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Theory article

Global boundedness of a higher-dimensional chemotaxis system on alopecia areata

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Abstract: This paper mainly focuses on the dynamics behavior of a three-component chemotaxis system on alopecia areata

$$\begin{cases} u_t = \Delta u - \chi_1 \nabla\cdot(u\nabla w) + w - \mu_1 u^2, & x\in\Omega, t>0,\\ v_t = \Delta v - \chi_2 \nabla\cdot(v\nabla w) + w + ruv - \mu_2 v^2, & x\in\Omega, t>0,\\ w_t = \Delta w + u + v - w, & x\in\Omega, t>0,\\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0, & x\in\partial\Omega, t>0,\\ u(x,0)=u_0(x),\ v(x,0)=v_0(x),\ w(x,0)=w_0(x), & x\in\Omega, \end{cases}$$

where $\Omega\subset\mathbb{R}^n$ $(n\geq 4)$ is a bounded convex domain with smooth boundary $\partial\Omega$, the parameters χ_i, μ_i $(i=1,2)$, and r are positive. We show that this system exists a globally bounded classical solution if μ_i $(i=1,2)$ is large enough. This result extends the corresponding results which were obtained by Lou and Tao (JDE, 2021) to the higher-dimensional case.

Keywords: chemotaxis; alopecia areata; global existence

1. Introduction

In this paper, we consider the spatio-temporal dynamics of a three-component chemotaxis system

$$\begin{cases} u_t = \Delta u - \chi_1 \nabla\cdot(u\nabla w) + w - \mu_1 u^2, & x\in\Omega, t>0,\\ v_t = \Delta v - \chi_2 \nabla\cdot(v\nabla w) + w + ruv - \mu_2 v^2, & x\in\Omega, t>0,\\ w_t = \Delta w + u + v - w, & x\in\Omega, t>0,\\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0, & x\in\partial\Omega, t>0,\\ u(x,0)=u_0(x),\ v(x,0)=v_0(x),\ w(x,0)=w_0(x), & x\in\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ ($n \geq 4$) is a bounded convex domain with smooth boundary $\partial\Omega$ and ν denotes the outer normal vector to $\partial\Omega$. Here, the parameters χ_i, μ_i ($i = 1, 2$) and r are positive constants. The initial data (u_0, v_0, w_0) satisfies

$$\begin{cases} u_0 \in C(\overline{\Omega}), & u_0 \geq 0 \text{ and } u_0 \not\equiv 0 \text{ in } \overline{\Omega}, \\ v_0 \in C(\overline{\Omega}), & v_0 \geq 0 \text{ and } v_0 \not\equiv 0 \text{ in } \overline{\Omega}, \\ w_0 \in W^{1,\infty}(\Omega), & w_0 \geq 0 \text{ in } \overline{\Omega}. \end{cases} \tag{1.2}$$

This system was originally proposed by Dobreva et al. [1] to describe the complex dynamic behavior of alopecia areata (AA). Alopecia areata is mainly manifested as hair loss, which is caused by the attack of the immune system on the hair follicle. Previous investigations [2, 3] have shown that the development of AA is usually initiated by abnormally high production of pro-inflammatory cytokines, such as interferon-gamma (IFN-γ) which is the most influential inducer of hair follicles immune privilege (HF IP) collapse. IFN-γ is secreted by the two types of immune cells which are CD4$^+$ T cells and CD8$^+$ T cells, and it diffuses and degrades, moreover, it is also the chemoattractant for CD4$^+$ T cells and CD8$^+$ T cells; CD4$^+$ T cells which move randomly are triggered by IFN-γ and decrease on density-dependent death; CD8$^+$ T cells which also move randomly, are triggered by IFN-γ, proliferate with the help of CD4$^+$ T cells, and undergo density-dependent death. Based on the above biological mechanism, in the system (1.1), the unknown functions $u(x, t)$, $v(x, t)$ and $w(x, t)$ respectively denote the density of CD4$^+$ T cells, the density of CD8$^+$ T cells and the concentration of IFN-γ.

From the mathematical perspective, Lou and Tao [4] recently studied global boundedness and the asymptotic stability of the solution for (1.1), when $n = 2$ or $n = 3$ and

$$\mu_1 > 8\chi_1^2 + \frac{r}{2} + 16, \ \mu_2 > 8\chi_2^2 + \frac{r}{2} + 16 \text{ as well as } \mu_1\mu_2^2 > \frac{4}{27}r^3,$$

this system admitted a global boundedness classical solution. Furthermore, as

$$\mu_1 < \mu_2 < 3\mu_1, \ r = \mu_2 - \mu_1 \text{ and } \sqrt{\chi_1^2 + \chi_2^2} < \chi_0,$$

where χ_0 was a positive constant, the classical solution was globally asymptotic stability. Subsequently, Tao and Xu [5] showed the spatio-temporal evolution of IFN-γ describing a quasi-steady-state approximation of the equation, and got the dimensionless parabolic-parabolic-elliptic version of (1.1) by replacing the third equation in (1.1) with $0 = \Delta w + u + v - w$, they proved the global boundedness of its solution when

$$\mu_1 > \frac{(n-2)_+}{n}(2\chi_1 + \frac{\chi_2}{2}) + \frac{r}{2} \text{ and } \mu_2 > \frac{(n-2)_+}{n}(2\chi_2 + \frac{\chi_1}{2}) + r,$$

then the large-time behavior of (1.1) was also gained as

$$\mu_1 < \mu_2 < 3\mu_1, \ r = \mu_2 - \mu_1 \text{ and } \chi_1^2 + \chi_2^2 < 2\mu_1(3\mu_1 - \mu_2).$$

To better understand the system (1.1), the previously existing two-component chemotactic system should be mentioned

$$\begin{cases} u_t = \Delta u - \chi\nabla \cdot (u\nabla v) + f(u), & x \in \Omega, t > 0, \\ v_t = \Delta v + u - v, & x \in \Omega, t > 0. \end{cases} \tag{1.3}$$

Many known results about the system (1.3) have been obtained in past studies. For instance, as $f(u) = 0$, system (1.3) corresponds to the classical KS system [6], its global boundedness has been testified in [7], and the blow-up behavior has also been constructed in a finite or infinite time (cf. [8–10], for instance) which strongly relied on the spatial dimension and the domain Ω. However, the logical damping term $f(u) = \mu u(1-u)$ plays an important role in preventing the blow-up phenomenon. Then, as $n \leq 2$, for an arbitrarily small coefficient $\mu > 0$, the system (1.3) can admit a uniformly bounded classical solution and prevent the blow-up phenomenon [11–13]. Whereas, $n \geq 3$, the global classical solution of (1.3) exists and remains uniformly bounded if the coefficient μ is large enough [14–17]. Moreover, it is worth mentioning that a variant of system (1.3) describes the movement of cells driven in an incompressible fluid under the influence of chemoattractant which is a different but interesting system called chemotaxis-Navier-Stokes system. The analysis of the solution of this system and its variants have attracted wide attention in recent years, such as Winkler [18, 19] and Zheng [20–24]. For a more detailed discussion on (1.3) or its variants, we refer the reader to [8, 25] and the references therein.

Main results and ideas.

Compared with (1.3) or other pre-existing chemotaxis systems [26–28], the main differences come from the terms w and ruv in the system (1.1), it is worth emphasizing that the nonlinear generation term ruv is obviously different from the nonlinear proliferation term appearing in [29] or [30], which is composed of the product of signal concentration and cell density. Inspired by Winkler [17] and Xiang [16], we can deal with the three-component chemotactic system (1.1) by some appropriate improvements for the functional $y(t) := \int_\Omega u^p + \int_\Omega |\nabla v|^{2q}$ for any $p > 1$, $q > 1$ and $t > 0$. Motivated by the above ideas, we shall study the evolution of the combined integral

$$\int_\Omega u^p + \int_\Omega v^p + \int_\Omega w^p + \int_\Omega |\nabla w|^{2q}$$

to address the difficulties from the activated term w and nonlinear production term ruv. We find that the behavior of solution can be impacted by nonlinear diffusion, the nonlinear zero-order production term ruv and logistic damping. Our main results are started as follows.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^n$ ($n = 4, 5$) be a bounded convex domain with smooth boundary, $\chi_i, \mu_i > 0$ ($i = 1, 2$) and $r > 0$. For any $\epsilon \in (0, 1)$, if*

$$\mu_1 > \frac{1}{3}\left[\sqrt{\frac{3}{\epsilon^2}} + \sqrt{\frac{1}{\epsilon}} + \sqrt{\frac{3n}{2\epsilon}} + 2\sqrt{n+8}\left(\sqrt{2} + \frac{\sqrt{2}}{\sqrt{1-\epsilon}}\right)\right]\chi_1 + \left(\frac{\sqrt{2}}{3} + \frac{1}{2}\right)r + 1$$

and

$$\mu_2 > \frac{1}{3}\left[\sqrt{\frac{3}{\epsilon^2}} + \sqrt{\frac{1}{\epsilon}} + \sqrt{\frac{3(n+4)}{2\epsilon}} + 2\sqrt{n+8}\left(\sqrt{2} + \frac{\sqrt{2}}{\sqrt{1-\epsilon}}\right)\right]\chi_2 + \frac{1}{2}r + 1,$$

then for any initial data (u_0, v_0, w_0) satisfying (1.2), then system (1.1) admits a globally uniformly bounded solution, there also exists a constant $K > 0$ such that

$$\|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{L^\infty(\Omega)} + \|w(\cdot, t)\|_{W^{1,\infty}(\Omega)} \leq K \quad \textit{for all } t > 0.$$

Remark 1.1. *It is worth noting that we add the convexity of* Ω *to simplify the total process. However, this assumption can be dropped by the methods in* [31]. *In fact, we first need to establish the estimates on* $\int_\Omega u^2$, $\int_\Omega v^2$, $\int_\Omega w^2$ *and* $\int_\Omega |\nabla w|^4$ *on non-convex domain* Ω*, this will lead to stronger restrictions for the parameters* μ_i $(i = 1, 2)$.

Remark 1.2. *This method can also be used to deal with the higher-dimensional case, however, the estimates of lower bound for* μ_i $(i = 1, 2)$ *will be more complex, we omit here.*

Compared with the asymptotic stability of some classical KS system in [32] or [33], it seems that the nonlinear term *ruv* greatly changed the large-time behavior of the smooth solutions of (1.1). To see this more intuitively, we first write

$$u_* := \frac{a+1}{\mu_1}, \quad v_* := au_* \;\; \text{and} \;\; w_* := \mu_1 u_*^2, \tag{1.4}$$

where $a := \frac{r+\sqrt{r^2+4\mu_1\mu_2}}{2\mu_2} > 0$. Then we have the following theorem.

Theorem 1.2. *Assume the conditions in Theorem 1.1 hold. Let*

$$\mu_1 < \mu_2 < 3\mu_1, \tag{1.5}$$

and

$$r := \mu_2 - \mu_1. \tag{1.6}$$

If

$$\sqrt{\chi_1^2 + \chi_2^2} < M$$

holds, where $M = M(\mu_1, \mu_2, u_0, v_0, w_0)$ *is a positive constant, then for any global classical solution* (u, v, w) *of (1.1) with the initial data fulfilling (1.2), it satisfies the following property*

$$u(\cdot, t) \to u_*, \;\; v(\cdot, t) \to v_* \;\; \text{and} \;\; w(\cdot, t) \to w_* \qquad \text{in } L^\infty(\Omega) \text{ as } t \to \infty. \tag{1.7}$$

Remark 1.3. *According to (1.6) and in view of (1.4), we obverse that*

$$u(\cdot, t) \to \frac{2}{\mu_1}, \;\; v(\cdot, t) \to \frac{2}{\mu_1} \;\; \text{and} \;\; w(\cdot, t) \to \frac{4}{\mu_1} \qquad \text{in } L^\infty(\Omega) \text{ as } t \to \infty. \tag{1.8}$$

The structure of this paper is as follows: In Section 2, we provide some crucial lemmas which will be used in the following context. In Section 3, we make some basal estimates which will help us to deal with the boundary integrals on Ω in the next section. Then we show some priori estimates in Section 4 and prove the global boundedness of the classical solution of (1.1) in Section 5. In Section 6, we mainly analyze the large-time behavior of the solution for (1.1).

2. Preliminaries

In this section, we start from local in time existence which is a crucial lemma for the existence of globally bounded classical solutions.

Lemma 2.1. [34] *Let $\Omega \subset \mathbb{R}^n$ $(n \geq 1)$ be a bounded domain with smooth boundary, $\chi_i, \mu_i > 0$ $(i = 1, 2)$ and $r > 0$. Then for any initial data (u_0, v_0, w_0) fulfilling (1.2), there exists $T_{max} \in (0, \infty]$ and a unique nonnegative solution (u, v, w) of (1.1) which satisfies $u, v \in C^0\left(\overline{\Omega} \times [0, T_{max})\right) \cap C^{2,1}\left(\overline{\Omega} \times (0, T_{max})\right)$ and $w \in C^0\left([0, T_{max}); W^{1,p}(\Omega)\right) \cap C^{2,1}\left(\overline{\Omega} \times (0, T_{max})\right)$ for any $p > n$. Furthermore, if $T_{max} < \infty$, then*

$$\lim_{t\to T_{max}} \sup\left\{\|u(\cdot,t)\|_{L^\infty(\Omega)} + \|v(\cdot,t)\|_{L^\infty(\Omega)} + \|w(\cdot,t)\|_{W^{1,\infty}(\Omega)}\right\} = \infty. \tag{2.1}$$

Proof. The local in time existence of the classical solution to (1.1) follows from well-known standard methods in [34, 35], we omit it here. □

Next, we present a estimate of the Neumann heat semigroup which will be used in section 5.

Lemma 2.2. [36] *Let $(e^{t\Delta})_{t\geq 0}$ be the Neumann heat semigroup in Ω, and let $p \in (0, \infty]$. Then there exists $C > 0$ such that for all $\varphi \in C^1\left(\overline{\Omega}; \mathbb{R}^n\right)$ satisfying $\varphi \cdot \nu = 0$ on $\partial\Omega$, we have*

$$\left\|e^{t\Delta}\nabla \cdot \varphi\right\|_{L^\infty(\Omega)} \leq Ct^{-\frac{1}{2}-\frac{n}{2p}} \|\varphi\|_{L^p(\Omega)} \quad \text{for all } t \in (0, T_{max}). \tag{2.2}$$

Then, we give some basic estimates for our following work.

Lemma 2.3. *There exist $m > 0$ and $h > 0$ such that the solution (u, v, w) of (1.1) fulfills*

$$\int_\Omega u(\cdot,t) \leq m, \quad \int_\Omega v(\cdot,t) \leq m \text{ and } \int_\Omega w(\cdot,t) \leq m \quad \text{for all } t \in (0, T_{max}), \tag{2.3}$$

Proof. This lemma can be proved by integrating three equations in (1.1) over Ω, then applying the ODE comparison can complete it easily. The interested readers can get detailed proof from Lemma 2.2 of [4]. □

3. Estimates on $\int_\Omega u^3$, $\int_\Omega v^3$, $\int_\Omega w^3$ and $\int_\Omega |\nabla w|^6$

Refer to some previous results in [25] or [16], when $n = 4$ or 5, the global existence of classical solution for (1.1) will be obtained by a priori bounds on $\int_\Omega u^3$, $\int_\Omega v^3$, $\int_\Omega w^3$ and $\int_\Omega |\nabla w|^6$. In order to establish those estimates, we begin with the following energy inequalities.

Lemma 3.1. *Let (u, v, w) be a solution of (1.1), then for any $\epsilon \in (0, 1)$ there holds that*

$$\begin{aligned}
&\frac{d}{dt}\left\{\int_\Omega u^3 + \int_\Omega v^3 + \int_\Omega w^3\right\} + 6(1-\epsilon)\left\{\int_\Omega u|\nabla u|^2 + \int_\Omega v|\nabla v|^2\right\} \\
&+ 6\int_\Omega w|\nabla w|^2 + 3\mu_1\int_\Omega u^4 + 3\mu_2\int_\Omega v^4 + 3\int_\Omega w^3 \\
&\leq 3\int_\Omega u^2 w + 3\int_\Omega v^2 w + 3r\int_\Omega uv^3 + 3\int_\Omega uw^2 + 3\int_\Omega vw^2 \\
&+ \frac{3\chi_1^2}{2\epsilon}\int_\Omega u^3|\nabla w|^2 + \frac{3\chi_2^2}{2\epsilon}\int_\Omega v^3|\nabla w|^2
\end{aligned} \tag{3.1}$$

for all $t \in (0, T_{max})$.

Proof. Multiplying the first equation in (1.1) by u^2, integrating by parts, we obtain

$$\frac{1}{3}\frac{d}{dt}\int_\Omega u^3 = -2\int_\Omega u|\nabla u|^2 + 2\chi_1\int_\Omega u^2\nabla u\cdot\nabla w + \int_\Omega u^2 w - \mu_1\int_\Omega u^4 \tag{3.2}$$

for all $t \in (0, T_{max})$. Applying Young's inequality with any $\epsilon \in (0, 1)$, we estimate

$$2\chi_1\int_\Omega u^2\nabla u\cdot\nabla w \le 2\epsilon\int_\Omega u|\nabla u|^2 + \frac{\chi_1^2}{2\epsilon}\int_\Omega u^3|\nabla w|^2 \tag{3.3}$$

for all $t \in (0, T_{max})$. Substituting (3.3) into (3.2) yields that

$$\frac{d}{dt}\int_\Omega u^3 + 6(1-\epsilon)\int_\Omega u|\nabla u|^2 + 3\mu_1\int_\Omega u^4 \le 3\int_\Omega u^2 w + \frac{3\chi_1^2}{2\epsilon}\int_\Omega u^3|\nabla w|^2 \tag{3.4}$$

for all $t \in (0, T_{max})$. Similarly, testing the second equation of (1.1) by v^2, we also have

$$\frac{1}{3}\frac{d}{dt}\int_\Omega v^3 + 2(1-\epsilon)\int_\Omega v|\nabla v|^2 + \mu_2\int_\Omega v^4 \le \int_\Omega v^2 w + r\int_\Omega uv^3 + \frac{\chi_2^2}{2\epsilon}\int_\Omega v^3|\nabla w|^2 \tag{3.5}$$

for all $t \in (0, T_{max})$. Finally, integrating the third equation of (1.1) behind multiplying w^2, we know

$$\frac{1}{3}\frac{d}{dt}\int_\Omega w^3 = -2\int_\Omega w|\nabla w|^2 + \int_\Omega uw^2 + \int_\Omega vw^2 - \int_\Omega w^3 \tag{3.6}$$

for all $t \in (0, T_{max})$. Combining with (3.4)–(3.6), we immediately deduce (3.1). □

Next, we make a priori estimate for $\int_\Omega |\nabla w|^6$.

Lemma 3.2. *Let Ω be convex, then we have*

$$\begin{aligned}&\frac{d}{dt}\int_\Omega |\nabla w|^6 + 3\int_\Omega |\nabla w|^2|\nabla|\nabla w|^2|^2 + 6\int_\Omega |\nabla w|^6\\ &\le 3(n+8)\int_\Omega u^2|\nabla w|^4 + 3(n+8)\int_\Omega v^2|\nabla w|^4 \quad \textit{for all } t\in(0,T_{max}).\end{aligned} \tag{3.7}$$

Proof. Using the third equation of (1.1) and $2\nabla w\cdot\nabla\Delta w = \Delta|\nabla w|^2 - 2|D^2 w|^2$, we have

$$\begin{aligned}\frac{1}{3}\frac{d}{dt}\int_\Omega |\nabla w|^6 &= 2\int_\Omega |\nabla w|^4\nabla w\cdot\nabla(\Delta w + u + v - w)\\ &= \int_\Omega |\nabla w|^4\Delta|\nabla w|^2 - 2\int_\Omega |\nabla w|^4|D^2 w|^2 + 2\int_\Omega |\nabla w|^4\nabla u\cdot\nabla w\\ &\quad + 2\int_\Omega |\nabla w|^4\nabla v\cdot\nabla w - 2\int_\Omega |\nabla w|^6\\ &= \int_{\partial\Omega} |\nabla w|^4\frac{\partial|\nabla w|^2}{\partial\nu} - 2\int_\Omega |\nabla w|^2|\nabla|\nabla w|^2|^2 - 2\int_\Omega |\nabla w|^4|D^2 w|^2\\ &\quad + 2\int_\Omega |\nabla w|^4\nabla u\cdot\nabla w + 2\int_\Omega |\nabla w|^4\nabla v\cdot\nabla w - 2\int_\Omega |\nabla w|^6\end{aligned} \tag{3.8}$$

for all $t \in (0, T_{max})$. Thanks to the convexity assumption on Ω and $\frac{\partial w}{\partial \nu}\Big|_{\partial\Omega} = 0$ [37], we ensure that

$$\frac{\partial |\nabla w|^2}{\partial \nu} \leq 0 \quad \text{on } \partial\Omega \times (0, T_{max}). \tag{3.9}$$

Moreover, due to $|\Delta w|^2 \leq n|D^2 w|^2$, we know

$$\begin{aligned} 2\int_\Omega |\nabla w|^4 \nabla u \cdot \nabla w &= -2\int_\Omega u|\nabla w|^4 \Delta w - 4\int_\Omega u|\nabla w|^2 \nabla w \cdot \nabla |\nabla w|^2 \\ &\leq n\int_\Omega u^2|\nabla w|^4 + \int_\Omega |\nabla w|^4 |D^2 w|^2 \\ &\quad + \frac{1}{2}\int_\Omega |\nabla w|^2 |\nabla|\nabla w|^2|^2 + 8\int_\Omega u^2 |\nabla w|^4 \end{aligned} \tag{3.10}$$

for all $t \in (0, T_{max})$. Similarly, we get

$$\begin{aligned} 2\int_\Omega |\nabla w|^4 \nabla v \cdot \nabla w &= -2\int_\Omega v|\nabla w|^4 \Delta w - 4\int_\Omega v|\nabla w|^2 \nabla w \cdot \nabla |\nabla w|^2 \\ &\leq n\int_\Omega v^2|\nabla w|^4 + \int_\Omega |\nabla w|^4 |D^2 w|^2 \\ &\quad + \frac{1}{2}\int_\Omega |\nabla w|^2 |\nabla|\nabla w|^2|^2 + 8\int_\Omega v^2 |\nabla w|^4 \end{aligned} \tag{3.11}$$

for all $t \in (0, T_{max})$. At the end, collecting the above results yields (3.7). □

The integrals $\int_\Omega u^2|\nabla w|^4$ and $\int_\Omega v^2|\nabla w|^4$ appearing on the right-hand side of (3.7) can be treated by utilizing the logical damping terms in (1.1) properly. To obtain this end, we shall establish the following similar integrals $\int_\Omega u|\nabla w|^4$ and $\int_\Omega v|\nabla w|^4$.

Lemma 3.3. *Let Ω be convex, then there exist two positive constants ϵ_1 and ϵ_2 independent of system parameters μ_i, χ_i $(i = 1, 2)$ and r such that*

$$\begin{aligned} &\frac{d}{dt}\int_\Omega u|\nabla w|^4 + 4\int_\Omega u|\nabla w|^4 + \int_\Omega u|\nabla|\nabla w|^2|^2 + \left(\mu_1 - \frac{\chi_1^2}{2\epsilon_1}\right)\int_\Omega u^2|\nabla w|^4 \\ &\leq 2(\epsilon_1 + \epsilon_2)\int_\Omega |\nabla w|^2|\nabla|\nabla w|^2|^2 + \frac{2}{\epsilon_2}\int_\Omega |\nabla u|^2|\nabla w|^2 + \int_\Omega w|\nabla w|^4 \\ &+ \left(1 + \frac{n}{4}\right)\int_\Omega u^3|\nabla w|^2 + 4\int_\Omega u|\nabla w|^2 \nabla v \cdot \nabla w \end{aligned} \tag{3.12}$$

and

$$\begin{aligned} &\frac{d}{dt}\int_\Omega v|\nabla w|^4 + 4\int_\Omega v|\nabla w|^4 + \int_\Omega v|\nabla|\nabla w|^2|^2 + \left(\mu_2 - \frac{\chi_2^2}{2\epsilon_1}\right)\int_\Omega v^2|\nabla w|^4 \\ &\leq 2(\epsilon_1 + \epsilon_2)\int_\Omega |\nabla w|^2|\nabla|\nabla w|^2|^2 + \frac{2}{\epsilon_2}\int_\Omega |\nabla v|^2|\nabla w|^2 + \int_\Omega w|\nabla w|^4 \\ &+ \left(1 + \frac{n}{4}\right)\int_\Omega v^3|\nabla w|^2 + 4\int_\Omega v|\nabla w|^2 \nabla u \cdot \nabla w + r\int_\Omega uv|\nabla w|^4 \end{aligned} \tag{3.13}$$

for all $t \in (0, T_{max})$.

Proof. Applying the u and w-equations of (1.1), then integrating it over Ω, we know

$$
\begin{aligned}
\frac{d}{dt}\int_\Omega u|\nabla w|^4 &= \int_\Omega |\nabla w|^4\left(\Delta u - \chi_1\nabla\cdot(u\nabla w) + w - \mu_1 u^2\right)\\
&\quad + 4\int_\Omega u|\nabla w|^2\nabla w\cdot\nabla\left(\Delta w + u + v - w\right)\\
&= -4\int_\Omega |\nabla w|^2\nabla u\cdot\nabla|\nabla w|^2 + 2\chi_1\int_\Omega u|\nabla w|^2\nabla w\cdot\nabla|\nabla w|^2\\
&\quad + \int_\Omega w|\nabla w|^4 - \mu_1\int_\Omega u^2|\nabla w|^4 + 2\int_{\partial\Omega} u|\nabla w|^2\frac{\partial|\nabla w|^2}{\partial\nu}\\
&\quad - 2\int_\Omega u\left|\nabla|\nabla w|^2\right|^2 - 4\int_\Omega u|\nabla w|^2\left|D^2 w\right|^2 - 4\int_\Omega u|\nabla w|^4\\
&\quad + 4\int_\Omega u|\nabla w|^2\nabla u\cdot\nabla w + 4\int_\Omega u|\nabla w|^2\nabla v\cdot\nabla w
\end{aligned}
\tag{3.14}
$$

for all $t \in (0, T_{max})$. Here, similar to (3.9), we get

$$
\frac{\partial|\nabla w|^2}{\partial\nu} \le 0 \quad \text{on } \partial\Omega\times(0, T_{max}). \tag{3.15}
$$

Moreover, we observe that

$$
2\chi_1\int_\Omega u|\nabla w|^2\nabla w\cdot\nabla|\nabla w|^2 \le 2\epsilon_1\int_\Omega |\nabla w|^2\left|\nabla|\nabla w|^2\right|^2 + \frac{\chi_1^2}{2\epsilon_1}\int_\Omega u^2|\nabla w|^4 \tag{3.16}
$$

and

$$
-4\int_\Omega |\nabla w|^2\nabla u\cdot\nabla|\nabla w|^2 \le 2\epsilon_2\int_\Omega |\nabla w|^2\left|\nabla|\nabla w|^2\right|^2 + \frac{2}{\epsilon_2}\int_\Omega |\nabla u|^2|\nabla w|^2 \tag{3.17}
$$

for all $t \in (0, T_{max})$ with some $\epsilon_1 > 0$ and $\epsilon_2 > 0$ independent of system parameters μ_i, χ_i $(i = 1, 2)$ and r. Using the similar computation in (3.10), we have

$$
\begin{aligned}
4\int_\Omega u|\nabla w|^2\nabla u\cdot\nabla w &= -2\int_\Omega u^2|\nabla w|^2\Delta w - 2\int_\Omega u^2\nabla w\cdot\nabla|\nabla w|^2\\
&\le \frac{n}{4}\int_\Omega u^3|\nabla w|^2 + 4\int_\Omega u|\nabla w|^2\left|D^2 w\right|^2\\
&\quad + \int_\Omega u\left|\nabla|\nabla w|^2\right|^2 + \int_\Omega u^3|\nabla w|^2
\end{aligned}
\tag{3.18}
$$

for all $t \in (0, T_{max})$. Combining with (3.14)–(3.18), we obtain (3.12). In the same way, by the combination of v and w-equations in (1.1), we can also prove (3.13) easily. □

In order to deal with the terms $\int_\Omega |\nabla u|^2|\nabla w|^2$ and $\int_\Omega |\nabla v|^2|\nabla w|^2$ in the right-hand sides of (3.12) and (3.13), let us establish estimates for $\int_\Omega u^2|\nabla w|^2$ and $\int_\Omega v^2|\nabla w|^2$, respectively.

Lemma 3.4. *Let Ω be convex. Then for any $\epsilon \in (0,1)$, there exist two constants $\epsilon_3 > 0$ and $\epsilon_4 > 0$ independent of system parameters μ_i, χ_i $(i = 1, 2)$ and r such that*

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega u^2|\nabla w|^2 + 2\left(1-\frac{2\epsilon}{3}\right)\int_\Omega |\nabla u|^2|\nabla w|^2 + 2\int_\Omega u^2|\nabla w|^2 + 2\left(\mu_1 - \frac{\chi_1^2}{4\epsilon_3}\right)\int_\Omega u^3|\nabla w|^2 \\
&\le \frac{3\chi_1^2}{2\epsilon}\int_\Omega u^2|\nabla w|^4 + \frac{2}{\epsilon_4}\int_\Omega u|\nabla u|^2 + \frac{3}{\epsilon}\int_\Omega u^4 + 2(\epsilon_3+\epsilon_4)\int_\Omega u\left|\nabla|\nabla w|^2\right|^2 \\
&+ 2\int_\Omega uw|\nabla w|^2 + \frac{2\epsilon}{3}\int_\Omega |\nabla v|^2|\nabla w|^2
\end{aligned} \tag{3.19}$$

and

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega v^2|\nabla w|^2 + 2\left(1-\frac{2\epsilon}{3}\right)\int_\Omega |\nabla v|^2|\nabla w|^2 + 2\int_\Omega v^2|\nabla w|^2 + 2\left(\mu_2 - \frac{\chi_2^2}{4\epsilon_3}\right)\int_\Omega v^3|\nabla w|^2 \\
&\le \frac{3\chi_2^2}{2\epsilon}\int_\Omega v^2|\nabla w|^4 + \frac{2}{\epsilon_4}\int_\Omega v|\nabla v|^2 + \frac{3}{\epsilon}\int_\Omega v^4 + 2(\epsilon_3+\epsilon_4)\int_\Omega v\left|\nabla|\nabla w|^2\right|^2 \\
&+ 2\int_\Omega vw|\nabla w|^2 + \frac{2\epsilon}{3}\int_\Omega |\nabla u|^2|\nabla w|^2 + 2r\int_\Omega uv^2|\nabla w|^2
\end{aligned} \tag{3.20}$$

for all $t \in (0, T_{max})$.

Proof. By directly computing the coupling of u^2 and $|\nabla w|^2$, we obtain

$$\begin{aligned}
\frac{d}{dt}\int_\Omega u^2|\nabla w|^2 &= 2\int_\Omega u|\nabla w|^2\left(\Delta u - \chi_1\nabla\cdot(u\nabla w) + w - \mu_1 u^2\right) + 2\int_\Omega u^2\nabla w\cdot\nabla(\Delta w + u + v - w) \\
&= -2\int_\Omega |\nabla u|^2|\nabla w|^2 - 4\int_\Omega u\nabla u\cdot\nabla|\nabla w|^2 + 2\chi_1\int_\Omega u|\nabla w|^2\nabla u\cdot\nabla w \\
&\quad + 2\chi_1\int_\Omega u^2\nabla w\cdot\nabla|\nabla w|^2 + 2\int_\Omega uw|\nabla w|^2 - 2\mu_1\int_\Omega u^3|\nabla w|^2 + 2\int_\Omega u^2\nabla u\cdot\nabla w \\
&\quad + 2\int_\Omega u^2\nabla v\cdot\nabla w - 2\int_\Omega u^2|\nabla w|^2 + \int_{\partial\Omega} u^2\frac{\partial|\nabla w|^2}{\partial\nu} - 2\int_\Omega u^2|D^2 w|^2
\end{aligned} \tag{3.21}$$

for all $t \in (0, T_{max})$. As before, we have

$$\frac{\partial|\nabla w|^2}{\partial\nu} \le 0 \quad \text{on } \partial\Omega\times(0,T_{max}), \tag{3.22}$$

and by applying Young's inequality, we derive

$$2\chi_1\int_\Omega u^2\nabla w\cdot\nabla|\nabla w|^2 \le 2\epsilon_3\int_\Omega u\left|\nabla|\nabla w|^2\right|^2 + \frac{\chi_1^2}{2\epsilon_3}\int_\Omega u^3|\nabla w|^2 \tag{3.23}$$

and

$$-4\int_\Omega u\nabla u\cdot\nabla|\nabla w|^2 \le 2\epsilon_4\int_\Omega u\left|\nabla|\nabla w|^2\right|^2 + \frac{2}{\epsilon_4}\int_\Omega u|\nabla u|^2 \tag{3.24}$$

for all $t \in (0, T_{max})$ with some $\epsilon_3 > 0$ and $\epsilon_4 > 0$ independent of system parameters μ_i, χ_i $(i = 1, 2)$ and r. Moreover, we also have

$$
\begin{aligned}
2\int_\Omega u^2 \nabla u \cdot \nabla w &\le \frac{2\epsilon}{3}\int_\Omega |\nabla u|^2|\nabla w|^2 + \frac{3}{2\epsilon}\int_\Omega u^4, \\
2\int_\Omega u^2 \nabla v \cdot \nabla w &\le \frac{2\epsilon}{3}\int_\Omega |\nabla v|^2|\nabla w|^2 + \frac{3}{2\epsilon}\int_\Omega u^4
\end{aligned} \tag{3.25}
$$

and

$$
2\chi_1 \int_\Omega u|\nabla w|^2 \nabla u \cdot \nabla w \le \frac{2\epsilon}{3}\int_\Omega |\nabla u|^2|\nabla w|^2 + \frac{3\chi_1^2}{2\epsilon}\int_\Omega u^2|\nabla w|^4 \tag{3.26}
$$

for all $t \in (0, T_{max})$ with $\epsilon \in (0, 1)$. In addition, it is obvious that

$$
-2\int_\Omega u^2|D^2 w|^2 \le 0 \quad \text{for all } t \in (0, T_{max}). \tag{3.27}
$$

Then (3.19) can be proved by taking (3.23)–(3.27) into (3.21). For the similarity of (3.20) and (3.19), the same operation can be done to testify (3.20). □

As usual, we rely on Gronwall's inequality to get L^3–boundedness of u, v and w and L^6–boundedness of ∇w. For this purpose, set

$$
\begin{aligned}
y(t) := \ & \delta_1\left\{\int_\Omega u^3 + \int_\Omega v^3 + \int_\Omega w^3\right\} + \delta_2\left\{\int_\Omega u^2|\nabla w|^2 + \int_\Omega v^2|\nabla w|^2\right\} \\
& + \delta_3\left\{\int_\Omega u|\nabla w|^4 + \int_\Omega v|\nabla w|^4\right\} + \delta_4\int_\Omega |\nabla w|^6
\end{aligned} \tag{3.28}
$$

for all $t \in [0, T_{max})$ with $\delta_i > 0$ $(i = 1, 2, 3, 4)$ independent of system parameters μ_i, χ_i $(i = 1, 2)$ and r. Then we combine Lemmas 3.1–3.4 and choose appropriate parameters to get the following lemma.

Lemma 3.5. *Let Ω be convex and and the initial data (u_0, v_0, w_0) fulfill (1.2). Then for any $\epsilon \in (0, 1)$, assume that μ_1, μ_2 satisfy*

$$
\mu_1 > \frac{1}{3}\left[\sqrt{\frac{3}{\epsilon^2}} + \sqrt{\frac{1}{\epsilon}} + \sqrt{\frac{3n}{2\epsilon}} + 2\sqrt{n+8}\left(\sqrt{2} + \frac{\sqrt{2}}{\sqrt{1-\epsilon}}\right)\right]\chi_1 + \left(\frac{\sqrt{2}}{3} + \frac{1}{2}\right)r + 1 \tag{3.29}
$$

and

$$
\mu_2 > \frac{1}{3}\left[\sqrt{\frac{3}{\epsilon^2}} + \sqrt{\frac{1}{\epsilon}} + \sqrt{\frac{3(n+4)}{2\epsilon}} + 2\sqrt{n+8}\left(\sqrt{2} + \frac{\sqrt{2}}{\sqrt{1-\epsilon}}\right)\right]\chi_2 + \frac{1}{2}r + 1, \tag{3.30}
$$

then we find a positive constant K_1 to ensure that

$$
\|u\|_{L^3(\Omega)} + \|v\|_{L^3(\Omega)} + \|w\|_{L^3(\Omega)} + \|\nabla w\|_{L^6(\Omega)} \le K_1 \quad \textit{for all } t \in (0, T_{max}). \tag{3.31}
$$

And there also exists a constant $K_2 > 0$ such that

$$
\|w\|_{L^\infty(\Omega)} \le K_2 \quad \textit{for all } t \in (0, T_{max}). \tag{3.32}
$$

Proof. For the convenience of readers, we divide the proof into the following two steps:

Step 1. In this step, we deal with the coupling terms appearing on the right-hand side of Lemma 3.1–Lemma 3.4 to approach Gronwall's inequality.

A direct linear combination Lemma 3.1–Lemma 3.4 after multiplying $\delta_i > 0$ $(i = 1, 2, 3, 4)$, we have

$$\begin{aligned}
&y'(t) + 2\delta_1 \left\{ \int_\Omega u^3 + \int_\Omega v^3 + \frac{3}{2}\int_\Omega w^3 \right\} + 2\delta_2 \left\{ \int_\Omega u^2|\nabla w|^2 + \int_\Omega v^2|\nabla w|^2 \right\} \\
&+ 4\delta_3 \left\{ \int_\Omega u|\nabla w|^4 + \int_\Omega v|\nabla w|^4 \right\} + 6\delta_1 \int_\Omega w|\nabla w|^2 + A_1 \int_\Omega u|\nabla u|^2 + A_1 \int_\Omega v|\nabla v|^2 \\
&+ A_2 \int_\Omega |\nabla u|^2|\nabla w|^2 + A_2 \int_\Omega |\nabla v|^2|\nabla w|^2 + A_3 \int_\Omega u\left|\nabla|\nabla w|^2\right|^2 + A_3 \int_\Omega v\left|\nabla|\nabla w|^2\right|^2 \\
&+ A_4 \int_\Omega |\nabla w|^2\left|\nabla|\nabla w|^2\right|^2 + A_5 \int_\Omega u^4 + A_5^* \int_\Omega v^4 + A_6 \int_\Omega u^3|\nabla w|^2 + A_6^* \int_\Omega v^3|\nabla w|^2 \\
&+ A_7 \int_\Omega u^2|\nabla w|^4 + A_7^* \int_\Omega v^2|\nabla w|^4 + 6\delta_4 \int_\Omega |\nabla w|^6 \\
&\le 2\delta_1 \int_\Omega u^3 + 2\delta_1 \int_\Omega v^3 + 3\delta_1 \int_\Omega u^2 w + 3\delta_1 \int_\Omega v^2 w + 3\delta_1 \int_\Omega uw^2 + 3\delta_1 \int_\Omega vw^2 \\
&+ 3r\delta_1 \int_\Omega uv^3 + 2r\delta_2 \int_\Omega uv^2|\nabla w|^2 + 2\delta_2 \int_\Omega uw|\nabla w|^2 + 2\delta_2 \int_\Omega vw|\nabla w|^2 \\
&+ r\delta_3 \int_\Omega uv|\nabla w|^4 + 2\delta_3 \int_\Omega w|\nabla w|^4 + 4\delta_3 \int_\Omega u|\nabla w|^2 \nabla v \cdot \nabla w + 4\delta_3 \int_\Omega v|\nabla w|^2 \nabla u \cdot \nabla w
\end{aligned} \tag{3.33}$$

for all $t \in (0, T_{max})$, where

$$\begin{cases}
A_1 := 6(1-\epsilon)\delta_1 - \frac{2}{\epsilon_4}\delta_2, & A_2 := 2(1-\epsilon)\delta_2 - \frac{2}{\epsilon_2}\delta_3, \\
A_3 := \delta_3 - 2(\epsilon_3 + \epsilon_4)\delta_2, & A_4 := 3\delta_4 - 4(\epsilon_1 + \epsilon_2)\delta_3, \\
A_5 := 3\mu_1\delta_1 - \frac{3}{\epsilon}\delta_2, & A_5^* := 3\mu_2\delta_1 - \frac{3}{\epsilon}\delta_2, \\
A_6 := 2\left(\mu_1 - \frac{\chi_1^2}{4\epsilon_3}\right)\delta_2 - \left(1 + \frac{n}{4}\right)\delta_3 - \frac{3\chi_1^2}{2\epsilon}\delta_1, & A_6^* := 2\left(\mu_2 - \frac{\chi_2^2}{4\epsilon_3}\right)\delta_2 - \left(1 + \frac{n}{4}\right)\delta_3 - \frac{3\chi_2^2}{2\epsilon}\delta_1, \\
A_7 := \left(\mu_1 - \frac{\chi_1^2}{2\epsilon_1}\right)\delta_3 - 3(n+8)\delta_4 - \frac{3\chi_1^2}{2\epsilon}\delta_2, & A_7^* := \left(\mu_2 - \frac{\chi_2^2}{2\epsilon_1}\right)\delta_3 - 3(n+8)\delta_4 - \frac{3\chi_2^2}{2\epsilon}\delta_2.
\end{cases} \tag{3.34}$$

Then applying Young's inequality, we estimate

$$\begin{aligned}
&2\delta_1 \int_\Omega u^3 \le \frac{3r}{4}\delta_1 \int_\Omega u^4 + C_1, \\
&2\delta_1 \int_\Omega v^3 \le \frac{3r}{4}\delta_1 \int_\Omega v^4 + C_1, \\
&3\delta_1 \int_\Omega u^2 w \le \frac{3}{\epsilon}\delta_2 \int_\Omega u^4 + \frac{1}{3}\delta_3 \int_\Omega w^3 + C_2, \\
&3\delta_1 \int_\Omega v^2 w \le \frac{3}{\epsilon}\delta_2 \int_\Omega v^4 + \frac{1}{3}\delta_3 \int_\Omega w^3 + C_2, \\
&3\delta_1 \int_\Omega uw^2 \le \frac{1}{3}\delta_3 \int_\Omega w^3 + \left(\frac{6r}{4}\delta_1 + 2\delta_2\right)\int_\Omega u^4 + C_3,
\end{aligned} \tag{3.35}$$

$$
\begin{aligned}
&3\delta_1 \int_\Omega vw^2 \le \frac{1}{3}\delta_3 \int_\Omega w^3 + \delta_2 \int_\Omega v^4 + C_4,\\
&2\delta_2 \int_\Omega uw|\nabla w|^2 \le \frac{3\chi_1^2}{2\epsilon}\delta_2 \int_\Omega u^2|\nabla w|^4 + \frac{2}{3}\delta_3 \int_\Omega w^3 + C_5
\end{aligned}
$$

and

$$
2\delta_2 \int_\Omega vw|\nabla w|^2 \le \frac{3\chi_2^2}{2\epsilon}\delta_2 \int_\Omega v^2|\nabla w|^4 + \frac{1}{3}\delta_3 \int_\Omega w^3 + C_6 \tag{3.36}
$$

for all $t \in (0, T_{max})$ with some positive constants C_i $(i = 1, 2, 3, 4, 5, 6)$. Apart from those, we also have

$$
\begin{aligned}
&3r\delta_1 \int_\Omega uv^3 \le \frac{9r}{4}\delta_1 \int_\Omega u^4 + \frac{3r}{4}\delta_1 \int_\Omega v^4,\\
&2r\delta_2 \int_\Omega uv^2|\nabla w|^2 \le r^2\delta_2 \int_\Omega u^2|\nabla w|^4 + \delta_2 \int_\Omega v^4,\\
&2\delta_3 \int_\Omega w|\nabla w|^4 \le \frac{4}{3}\delta_3 \int_\Omega |\nabla w|^6 + \frac{2}{3}\delta_3 \int_\Omega w^3,\\
&r\delta_3 \int_\Omega uv|\nabla w|^4 \le \frac{r}{2}\delta_3 \int_\Omega u^2|\nabla w|^4 + \frac{r}{2}\delta_3 \int_\Omega v^2|\nabla w|^4,\\
&4\delta_3 \int_\Omega u|\nabla w|^2\nabla v \cdot \nabla w \le 3\delta_3 \int_\Omega u^2|\nabla w|^4 + \frac{4}{3}\delta_3 \int_\Omega |\nabla v|^2|\nabla w|^2
\end{aligned} \tag{3.37}
$$

and

$$
4\delta_3 \int_\Omega v|\nabla w|^2\nabla u \cdot \nabla w \le 3\delta_3 \int_\Omega v^2|\nabla w|^4 + \frac{4}{3}\delta_3 \int_\Omega |\nabla u|^2|\nabla w|^2 \tag{3.38}
$$

for all $t \in (0, T_{max})$. Together with (3.33)–(3.38), we can find a positive constant C to ensure that

$$
\begin{aligned}
&y'(t) + 2\delta_1 \left\{\int_\Omega u^3 + \int_\Omega v^3 + 3\int_\Omega w^3\right\} + 2\delta_2 \left\{\int_\Omega u^2|\nabla w|^2 + \int_\Omega v^2|\nabla w|^2\right\}\\
&+ 4\delta_3 \left\{\int_\Omega u|\nabla w|^4 + \int_\Omega v|\nabla w|^4\right\} + 6\delta_4 \int_\Omega |\nabla w|^6 + 6\delta_1 \int_\Omega w|\nabla w|^2\\
&+ B_1 \int_\Omega u^3|\nabla w|^2 + B_1^* \int_\Omega v^3|\nabla w|^2 + B_2 \int_\Omega u^2|\nabla w|^4 + B_2^* \int_\Omega v^2|\nabla w|^4 + B_3 \int_\Omega u^4\\
&+ B_3^* \int_\Omega v^4 + B_4 \int_\Omega |\nabla u|^2|\nabla w|^2 + B_4 \int_\Omega |\nabla v|^2|\nabla w|^2 + B_5 \int_\Omega u|\nabla u|^2 + B_5 \int_\Omega v|\nabla v|^2\\
&+ B_6 \int_\Omega u\left|\nabla|\nabla w|^2\right|^2 + B_6 \int_\Omega v\left|\nabla|\nabla w|^2\right|^2 + B_7 \int_\Omega |\nabla w|^2\left|\nabla|\nabla w|^2\right|^2\\
&\le C \quad \text{for all } t \in (0, T_{max}),
\end{aligned} \tag{3.39}
$$

where

$$\begin{cases} B_1 := 2\left(\mu_1 - \frac{\chi_1^2}{4\epsilon_3}\right)\delta_2 - \left(1 + \frac{n}{4}\right)\delta_3 - \frac{3\chi_1^2}{2\epsilon}\delta_1, \\ B_1^* := 2\left(\mu_2 - \frac{\chi_2^2}{4\epsilon_3}\right)\delta_2 - \left(1 + \frac{n}{4}\right)\delta_3 - \frac{3\chi_2^2}{2\epsilon}\delta_1, \\ B_2 := \left(\mu_1 - \frac{\chi_1^2}{2\epsilon_1} - \frac{1}{2}r - 3\right)\delta_3 - 3(n+8)\delta_4 - \left(\frac{3\chi_1^2}{\epsilon} + r^2\right)\delta_2, \\ B_2^* := \left(\mu_2 - \frac{\chi_2^2}{2\epsilon_1} - \frac{1}{2}r - 3\right)\delta_3 - 3(n+8)\delta_4 - \frac{3\chi_2^2}{\epsilon}\delta_2, \\ B_3 := 3(\mu_1 - r)\delta_1 - \left(2 + \frac{6}{\epsilon}\right)\delta_2, \qquad B_3^* := 3(\mu_2 - r)\delta_1 - \left(2 + \frac{6}{\epsilon}\right)\delta_2, \\ B_4 := 2(1-\epsilon)\delta_2 - \left(\frac{2}{\epsilon_2} + \frac{4}{3}\right)\delta_3, \qquad B_5 := 6(1-\epsilon)\delta_1 - \frac{2}{\epsilon_4}\delta_2, \\ B_6 := \delta_3 - 2(\epsilon_3 + \epsilon_4)\delta_2, \qquad B_7 := 3\delta_4 - 4(\epsilon_1 + \epsilon_2)\delta_3. \end{cases} \tag{3.40}$$

Step 2. To deal with the extra terms on the left of (3.39), we choose to ignore them if the coefficient in front of them is positive. This requires that the parameters μ_i $(i = 1, 2)$ satisfy certain conditions. Here we need $B_i \geq 0$ $(i = 1, 2, \cdots, 7)$ and $B_j^* \geq 0$ $(j = 1, 2, 3)$, and from (3.40), we find that the fourth constraint B_4 has something special in common with the sixth constraint B_6, then we have

$$2(\epsilon_3 + \epsilon_4) \leq \frac{1-\epsilon}{\frac{2}{3} + \frac{1}{\epsilon_2}} \Leftrightarrow \frac{2}{(1-\epsilon)} \leq \frac{\epsilon_2}{(\epsilon_3 + \epsilon_4)\left(1 + \frac{2}{3}\epsilon_2\right)} < \frac{\epsilon_2}{\epsilon_3}. \tag{3.41}$$

We use the fact that

$$a^2 + b^2 \geq 2\sqrt{ab} \quad \text{for all } a \geq 0 \text{ and } b \geq 0,$$

then the inequality manipulations from (3.41) and $B_i \geq 0$ $(i = 1, 2, 3, 7)$ show that

$$\begin{aligned} 3\mu_1 &\geq \frac{2}{\epsilon}\frac{\delta_2}{\delta_1} + \frac{3\chi_1^2}{8\epsilon}\frac{\delta_1}{\delta_2} + \frac{2}{3}\frac{\delta_2}{\delta_1} + \frac{3\chi_1^2}{8\epsilon}\frac{\delta_1}{\delta_2} + \frac{n}{8}\frac{\delta_3}{\delta_2} + \frac{3\chi_1^2}{\epsilon}\frac{\delta_2}{\delta_3} + r^2\frac{\delta_2}{\delta_3} + \frac{1}{2}\frac{\delta_3}{\delta_2} \\ &\quad + \frac{\chi_1^2}{2\epsilon_1} + \frac{\chi_1^2}{4\epsilon_3} + 3(n+8)\frac{\delta_4}{\delta_3} + \frac{3}{2}r + 3 \\ &> \left(\sqrt{\frac{3}{\epsilon^2}} + \sqrt{\frac{1}{\epsilon}} + \sqrt{\frac{3n}{2\epsilon}}\right)\chi_1 + \frac{\chi_1^2}{2\epsilon_1} + \frac{\chi_1^2}{4\epsilon_3} + 4(n+8)(\epsilon_1 + \epsilon_2) + \left(\sqrt{2} + \frac{3}{2}\right)r + 3 \\ &> \left[\sqrt{\frac{3}{\epsilon^2}} + \sqrt{\frac{1}{\epsilon}} + \sqrt{\frac{3n}{2\epsilon}} + 2\sqrt{n+8}\left(\sqrt{2} + \frac{\sqrt{2}}{\sqrt{1-\epsilon}}\right)\right]\chi_1 + \left(\sqrt{2} + \frac{3}{2}\right)r + 3, \end{aligned} \tag{3.42}$$

and the following inequality comes from (3.41), $B_j^* \geq 0$ $(j = 1, 2, 3)$ and $B_7 \geq 0$

$$3\mu_2 > \left[\sqrt{\frac{3}{\epsilon^2}} + \sqrt{\frac{1}{\epsilon}} + \sqrt{\frac{3(n+4)}{\epsilon}} + 2\sqrt{(n+8)}\left(\sqrt{2} + \frac{\sqrt{2}}{\sqrt{1-\epsilon}}\right)\right]\chi_2 + \frac{3}{2}r + 3. \tag{3.43}$$

After fixing ϵ_i $(i = 1, 2, 3, 4)$, it is possible for us to ignore some of the terms on the left of (3.39) if μ_i $(i = 1, 2)$ are sufficiently large to obtain

$$\begin{aligned} &y'(t) + 2\delta_1\left\{\int_\Omega u^3 + \int_\Omega v^3 + \frac{3}{2}\left(1 - \frac{\delta_3}{\delta_1}\right)\int_\Omega w^3\right\} + 2\delta_2\left\{\int_\Omega u^2|\nabla w|^2 + \int_\Omega v^2|\nabla w|^2\right\} \\ &+ 4\delta_3\left\{\int_\Omega u|\nabla w|^4 + \int_\Omega v|\nabla w|^4\right\} + 2\delta_4\left(3 - \frac{2\delta_3}{3\delta_4}\right)\int_\Omega |\nabla w|^6 \\ &\leq C \quad \text{for all } t \in (0, T_{max}), \end{aligned} \tag{3.44}$$

then we obverse that $y(t)$ fulfills

$$y'(t) + \xi y(t) \le C \quad \text{for all } t \in (0, T_{max}).$$

where $\xi = \min\left\{2, 3\left(1 - \frac{\delta_3}{\delta_1}\right), 2\left(3 - \frac{2\delta_3}{3\delta_4}\right)\right\}$. Applying Gronwall's inequality will yield

$$y(t) \le K_1 = \max\left\{y(0), \frac{C}{\xi}\right\} \quad \text{for all } t \in (0, T_{max}),$$

which means

$$\|u\|_{L^3(\Omega)} + \|v\|_{L^3(\Omega)} + \|w\|_{L^3(\Omega)} + \|\nabla w\|_{L^6(\Omega)} \le K_1 \quad \text{for all } t \in (0, T_{max}).$$

Then there exists a constant $K_2 > 0$ such that

$$\|w\|_{L^\infty(\Omega)} \le K_2 \quad \text{for all } t \in (0, T_{max}), \tag{3.45}$$

due to the parabolic regularity [34, 38]. This completes the proof. □

4. Proof of Theorem 1.1

In Section 3, we get L^∞-boundedness of w which plays an important role in our subsequent proof. In this section, we will use heat semigroup theory to prove Theorem 1.1.

Lemma 4.1. *Let $\Omega \subset \mathbb{R}^n$ $(n \le 5)$ be a convex domain and the initial data (u_0, v_0, w_0) satisfy (1.2), then there exists $K_3 = K_3(n, m, K_1, K_2) > 0$ such that*

$$\|u(\cdot, t)\|_{L^\infty(\Omega)} \le K_3 \tag{4.1}$$

and

$$\|v(\cdot, t)\|_{L^\infty(\Omega)} \le K_3 \tag{4.2}$$

as well as

$$\|w(\cdot, t)\|_{W^{1,\infty}(\Omega)} \le K_3 \tag{4.3}$$

for all $t \in (0, T_{max})$.

Proof. The proof is based on [31]. Given $T \in (0, T_{max})$, write

$$M(T) := \sup_{t \in (0,T)} \|u(\cdot, t)\|_{L^\infty(\Omega)}.$$

Since $u_t = \Delta u - \chi_1 \nabla \cdot (u \nabla w) + w - \mu_1 u^2$ in $\Omega \times (0, T_{max})$, we represent $u(\cdot, t)$ for each $t \in (0, T_{max})$ according to

$$\begin{aligned} u(\cdot, t) &= e^{(t-t_0)\Delta} u(\cdot, t_0) - \chi_1 \int_{t_0}^t e^{(t-s)\Delta} \nabla \cdot (u(\cdot, s) \nabla w(\cdot, s)) ds + \int_{t_0}^t e^{(t-s)\Delta} (w(\cdot, s) - \mu_1 u^2(\cdot, s)) ds \\ &=: u_1(\cdot, t) + u_2(\cdot, t) + u_3(\cdot, t), \end{aligned} \tag{4.4}$$

where $t_0 := (t-1)_+$. Here by the maximum principle, we can estimate

$$\|u_1(\cdot,t)\|_{L^\infty(\Omega)} \le \|u_0\|_{L^\infty(\Omega)} \quad \text{if } t \in (0,1], \tag{4.5}$$

whereas if $t > 1$, then standard $L^p - L^q$ estimates for the Neumann heat semigroup (cf. [7, Lemma 1.3 (i)], for instance) provide $C_1 > 0$ such that

$$\|u_1(\cdot,t)\|_{L^\infty(\Omega)} \le C_1(t-t_0)^{-\frac{n}{2}}\|u(\cdot,t_0)\|_{L^1(\Omega)} = C_1\|u(\cdot,t_0)\|_{L^1(\Omega)} \le C_1 m \tag{4.6}$$

holds because of (2.3).

Next, according to (3.32), $\sup_{x\in\Omega}|w(x,t)| \le K_2$ for all $t \in (0,T_{max})$ can be obtained. Again by the maximum principle, we have

$$u_3(\cdot,t) \le \int_{t_0}^t e^{(t-s)\Delta}w(\cdot,s)ds \le \int_{t_0}^t e^{(t-s)\Delta}K_2 ds = K_2(t-t_0) \le K_2. \tag{4.7}$$

At the end, to estimate u_2, we choose an arbitrary $p \in (n,6)$. Then invoking known smoothing properties of $(e^{\eta\Delta})_{\eta\ge 0}$ ([7, Lemma 1.3 (iv)]) and applying the Hölder inequality to find a constant $C_2 > 0$ such that

$$\begin{aligned}
\|u_2(\cdot,t)\|_{L^\infty(\Omega)} &\le C_2\int_{t_0}^t (t-s)^{-\frac{1}{2}-\frac{n}{2p}}\|u(\cdot,s)\nabla w(\cdot,s)\|_{L^p(\Omega)}ds \\
&\le C_2\int_{t_0}^t (t-s)^{-\frac{1}{2}-\frac{n}{2p}}\|u(\cdot,s)\|_{L^{\frac{6p}{6-p}}(\Omega)}\|\nabla w(\cdot,s)\|_{L^6(\Omega)}ds \\
&\le C_2\int_{t_0}^t (t-s)^{-\frac{1}{2}-\frac{n}{2p}}\|u(\cdot,s)\|^a_{L^\infty(\Omega)}\|u(\cdot,s)\|^{1-a}_{L^1(\Omega)}\|\nabla w(\cdot,s)\|_{L^6(\Omega)}ds,
\end{aligned}$$

where $a := \frac{7p-6}{6p} \in (0,1)$. In view of (2.3), (3.31) and the definition of $M(T)$, this yields that

$$\|u_2(\cdot,t)\|_{L^\infty(\Omega)} \le C_2K_1m^{1-a}\int_{t_0}^t \eta^{-\frac{1}{2}-\frac{2}{p}}d\eta\cdot M^a(T),$$

so that since $\frac{1}{2}+\frac{n}{2p} < 1$ according to our limitation $p > n$. Combining with (4.4)–(4.7), we obtain $C_3 > 0$ such that

$$\begin{aligned}
\|u(\cdot,t)\|_{L^\infty(\Omega)} = \sup_{x\in\Omega} u(x,t) &\le \sup_{x\in\Omega} u_1(x,t) + \sup_{x\in\Omega} u_2(x,t) + \sup_{x\in\Omega} u_3(x,t) \\
&\le C_3 + C_3M^a(T)
\end{aligned}$$

for all $t \in (0,T_{max})$. Therefore, we know

$$M(T) \le C_3 + C_3M^a(T) \qquad \text{for all } t \in (0,T_{max}),$$

this implies

$$M(T) \le \max\left\{1,(2C_3)^{\frac{1}{1-a}}\right\} \qquad \text{for all } t \in (0,T_{max}),$$

this completes the proof of (4.1). Then we can get

$$ruv - \mu_2v^2 \le rK_3v - \mu_2v^2 \le \frac{(rK_3)^2}{4\mu_2} \qquad \text{for all } t \in (0,T_{max}).$$

With this, (4.2) can be proved by the same operations as u. Then we obtain (4.3) due to the parabolic regularity. Collecting (4.1)–(4.3), we can easily find a positive constant K to prove Theorem 1.1. □

5. Proof of Theorem 1.2

Theorem 1.1 claims the global existence and boundedness of the classical solution of (1.1) if μ_i ($i = 1, 2$) are sufficiently large. In this section, we shall show the proof of Theorem 1.2, which relies on constructing the Lyapunov functional

$$\begin{aligned}\Gamma(t) :=& \int_\Omega \left\{ u(\cdot,t) - u_* - u_* \ln \frac{u(\cdot,t)}{u_*} \right\} + \int_\Omega \left\{ v(\cdot,t) - v_* - v_* \ln \frac{v(\cdot,t)}{v_*} \right\} \\ &+ 2 \int_\Omega \left\{ w(\cdot,t) - w_* - w_* \ln \frac{w(\cdot,t)}{w_*} \right\} \quad \text{for all } t > 0,\end{aligned} \tag{5.1}$$

where u_*, v_* *and* w_* are given in (1.4). As a matter of fact, the above Lyapunov functional has been widely used in asymptotic analysis (cf. [33, 39], for instance). Similar to the previous work [4], we begin with some basic calculations and outline the main ideas of the proof.

Lemma 5.1. *Let*

$$\mu_1 < \mu_2 < 3\mu_1 \tag{5.2}$$

and

$$r = \mu_2 - \mu_1. \tag{5.3}$$

and let $L_w > 0$*. Then whenever* $\sqrt{\chi_1^2 + \chi_2^2} < M$*, where* $M = M(\mu_1, \mu_2, u_0, v_0, w_0) > 0$*, and* (u, v, w) *is a positive global classical solution of (1.1) in* $\overline{\Omega} \times (0, \infty)$ *with the initial data* (u_0, v_0, w_0) *fulfill (1.2) and*

$$\|w(\cdot,t)\|_{L^\infty(\Omega)} \le L_w \quad \textit{for all } t > 0, \tag{5.4}$$

then the Lyapunov functional (5.1) holds the decay property:

$$\Gamma'(t) \le -\frac{3\mu_1 - \mu_2}{2} \int_\Omega (u - u_*)^2 - \frac{\mu_1 + \mu_2}{2} \int_\Omega (v - v_*)^2 \quad \textit{for all } t > 0, \tag{5.5}$$

which implies that

$$u(\cdot,t) \to u_*, \;\; v(\cdot,t) \to v_* \;\; \textit{and} \;\; w(\cdot,t) \to w_* \quad \textit{in } L^\infty(\Omega) \textit{ as } t \to \infty. \tag{5.6}$$

Proof. Starting from structuring the functional (5.1), we divide the proof process into the following steps:

Step 1. Using the three equations in (1.1), and computing it straightforward to obtain

$$\begin{aligned}\frac{d}{dt}\Gamma(t) \le& \left\{ \left(\mu_1 u_*^2 + \mu_2 v_*^2 - r u_* v_* \right) - 2\left(\sqrt{2u_* w_*} + \sqrt{2v_* w_*} - w_* \right) \right\} \cdot |\Omega| \\ &+ (1 - a) \int_\Omega u + \left(1 - \frac{1}{a}\right) \int_\Omega v - \int_\Omega \left(\frac{2w_*}{w^2} - \frac{\chi_1^2 u_* + \chi_2^2 v_*}{4} \right) |\nabla w|^2 \\ &- \left(\mu_1 - \frac{r}{2} \right) \int_\Omega (u - u_*)^2 - \left(\mu_2 - \frac{r}{2} \right) \int_\Omega (v - v_*)^2 \quad \text{for all } t > 0.\end{aligned} \tag{5.7}$$

Under the assumption of (5.3) and the definition of a in (1.4), we obverse that

$$0 < a = \frac{\mu_2 - \mu_1 + \sqrt{(\mu_2 - \mu_1)^2 + 4\mu_1\mu_2}}{2\mu_2} = 1, \tag{5.8}$$

then we have

$$u_* = \frac{2}{\mu_1}, \quad v_* = \frac{2}{\mu_1}, \quad w_* = \frac{4}{\mu_1} \tag{5.9}$$

and

$$\begin{aligned}
&\mu_1 u_*^2 + \mu_2 v_*^2 - r u_* v_* - 2(\sqrt{2u_* w_*} + \sqrt{2 v_* w_*} - w_*) \\
&= 2\mu_1 \cdot \frac{4}{\mu_1^2} - 2\left(2\sqrt{2 \cdot \frac{2}{\mu_1} \cdot \frac{4}{\mu_1}} - \frac{4}{\mu_1}\right) = 0.
\end{aligned} \tag{5.10}$$

Let $M := \frac{4}{L_w}$, and in view of (5.9), we get

$$\begin{aligned}
-\left(\frac{2w_*}{w^2} - \frac{\chi_1^2 u_* + \chi_2^2 v_*}{4}\right) &\leq -\left(\frac{8}{\mu_1 L_w^2} - \frac{\chi_1^2 + \chi_2^2}{2\mu_1}\right) \\
&= -\frac{1}{2\mu_1}\left[M^2 - (\chi_1^2 + \chi_2^2)\right] \\
&\leq 0.
\end{aligned} \tag{5.11}$$

Without losing generality, we let $0 < \chi_1, \chi_2 < 1$. Then since the L^∞ of w only depends on the upper bound of χ_i $(i = 1, 2)$ and on the lower bound of μ_i $(i = 1, 2)$ as well as on the initial data (u_0, v_0, w_0), we notice that L_w has the following dependence relationship:

$$L_w = L_w(\mu_1, \mu_2, u_0, v_0, w_0).$$

Therefore, in view of (5.4), it is possible for us to fix some $M = M(\mu_1, \mu_2, u_0, v_0, w_0) > 0$ such that whenever $\sqrt{\chi_1^2 + \chi_2^2} < \min\{1, M\}$, (5.11) holds. Then collecting (5.7)–(5.11), we obtain (5.5).

Step 2. Applying Hölder regularity [42, Theorem 1.3] and [40] to establish weak convergence of u and v in $L^\infty(\Omega)$. According to (1.1), there exist $\theta \in (0, 1)$ and $C > 0$ such that

$$\|u\|_{C^{\theta,\frac{\theta}{2}}(\Omega\times[t,t+1])} + \|v\|_{C^{\theta,\frac{\theta}{2}}(\Omega\times[t,t+1])} + \|w\|_{C^{1+\theta,\frac{1+\theta}{2}}(\Omega\times[t,t+1])} \leq C \tag{5.12}$$

for all $t > 1$.

Step 3. Due to the fact that $s - 1 - \ln s \geq 0$ for all $s > 0$, Γ is non-negative. In view of (5.5), we obverse that $\Gamma(1)$ is finite due to positivity of $(u(\cdot, 1), v(\cdot, 1), w(\cdot, 1))$ in $\overline{\Omega}$, then integrating (5.5) in time shows that

$$\int_1^\infty \int_\Omega (u - u_*)^2 + \int_1^\infty \int_\Omega (v - v_*)^2 < \infty, \tag{5.13}$$

which implies that

$$u(\cdot, t) \to u_* \; and \; v(\cdot, t) \to v_* \quad \text{in } L^\infty(\Omega) \text{ as } t \to \infty, \tag{5.14}$$

due to the contradiction argument (cf. [41], for instance).

Step 4. Multiplying the third equation in (1.1) by $w - w_*$, using Young's inequality again to obtain

$$\int_1^\infty \int_\Omega (w - w_*)^2 < \infty, \tag{5.15}$$

which implies

$$w(\cdot, t) \to w_* \quad \text{in } L^\infty(\Omega) \text{ as } t \to \infty. \tag{5.16}$$

This completes the proof.

We are now in a position to prove Theorem 1.2.

Proof of Theorem 1.2. Theorem 1.2 is a direct consequence of Lemma 5.1, and the interested readers may refer to [4, Lemma 6.1] for a detailed proof. □

6. Conclusions

In this work, we considered the dynamics behavior of a three-component chemotaxis system on alopecia areata in the higher-dimensional case. We mainly proved the existence of the global bounded classical solution for the discussed chemotaxis system if μ_i $(i = 1, 2)$ is large enough. In the further work, we will consider this chemotaxis system with the nonlinear self-diffusion, nonlinear chemotactic sensitivity and the generalized logistic sources. Moreover, the chemotaxis system with singular chemotactic sensitivity can also be discussed.

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Conflict of interest

The authors declare there is no conflict of interest.

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