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Research article

A sustainable smart production model for partial outsourcing and reworking

Raj Kumar Bachar[1], **Shaktipada Bhuniya**[2], **Ali AlArjani**[3], **Santanu Kumar Ghosh**[1] **and Biswajit Sarkar**[4,5,*]

[1] Department of Mathematics, Kazi Nazrul University, Asansol 713340, West Bengal, India

[2] Department of Mathematics & Statistics, Banasthali Vidyapith, Rajasthan 304 022, India

[3] Department of Industrial Engineering, College of Engineering, Prince Sattam Bin Abdulaziz University, Al Kharj 11942, Saudi Arabia

[4] Department of Industrial Engineering, Yonsei University, 50 Yonsei-ro, Sinchon-dong, Seodaemun-gu, Seoul 03722, South Korea

[5] Center for Transdisciplinary Research (CFTR), Saveetha Dental College, Saveetha Institute of Medical and Technical Sciences, Saveetha University, 162, Poonamallee High Road, Velappanchavadi, Chennai 600077, Tamil Nadu, India

* **Correspondence:** Email: bsbiswajitsarkar@gmail.com; Tel: +82-10-7498-1981.

Abstract: Smart production plays a significant role to maintain good business terms among supply chain players in different situations. Adjustment in production uptime is possible because of the smart production system. The management may need to reduce production uptime to deliver products ontime. But, a decrement in production uptime reduces the projected production quantity. Then, the management uses a limited investment for pursuing possible alternatives to maintain production schedules and the quality of products. This present study develops a mathematical model for a smart production system with partial outsourcing and reworking. The market demand for the product is price dependent. The study aims to maximize the total profit of the production system. Even in a smart production system, defective production rate may be less but unavoidable. Those defective products are repairable. The model is solved by classical optimization. Results show that the application of a variable production rate of the smart production for variable market demand has a higher profit than a traditional production (52.65%) and constant demand (12.45%).

Keywords: smart production; remanufacturing; defective items; reworking; product outsourcing; carbon emissions

1. Introduction

The growth of smart production system increases in the past few years. We have seen different policy changes in the business industry in the last few years, and corresponding production industries have changed their strategies too. Several traditional products become obsolete whereas digital technology-based products become an important part of life. Online business and product delivery system through (omni channel) make a lot of changes in the industrial process and as a result, market demand for products become sensitive. A smart production system is a combination of automated smart machines which can take decisions based on the information. All automated machines of the smart production system are connected to the same data source through cloud computing. Finally, automated decisions are verified by skilled workers. The flexibility of a smart production system is able to control the manufacturing process autonomously with more accuracy.

Even with high accuracy, any machine in the smart production system can produce defective products. There are many reasons for producing defective products from a smart production system. Some of the reasons can be listed as follows: information error and wrong data analysis, machinery breakdown of any machine, error in machine setup, and tool/die of the machinery system. Now, due to the machine setup error, a percentage of the produced product becomes defective from the starting of the production process. To maintain the situation, the manufacturer runs a reworking process after finishing the production process. The produced defective products are sent for reworking immediately. Now, due to the continuous production of defective products, the manufacturer ends the in-time production after a certain time. But, the order is not fulfilled yet due to defective products. Reworked products support the delivery process of ordered quantity, but that is not sufficient. Thus, the manufacturer decides to outsource the rest amount of the ordered products to handle the situation. Outsourced products have a similar quality to the in-house products. Meanwhile, the reworking process supports the production process to fulfill the order.

The proposed research considers a smart production system under partial outsourcing and rework of defective items. Maximum studies [1] considered constant production rate and demand for optimal batch size considering partial outsourcing and rework. To fulfill the research gap, the proposed research gives a new direction for dealing with smart production when it produces defective items from the beginning of the production system.

1.1. Research gap

The following conclusion can be drawn based on the gaps in the existing literature.

- Studies on variable production rate (VPR) and demand exist in the literature. But, a defective production scenario for a smart production system with a reworking within the same cycle has not been studied yet.
- Many production models are developed under partial product outsourcing. Some of those studies consider variable production system under the replenishment strategy. But, the replenish by reworked products with partial product outsourcing under a smart production system make a significant contribution to the literature.
- Several research papers have studied different production management strategies and production uptime-downtime concepts. But, a defective smart production system with rework, partial out-

sourcing with similar quality, the separate holding cost for different category products, and selling price-dependent demand have rarely been investigated.

1.2. Contribution

The present study has the following contributions to the literature.

- A profit maximization model is studied based on the random defective rate of smart production. A parallel reworking system works within the same cycle for defective products. As the precision level is high for a smart production system, the manufacturer uses the reworked products as the new ones to fulfill the order.
- The manufacturer reduces the production up-time because of the defective production rate. Outsourcing of products is not started immediately after stopping the in-time production. The demand during that time lapse is satisfied by the reworked products with a reworking rate P_1.
- The manufacturer uses a partial product outsourcing policy for the production system. After finishing the in-time production, the manufacturer outsources the rest amount of the products. A variable outsourcing cost makes each outsourced product similar in quality to the new product.
- The proposed model considers a VPR with a variable demand that is dependent on the selling price as well as a unit production cost (UPC). The UPC includes the tool/die cost, raw material cost, and development cost.

This model considers two different examples with different parameters. In each example, smart production with a variable production rate shows the maximum profit compared to the other products. The focus is on determining the maximum profit of the total production system. The optimal profit is verified both analytically and numerically.

1.3. Structure of this study

The remainder of this paper is organized as follows. The past research details are discussed in Section 2 literature review. Section 3 presents the purpose of the problem, related mathematical symbols, and associated hypotheses. Mathematical modeling is presented in Section 4. Section 5 presents the methodology used to determine the solution. A numerical application is described in Section 6, and Section 7 presents the sensitivity analysis. Section 8 provides managerial insights, and Section 9 presents the conclusions of this study.

2. Literature review

Contributions of previous research and the gaps in the literature are discussed in this section. The contributions of previous researchers to variable production rate research are described in the first subsection. Further, defective items, reworking, outsourcing, and selling-price-dependent demand are important keywords related to this model. Existing research about the keywords is discussed in this section. However, for a better understanding of the research gap, a research gap is provided in Table 1.

2.1. Smart production

In different unavoidable situations, there is a need for some extra products in a short interval of time to fulfill the customer demand. To control such a situation and to fulfill the variable market demand, smart production is very much essential for the production industry. Many studies had been published on defective products from a flexible production system. Khouja and Mehrez [2] considered a flexible production under a variable production rate by optimizing the variable type of the production rate. They considered an imperfect production system. Eiamkanchanalai & Banerjee [3] considered a model by assuming a flexible production policy and determined the optimal time variable, separate goods rate of production. They considered an iterative solution procedure with the rate of production as a quadratic function.

Giri & Dohi [4] explored different failure rates and improved their previous model by considering variable production rates. Glock [5] proposed a research model that described the significant effects of VPR on inventory systems. The author illustrated that a variable production rate reduces inventory-carrying costs. Kim & Glock [6] studied multiple types of parallel-machine problems for production planning. Their model considered different production strategies. Mridha et al. [7] developed a smart production system for green products. They used an automated inspection policy for inspecting products without inspection errors but did not consider outsourcing. Khan et al. [8] discussed a flexible production system with service level constraints and found decisions by Nash game. Saxena & Sarkar [9] derived an unreliable supply chain management for optimum replenishment policy. They used a radio frequency identification (RFID) for product safety but did not consider product outsourcing policy.

It is clear from the history of all these studies mentioned earlier that maximum studies are based on a flexible production system. Studies, based on smart production, do not discuss the scenario of defective products and their replenishment policy. Very less research has been found focusing on the defective production rate-based smart production system with product outsourcing planning. Thus, in the next section, the effects of defective items are described.

2.2. Defective items

The occurrence of machine failure and defective production are common issues in production systems. Therefore, alternative ways must be designed to reduce defective production or rework defective products to fulfill customer demand. Shih [10] presented a study based on the optimal inventory policies for defective production. They calculated optimal value of a large number of perfect and imperfect products. Rosenblatt & Lee [11] studied a defective production by considering a linear and exponential deterioration model to optimize the total annual cost. Boone et al. [12] studied the effects of defective items on the production process. They investigated the relationship between defective items and variable production rate. Sana et al. [13] formulated a model based on a volume-flexible imperfect process. They maximized profits using the interior penalty function method for constrained optimization.

Chakraborty & Giri [14] formulated a deteriorating model under inspections and reworks with maintenance. They considered a preventive maintenance for in-control defective items. Jawlaa and Singh [15] introduced a preservation technology and a learning environment to minimize the total cost. Sarkar et al. [16] determined three-echelon supply chain model where all products are biodegradable and products were delivered through three different transportation mode. Marchi et al. ([17, 18]) and Bazan et al. [19] focussed on an EPQ model in which learning improvement in reliability, quality, and machine failure, reverse logistics production rate, consignment stock, and sustainability were considered. They investigated learning effects on imperfect production, energy efficiency and its impact on lot sizing, and the impact of production rate on greenhouse gas emissions.

From the past research history, we can conclude that every research discussed defective items in their production system. But the production system is not smart. However, defective items face a huge loss to the production company. In this sense, how the reworking connects with defective items is discussed in detail in the next section.

2.3. Reworking

Rework is a part of the production process that involves the modification of defective products. It may be applied to deteriorated products, other defective products, or green products. Many studies have focused on this topic. Flapper & Teunter [20] reported a study that discussed deteriorating items and their reworking policies. They indicated that a reworkable state occurred in defective lots and that took place over time. The time and cost of the rework were included. Biswas & Sarker [21] studied a single-stage production system with scrap product detection and reworking. They optimized the batch quantity using an in-cycle rework and scrap detection. Taleizadeh et al. [22] presented a multi-production model with repair failure. They considered service-level constraints for repair failures and scrap products. Khanna et al. [23] considered a strategic production model with a rework process and different policies. They introduced a two-level trade-credit under sales returns. Bachar et al. [24] derived a defective smart production system with reworking and outsourcing but without emissions from the system. Padiyar et al. [25] derived a multi-echelon supply chain model for deteriorating multi-item under the imprecise and inflationary environment with imperfect production. They considered joint replenishment strategy and triangular fuzzy number in their model. Das et al. [26] developed a defective production with a stochastic credit period. The manufacturer offered a credit period to the retailer for payback the wholesale price.

From the past research history, it is shown that no research focused on the reworking of defective items under smart production and partial outsourcing. Hence, there is a big research gap on the effect of reworking defective products in a smart production system. This proposed study tries to overcome this gap. Moreover, the outsourcing strategy with defective items and reworking needs further investigation. This research gap is discussed in the next section.

2.4. Outsourcing

For an imperfect production system, outsourcing is one of the profitable policies. This may include partial or full outsourcing, depending on the product quality. In most situations, partial outsourcing results in greater profitability for any production system. Coman and Ronen [27] presented a linear programming model with a production outsourcing cost (OC). Their model considered the theory of constraints to maximize the net profit. In addition, Sarkar et al. [28] developed a warehouse-based supply chain model with RFID where supply chain players are untrustable. They did not consider any kind of outsourcing for their product. Hahn et al. [29] presented aggregate planning of a stochastic model with outsourcing of the manufacturing system. They considered a multi-criteria approach for coordinating internal and external manufacturing decisions. Chiu et al. [1] presented a model considering partial outsourcing on an imperfect inventory model and optimized the lot size. They considered reworking at a constant rate. Chiu et al. [30] offered an economic manufacturing quantity-based replenishment system to satisfy the constant market demand. In addition, they introduced scrapped items in a reworkable system. Besides, a proper data analysis [31] on outsourcing of products can provide a prediction of money flow about the scenario.

From the past research history, it can be concluded that very less research included outsourcing in their manufacturing system to deal with reworking and shortage. Then, it can make a sustainable production system. In this sense, how sustainability connects with outsourcing, and reworking for the smart production system is discussed in detail in the next section.

2.5. Sustainability

With learning from the past socio-economic stagnation of society, sustainable development is a vital step of advanced thinking toward a stable plan. Social, economic, and environmental matters are the three pillars of the development indicator. The environmental benefit under different profit-based business strategies makes the eco-friendly approach for the production management team. There are many research articles in past research on the sustainability approach. Kar et al. [32] considered an emissions-controlled production system where the manufacturer uses a hybrid channel to sell a single type of product. They used a carbon cap but did not discuss sustainability in detail. Bachar et al. [33] discussed a sustainable green production model for environmental benefit. All this research considered the sustainability approach, but there was a lack of consideration for outsourcing benefits for smart production system and different beneficiary strategies. However, an FPR can handle a variable type of market demand. In this sense, how the selling-price dependent demand (SPDD) connects with outsourcing and reworking under VPR is discussed in the next section.

2.6. Selling price-dependent demand

The most important aspect of a business process is customer demand. Variable demand is more suitable than constant demand. Thus, to make the demand variable, many studies consider different functions, such as demand functions dependent on advertisements, selling price, and stock. This research model considers a demand function based on the average selling price. Abad & Jaggi [34] presented a paper with game strategies of coordination and non-coordination relationships between different types of vendors and buyers. They suggested that the selling price is price-sensitive. It had an additional benefit depending on trade-credit. Pal et al. [35] studied supply chain management with price, quality, and promotional efforts with price-sensitive demand. Using different game policies, they considered a strategy of providing warranty policies to attract more customers. In their model, both participating supply chain members shared a cost-warranty policy. Bhunia & Shaikh [36] considered a non-linear programming model based on deteriorating items. They considered an inventory model for deteriorating items with a selling price-dependent demand and Weibull distribution. Alfares & Ghaithan [37] considered a research model based on inventory and selling price-dependent demand. Their special contributions were holding costs and discounts, which varied over time. Sarkar et al. [38] developed a smart production model considering dual channel retailing and autonomation technology for a selling price-dependent model.

Several researchers have developed models in which market demand depends on the selling price of the products, but price-dependent demand in a partial outsourcing inventory is not well-discussed. Therefore, to the best of our knowledge, this is the first study that focuses on this direction. A comparison between previous studies and this study is presented in Table 1.

Previous research details, stated in this section, mainly focused on variable demand with a flexible and constant production system. This research considers the effects of defective production on the

Table 1. Contribution of the authors.

Author(s)	Production Rate	Demand Rate	Defective Items	Outsourcing	Rework	Model Type
Giri & Dohi [4]	variable	Constant	NA	NA	NA	EMQ
Kim & Glock [6]	Variable	Constant	NA	NA	NA	Inventory
Sana et al. [13]	Constant	Constant	Yes	NA	NA	Inventory
Taleizadeh et al. [22]	Constant	Constant	Yes	NA	Yes	EPQ
Hahn et al. [29]	Constant	Random	NA	Yes	NA	Inventory
Chiu et al. [1]	Constant	Constant	Yes	Yes	Yes	Inventory
Chiu et al. [30]	Constant	Constant	Yes	Yes	Yes	Inventory
Pal et al. [35]	Constant	Constant	Yes	NA	yes	SCM
Bhunia & Shaikh [36]	NA	SPDD	Yes	NA	Yes	Inventory
Paper	Variable	SPDD	Yes	Yes	Yes	Smart Production

SCM: supply chain management; EPQ: economic production quantity; EMQ: economic manufacturing quantity; NA: not applicable

sustainable smart production system, reworking, and outsourcing.

3. Model purpose, notation, and assumptions

This section provides notation and assumptions for the proposed model. The purpose and a brief description of the model are described below.

3.1. Model purpose

The proposed model studies a smart production system with a random defective rate. The demand is considered a function of the selling price, having a minimum and maximum price range. The production rate is variable and depends on the raw material cost, development cost, and tool/die cost of the machine. In addition, partial outsourcing is an important responsible factor for the efficient operation of business when the production system meets shortages. Chiu et al. [1] considered a model by assuming partial outsourcing and defective production with rework, but they considered a constant demand and constant production rate. Therefore, this research model is an improvement of the previously stated model. The aim is to determine the maximum profit by considering partial product outsourcing with reworking. In addition, this study considers special cases to show different scenarios of the model. The total profit of the system is proved both numerically and analytically.

3.2. Notation

The model depends on the following variables and parameters.

Decision variables

P production rate (unit/unit time)
Q production lot size (units/cycle)
p unit selling price of products($/unit)

Parameters

K setup cost (in-house) (\$/setup)
K^c carbon emissions cost due to in-house setup (\$/setup)
h^c carbon emissions cost for holding products per unit per unit time (\$/unit/unit time)
h_1^c carbon emissions cost for holding reworked products per unit per unit time (\$/unit/unit time)
C_R^c carbon emissions cost from reworking process (\$/unit)
h holding cost of perfect product (\$/unit/unit time)
h_1 holding cost in each reworked products (\$/unit/unit time)
γ_1 scaling parameter of raw material cost for manufacturing system
γ_2 scaling parameter of development cost for the product
γ_3 scaling parameter of tool/die cost
C_R reworking cost (\$/unit)
K_π constant type outsourcing cost (\$/unit)
C_π unit variable outsourcing cost (\$/unit)
P_1 reworking rate (units/unit time)
$C(P)$ unit production cost (\$/unit)
π outsourcing portion of item in a lot size ($0 < \pi < 1$)
β_1 connecting variable between K_π and in-house production cost, K_π, where $K_\pi = (1+\beta_1)(K+K^c)$ and $-1 \leq \beta_1 \leq 0$
β_2 connecting variable between unit production cost and C_π, where $C_\pi = (1+\beta_2)C(P)$ and $\beta_2 \geq 0$
T_π replenishment cycle time (time unit)
H_1 maximum inventory level of perfect product production comes to an end (unit)
H_2 inventory level of the reworking of the defective product comes to an end
H maximum inventory level of perfect products when outsourced products are received
$t_{1\pi}$ production uptime when $\pi = 0$ (year)
$t_{2\pi}$ reworking time $\pi = 0$ (year)
$t_{3\pi}$ production downtime when $\pi = 0$ (year)
T cycle time if $\pi = 0$ (year)
TC total operating cost per cycle (\$/year)
p_{max} maximum selling price of unit product (\$/unit)
p_{min} minimum selling price of unit product (\$/unit)
ξ_1 scaling parameter of market demand
x portion of repairable defective products randomly produced during the production
$E[x]$ expected value of x
TEP total expected profit (\$/cycle)

3.3. Assumptions

The model is formulated with the following assumptions.

1) A smart production model with defective items is used. Rework is performed within the same cycle as production. Rework is possible only with additional costs. The defect production rate is random and among the defective items, only repairable items are reworked (Biswas & Sarker [21]). Repaired products have the same quality as manufactured products.

2) A fixed portion π of the optimal lot size quantity Q $(0 < \pi < 1)$ is outsourced, i.e., partial outsourcing is considered here. Outsourced products are the same quality as manufactured products. The outsourced products are delivered to the market right after receiving those products.
3) Here, market demand is variable and selling price-dependent (Bhunia & Shaikh [36]). The demand is considered as $D = \xi_1 \frac{(p_{max}-p)}{(p-p_{min})}$.
4) Consider a unit production cost, which is a function of the variable production rate. From the function of the unit production cost, it is observed that the raw material cost is fixed, the development cost is inversely proportional to the variable production rate, and the tool/die cost is directly proportional to the variable production rate. The unit production cost function is $C(P) = (\gamma_1 + \frac{\gamma_2}{P} + \gamma_3 P)$ (Kim & Glock [6], Bhuniya et al. [39]).
5) In the proposed model economic dimension has been included by optimizing the profit function. The social dimension has been focused by considering labor cost in tool/die cost. Environmental dimension has been studied in this model by considering carbon emissions cost in-house setup formation(K^c), holding perfect product (h^c), holding reworked products (h_1^c), and reworking defective products (C_R^c).

4. Model formulation

In this section, different costs are considered for formulating the proposed model in detail. The manufacturer produces a single type of product using a smart production system. But, defective products are produced from the system at a random rate. At the end of the in-house production cycle, the reworking of faulty products begins. To avoid shortage, π $(0 < \pi < 1)$ portion of the production lot size is outsourced. The manufacturer assumes that outsourced products are perfect as manufactured products. If $\pi = 1$, the proposed system will be fully outsourced. If $\pi = 0$, the system will be an in-house production system (Figure 1). Defective products produce at a random rate d. From Figure 1, the following formulas can be obtained.

The level of perfect-quality on-hand inventory after the completion of in-house production is obtained by subtracting the defective rate and demand rate of products from the production rate using the following formula

$$H_1 = (P - d - D)t_{1\pi}. \tag{4.1}$$

The level of the perfect-quality on-hand inventory when the reworking process ends is obtained by the sum of H_1 and the remaining reworked items that cover the market demand in parallel. The inventory becomes

$$H_2 = H_1 + (P_1 - D)t_{2\pi}. \tag{4.2}$$

The maximum level of perfect-quality on-hand inventory when the outsourcing items are received is obtained by the sum of H_2 with the outsourcing products, using the following formula

$$H = H_2 + \pi Q = Dt_{3\pi}. \tag{4.3}$$

The following time indicates the production uptime $t_{1\pi}$, reworking time $t_{2\pi}$, and production downtime $t_{3\pi}$ when the outsourced products continue to fulfill the market demand. In addition to the relationship with the perfect quality inventory, the reworking item inventory is presented here. Thus, the

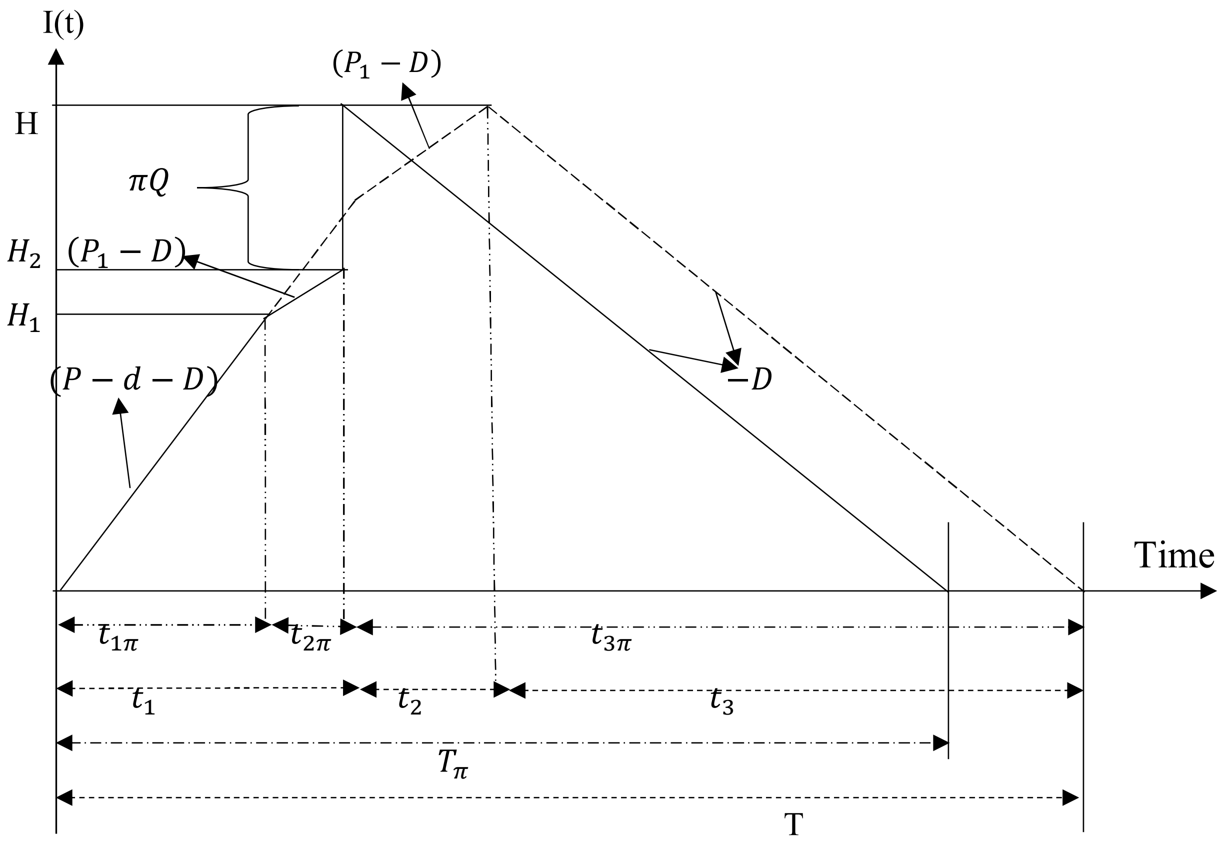

Figure 1. Inventory position of the proposed system with partial product outsourcing (solid line) versus inventory position of the system without a partial product outsourcing strategy (dotted line) [24].

required formula is as follows:

$$t_{1\pi} = \frac{H_1}{(P - d - D)} = \frac{(1 - \pi)Q}{P}, \tag{4.4}$$

$$t_{2\pi} = \frac{x[(1 - \pi)Q]}{P_1}, \tag{4.5}$$

$$t_{3\pi} = \frac{H}{D} = \frac{H_2 + \pi Q}{D}. \tag{4.6}$$

The cycle time is the sum of the perfect-quality item production time, which is known as the production uptime, reworking of defective products, and production downtime time. In general, the cycle time is calculated by dividing the number of lot sizes by market demand. Hence, the cycle time and repairable defective products are considered as follows:

$$T_\pi = t_{1\pi} + t_{2\pi} + t_{3\pi} = \frac{Q}{D}, \tag{4.7}$$

$$dt_{1\pi} = xPt_{1\pi} = x[(1 - \pi)Q]. \tag{4.8}$$

Associative costs are given below.

4.1. Carbon emissions cost

The cost associated with the reduction of carbon emission during the production of any item is known as carbon emission cost. This model considers carbon emission cost due to in-house setup formation(K^c), holding the product per holding (h^c), holding reworked products per holding (h_1^c), and reworking the product (C_R^c).

4.2. Production setup cost (PSC)

Through investment in the production setup, machines can be prepared for smart production to processing the different batches of products. This is one of the basic costs of starting a business and running it efficiently. Depending on the setup, the production process improves rapidly. Here, a fixed setup cost is considered as a combination of setup cost and carbon emissions cost due to in-house setup formation as

$$PSC = K + K^c. \tag{4.9}$$

4.3. Variable production cost (VPC)

A variable production rate is considered for the smart production system. The unit production cost of products depends upon the production rate. The unit production cost depends upon the tool/die, development, and raw material costs. This production cost is applicable for the in-house batch size of the total production lot size.

$$VPC = (\gamma_1 + \frac{\gamma_2}{P} + \gamma_3 P)(1 - \pi)Q. \tag{4.10}$$

4.4. Fixed outsourcing cost (FOC)

π percentage of the total production batch size is outsourced. In this production model, two types of outsourcing costs are considered: fixed and variable. The fixed outsourcing cost is related to the in-house setup cost, linking with a negative parameter β_1 ($-1 \leq \beta_1 \leq 0$), which is less than unity. This implies that the fixed outsourcing cost is less than the in-house setup cost. Therefore, the fixed outsourcing cost is

$$FOC = K_\pi = (1 + \beta_1)(K + K^c). \tag{4.11}$$

4.5. Variable outsourcing cost (VOC)

Variable outsourcing cost is related to the unit production cost by linking the positive variable β_2 ($\beta_2 \geq 0$). This implies that the unit outsourcing cost is more than the unit production cost. That is, the manufacturer pays more cost for outsourcing products than producing a similar amount of in-house products. The variable outsourcing cost is

$$VOC = C_\pi(\pi Q) = (1 + \beta_2)C(P)(\pi Q). \tag{4.12}$$

4.6. Reworking cost (RC)

x percentage of products (in-house) is defective and the rate x is random. As these defective products are reworkable, reworking is used to make them similar quality as new products. All reworkable products are represented by $x[(1-\pi)Q]$. Reworking cost includes carbon emission cost from the reworking process and is expressed as

$$RC = (C_R + C_R^c)x[(1-\pi)Q]. \tag{4.13}$$

4.7. Holding cost of the reworked products (HCR)

All unsold products are stored through this type of investment. Here, two different holding costs are calculated: for reworked products and total in-house produced products. During the time interval $t_{2\pi}$, only the rework is completed. Thus, the holding cost of the reworked goods includes the holding cost of each reworked product and the carbon emission cost for holding reworked products. This can be expressed as

$$HCR = (h_1 + h_1^c)\frac{dt_{1\pi}}{2}(t_{2\pi}). \tag{4.14}$$

4.8. Holding cost for perfect and defective products (HCPD)

All in-house products have the same unit holding cost. The total time interval is separated into three parts: $t_{1\pi}$, $t_{2\pi}$, and $t_{3\pi}$. Therefore, holding cost for perfect quality and defective items include holding cost and corresponding carbon emissions cost for holding those products. Thus, the holding cost is

$$HCPD = (h + h^c)\left[\frac{H_1 + dt_{1\pi}}{2}(t_{1\pi}) + \frac{H_1 + H_2}{2}(t_{2\pi}) + \frac{H}{2}(t_{3\pi})\right]. \tag{4.15}$$

4.9. Total cost (TC)

The total cost for this system $TC(P, Q, p)$ includes the setup cost, variable production cost, fixed-type outsourcing cost, variable-type outsourcing cost, reworking cost, holding cost for reworked goods, holding cost for in-house products in $t_{1\pi}, t_{2\pi}$, and $t_{3\pi}$. Therefore, $TC(P, Q, p)$ is given as follows:

$$\begin{aligned} TC(P,Q,p) &= (PSC + VPC + FOC + VOC + RC + HCR + HCPD) \\ &= (K + K^c) + \left(\gamma_1 + \frac{\gamma_2}{P} + \gamma_3 P\right)(1-\pi)Q + K_\pi + C_\pi(\pi Q) + (C_R + C_R^c)x[(1-\pi)Q] \\ &+ (h_1 + h_1^c)\frac{dt_{1\pi}}{2}(t_{2\pi}) + (h + h^c)\left[\frac{H_1 + dt_{1\pi}}{2}(t_{1\pi}) + \frac{H_1 + H_2}{2}(t_{2\pi}) + \frac{H}{2}(t_{3\pi})\right]. \end{aligned} \tag{4.16}$$

By substituting K_π and C_π in Eq (4.16), the total cost per cycle, $TC(P, Q, p)$, can be calculated as follows:

$$\begin{aligned} TC(P,Q,p) &= (K + K^c) + \left(\gamma_1 + \frac{\gamma_2}{P} + \gamma_3 P\right)(1-\pi)Q + (K + K^c)(1+\beta_1) \\ &+ (1+\beta_2)\left(\gamma_1 + \frac{\gamma_2}{P} + \gamma_3 P\right)(\pi Q) + (C_R + C_R^c)x[(1-\pi)Q] \\ &+ (h_1 + h_1^c)\frac{dt_{1\pi}}{2}(t_{2\pi}) + (h + h^c)\left[\frac{H_1 + dt_{1\pi}}{2}(t_{1\pi}) + \frac{H_1 + H_2}{2}(t_{2\pi}) + \frac{H}{2}(t_{3\pi})\right]. \end{aligned}$$

The expected value of random defective rate x is $E[x] = \zeta$. Then, the expected total cost per cycle $E[TCU(P, Q, p)]$ is expressed as

$$E[TCU(P,Q,p)] = \frac{E[TC(Q,P,p)]}{E[T_\pi]}$$

$$
\begin{aligned}
&= \frac{D}{Q}\Big[(K+K^c)+Q\Big(\gamma_1+\frac{\gamma_2}{P}+\gamma_3 P\Big)(1-\pi)+(K+K^c)(1+\beta_1)\\
&+ Q\pi(1+\beta_2)\Big(\gamma_1+\frac{\gamma_2}{P}+\gamma_3 P\Big)+Q(1-\pi)\zeta(C_R+C_R^c)\\
&+ \frac{Q^2((h_1+h_1^c)-(h+h^c))}{2}\Big(\frac{\zeta^2(1-\pi)^2}{P_1}\Big)\\
&+ \frac{(h+h^c)Q^2}{2}\Big(\frac{1}{D}-\Big(\frac{1-\pi^2}{P}\Big)+\frac{\zeta(1-\pi)}{P_1}(-2\pi)\Big)\Big]\\
&= \frac{1}{Q}\Big(\xi_1\frac{(p_{max}-p)}{(p-p_{min})}\Big)\Big[(K+K^c)+Q\Big(\gamma_1+\frac{\gamma_2}{P}+\gamma_3 P\Big)(1-\pi)+(K+K^c)(1+\beta_1)\\
&+ Q\pi(1+\beta_2)(\gamma_1+\frac{\gamma_2}{P}+\gamma_3 P)+Q(1-\pi)\zeta(C_R+C_R^c)\\
&+ \frac{Q^2((h_1+h_1^c)-(h+h^c))}{2}\Big(\frac{\zeta^2(1-\pi)^2}{P_1}\Big)\\
&+ \frac{(h+h^c)Q^2}{2}\Big(\frac{1}{D}-\Big(\frac{1-\pi^2}{P}\Big)+\frac{\zeta(1-\pi)}{P_1}(-2\pi)\Big)\Big]. \qquad (4.17)
\end{aligned}
$$

4.10. *Total expected profit (TEP)*

The revenue is calculated as follows: Revenue = pD. Thus, the total expected profit is

$$
\begin{aligned}
TEP(P,Q,p) &= \text{Revenue}-\frac{1}{Q}\Big(\xi_1\frac{(p_{max}-p)}{(p-p_{min})}\Big)\Big[(K+K^c)+Q\Big(\gamma_1+\frac{\gamma_2}{P}+\gamma_3 P\Big)(1-\pi)\\
&+ Q(K+K^c)(1+\beta_1)+\pi(1+\beta_2)\Big(\gamma_1+\frac{\gamma_2}{P}+\gamma_3 P\Big)+Q(1-\pi)\zeta(C_R+C_R^c)\\
&+ \frac{Q^2((h_1+h_1^c)-(h+h^c))}{2}\Big(\frac{\zeta^2(1-\pi)^2}{P_1}\Big)+\frac{(h+h^c)Q^2}{2}\Big(\frac{1}{D}-(\frac{1-\pi^2}{P})+\frac{\zeta(1-\pi)}{P_1}(-2\pi)\Big)\Big]\\
&= p\Big(\xi_1\frac{(p_{max}-p)}{(p-p_{min})}\Big)-\frac{1}{Q}\Big(\xi_1\frac{(p_{max}-p)}{(p-p_{min})}\Big)\Big[(K+K^c)+Q\Big(\gamma_1+\frac{\gamma_2}{P}+\gamma_3 P\Big)(1-\pi)\\
&+ (K+K^c)(1+\beta_1)+Q\pi(1+\beta_2)\Big(\gamma_1+\frac{\gamma_2}{P}+\gamma_3 P\Big)+Q(1-\pi)\zeta(C_R+C_R^c)\\
&+ \frac{Q^2((h_1+h_1^c)-(h+h^c))}{2}\Big(\frac{\zeta^2(1-\pi)^2}{P_1}\Big)\\
&+ \frac{(h+h^c)Q^2}{2}\Big(\frac{1}{D}-\Big(\frac{1-\pi^2}{P}\Big)\frac{\zeta(1-\pi)}{P_1}(-2\pi)\Big)\Big]. \qquad (4.18)
\end{aligned}
$$

5. Solution methodology

To solve the mathematical model analytically, the classical optimization method is considered. The decision variables P, Q, and p are optimized. Because there are multiple decision variables, the Hessian matrix is used to test the global optimum of the solution. First, the total expected profit (Eq (4.18)) is partially differentiated with respect to the decision variables. Then make the first-order derivatives equal to zero. Thus, the values of decision variables P^*, Q^*, and p^* are as follows:

$$
P^* = \sqrt{\frac{2\gamma_2(1-\pi)+2\pi(1+\beta_2)\gamma_2-(h+h^c)Q(1-\pi^2)}{\gamma_3(1-\pi)+\pi\gamma_3(1+\beta_2)}},
$$

$$Q^* = \frac{\Psi - \left(\xi_1 \frac{(p_{max}-p)}{(p-p_{min})}\right)\left[(\gamma_1 + \frac{\gamma_2}{P} + \gamma_3 P)(1-\pi) + \pi(1+\beta_2)(\gamma_1 + \frac{\gamma_2}{P} + \gamma_3 P) + (1-\pi)\zeta(C_R + C_R^c)\right]}{\left(\xi_1 \frac{(p_{max}-p)}{(p-p_{min})}\right)\left[Q((h_1+h_1^c) - (h+h^c))\left(\frac{\zeta^2(1-\pi)^2}{P_1}\right) + (h+h^c)Q\left(\frac{1}{D} - (\frac{1-\pi^2}{P}) + \frac{\zeta(1-\pi)}{P_1}(-2\pi)\right)\right]},$$

$$p^* = \frac{\sqrt{\Upsilon^2 - 4\zeta p^2 \xi_1\left[1 - \frac{(h+h^c)\xi_1 Q}{D^2}\frac{(p_{min}-p_{max})}{(p-p_{min})^2}\right]} - \Upsilon}{2p^2(\xi_1)\left[1 - \frac{(h+h^c)\xi_1 Q}{D^2}\frac{(p_{min}-p_{max})}{(p-p_{min})^2}\right]}. \tag{5.1}$$

(See Appendix A for the calculations of the first-order derivatives.)

Here, sufficient conditions of the optimum results of the decision variables are shown. To prove the global optimality and satisfy sufficient conditions, some propositions are utilized. The propositions are as follows:

Proposition 1. The first-order principal minor of the Hessian matrix at P^*, Q^*, p^* of the total expected profit function is less than zero if $2\gamma_2(1-\pi) + 2(1+\beta_2)\gamma_2 > hQ(1-\pi^2)$.

Proof. Please see Appendices B and C.

Proposition 2. The second-order principal minor of the Hessian matrix at optimum values P^*, Q^*, and p^* of the total expected profit function is greater than zero if $\psi\chi > \sigma^2$.

Proof. Please see Appendices B and D.

Proposition 3. The third-order principal minor of the Hessian matrix at P^*, Q^*, and p^* of the total expected profit function is less than zero if $\psi(\chi\varphi - \vartheta^2) + \rho(\sigma\vartheta - \rho\chi) < \sigma(\sigma\varphi - \rho\vartheta)$.

Proof. Please see Appendices B and E.

Proposition 4. The total expected profit function is a convex function at optimum values P^*, Q^*, and p^* if $\psi < 0$, $\psi\chi > \sigma^2$, and $\psi(\chi\varphi - \vartheta^2) + \rho(\sigma\vartheta - \rho\chi) < \sigma(\sigma\varphi - \rho\vartheta)$.

Proof. Please see Appendices B–E.

6. Numerical experiment

Different examples are proposed here to validate the developed mathematical model. Using Mathematica 11.3.0 and the parametric values from Chiu et al. [1], the optimum outcomes are obtained and verified.

6.1. Example 1

The following input parameter values are considered to illustrate the numerical example. Here, K = 4,998 (\$/setup); K^c=2 (\$/setup); $\gamma_1 = 320$; $\gamma_2 = 11{,}910$; $\gamma_3 = 0.009$; $C_R = 48$ (\$/unit); C_R^c= 2 (\$/unit); β_1 = -0.3; $\beta_2 = 0.3$; $p_{max} = 900$ (\$/unit); $p_{min} = 400$ (\$/unit); $\xi_1 = 20$; $h_1 = 25$ (\$/unit/unit time); h_1^c=0.01(\$/unit/unit time); $h = 15$ (\$/unit/unit time); h^c= 0.9(\$/unit/unit time); $E[x] = 0.2$; $\pi = 0.05$; and $P_1 = 110$ (units/unit time).

The optimal results of the decision variable are as follows: P^* = 1,139.50 (unit/unit time); Q^* = 188.25 (unit/cycle); p^* = 594.26 (\$/unit), and TEP = 2,242.59 (\$/cycle).

Now, the optimality of the result is checked numerically. Here, H_{11} = -0.000631505 < 0, H_{22} = 0.0000506551 > 0, and H_{33} = -7.6019×10^{-6} < 0. Figure 2 provides a graphical representation. The concave 3D figure graphically supports the optimal results of TEP.

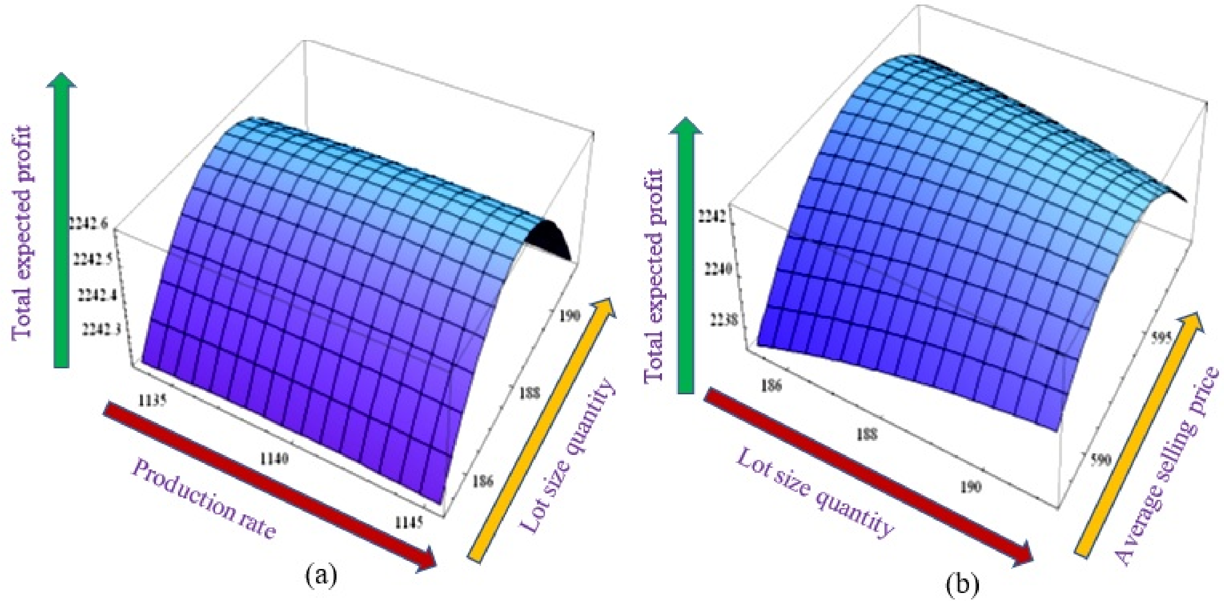

Figure 2. (a) TEP versus variable production rate and lot size quantity. (b) TEP versus lot size quantity and selling price.

6.2. Example 2

Another example is provided to test the model. The following input parametric values are considered to illustrate the numerical example. K = 498 ($/setup); K^c=2 ($/setup); γ_1 = 320 ; γ_2 = 900 ; γ_3 = 0.02 ; C_R = 98 ($/unit); C_R^c= 2 ($/unit); β_1 = -0.3; β_2 = 0.3; p_{max} = 900 ($/unit); p_{min} = 400 ($/unit); ξ_1 = 10; h_1 = 8 ($/unit/unit time); h_1^c=1 ($/unit/unit time); h = 0.09 ($/unit/unit time); h^c=0.01 ($/unit/unit time); $E[x]$ = 0.62; π = 0.05, and P_1 = 50 (units/year).

The optimal results of the decision variables are as follows: P^* = 211.17 (unit/unit time); Q^* = 165.69 (unit/cycle); p^* = 435.42 ($/unit), and at this optimal value, TEP = 4,308.34 ($/cycle).

The optimality of the results is checked numerically too. Here, H_{11} = -0.0252149 < 0, H_{22} = 0.0012359 > 0, and H_{33} = -0.000697724 < 0. Concave 3D Figure 3 graphically supports the optimal results of TEP.

6.3. Special observations

In this section, some special observations are described based on the proposed research. The results and comparative studies validate the research.

6.3.1. Fixed production rate

A special observation is made for the TEP for a fixed production rate instead of a VPR. Keeping the parametric values of Example 1 fixed and using the fixed production rate of P = 200 units per unit time, the optimum outcomes are Q^* = 139.52 (unit/cycle), p^* = 667.03 ($/unit), and at this optimal value, TEP = 1,061.87 ($/year). Here the TEP is less than the original model. Besides, the selling price increases for the fixed production rate. That is, more selling price with less production rate provides less profit. In both cases of production rate and selling price, the smart production system is

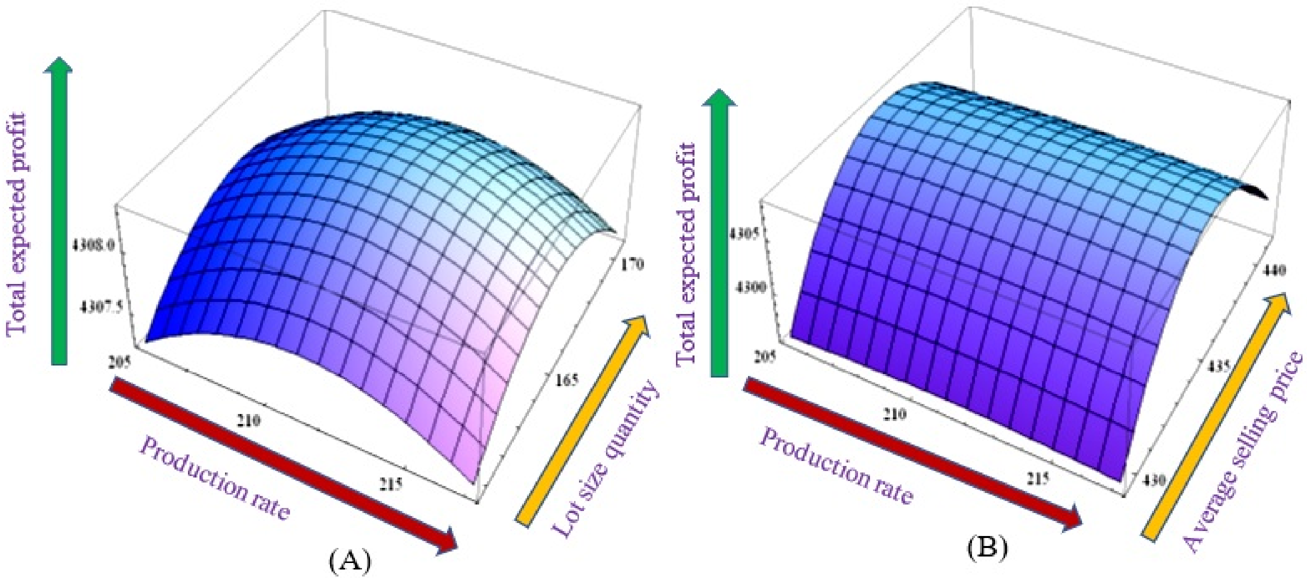

Figure 3. (a) TEP versus variable production rate and production lot size quantity. (b) TEP versus variable production rate and selling price.

better than a traditional production system with a constant production rate. Here, a statistical analysis is considered to achieve the results with confidence for the fixed production rate. The values of the principal minor are such that $H_{11} = -0.109231 < 0$, $H_{22} = +0.0122797 > 0$. Hence, the TEP is a global maximum, as the values of the Hessian at the optimal values of the decision variables are alternate in sign.

6.3.2. Fixed demand and selling price

Another special observation is made for the TEP as a fixed selling price and demand instead of a variable. Keeping the same parametric values of Example 1 and using the fixed selling price of $p = 500$ units per unit and demand $D = 500$ unit per unit time, the optimum outcomes are $P^* = 1{,}129.69$ (unit/unit time), $q^* = 356.74$ (unit/cycle), and at this optimal value, $TEP = 1{,}963.49$ (\$/cycle). It is seen that even if the selling price is more than the variable demand and selling price, the total expected profit is less than the variable demand and selling price. This implies that smart production is more valuable when the market has a variable demand and the manufacturer decides the selling price. Hence, it may be concluded that variable selling price and variable demand are better for the smart production system than fixed demand and selling price. The values of the principal minors are $H_{11} = -0.00202357 < 0$ and $H_{22} = +0.0000757578 > 0$. Hence, the TEP is a global maximum, as the values of the Hessian at the optimal values of the decision variables are alternate in sign.

6.3.3. Discussions

From the above numerical experiment and their special observation, it can be concluded that the TEP is the maximum for the originally proposed model. All special observations are numerically expressed using Mathematica 11.3.0 software. Figure 4 shows the total expected profit for different special cases. Comparing the results of the special cases and from Figure 4 we accept the proposed

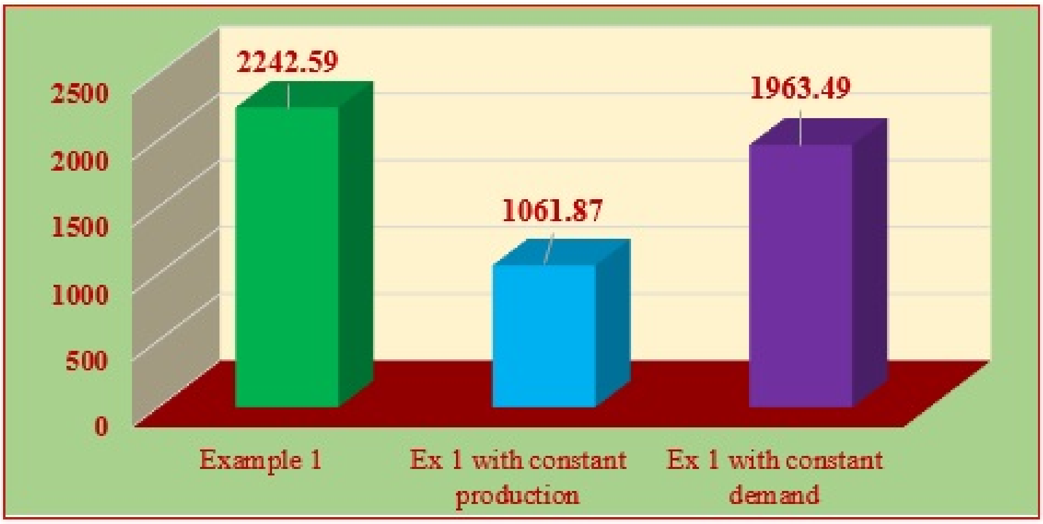

Figure 4. Comparison among the total expected profit of example 1 and its special cases.

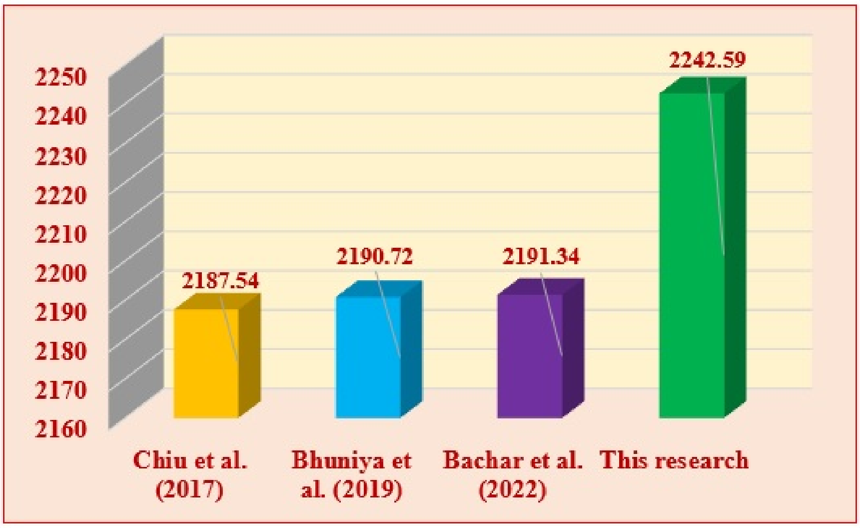

Figure 5. Comparison among the total expected profit of Example 1 and other studies in the literature review.

research of a smart production system under rework, outsourcing, and variable demand is more profitable.

A thorough discussion of this research reveals that variable production rate and variable selling price under variable demand increase the profit margins, shown in the unique case portion. Moreover, it is concluded that the originally proposed model's profit is higher in both cases as compared to the other cases. In the case of variable production rate, the case has shown a higher profit (52.65%) than the fixed production case and a higher profit (12.45%)than the constant demand case. Hence, the special observations help in the validation of the original research on the smart production system.

From the above numerical experiments and their comparison among the previous research articles, it can be concluded that the TEP is the maximum for the proposed model. All cost amounts are numerically expressed using MATHEMATICA 11.3.0 software. Figure 5 shows the comparison among the TEP of Example 1 of the proposed research, Chiu et al. [1], Bhuniya et al. [39], and Bachar et al. [33]. In the research article of Chiu et al. [1], they considered partial outsourcing policies with a constant production rate and constant demand. Their total profit was $ 2187.54 per cycle. In addition to the previously stated research, the partial outsourcing concept of Bachar et al. [33] gives a total profit $2191.34 per cycle, and Bhuniya et al. [39] give a total profit $2190.72 per cycle. In comparison to this previous research, the proposed model Example 1 gives a total profit $2242.59 per cycle.

7. Sensitivity analysis

Significant observations for the costs and scaling parameters are numerically calculated, and the effects of the changes in these parameters are presented in Table 2 and in Figure 6.

Table 2 shows how the costs and scaling parameters affect the TEP owing to changes such as (−50%, −25%, +25%, +50%). From the following sensitivity table, the following conclusions can be made.

Table 2. Sensitivity analysis table.

Parameters	Changes (%)	TEP (%)	Parameters	Changes (%)	TEP (%)
	-50%	–		-50%	+06.44
	-25%	+05.20		-25%	+02.95
γ_2	+25%	–	γ_3	+25%	-02.61
	+50%	-7.34		+50%	-04.96
	-50%	+06.36		-50%	–
	-25%	+03.16		-25%	–
C_R	+25%	-03.14	ξ_1	+25%	+25.20
	+50%	-06.26		+50%	+49.51
	-50%	+25.99		-50%	+10.39
	-25%	+11.19		-25%	+05.05
h	+25%	-09.04	K	+25%	-49.89
	+50%	-16.67		+50%	-51.10

− Not found

1) The sensitivity table clearly shows that the scaling parameter of the demand function strongly affects the TEP. A decrease in the value of the scaling parameter of the demand function is not applicable, but an increase in its value increases TEP.

2) Next most important parameter is the unit holding cost of in-house products. It has inverse properties that correspond to the profit of the system. An increase in the value of the unit holding cost decreases TEP, and a decrease in its value increases TEP. Decreasing the value of this holding cost is more profitable than increasing the cost.

3) Smart production setup cost is the third most important parameter for the total profit. The sensitivity table demonstrates that an increase in the value of the in-house setup cost decreases TEP, and a decrease in its value increases TEP. Increasing the value of setup cost by 50% causes more than 50% profit reduction of the system.

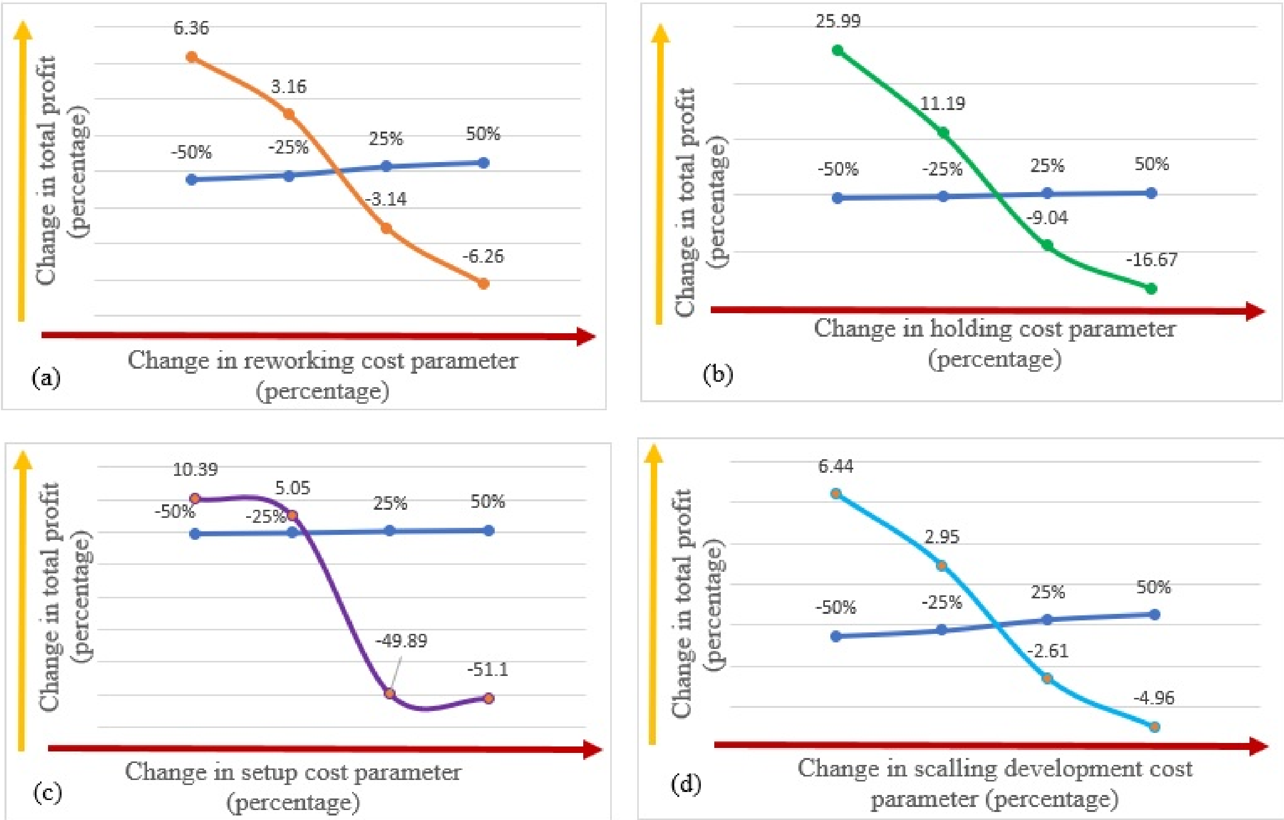

Figure 6. Effects of changes in parametric values versus total expected profit.

4) The unit reworking cost is the next sensitive parameter for the system. An increase in its value decreases TEP, and a decrease in its value increases TEP. Increasing and decreasing reworking costs has almost a similar impact.

5) The scaling parameters, which indicate the development cost, have a discrete type relationship with TEP. However, the table clearly shows that an increase in value sometimes decreases TEP and sometimes has no effect. To put it another way, a decrease in the value of the scaling parameters sometimes increases TEP and sometimes has no effect.

8. Managerial insights

The proposed model gives some valuable recommendations through analytical expression and numerical discussion. The manager of any industry can make several significant decisions based on the current study to optimize the cost/profit of the industry. To support the research managerial insights are generated from the analytical as well as numerical results. Further discussion involving the industrial practice with the considerations of purely in-house production, purchasing system, and partial outsourcing are presented here.

8.1. Managerial insights from analytic results

The analysis of the theoretical model and the significance of analytical results for any decision is important to make decisions. Industry managers may obtain more benefits through several findings of the research articles. Some of the recommendations, obtained through the analytic findings of this study, are stated in the following discussions.

8.1.1. Purely in-house production for a traditional production system

If anyone considers the purely in-house production system for constant production, then $\pi = 0$. Then, the new system is a constant production system with no outsourcing and constant selling price. TEP of this system is as follows.

$$\begin{aligned} TEP(Q) &= p\Big(\xi_1\frac{(p_{max}-p)}{(p-p_{min})}\Big) - \frac{1}{Q}\Big(\xi_1\frac{(p_{max}-p)}{(p-p_{min})}\Big)\Big[(K+K^c) + Q\Big(\gamma_1 + \frac{\gamma_2}{P} + \gamma_3 P\Big) \\ &+ (K+K^c)(1+\beta_1) + Q\zeta(C_R + C_R^c) + \frac{Q^2((h_1+h_1^c)-(h+h^c))}{2}\Big(\frac{\zeta^2}{P_1}\Big) \\ &+ \frac{(h+h^c)Q^2}{2}\Big(\frac{1}{D}-\frac{1}{P}\Big)\Big]. \end{aligned} \tag{8.1}$$

In this case, the total profit is \$1879.37 per cycle. The profit is less than the proposed outsourcing policy (Example 1). This implies that reworking strategy can satisfy demand but it requires extra cycle time (Figure 1). Due to a long cycle time, only the reworking process is secures less profit than the partial outsourcing policy. The result shows that the partial outsourcing policy within a smart production system is more profitable (16.2%) in a trade-off with only a reworking strategy for a constant production system. On the other hand, a pure in-house system for a constant production system is more profitable (43.5%) for partial outsourcing within a constant production system (Special Observation: 6.3.1). Thus, only reworking is not enough to satisfy the market demand if the manufacturer uses a constant production system. But, if the manufacturer uses a constant production system, a pure in-house production system is more profitable rather than outsourcing.

8.1.2. Purely in-house production under a smart production system

The section considers a purely in-house smart production system, i.e., $\pi = 0$. The TEP function $TEP(P, Q, p)$ is similar as Eq (8.1) as

$$\begin{aligned} TEP(P,Q,p) &= p\Big(\xi_1\frac{(p_{max}-p)}{(p-p_{min})}\Big) - \frac{1}{Q}\Big(\xi_1\frac{(p_{max}-p)}{(p-p_{min})}\Big)\Big[(K+K^c) + Q(\gamma_1 + \frac{\gamma_2}{P} + \gamma_3 P) \\ &+ (K+K^c)(1+\beta_1) + Q\zeta(C_R + C_R^c) + \frac{Q^2((h_1+h_1^c)-(h+h^c))}{2}\Big(\frac{\zeta^2}{P_1}\Big) \\ &+ \frac{(h+h^c)Q^2}{2}\Big(\frac{1}{D}-\frac{1}{P}\Big)\Big]. \end{aligned} \tag{8.2}$$

In this case, the total profit is \$2190.72 per cycle. The concept of purely in-house production under the smart purchasing system gives 16.56% more profit rather than purely in-house constant production planning. Besides, the outsourcing cost is more than the unit production cost. But, the partial outsourcing policy is 2.32% more profitable than this case. This implies that only a reworking policy is efficient to control the market demand under a smart production but, still the proposed outsourcing policy for a smart production is more profitable for the manufacturer.

8.2. Overall recommendations

The following recommendations are provided for the industry.

1) The manager can avoid uncertainties regarding customer issues using a smart production system rather than a traditional production system with constant production rate. Results prove that every combination is profitable but the proposed policy is economically most beneficial.

2) In addition, the fixed outsourcing cost is less than the in-house production setup cost. This implies that the manager can outsource more products with a less setup cost than the smart production to satisfy the market demand as well as customer satisfaction. Management of any company always tries to enhance their goodwill by fulfilling customer satisfaction and maintaining the quality of the product.

3) The manager should focus on the reworking of defective products and should carefully monitor the quality of the outsourced products. No defective products should be outsourced because they could cause a bad reputation for the production system. Reworking of defective products reduces the overall cost of the production system.

4) The manager should ensure that only repairable items are reworked; otherwise, the production system will be hampered with unnecessary work and cost.

5) Outsourcing is considered in our present study. The Impacts of outsourcing on the organization are defined as it reduces the overall cost of the system, adding time to focus on scaling the business, speeding up shipping times, increasing flexibility and adaptability, increasing accuracy in order processing, and boosting customer satisfaction.

9. Conclusions

This model proved that a smart production system is efficient to handle market demand even if it produced defective products. Results proved that partial product outsourcing was one of the most strategic decisions for the management. A smart production system with partial outsourcing was more beneficial but a traditional production system with an in-house production system was more profitable. That is, the adjustment in production rate reduces the system cost, related to production, reworking, and holding. Besides, the production uptime and downtime strategy played an essential role in the production system. The rework of defective items had a significant contribution to satisfy the market demand. It helped to maintain a good reputation of the manufacturer. Through optimizing the decision variables, the target was to determine the maximize the total profit. Numerical tools such as Mathematica 11.3.0 was used for the numerical results and to prove global optimality. Finally, the proposed model outlined a profitable business strategy considering a smart production system with partial outsourcing and rework. This model can further be extended by considering unreliability, transportation [40], single-setup-multiple-unequal-delivery-policy, green technology, preservation technology with promotion and time-dependent deterioration [41, 42]. Production disruptions and different pandemic situations are the most significant limitations of this model. In the present situation, the global business procedure easily fulfills and satisfies customer demand through online or offline shopping systems. Another direction for the development is to incorporate inspection costs and errors during the inspection.

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Conflict of interest

The authors declare no conflict of interest.

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Appendix A

$TEP(.) = TEP(P, Q, p)$

The first-order partial derivatives with respect to the decision variables are as follows:

$$\frac{\partial TEP(.)}{\partial P} = -\frac{1}{Q}\Big(\xi_1\frac{(p_{max}-p)}{(p-p_{min})}\Big)\Big[Q\Big(\frac{-\gamma_2}{P^2}+\gamma_3\Big)(1-\pi)+\pi Q(1+\beta_2)\Big(\frac{-\gamma_2}{P^2}+\gamma_3\Big)+\frac{(h+h^c)Q^2(1-\pi^2)}{2P^2}\Big]$$

By equating the above equation zero, we get

$$0 = -\frac{1}{Q}\Big(\xi_1\frac{(p_{max}-p)}{(p-p_{min})}\Big)\Big[Q\Big(\frac{-\gamma_2}{P^2}+\gamma_3\Big)(1-\pi)+\pi Q(1+\beta_2)\Big(\frac{-\gamma_2}{P^2}+\gamma_3\Big)+\frac{(h+h^c)Q^2(1-\pi^2)}{2P^2}\Big]$$

After simplifying the above equation, we get

$$P^* = \sqrt{\frac{2\gamma_2(1-\pi)+2\pi(1+\beta_2)\gamma_2-(h+h^c)Q(1-\pi^2)}{\gamma_3(1-\pi)+\pi\gamma_3(1+\beta_2)}}$$

$$\begin{aligned}
\frac{\partial TEP(.)}{\partial Q} &= \frac{1}{Q^2}\Big(\xi_1\frac{(p_{max}-p)}{(p-p_{min})}\Big)\Big[(K+K^c)+Q(\gamma_1+\frac{\gamma_2}{P}+\gamma_3 P)(1-\pi)+(K+K^c)(1+\beta_1)\\
&+ Q\pi(1+\beta_2)(\gamma_1+\frac{\gamma_2}{P}+\gamma_3 P)+Q(1-\pi)\zeta(C_R+C_R^c)+\frac{Q^2((h_1+h_1^c)-(h+h^c))}{2}\Big(\frac{\zeta^2(1-\pi)^2}{P_1}\Big)\\
&+ \frac{(h+h^c)Q^2}{2}\Big(\frac{1}{D}-(\frac{1-\pi^2}{P})+\frac{\zeta(1-\pi)}{P_1}(-2\pi)\Big)\Big]\\
&- \frac{1}{Q}\Big(\xi_1\frac{(p_{max}-p)}{(p-p_{min})}\Big)\Big[(\gamma_1+\frac{\gamma_2}{P}+\gamma_3 P)(1-\pi)+\pi(1+\beta_2)(\gamma_1+\frac{\gamma_2}{P}+\gamma_3 P)\\
&+ (1-\pi)\zeta(C_R+C_R^c)+Q((h_1+h_1^c)-(h+h^c))\Big(\frac{\zeta^2(1-\pi)^2}{P_1}\Big)+(h+h^c)Q\Big(\frac{1}{D}-(\frac{1-\pi^2}{P})\\
&+ \frac{\zeta(1-\pi)}{P_1}(-2\pi)\Big)\Big]
\end{aligned}$$

By equating the above equation zero, we obtained

$$0 = \frac{1}{Q^2}\Big(\xi_1\frac{(p_{max}-p)}{(p-p_{min})}\Big)\Big[(K+K^c)+Q(\gamma_1+\frac{\gamma_2}{P}+\gamma_3 P)(1-\pi)+(K+K^c)(1+\beta_1)$$

$$
\begin{aligned}
&+ \quad Q\pi(1+\beta_2)(\gamma_1+\frac{\gamma_2}{P}+\gamma_3 P)+Q(1-\pi)\zeta(C_R+C_R^c)+\frac{Q^2((h_1+h_1^c)-(h+h^c))}{2}\Big(\frac{\zeta^2(1-\pi)^2}{P_1}\Big)\\
&+ \quad \frac{(h+h^c)Q^2}{2}\Big(\frac{1}{D}-(\frac{1-\pi^2}{P})+\frac{\zeta(1-\pi)}{P_1}(-2\pi)\Big)\Big]\\
&- \quad \frac{1}{Q}\Big(\xi_1\frac{(p_{max}-p)}{(p-p_{min})}\Big)\Big[(\gamma_1+\frac{\gamma_2}{P}+\gamma_3 P)(1-\pi)+\pi(1+\beta_2)(\gamma_1+\frac{\gamma_2}{P}+\gamma_3 P)\\
&+ \quad (1-\pi)\zeta(C_R+C_R^c)+Q((h_1+h_1^c)-(h+h^c))\Big(\frac{\zeta^2(1-\pi)^2}{P_1}\Big)+(h+h^c)Q\Big(\frac{1}{D}-(\frac{1-\pi^2}{P})\\
&+ \quad \frac{\zeta(1-\pi)}{P_1}(-2\pi)\Big)\Big]
\end{aligned}
$$

After simplifying the above equation, we get,

$$
Q^* = \frac{\Psi-\Big(\xi_1\frac{(p_{max}-p)}{(p-p_{min})}\Big)\Big[(\gamma_1+\frac{\gamma_2}{P}+\gamma_3 P)(1-\pi)+\pi(1+\beta_2)(\gamma_1+\frac{\gamma_2}{P}+\gamma_3 P)+(1-\pi)\zeta(C_R+C_R^c)\Big]}{\Big(\xi_1\frac{(p_{max}-p)}{(p-p_{min})}\Big)\Big[Q((h_1+h_1^c)-(h+h^c))\Big(\frac{\zeta^2(1-\pi)^2}{P_1}\Big)+(h+h^c)Q\Big(\frac{1}{D}-(\frac{1-\pi^2}{P})+\frac{\zeta(1-\pi)}{P_1}(-2\pi)\Big)\Big]}
$$

$$
\begin{aligned}
\frac{\partial TEP(.)}{\partial p} &= \xi_1\frac{(p_{min}-p_{max})}{(p-p_{min})^2}\Big[p-\frac{1}{Q}\Big\{(K+K^c)+Q(\gamma_1+\frac{\gamma_2}{P}+\gamma_3 P)(1-\pi)+(K+K^c)(1+\beta_1)\\
&+ \quad Q\pi(1+\beta_2)(\gamma_1+\frac{\gamma_2}{P}+\gamma_3 P)+Q(1-\pi)\zeta(C_R+C_R^c)+\frac{Q^2((h_1+h_1^c)-(h+h^c))}{2}\Big(\frac{\zeta^2(1-\pi)^2}{P_1}\Big)\\
&+ \quad \frac{(h+h^c)Q^2}{2}\Big(\frac{1}{D}-(\frac{1-\pi^2}{P})+\frac{\zeta(1-\pi)}{P_1}(-2\pi)\Big)\Big\}\Big]\\
&+ \quad \Big(\xi_1\frac{(p_{max}-p)}{(p-p_{min})}\Big)\Big[1-\frac{(h+h^c)\xi_1 Q}{D^2}\frac{(p_{min}-p_{max})}{(p-p_{min})^2}\Big]
\end{aligned}
$$

After equating the above equation zeo, we get

$$
\begin{aligned}
0 &= \xi_1\frac{(p_{min}-p_{max})}{(p-p_{min})^2}\Big[p-\frac{1}{Q}\Big\{(K+K^c)+Q(\gamma_1+\frac{\gamma_2}{P}+\gamma_3 P)(1-\pi)+(K+K^c)(1+\beta_1)\\
&+ \quad Q\pi(1+\beta_2)(\gamma_1+\frac{\gamma_2}{P}+\gamma_3 P)+Q(1-\pi)\zeta(C_R+C_R^c)+\frac{Q^2((h_1+h_1^c)-(h+h^c))}{2}\Big(\frac{\zeta^2(1-\pi)^2}{P_1}\Big)\\
&+ \quad \frac{(h+h^c)Q^2}{2}\Big(\frac{1}{D}-(\frac{1-\pi^2}{P})+\frac{\zeta(1-\pi)}{P_1}(-2\pi)\Big)\Big\}\Big]\\
&+ \quad \Big(\xi_1\frac{(p_{max}-p)}{(p-p_{min})}\Big)\Big[1-\frac{(h+h^c)\xi_1 Q}{D^2}\frac{(p_{min}-p_{max})}{(p-p_{min})^2}\Big]
\end{aligned}
$$

After simplifying the above equation we obtained

$$
p^* = \frac{\sqrt{\Upsilon^2-4\zeta p^2\xi_1\Big[1-\frac{(h+h^c)\xi_1 Q}{D^2}\frac{(p_{min}-p_{max})}{(p-p_{min})^2}\Big]}-\Upsilon}{2p^2(\xi_1)\Big[1-\frac{(h+h^c)\xi_1 Q}{D^2}\frac{(p_{min}-p_{max})}{(p-p_{min})^2}\Big]}
$$

where

$$\Psi = \frac{1}{Q}\Big(\xi_1\frac{(p_{max}-p)}{(p-p_{min})}\Big)\Big[(K+K^c)+Q(\gamma_1+\frac{\gamma_2}{P}+\gamma_3 P)(1-\pi)+(K+K^c)(1+\beta_1)$$
$$+ Q\pi(1+\beta_2)(\gamma_1+\frac{\gamma_2}{P}+\gamma_3 P)+Q(1-\pi)\zeta(C_R+C_R^c)+\frac{Q^2((h_1+h_1^c)-(h+h^c))}{2}\Big(\frac{\zeta^2(1-\pi)^2}{P_1}\Big)$$
$$+ \frac{(h+h^c)Q^2}{2}\Big(\frac{1}{D}-(\frac{1-\pi^2}{P})+\frac{\zeta(1-\pi)}{P_1}(-2\pi)\Big)\Big]$$

and

$$\Upsilon = p\xi_1(p_{min}-p_{max})+p\xi_1(p_{max}+p_{min})\Big[1-\frac{(h+h^c)\xi_1 Q}{D^2}\frac{(p_{min}-p_{max})}{(p-p_{min})^2}\Big]$$
$$\zeta = \xi_1(p_{min}-p_{max})\Psi+p_{min}\xi_1 p_{max}\Big[1-\frac{(h+h^c)\xi_1 Q}{D^2}\frac{(p_{min}-p_{max})}{(p-p_{min})^2}\Big]$$

Appendix B

The second-order partial derivatives with respect to the decision variables are as follows:

$$\frac{\partial^2 TEP(.)}{\partial P^2} = -\frac{\Big(\xi_1\frac{(p_{max}-p)}{(p-p_{min})}\Big)}{P^3}\Big[2\gamma_2(1-\pi)+2(1+\beta_2)\gamma_2-(h+h^c)Q(1-\pi^2)\Big] = \psi(say)$$

$$\frac{\partial^2 TEP(.)}{\partial Q^2} = -\frac{2}{Q^3}\Big(\xi_1\frac{(p_{max}-p)}{(p-p_{min})}\Big)\Big[(K+K^c)+Q(\gamma_1+\frac{\gamma_2}{P}+\gamma_3 P)(1-\pi)+(K+K^c)(1+\beta_1)$$
$$+ Q\pi(1+\beta_2)(\gamma_1+\frac{\gamma_2}{P}+\gamma_3 P)+Q(1-\pi)\zeta(C_R+C_R^c)+\frac{Q^2((h_1+h_1^c)-(h+h^c))}{2}\Big(\frac{\zeta^2(1-\pi)^2}{P_1}\Big)$$
$$+ \frac{(h+h^c)Q^2}{2}\Big(\frac{1}{D}-(\frac{1-\pi^2}{P})+\frac{\zeta(1-\pi)}{P_1}(-2\pi)\Big)\Big]$$
$$+ \frac{2\Big(\xi_1\frac{(p_{max}-p)}{(p-p_{min})}\Big)}{Q^2}\Big[(\gamma_1+\frac{\gamma_2}{P}+\gamma_3 P)(1-\pi)+\pi(1+\beta_2)(\gamma_1+\frac{\gamma_2}{P}+\gamma_3 P)$$
$$+ (1-\pi)\zeta(C_R+C_R^c)+Q((h_1+h_1^c)-(h+h^c))\Big(\frac{\zeta^2(1-\pi)^2}{P_1}\Big)+(h+h^c)Q\Big(\frac{1}{D}-(\frac{1-\pi^2}{P})$$
$$+ \frac{\zeta(1-\pi)}{P_1}(-2\pi)\Big)\Big]-\frac{\Big(\xi_1\frac{(p_{max}-p)}{(p-p_{min})}\Big)}{Q}\Big[\frac{((h_1+h_1^c)-(h+h^c))\zeta^2(1-\pi)^2}{P_1}+(h+h^c)\Big(\frac{1}{D}-\frac{(1-\pi^2)}{P}$$
$$+ \frac{\zeta(1-\pi)}{P_1}(-2\pi)\Big)\Big] = \chi(say)$$

$$\frac{\partial^2 TEP(.)}{\partial p^2} = -2\xi_1\frac{(p_{min}-p_{max})}{(p-p_{min})^3}\Big[p-\frac{1}{Q}\Big\{(K+K^c)+Q(\gamma_1+\frac{\gamma_2}{P}+\gamma_3 P)(1-\pi)+(K+K^c)(1+\beta_1)$$
$$+ Q\pi(1+\beta_2)(\gamma_1+\frac{\gamma_2}{P}+\gamma_3 P)+Q(1-\pi)\zeta(C_R+C_R^c)+\frac{Q^2((h_1+h_1^c)-(h+h^c))}{2}\Big(\frac{\zeta^2(1-\pi)^2}{P_1}\Big)$$

$$
\begin{aligned}
&+ \quad \frac{(h+h^c)Q^2}{2}\Big(\frac{1}{D} - (\frac{1-\pi^2}{P}) + \frac{\zeta(1-\pi)}{P_1}(-2\pi)\Big)\Big\}\Big]\\
&+ \quad 2\xi_1 \frac{(p_{min}-p_{max})}{(p-p_{min})^2}\Big[1 - \frac{(h+h^c)\xi_1 Q}{D^2}\frac{(p_{min}-p_{max})}{(p-p_{min})^2}\Big]\\
&+ \quad \Big(\xi_1 \frac{(p_{max}-p)}{(p-p_{min})}\Big)\frac{2(h+h^c)\xi_1 Q(p_{min}-p_{max})}{D^2(p-p_{min})^3} = \varphi(say)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 TEP(.)}{\partial P \partial Q} &= \frac{\partial^2 TEP(.)}{\partial Q \partial P} = \frac{1}{Q^2}\Big(\xi_1 \frac{(p_{max}-p)}{(p-p_{min})}\Big)\Big[Q\Big(\frac{-\gamma_2}{P^2} + \gamma_3\Big)(1-\pi)\\
&+ \quad \pi Q(1+\beta_2)\Big(\frac{-\gamma_2}{P^2} + \gamma_3\Big) + \frac{(h+h^c)Q^2(1-\pi^2)}{2P^2}\Big] - \frac{1}{Q}\Big(\xi_1 \frac{(p_{max}-p)}{(p-p_{min})}\Big)\Big[\Big(\frac{-\gamma_2}{P^2} + \gamma_3\Big)(1-\pi)\\
&+ \quad \pi(1+\beta_2)\Big(\frac{-\gamma_2}{P^2} + \gamma_3\Big) + \frac{(h+h^c)Q(1-\pi^2)}{P^2}\Big] = \sigma(say)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 TEP(.)}{\partial P \partial p} &= \frac{\partial^2 TEP(.)}{\partial p \partial P} = -\frac{\xi_1}{Q}\frac{(p_{min}-p_{max})}{(p-p_{min})^2}\Big[Q\Big(\frac{-\gamma_2}{P^2} + \gamma_3\Big)(1-\pi)\\
&+ \quad \pi Q(1+\beta_2)\Big(\frac{-\gamma_2}{P^2} + \gamma_3\Big) + \frac{(h+h^c)Q^2(1-\pi^2)}{2P^2}\Big] = \rho(say)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 TEP(.)}{\partial Q \partial p} &= \frac{\partial^2 TEP(.)}{\partial p \partial Q} = \xi_1 \frac{(p_{min}-p_{max})}{(p-p_{min})^2}\frac{1}{Q^2}\Big[(K+K^c) + Q(\gamma_1 + \frac{\gamma_2}{P} + \gamma_3 P)(1-\pi) + (K+K^c)(1+\beta_1)\\
&+ \quad Q\pi(1+\beta_2)(\gamma_1 + \frac{\gamma_2}{P} + \gamma_3 P) + Q(1-\pi)\zeta(C_R + C_R^c) + \frac{Q^2((h_1+h_1^c)-(h+h^c))}{2}\Big(\frac{\zeta^2(1-\pi)^2}{P_1}\Big)\\
&+ \quad \frac{(h+h^c)Q^2}{2}\Big(\frac{1}{D} - (\frac{1-\pi^2}{P}) + \frac{\zeta(1-\pi)}{P_1}(-2\pi)\Big)\Big]\\
&- \quad \frac{\xi_1}{Q}\frac{(p_{min}-p_{max})}{(p-p_{min})^2}\Big[(\gamma_1 + \frac{\gamma_2}{P} + \gamma_3 P)(1-\pi) + \pi(1+\beta_2)(\gamma_1 + \frac{\gamma_2}{P} + \gamma_3 P) + (1-\pi)\zeta(C_R + C_R^c)\\
&+ \quad \frac{((h_1+h_1^c)-(h+h^c))}{2}\Big(\frac{\zeta^2(1-\pi)^2}{P_1}\Big) + (h+h^c)Q\Big(\frac{1}{D} - (\frac{1-\pi^2}{P}) + \frac{\zeta(1-\pi)}{P_1}(-2\pi)\Big)\Big]\\
&- \quad \Big(\xi_1 \frac{(p_{max}-p)}{(p-p_{min})}\Big)\frac{(h+h^c)\xi_1}{D^2}\frac{(p_{min}-p_{max})}{(p-p_{min})^2} = \vartheta(say)
\end{aligned}
$$

Appendix C

$$
\mid H_{11} \mid = \Big| \frac{\partial^2 TEP(.)}{\partial P^2} \Big| = -\frac{\Big(\xi_1 \frac{(p_{max}-p)}{(p-p_{min})}\Big)}{P^3}\Big[2\gamma_2(1-\pi) + 2(1+\beta_2)\gamma_2 - (h+h^c)Q(1-\pi^2)\Big] = \psi
$$

Appendix D

$$
\begin{aligned}
\mid H_{22} \mid = \begin{vmatrix} \frac{\partial^2 TEP(.)}{\partial P^2} & \frac{\partial^2 TEP(.)}{\partial P \partial Q} \\ \frac{\partial^2 TEP(.)}{\partial Q \partial P} & \frac{\partial^2 TEP(.)}{\partial Q^2} \end{vmatrix} &= \frac{\partial^2 TEP(.)}{\partial P^2}\frac{\partial^2 TEP(.)}{\partial Q^2} - \Big(\frac{\partial^2 TEP(.)}{\partial P \partial Q}\Big)^2\\
&= \psi\chi - \sigma^2
\end{aligned}
$$

Appendix E

$$\left|H_{33}\right| = \begin{vmatrix} \frac{\partial^2 TEP(.)}{\partial P^2} & \frac{\partial^2 TEP(.)}{\partial P\partial Q} & \frac{\partial^2 TEP(.)}{\partial P\partial p} \\ \frac{\partial^2 TEP(.)}{\partial Q\partial P} & \frac{\partial^2 TEP(.)}{\partial Q^2} & \frac{\partial^2 TEP(.)}{\partial Q\partial p} \\ \frac{\partial^2 TEP(.)}{\partial p\partial P} & \frac{\partial^2 TEP(.)}{\partial p\partial Q} & \frac{\partial^2 TEP(.)}{\partial p^2} \end{vmatrix} = \psi(\chi\varphi - \vartheta^2) - \sigma(\sigma\varphi - \rho\vartheta) + \rho(\sigma\vartheta - \rho\chi).$$