 0.002, \sigma_2 = 0.001, \sigma_3 = 0.002, \sigma_4 = 0.003$. This shows that the stationary distribution of the densities of forest trees, human population, vapor clouds and raindrops follow the standard normal distribution with mean values 9996.79, 8000.10, 1218.05, and 2020.74, respectively. Further, by setting the intensities of the noise terms to $\sigma_1 = 0.3, \sigma_2 = 0.2, \sigma_3 = 0.09, \sigma_4 = 0.09$, Figure 10 depicts the changes in the mean values of the densities of forest trees, human population, vapor clouds, and raindrops as well as in the skewness. Here, it is noteworthy that at $\sigma_1 = 0.002, \sigma_2 = 0.001, \sigma_3 = 0.002, \sigma_4 = 0.003$, the stationary distribution of the densities of forest trees, human population, vapor clouds, and raindrops follow a standard normal distribution, and at the

increased intensities of noise terms $\sigma_1 = 0.3, \sigma_2 = 0.2, \sigma_3 = 0.09, \sigma_4 = 0.09$, the stationary distribution of the densities of forest trees, human population, vapor clouds, and raindrops is positively skewed.

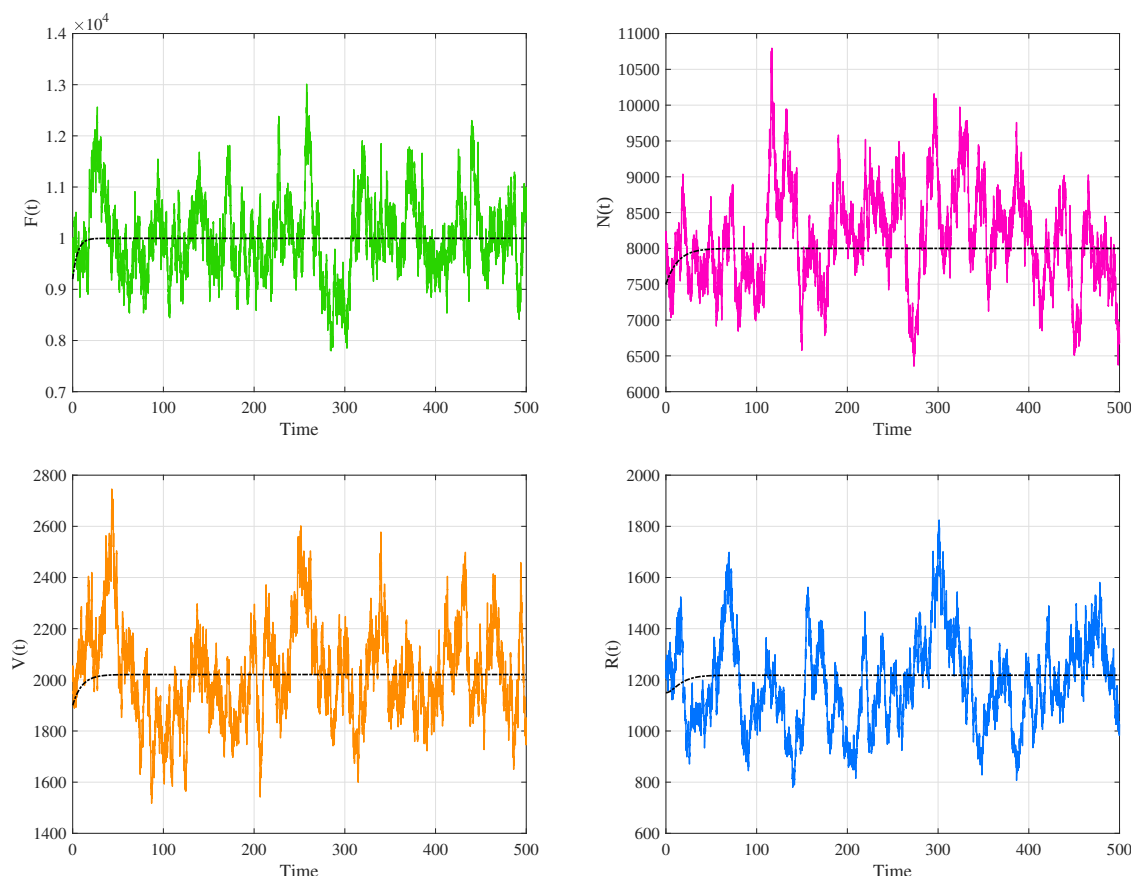


Figure 8. Solution trajectories of stochastic model system (4.1) with $\sigma_1 = 0.05, \sigma_2 = 0.04, \sigma_3 = 0.05, \sigma_4 = 0.06$, and deterministic model system (2.1).

A boxplot for the density of raindrops ($R(t)$) corresponding to the increasing values of transpiration rate (ν) with the set $\nu = \{0.0002, 0.0004, 0.0006, 0.0008\}$ is presented in Figure 11. For this, we set the intensities of the noise terms at $\sigma_1 = 0.002, \sigma_2 = 0.001, \sigma_3 = 0.002, \sigma_4 = 0.003$, and assign the rest of the parameter values the same as specified in Table 1. This plot visualizes the distribution of values of the densities of raindrops. For the transpiration rate $\nu = 0.0002$, the median rainfall is 1217.88, which represents the central tendency of the data, indicating that the typical rainfall value lies near this point. The interquartile range (IQR) extends from 1212.35 (Q_1) to 1223.35 (Q_3), encompassing the central 50% of the observed rainfall values. The whiskers extend from 1194.54 to 1239.83, capturing the range of typical rainfall values within 1.5 times the IQR. Outliers exceeding 1239.83 are present, suggesting the occurrence of more rainfall that deviates significantly from the general distribution. Similarly, for the higher transpiration rate $\nu = 0.0004, 0.0006, 0.0008$, the median rainfall is 1229.82, 1242.00, and 1253.66, respectively, and the third quartile range (Q_3) is 1235.40, 1247.59, and 1259.45, respectively, as shown in Figure 11. The physical significance of the box plot of rainfall corresponding to the increasing values of ν is that it offers a compact summary of the distribution of rainfall, revealing

both the typical behavior (central tendency) and the potential for extreme rainfall events (outliers) for varying transpiration rates from forest trees. The median assesses the average amount of rain typically experienced, which is critical for crop planning, water storage, and flood control measures. The IQR shows the typical range of rainfall we can expect, helping infrastructure plans, such as drainage systems or reservoirs, and outliers might represent extremely low or high rain, which could inform emergency preparedness and resource allocation for unexpected events, like floods and droughts.

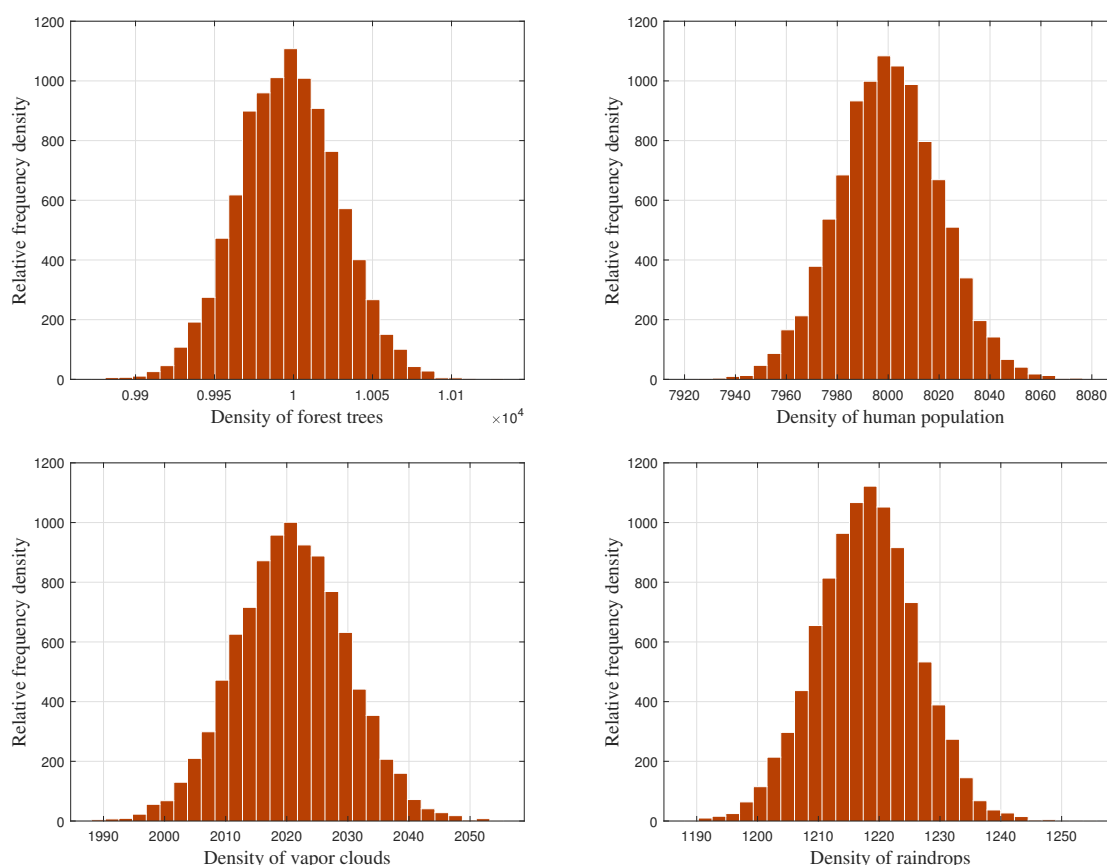


Figure 9. Stationary distribution of all dynamic variables for stochastic model system (4.1) with $\sigma_1 = 0.002$, $\sigma_2 = 0.001$, $\sigma_3 = 0.002$, $\sigma_4 = 0.003$, at $t = 1000$ from 10,000 simulations.

6.3. Limitations and prospects

The results of this research present the interconnected role of human-induced development activities, forest trees, environmental processes (such as evaporation and condensation–nucleation of vapor clouds) on the regulation of the hydrologic cycle. We analyzed the role of transpiration from forest trees in influencing rainfall dynamics. However, it may be noted that the transpiration by forest trees is also influenced by factors such as canopy height and surface roughness, and wind flow and turbulence, which are not accounted for in our modeling assumptions as inclusion of these factors in the modeling phenomena can deviate the results. The model does not incorporate transpiration from other vegetation types, including agricultural crops explicitly, and this may have also some effects on the hydrologic cycle. Additionally, rainfall patterns are subject to seasonal variations, which may influence the outcomes of our analysis. The establishment of new plantations with high transpiration

rates may involve time lags and require substantial financial investment. Since human-driven deforestation and other developmental activities such as urbanization, industrialization, and constructing impervious surfaces are often directed to fit comfort and economic need, it may be difficult to implement a sustainable approach in real-life to reduce the issues that they are posing in disturbing the natural hydrologic cycle and environment.

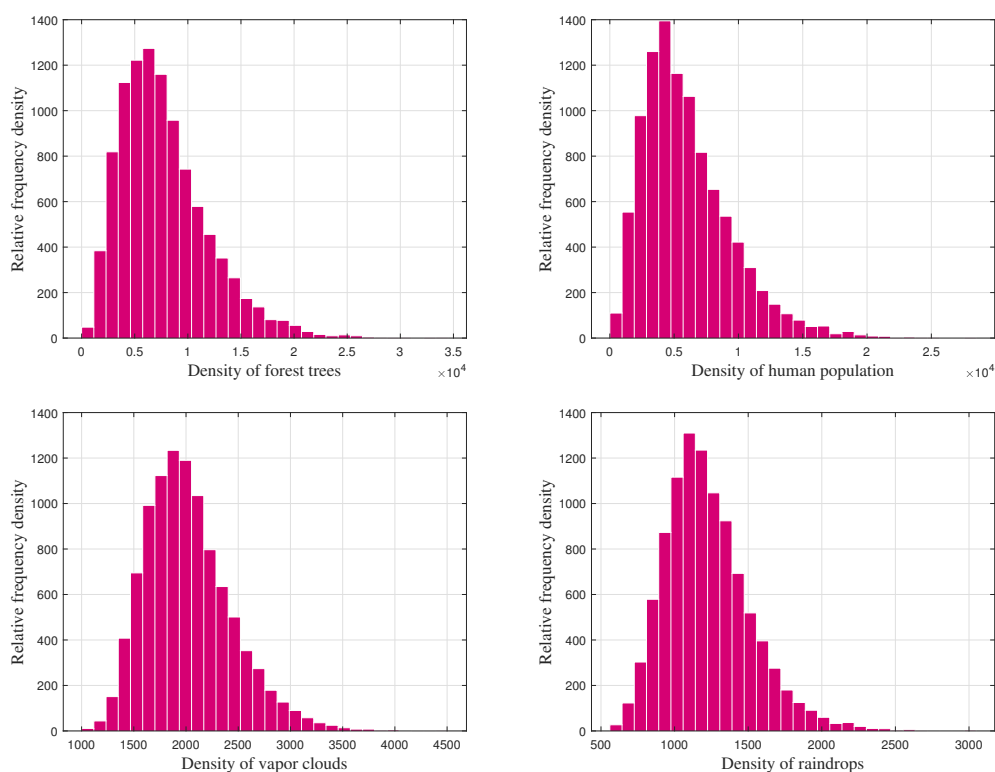


Figure 10. Stationary distribution of all dynamic variables for stochastic model system (4.1) with $\sigma_1 = 0.3, \sigma_2 = 0.2, \sigma_3 = 0.09, \sigma_4 = 0.09$, at $t = 1000$ from 10,000 simulations.

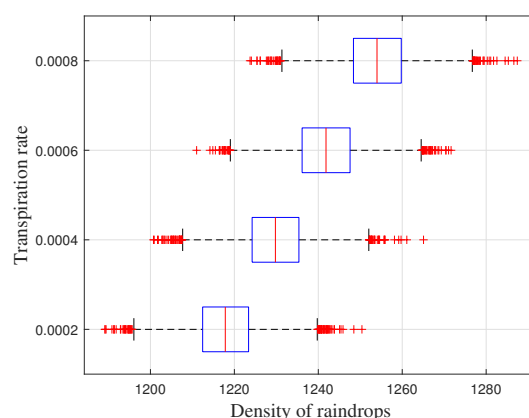


Figure 11. Box plot of rainfall for stochastic model system (4.1) with $\sigma_1 = 0.002, \sigma_2 = 0.001, \sigma_3 = 0.002, \sigma_4 = 0.003$ and varying transpiration rate (ν) with the set $\nu = \{0.0002, 0.0004, 0.0006, 0.0008\}$.

7. Conclusions

Human population growth and forest trees play a crucial role in altering the hydrologic cycle, particularly by influencing precipitation patterns within ecosystems. To assess this impact of human population and forestry trees on the hydrologic cycle, we proposed a mathematical model and analyzed it by employing the stability theory of differential equations.

The system manifests four feasible equilibria under certain conditions. The analysis reveals the instability of three equilibria: (i) the boundary equilibrium, when neither the density of forest trees nor the density of the human population are considered, is unstable, (ii) the equilibrium point, when only the density of forest trees is excluded, is unstable with certain conditions, and (iii) the equilibrium point, when only the density of the human population is excluded, is unstable. (iv) The interior equilibrium point, where the influence of all dynamic variables is visible, exhibits both local and global asymptotic stability under the specified conditions. By employing Lyapunov's stability theory, we established the existence of a unique global positive solution for the stochastic model system (4.1) and demonstrated its boundedness. The system is shown to have a unique stationary distribution characterized by the ergodic property. Moreover, in the presence of low/high intensities of noise terms, the paths of the densities of forest trees, human population, vapor clouds, and raindrops for system (4.1) display fluctuations around the paths of these variables for model (2.1) having low/high oscillations of small/large amplitudes. The stationary distribution of the considered dynamic variables follows the normal distribution with the mean values, whereas, for increased intensities of noise terms, the stationary distribution of these variables exhibits a slight shift in mean values, accompanied by a positive skewness. The key findings and their real-life implications are as follows:

- The increase in the reducing factor due to anthropogenic activities (such as urbanization, industrialization, replacement of natural land with impervious surfaces, water and air pollution) leads to a reduction in the densities of forest trees and vapor clouds. Thus, the reduced rainfall due to diminished vapor clouds hinders the density of trees and compromises the ability of trees to sustain a forest ecosystem, results in a disruption of the hydrologic cycle.
- An increase in the transpiration rate from the forest trees elevates the densities of vapor clouds and rainfall. This approach is crucial for maintaining the hydrological cycle, as it facilitates water recycling between the land, trees, and atmosphere. By enhancing precipitation, it supports forest growth and stabilizes local water availability. Increased rainfall replenishes soil moisture, which is essential for tree health, thereby promoting a robust forest ecosystem. This positive feedback loop helps sustain the hydrological cycle, ensuring the continued circulation of water within the environment.
- The simultaneous improvements in the efficient rainfall utilization by the forest trees as well as natural formation of vapor clouds are efficient strategy for enhancing forest density. This creates a self-sustaining ecosystem, ensuring sufficient natural evaporation from the Earth (without human interference) and effective utility of rain water for processes like photosynthesis, nutrient uptake, and soil moisture replenishment. This collectively increases the growth and density of forest trees.
- Improving the efficiency of clouds to naturally convert into raindrops can enhance precipitation, but this approach is most effective when natural vapor clouds' conversion is less impeded by human development activities.

- For increasing values of the formation rate of vapor clouds due to transpiration from trees, the rainfall pattern provides a concise summary of rainfall distribution, highlighting typical patterns (median and interquartile range) and extreme events (outliers). The median reflects average rainfall, aiding in crop planning, water storage, and flood control. The interquartile range indicates the typical range, useful for designing infrastructure like drainage systems. Outliers signal rare events, such as floods or droughts, guiding emergency preparedness and resource allocation.

Conclusively, it may be noted that environmental fluctuations, that may occur due to the growing human population and their developmental activities, affect the rainfall events significantly. Thus, nourishing a growing world population while protecting the environment requires landscape approaches that sustainably use vital ecosystem services and tackle ecological challenges. This research work provides sustainable directives to maintain the ecosystem. Countries should focus on the activities of a rapidly growing population and they should plan cities, towns, and all the infrastructure in a sustainable way. Policymakers should focus on the plantation of new trees having a higher transpiration rate, which is beneficial to maintaining the rainfall. They should also emphasize the need to implement a zero-deforestation policy for sustainable timber harvesting to protect and preserve those trees we already have, so that the relationship between rainfall and forest trees in maintaining the ecosystem remains intact.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

Gauri Agrawal is grateful to the Department of Science and Technology, Government of India, for providing financial support in the form of senior research fellowship (DST/INSPIRE Fellowship/2020/IF200224) and would like to acknowledge the DST-Centre for Interdisciplinary Mathematical Sciences, BHU, Varanasi, for providing the facilities and intellectual environment.

Conflict of interest

The authors declare there is no conflict of interest.

References

1. Z. Cui, Y. Zhang, A. Wang, J. Wu, Forest evapotranspiration trends and their driving factors under climate change, *J. Hydrol.*, **644** (2024), 132114. <https://doi.org/10.1016/j.jhydrol.2024.132114>
2. D. Sheil, How plants water our planet: Advances and imperatives, *Trends Plant Sci.*, **19** (2014), 209–211. <https://doi.org/10.1016/j.tplants.2014.01.002>
3. S. Jasechko, Z. Sharp, J. Gibson, S. J. Birks, Y. Yi, P. J. Fawcett, Terrestrial water fluxes dominated by transpiration, *Nature*, **496** (2013), 347–350. <https://doi.org/10.1038/nature11983>

4. J. E. Bagley, A. R. Desai, K. J. Harding, P. K. Snyder, J. A. Foley, Drought and deforestation: Has land cover change influenced recent precipitation extremes in the Amazon?, *J. Climate*, **27** (2014), 345–361. <https://doi.org/10.1175/JCLI-D-12-00369.1>
5. S. A. Spera, G. L. Galford, M. T. Coe, M. N. Macedo, J. F. Mustard, Land-use change affects water recycling in Brazil's last agricultural frontier, *Glob. Chang. Biol.*, **22** (2016), 3405–3413. <https://doi.org/10.1111/gcb.13298>
6. A. Staal, O. A. Tuinenburg, J. H. C. Bosmans, M. Holmgren, E. H. V. Nes, M. Scheffer, et al., Forest–rainfall cascades buffer against drought across the Amazon, *Nat. Clim. Change*, **8** (2018), 539–543. <https://doi.org/10.1038/s41558-018-0177-y>
7. R. Ranjan, Assessing the impact of mining on deforestation in India, *Resour. Policy*, **60** (2019), 23–35. <https://doi.org/10.1016/j.resourpol.2018.11.022>
8. M. Bologna, G. Aquino, Deforestation and world population sustainability: A quantitative analysis, *Sci. Rep.*, **10** (2020), 7631. <https://doi.org/10.1038/s41598-020-63657-6>
9. C. M. Jr Souza, J. V. Siqueira, M. H. Sales, A. V. Fonseca, J. G. Ribeiro, I. Numata, et al., Ten-year landsat classification of deforestation and forest degradation in the Brazilian Amazon, *Remote Sens.*, **5** (2013), 5493–5513. <https://doi.org/10.3390/rs5115493>
10. FAO, Global Forest Resources Assessment 2020-Key findings.
11. H. Boubekraoui, Y. Maouni, A. Ghallab, M. Draoui, A. Maouni, Spatio-temporal analysis and identification of deforestation hotspots in the Moroccan western Rif, *Trees For. People*, **12** (2023), 100388. <https://dx.doi.org/10.2139/ssrn.4327300>
12. C. Smith, J. C. Baker, D. V. Spracklen, Tropical deforestation causes large reductions in observed precipitation, *Nature*, **615** (2023), 270–275. <https://doi.org/10.1038/s41586-022-05690-1>
13. F. DeSales, T. Santiago, T. W. Biggs, K. Mullan, E. O. Sills, C. Monteverde, Impacts of protected area deforestation on dry-season regional climate in the Brazilian Amazon, *J. Geophys. Res. Atmos.*, **125** (2020), e2020JD033048 <https://doi.org/10.1029/2020JD033048>
14. Y. Mu, T. W. Biggs, F. De Sales, Forests mitigate drought in an agricultural region of the Brazilian Amazon: Atmospheric moisture tracking to identify critical source areas, *Geophys. Res. Lett.*, **48** (2021), e2020GL091380. <https://doi.org/10.1029/2020GL091380>
15. H. Douville, K. Raghavan, J. Renwick, R. P. Allan, P. A. Arias, M. Barlow, et al., Water cycle changes, in *Climate Change 2021: The Physical Science Basis*, (2023), 1055–1210.
16. M. Kravcik, J. Lambert, A global action plan for the restoration of natural water cycles and climate, 2017. Available from: https://bio4climate.org/downloads/Kravcik_Global_Action_Plan.pdf.
17. B. Bhagowati, K. U. Ahamad, A review on lake eutrophication dynamics and recent developments in lake modeling, *Ecohydrol. Hydrobiol.*, **19** (2019), 155–166. <https://doi.org/10.1016/j.ecohyd.2018.03.002>
18. D. Han, M. J. Currell, G. Cao, B. Hall, Alterations to groundwater recharge due to anthropogenic landscape change, *J. Hydrol.*, **554** (2017), 545–557. <https://doi.org/10.1016/j.jhydrol.2017.09.018>
19. E. Bilgic, A. Baba, Effect of urbanization on water resources: Challenges and prospects, in *Groundwater in Arid and Semi-Arid Areas*, Springer, (2023), 81–108.

20. T. T. Vo, L. Hu, L. Xue, Q. Li, S. Chen, Urban effects on local cloud patterns, *Proc. Natl. Acad. Sci.*, **120** (2023), e2216765120. <https://doi.org/10.1073/pnas.2216765120>
21. B. Dubey, S. Sharma, P. Sinha, J. B. Shukla, Modelling the depletion of forestry resources by population and population pressure augmented industrialization, *Appl. Math. Modell.*, **33** (2009), 3002–3014. <https://doi.org/10.1016/j.apm.2008.10.028>
22. K. Lata, A. K. Misra, J. B. Shukla, Modeling the effect of deforestation caused by human population pressure on wildlife species, *Nonlinear Anal. Modell. Control*, **23** (2018), 303–320. <https://doi.org/10.15388/NA.2018.3.2>
23. K. Lata, A. K. Misra, The influence of forestry resources on rainfall: A deterministic and stochastic model, *Appl. Math. Modell.*, **81** (2020), 673–689. <https://doi.org/10.1016/j.apm.2020.01.009>
24. O. G. Gaoue, C. N. Ngonghala, J. Jiang, M. Lelu, Towards a mechanistic understanding of the synergistic effects of harvesting timber and non-timber forest products, *Methods Ecol. Evol.*, **7** (2016), 398–406. <https://doi.org/10.1111/2041-210X.12493>
25. E. Karesdotter, G. Destouni, N. Ghajarnia, R. B. Lammers, Z. Kalantari, Distinguishing direct human-driven effects on the global terrestrial water cycle, *Earth's Future*, **10** (2022), e2022EF002848. <https://doi.org/10.1029/2022EF002848>
26. L. Tanika, C. Wamucii, L. Best, E. G. Lagneaux, M. Githinji, M. V. Noordwijk, Who or what makes rainfall? Relational and instrumental paradigms for human impacts on atmospheric water cycling, *Curr. Opin. Environ. Sustain.*, **63** (2023), 101300. <https://doi.org/10.1016/j.cosust.2023.101300>
27. A. K. Misra, G. Agrawal, K. Lata, Modeling the influence of human population and human population augmented pollution on rainfall, *Discrete Continuous Dyn. Syst. Ser. B*, **27** (2022), 2979–3003. <https://doi.org/10.3934/dcdsb.2021169>
28. G. Agrawal, A. K. Agrawal, A. K. Misra, Modeling the impacts of chemical substances and time delay to mitigate regional atmospheric pollutants and enhance rainfall, *Phys. D Nonlinear Phenom.*, **472** (2025), 134507. <https://doi.org/10.1016/j.physd.2024.134507>
29. S. Sundar, R. Naresh, A. K. Misra, J. B. Shukla, A nonlinear mathematical model to study the interactions of hot gases with cloud droplets and raindrops, *Appl. Math. Modell.*, **33** (2009), 3015–3024. <https://doi.org/10.1016/j.apm.2008.10.032>
30. Z. Huang, Q. Yang, J. Cao, A stochastic model for interactions of hot gases with cloud droplets and raindrops, *Nonlinear Anal. Real World Appl.*, **12** (2011), 203–214. <https://doi.org/10.1016/j.nonrwa.2010.06.008>
31. G. Agrawal, A. K. Agrawal, J. Dhar, Effects of human population and atmospheric pollution on rainfall: A modeling study, *J. Indian Math. Soc.*, **91** (2024), 550–664.
32. A. K. Misra, A. Tripathi, Stochastic stability of aerosols–stimulated rainfall model, *Phys. A*, **527** (2019), 121337. <https://doi.org/10.1016/j.physa.2019.121337>
33. G. Agrawal, A. K. Agrawal, J. Dhar, A. K. Misra, Modeling the impact of cloud seeding to rescind the effect of atmospheric pollutants on natural rainfall, *Model. Earth Syst. Environ.*, **10** (2024), 1573–1588. <https://doi.org/10.1007/s40808-023-01854-8>

34. A. K. Misra, G. Agrawal, A. Yadav, When to perform cloud seeding for maximum agricultural crop yields? A modeling study, *Int. J. Numer. Methods Heat Fluid Flow*, Forthcoming 2024. <https://doi.org/10.1108/HFF-09-2024-0711>
35. C. Salas-Eljatib, L. Mehtatalo, T. G. Gregoire, D. P. Soto, R. Vargas-Gatete, Growth equations in forest research: Mathematical basis and model similarities, *Curr. Forestry Rep.*, **7** (2021), 230–244. <https://doi.org/10.1007/s40725-021-00145-8>
36. F. Brauer, C. Castillo-Chavez, *Mathematical Models in Population Biology and Epidemiology*, Springer New York, 2011. <https://doi.org/10.1007/978-1-4614-1686-9>
37. R. A. Houze, Chapter 3-Cloud microphysics, *Int. Geophys.*, **104** (2014), 47–76. <https://doi.org/10.1016/B978-0-12-374266-7.00003-2>.
38. O. Boucher, *Atmospheric Aerosols: Properties and Climate Impacts*, Springer Dordrecht, Springer Nature, 2015. <https://doi.org/10.1007/978-94-017-9649-1>.
39. M. O. Andreae, D. Rosenfeld, Aerosol-cloud-precipitation interactions. Part 1. The nature and sources of cloud-active aerosols, *Earth-Sci. Rev.*, **89** (2008), 13–41. <https://doi.org/10.1016/j.earscirev.2008.03.001>
40. M. Ruiz-Vasquez, P. A. Arias, J. A. Martinez, J. C. Espinoza, Effects of Amazon basin deforestation on regional atmospheric circulation and water vapor transport towards tropical South America, *Clim. Dyn.*, **54** (2020), 4169–4189. <https://doi.org/10.1007/s00382-020-05223-4>
41. A. T. Leite-Filho, B. S. Soares-Filho, J. L. Davis, G. M. Abrahao, J. Borner, Deforestation reduces rainfall and agricultural revenues in the Brazilian Amazon, *Nat. Commun.*, **12** (2021), 2591. <https://doi.org/10.1038/s41467-021-22840-7>
42. A. M. Makarieva, A. V. Nefiodov, A. D. Nobre, D. Sheil, P. Nobre, J. Pokorny, et al., Vegetation impact on atmospheric moisture transport under increasing land-ocean temperature contrasts, *Heliyon*, **8** (2022), e11173. <https://doi.org/10.1016/j.heliyon.2022.e11173>
43. H. I. Freedman, J. W. H. So, Global stability and persistence of simple food chains, *Math. Biosci.*, **76** (1985), 69–86. [https://doi.org/10.1016/0025-5564\(85\)90047-1](https://doi.org/10.1016/0025-5564(85)90047-1)
44. J. K. Hale, *Ordinary Differential Equations*, Wiley-Interscience, New York, 1969.
45. Z. Huang, G. Huang, Mathematical analysis on deterministic and stochastic lake ecosystem models, *Math. Biosci. Eng.*, **16** (2019), 4723–4740. <https://doi.org/10.3934/mbe.2019237>
46. D. Bolatova, S. Kadyrov, A. Kashkynbayev, Mathematical modeling of infectious diseases and the impact of vaccination strategies, *Math. Biosci. Eng.*, **21** (2024), 7103–7123. <https://doi.org/10.3934/mbe.2024314>
47. S. Li, S. Wang, Analysis of a stochastic predator–prey model with disease in the predator and Beddington–DeAngelis functional response, *Adv. Differ. Equations*, **224** (2015). <https://doi.org/10.1186/s13662-015-0448-0>
48. X. Mao, *Stochastic Differential Equations and Applications*, Horwood, New York, 1997.
49. R. K. Upadhyay, R. D. Parshad, K. Antwi-Fordjour, E. Quansah, S. Kumari, Global dynamics of stochastic predator-prey model with mutual interference and prey defense, *J. Appl. Math. Comput.*, **60** (2019), 169–190. <https://doi.org/10.1007/s12190-018-1207-7>

50. D. J. Higham, An algorithmic introduction to numerical simulation of stochastic differential equations, *SIAM Rev.*, **43** (2001), 525–546. <https://doi.org/10.1137/S0036144500378302>
51. X. Mao, G. Marion, E. Renshaw, Environmental Brownian noise suppresses explosions in population dynamics, *Stochastic Process. Appl.*, **97** (2002), 95–110. [https://doi.org/10.1016/S0304-4149\(01\)00126-0](https://doi.org/10.1016/S0304-4149(01)00126-0)
52. N. Dalal, D. Greenhalgh, X. Mao, A stochastic model for internal HIV dynamics, *J. Math. Anal. Appl.*, **341** (2008), 1084–1101. <https://doi.org/10.1016/j.jmaa.2007.11.005>
53. L. Arnold, *Stochastic Differential Equations: Theory and Applications*, New York, 1974.
54. A. Friedman, *Stochastic Differential Equations and Applications*, Academic Press, New York, 1976.



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