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Research article

On a hyperbolic-parabolic chemotaxis system

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Abstract: Stability of steady state solutions associated with initial and boundary value problems of a coupled fluid-reaction-diffusion system in one space dimension is analyzed. It is shown that under Dirichlet-Dirichlet type boundary conditions, non-trivial steady state solutions exist and are locally stable when the system parameters satisfy certain constraints.

Keywords: hyperbolic-parabolic system; chemotaxis; initial-boundary value problem; classical solution; global existence; long-time behavior

1. Introduction

Chemotaxis is a phenomenon describing the influence of environmental chemical substances on the motion of various cells. Chemotaxis widely exists in various biological phenomena, such as cells aggregation [1], embryonic development [2], vascular network formation [3, 4], etc. This paper is concerned with the following hyperbolic-parabolic chemotaxis system describing vasculogenesis

$$\begin{cases} \partial_t\rho + \nabla\cdot(\rho\mathbf{u}) = 0, \\ \partial_t(\rho\mathbf{u}) + \nabla\cdot(\rho\mathbf{u}\otimes\mathbf{u}) + \nabla P(\rho) = -\alpha\rho\mathbf{u} + \beta\rho\nabla\Phi, \\ \tau\partial_t\Phi = d\Delta\Phi - a\Phi + b\rho, \end{cases} \tag{1.1}$$

where $(\mathbf{x}, t) \in \Omega\times(0,\infty)$. The model (1.1) was proposed in [5] to reproduce key features of experiments of *in vitro* formation of blood vessels showing that cells randomly spreading on a gel matrix autonomously organize to a connected vascular network (more extensive modeling details can be found in [6]), where the unknowns $\rho = \rho(\mathbf{x}, t) \geq 0$ and $\mathbf{u} = \mathbf{u}(\mathbf{x}, t) \in \Omega$ denote the density and velocity of endothelial cells, respectively, and $\Phi = \Phi(\mathbf{x}, t) \geq 0$ denotes the concentration of the chemoattractant secreted by the endothelial cells. The convection term $\nabla\cdot(\rho\mathbf{u}\otimes\mathbf{u})$ models the cell movement persistence (inertial effect), $P(\rho)$ is the cell-density dependent pressure function accounting for the fact that closely packed cells resist to compression due to the impenetrability of cellular matter, $-\alpha\rho\mathbf{u}$ corresponds to a damping (friction) force with coefficient $\alpha > 0$ as a result of the interaction between cells and the underlying substratum and the quantity $|\beta| > 0$ measures the intensity of cell response to the chemical concentration gradient, where $\beta > 0$ (resp. $\beta < 0$) means the chemotaxis is attractive (resp. repulsive) (cf. [7, 8]). In this paper, we consider attractive chemotaxis. $\tau \geq 0$ and $d > 0$ are the relaxation time scale and diffusion coefficient of the chemoattractant, respectively, and the positive constants a and b denote the death and secretion rates of the chemoattractant, respectively. In the literature (cf. [9]), some parabolic-hyperbolic chemotaxis models with different structures than (1.1) have also been studied.

Chavanis and Sire obtained through asymptotic analysis in [10] that when the damping coefficient β is large, the solution of (1.1) converges to the solution of the Keller-Segel model. Subsequently, this has also been verified from the mathematical analysis in [11]. Natalini et al. in [12] numerically studied the difference and connection between the model (1.1) and the Keller-Segel chemotaxis model. By adding a viscous term $\Delta\mathbf{u}$ to the second equation of (1.1), the linear stability of the constant ground state $[\bar{\rho}, 0, \bar{\Phi}]$ was obtained in [13] under the condition

$$bP'(\bar{\rho}) - a\alpha\bar{\rho} > 0. \tag{1.2}$$

When the initial value $[\rho_0, \mathbf{u}_0, \Phi_0] \in [H^s(\mathbb{R}^d)]^{d+2} (s > d/2 + 1)$ is a small perturbation of the constant ground state $[\bar{\rho}, \mathbf{0}, \bar{\Phi}]$ with $\bar{\rho} > 0$ sufficiently small, it was shown in [14, 15] that the system (1.1) admits global strong solutions without vacuum converging to $[\bar{\rho}, \mathbf{0}, \bar{\Phi}]$ with an algebraic rate $(1 + t)^{-\frac{3}{4}}$ as $t \to \infty$. In [16], when the pressure function P satisfies (1.2), the authors removed the limitation that the density is sufficiently small, obtained the global existence of classical solutions to (1.1), and improved the decay rates of the solutions. Subsequently, in [17], the authors also proved that the system (1.1) in $\mathbb{R}$ admits nonlinear diffusion waves which are stable against a small perturbation. Recently in [18], the well-posedness of global classical solutions to the Cauchy problem of (1.1) is established in homogeneous hybrid Besov spaces.

All the studies above are on the Cauchy problem of (1.1), and the problem becomes more complicated when boundary effects are considered. Subsequently, the stationary solutions of (1.1) with vacuum (bump solutions) in a bounded interval with zero-flux boundary condition were constructed in [19, 20]. Recently, the stability of transition layer solutions of (1.1) on $\mathbb{R}_+ = [0, \infty)$ was established in [21]. An interesting question is whether the stability of non-constant stationary solutions of (1.1) can be considered in bounded regions. In the following, we will consider the hyperbolic-parabolic chemotaxis system (1.1) on $I = [0, 1]$ with $P = A_0\rho^2$:

$$\begin{cases} \rho_t + (\rho u)_x = 0, & x \in I,\ t > 0, \\ \rho u_t + \rho u u_x + 2A_0\rho\rho_x = -\alpha\rho u + \beta\rho\Phi_x, & x \in I,\ t > 0, \\ \tau\Phi_t = d\Phi_{xx} - a\Phi + b\rho, & x \in I,\ t > 0; \\ (\rho, u, \Phi)(x, 0) = (\rho_0, u_0, \Phi_0)(x), & x \in I; \\ u|_{x=0,\,x=1} = 0, \quad \Phi|_{x=0,\,x=1} = 0, & t > 0, \end{cases} \tag{1.3}$$

where A_0 is a positive constant. In this paper, we shall first use delicate analysis to show that the system (1.3) has a unique non-constant stationary solution. Then we show that the stationary solution is locally asymptotically stable when the system parameters satisfy certain constraints.

2. Main results

2.1. Stationary solution

To identify the stationary solution associated with the initial-boundary value problem (1.3), we first note that because of the dissipation mechanism induced by linear damping and the boundary condition for u, it is reasonable to expect that the equilibrium velocity is zero. Denote respectively the equilibrium density and concentration by $\widehat{\rho}$ and $\widehat{\Phi}$. Taking into account the zero equilibrium velocity, we see that $\widehat{\rho}$ and $\widehat{\Phi}$ satisfy

$$\begin{cases} 2A_0\widehat{\rho}\widehat{\rho}_x = \beta\widehat{\rho}\widehat{\Phi}_x, & x \in I,\ t > 0, \\ d\widehat{\Phi}_{xx} - a\widehat{\Phi} + b\widehat{\rho} = 0, & x \in I,\ t > 0; \\ \widehat{\Phi}|_{x=0,\,x=1} = 0, & t > 0. \end{cases} \tag{2.1}$$

Lemma 2.1. *The boundary value problem* (2.1) *admits a unique solution* $(\widehat{\rho}, \widehat{\Phi})$. *Moreover,* $\frac{\mathrm{d}\widehat{\rho}(x)}{\mathrm{d}x}$ *and* $\frac{\mathrm{d}^2\widehat{\rho}(x)}{\mathrm{d}x^2}$ *are small, when d is large while a, b,* A_0*, and* β *are fixed.*

Proof. The first equation of (2.1) implies $2A_0\widehat{\rho} = \beta\widehat{\Phi}+\widehat{C}$, for some constant $\widehat{C}$ which is to be determined later. Substituting $\widehat{\rho} = \frac{\beta}{2A_0}\widehat{\Phi} + \frac{\widehat{C}}{2A_0}$ into the second equation of (2.1), we have

$$d\widehat{\Phi}_{xx} - a\widehat{\Phi} + \frac{b\beta}{2A_0}\widehat{\Phi} + \frac{b\widehat{C}}{2A_0} = 0. \tag{2.2}$$

Let

$$\Lambda \equiv \frac{b\beta - 2aA_0}{2dA_0}, \quad \widehat{D} \equiv \frac{b\widehat{C}}{b\beta - 2aA_0}, \quad \Psi \equiv \widehat{\Phi} - \widehat{D}. \tag{2.3}$$

Then we derive from (2.2) that

$$\Psi_{xx} - \Lambda\Psi = 0. \tag{2.4}$$

Now we discuss the three cases regarding the sign of Λ.

Case I. If $\Lambda = 0$ (i.e., $b\beta = 2aA_0$), (2.2) can be written directly as

$$\widehat{\Phi}_{xx} + \frac{b\widehat{C}}{2dA_0} = 0.$$

Then we obtain $\widehat{\Phi}(x) = -\frac{b\widehat{C}}{2dA_0}(\frac{x^2}{2} + Ax + B)$ for some constants A and B. Since $\widehat{\Phi} = 0$ when $x = 0$ and $x = 1$, we can deduce that $\widehat{\Phi}(x) = -\frac{b\widehat{C}}{4dA_0}(x^2 - x)$ for all $x \in [0, 1]$. This implies

$$\widehat{\rho}(x) = -\frac{b\beta\widehat{C}}{8dA_0^2}\left(x^2 - x\right) + \frac{\widehat{C}}{2A_0}. \tag{2.5}$$

Since the total cellular mass is conserved under the zero velocity boundary condition, we can show that

$$\int_0^1 \widehat{\rho}(x)\mathrm{d}x = \frac{b\beta\widehat{C}}{48dA_0^2} + \frac{\widehat{C}}{2A_0} = \int_0^1 \rho_0(x)\mathrm{d}x.$$

This implies, if the initial total mass is positive,

$$0 < A_0 \int_0^1 \rho_0(x)\mathrm{d}x < \widehat{C} < 4A_0 \int_0^1 \rho_0(x)\mathrm{d}x, \tag{2.6}$$

when $d > 0$ is sufficiently large while the other parameters are fixed. Taking into consideration that ρ represents the cell density, we should require $\widehat{\rho}(x) > 0$ for all $x \in [0, 1]$. Moreover, the reader will see from the asymptotic analysis presented below that $\widehat{\rho}$ needs to be bounded from above and below away from zero, in order to obtain the stability of the non-constant stationary solution. Combining (2.5) and (2.6), and noting $x - x^2 \leq \frac{1}{4}$ for all $x \in [0, 1]$, we see that

$$0 < \frac{1}{4} \int_0^1 \rho_0(x)\mathrm{d}x < \widehat{\rho}(x) < 4 \int_0^1 \rho_0(x)\mathrm{d}x, \quad \forall\, x \in [0, 1], \tag{2.7}$$

when d is relatively large compared with the other parameters. This gives us the desired property of the equilibrium density.

Meanwhile, in the asymptotic analysis we will require $\frac{\mathrm{d}\widehat{\rho}(x)}{\mathrm{d}x}$ and $\frac{\mathrm{d}^2\widehat{\rho}(x)}{\mathrm{d}x^2}$ to be relatively small compared with $\widehat{\rho}$. To fulfill such a requirement, we observe that

$$\begin{aligned} \frac{\mathrm{d}\widehat{\rho}(x)}{\mathrm{d}x} &= -\frac{b\beta\widehat{C}x}{4dA_0^2} + \frac{b\beta\widehat{C}}{8dA_0^2} \to 0, \quad \text{as} \quad d \to \infty, \\ \frac{\mathrm{d}^2\widehat{\rho}(x)}{\mathrm{d}x^2} &= -\frac{b\beta\widehat{C}}{4dA_0^2} \to 0, \quad \text{as} \quad d \to \infty. \end{aligned}$$

Hence, in the case $\Lambda = 0$, the smallness of $\frac{\mathrm{d}\widehat{\rho}(x)}{\mathrm{d}x}$ and $\frac{\mathrm{d}^2\widehat{\rho}(x)}{\mathrm{d}x^2}$ can be realized, when d is sufficiently large, while a, b, A_0 and β are fixed.

Case II. When $\Lambda > 0$, let $\lambda = \sqrt{\Lambda}$. Then we have from (2.4) that $\Psi(x) = Ae^{\lambda x} + Be^{-\lambda x}$ for some constants A and B, which implies $\widehat{\Phi}(x) = Ae^{\lambda x} + Be^{-\lambda x} + \widehat{D}$. Using the boundary conditions, we can show that

$$\widehat{\Phi}(x) = \frac{\widehat{D}}{e^{\lambda} + 1}(e^{\lambda} + 1 - e^{\lambda x} - e^{\lambda(1-x)}). \tag{2.8}$$

Since $\widehat{\rho} = \frac{\beta}{2A_0}\widehat{\Phi} + \frac{\widehat{C}}{2A_0}$, we obtain

$$\widehat{\rho}(x) = \frac{\beta\widehat{D}}{2A_0(e^{\lambda} + 1)}(e^{\lambda} + 1 - e^{\lambda x} - e^{\lambda(1-x)}) + \frac{\widehat{C}}{2A_0}.$$

Using the conservation of total mass again, we can show that

$$\int_0^1 \widehat{\rho}(x)\mathrm{d}x = \frac{\beta\widehat{D}(\lambda e^{\lambda} + \lambda - 2e^{\lambda} + 2)}{2A_0(e^{\lambda} + 1)\lambda} + \frac{\widehat{C}}{2A_0} = \int_0^1 \rho_0(x)\mathrm{d}x.$$

Recalling the definition of $\widehat{D}$ (see (2.3)), we have

$$\widehat{C}\left[\frac{b\beta(\lambda e^{\lambda} + \lambda - 2e^{\lambda} + 2)}{(b\beta - 2aA_0)(e^{\lambda} + 1)\lambda} + 1\right] = 2A_0 \int_0^1 \rho_0(x)\mathrm{d}x,$$

which yields

$$\widehat{C} = 2A_0\left(\int_0^1 \rho_0(x)\mathrm{d}x\right)\left[\frac{b\beta}{b\beta - 2aA_0}\, f_1(\lambda) + 1\right]^{-1},$$

where

$$f_1(\lambda) = \frac{\lambda e^{\lambda} + \lambda - 2e^{\lambda} + 2}{(e^{\lambda} + 1)\lambda}.$$

Therefore,

$$\widehat{\rho}(x) = \frac{\widehat{C}}{2A_0}\left[\frac{b\beta}{b\beta - 2aA_0}\, g_1(\lambda; x) + 1\right],$$

where

$$g_1(\lambda; x) = \frac{e^{\lambda} + 1 - e^{\lambda x} - e^{\lambda(1-x)}}{e^{\lambda} + 1}. \tag{2.9}$$

To guarantee $\widehat{\rho}$ is bounded from above and below away from zero, we first note $f_1(\lambda) \in (0, 1)$ for $\lambda \in (0, \infty)$ and $\lim_{\lambda\to 0} f_1(\lambda) = 0$. Second, observe that since $a > 0$, $b > 0$, $A_0 > 0$ and $d > 0$, then $\Lambda > 0$ if and only if $b\beta > 2aA_0 > 0$. This implies $\frac{b\beta}{b\beta - 2aA_0} > 0$. Hence, when a, b, A_0 and β are fixed, there exists a small number $\lambda_0 > 0$ such that

$$1 < \frac{b\beta}{b\beta - 2aA_0}\, f_1(\lambda) + 1 < 2, \quad \forall\, \lambda \in (0, \lambda_0).$$

This implies

$$A_0 \int_0^1 \rho_0(x)\mathrm{d}x < \widehat{C} < 2A_0 \int_0^1 \rho_0(x)\mathrm{d}x, \quad \forall\, \lambda \in (0, \lambda_0). \tag{2.10}$$

Moreover, note that for all $x \in [0, 1]$, it holds that

$$0 \le g_1(\lambda; x) \le g_1(\lambda; 1/2) = \frac{(e^{\frac{\lambda}{2}} - 1)^2}{e^{\lambda} + 1} \to 0, \quad \text{as} \quad \lambda \to 0.$$

Therefore, there exists a small number $\lambda_1 > 0$ such that

$$1 < \frac{b\beta}{b\beta - 2aA_0}\, g_1(\lambda; x) + 1 < 2, \quad \forall\, x \in [0, 1], \quad \forall\, \lambda \in (0, \lambda_1),$$

which implies

$$\frac{\widehat{C}}{2A_0} < \widehat{\rho}(x) < \frac{\widehat{C}}{A_0}, \quad \forall\, x \in [0, 1], \quad \forall\, \lambda \in (0, \lambda_1).$$

In view of (2.10), we see that

$$\frac{1}{2}\int_0^1 \rho_0(x)\mathrm{d}x < \widehat{\rho}(x) < 2\int_0^1 \rho_0(x)\mathrm{d}x, \quad \forall\, x \in [0, 1], \quad \forall\, \lambda \in (0, \min\{\lambda_0, \lambda_1\}). \tag{2.11}$$

Since $\lambda = \sqrt{\Lambda} = \sqrt{\frac{b\beta - 2aA_0}{2dA_0}}$, the smallness of λ can be realized when d is sufficiently large, while a, b, A_0 and β are fixed. We also observe that

$$\frac{\mathrm{d}\widehat{\rho}(x)}{\mathrm{d}x} = \frac{\widehat{C}}{2A_0} \cdot \frac{b\beta}{b\beta - 2aA_0} \cdot \frac{-\lambda e^{\lambda x} + \lambda e^{\lambda(1-x)}}{e^{\lambda} + 1} \to 0, \quad \text{as} \quad \lambda \to 0,$$
$$\frac{\mathrm{d}^2\widehat{\rho}(x)}{\mathrm{d}x^2} = \frac{\widehat{C}}{2A_0} \cdot \frac{b\beta}{b\beta - 2aA_0} \cdot \frac{-\lambda^2 e^{\lambda x} - \lambda^2 e^{\lambda(1-x)}}{e^{\lambda} + 1} \to 0, \quad \text{as} \quad \lambda \to 0.$$

Hence, the smallness of $\frac{\mathrm{d}\widehat{\rho}(x)}{\mathrm{d}x}$ and $\frac{\mathrm{d}^2\widehat{\rho}(x)}{\mathrm{d}x^2}$ can be realized when λ is sufficiently small, or equivalently, when d is large while a, b, A_0 and β are fixed.

Case III. When $\Lambda < 0$, let $\lambda = \sqrt{-\Lambda}$. Then we have $\widehat{\Phi}(x) = A\cos\lambda x + B\sin\lambda x + \widehat{D}$. Using the boundary conditions, we get $A + \widehat{D} = 0$ and $A\cos\lambda + B\sin\lambda + \widehat{D} = 0$. Note that B is uniquely determined if $\lambda \neq k\pi$ ($k \in \mathbb{N}$). In this case, we have $A = -\widehat{D}$ and $B = \frac{\cos\lambda - 1}{\sin\lambda}\widehat{D}$. Hence, $\widehat{\Phi}$ is given by

$$\widehat{\Phi}(x) = \widehat{D}\Big(\frac{\cos\lambda - 1}{\sin\lambda}\sin\lambda x - \cos\lambda x + 1\Big).$$

Following the same spirit as in **Case II**, we can show that

$$\widehat{\rho}(x) = \frac{\widehat{C}}{2A_0}\left[\frac{b\beta}{b\beta - 2aA_0}\, g_2(\lambda; x) + 1\right], \tag{2.12}$$

where

$$g_2(\lambda; x) = \frac{\sin\lambda - \sin\lambda\cos\lambda x + \cos\lambda\sin\lambda x - \sin\lambda x}{\sin\lambda},$$

and the constant $\widehat{C}$ is given by

$$\widehat{C} = 2A_0\left(\int_0^1 \rho_0(x)\mathrm{d}x\right)\left[\frac{b\beta}{b\beta - 2aA_0}\, f_2(\lambda) + 1\right]^{-1},$$

where

$$f_2(\lambda) = \frac{\lambda\sin\lambda - 2 + 2\cos\lambda}{\lambda\sin\lambda}.$$

Note that $\Lambda < 0$ if and only if $b\beta < 2aA_0$, and

$$\frac{b\beta}{b\beta - 2aA_0}\begin{cases} > 0 & \text{if} \quad a > 0 \quad \text{and} \quad \beta < 0, \\ < 0 & \text{if} \quad a > 0 \quad \text{and} \quad \beta > 0. \end{cases} \tag{2.13}$$

The function $f_2(\lambda)$ satisfies

$$\lim_{\lambda\to 0} f_2(\lambda) = 0 \quad \text{and} \quad f_2'(\lambda) = \frac{2(\cos\lambda - 1)(\lambda - \sin\lambda)}{\lambda^2\sin^2\lambda}.$$

These imply when λ is sufficiently close to zero, $f_2(\lambda) < 0$ is sufficiently small. Regarding the two cases in (2.13), we can show that there exists a small number $\lambda_2 > 0$, such that for all $\lambda \in (0, \lambda_2)$,

$$\left[\frac{b\beta}{b\beta - 2aA_0} f_2(\lambda) + 1\right] \in \begin{cases} (0.5, 1) & \text{if} \quad a > 0 \quad \text{and} \quad \beta < 0, \\ (1, 2) & \text{if} \quad a > 0 \quad \text{and} \quad \beta > 0. \end{cases}$$

In summary, the constant $\widehat{C}$ satisfies

$$A_0 \int_0^1 \rho_0(x)\mathrm{d}x < \widehat{C} < 4A_0 \int_0^1 \rho_0(x)\mathrm{d}x, \quad \forall\, \lambda \in (0, \lambda_2). \tag{2.14}$$

For $g_2(\lambda; x)$, we can show that

$$\begin{aligned} & g_2(\lambda; 0) = 0 = g_2(\lambda; 1), \\ & \frac{\mathrm{d}g_2(\lambda; x)}{\mathrm{d}x} = \frac{\lambda(\cos\lambda(1-x) - \cos\lambda x)}{\sin\lambda}, \\ & \frac{\mathrm{d}^2 g_2(\lambda; x)}{\mathrm{d}x^2} = \frac{\lambda^2(\sin\lambda(1-x) + \sin\lambda x)}{\sin\lambda}, \end{aligned} \tag{2.15}$$

which imply $\frac{\mathrm{d}g_2(\lambda;x)}{\mathrm{d}x} = 0$ when $x = 0.5$, and $g_2(\lambda; x)$ is convex for $x \in [0, 1]$ when $\lambda > 0$ is sufficiently small. These tell us $g_2(\lambda; x) \le 0$ for $x \in [0, 1]$ and

$$|g_2(\lambda; x)| \le -g_2(\lambda; 0.5) = \sec 0.5\lambda - 1,$$

when $\lambda > 0$ is sufficiently small. Hence, there exists a small number $\lambda_3 > 0$, such that for all $x \in [0, 1]$,

$$\left[\frac{b\beta}{b\beta - 2aA_0} g_2(\lambda; x) + 1\right] \in \begin{cases} (0.5, 1) & \text{if} \quad a > 0 \quad \text{and} \quad \beta < 0, \\ (1, 2) & \text{if} \quad a > 0 \quad \text{and} \quad \beta > 0. \end{cases}$$

Therefore,

$$\frac{\widehat{C}}{4A_0} < \widehat{\rho}(x) < \frac{\widehat{C}}{A_0}, \quad \forall\, x \in [0, 1], \quad \forall\, \lambda \in (0, \lambda_3). \tag{2.16}$$

Combining (2.14) and (2.16), we see that

$$\frac{1}{4} \int_0^1 \rho_0(x)\mathrm{d}x < \widehat{\rho}(x) < 4 \int_0^1 \rho_0(x)\mathrm{d}x, \quad \forall\, x \in [0, 1], \quad \forall\, \lambda \in (0, \min\{\lambda_2, \lambda_3\}).$$

In addition, we see from (2.12) and (2.15) that

$$\left|\frac{\mathrm{d}\widehat{\rho}(x)}{\mathrm{d}x}\right| \to 0 \quad \text{and} \quad \left|\frac{\mathrm{d}^2\widehat{\rho}(x)}{\mathrm{d}x^2}\right| \to 0 \quad \text{as} \quad \lambda \to 0.$$

Thus, the smallness of the derivatives of $\widehat{\rho}$ can be realized when $\lambda > 0$ is sufficiently small. Combining the conclusions of the above three cases, we have completed the proof of Lemma 2.1. □

2.2. Stability of stationary solution

With the preliminary discussions in §2.1, we now state the main results for (1.3). We first introduce some notations for convenience.

Notation 2.1. *Throughout this paper, we use* $\|\cdot\|_{L^2}$, $\|\cdot\|_{H^s}$, *and* $\|\cdot\|_{L^\infty}$ *to denote the norms of the standard Lebesgue space* $L^2((0,1))$, *Hilbert space* $H^s((0,1))$, *and Sobolev space* $L^\infty((0,1))$, *respectively. The total energy, of order* $\mathfrak{s} \in \mathbb{N}$, *of the function* f *is denoted by*

$$\|f(t)\|_{\mathfrak{s}}^2 \equiv \sum_{k=0}^{\mathfrak{s}} \|(\partial_t^k f)(t)\|_{H^{\mathfrak{s}-k}}^2. \tag{2.17}$$

In addition, we use $\|(f_1, f_2, ..., f_n)\|_\star^2$ *to denote* $\|f_1\|_\star^2 + \|f_2\|_\star^2 + \cdots \|f_n\|_\star^2$, *where* $\star$ *denotes either* L^2, H^s, L^∞, *or* $\mathfrak{s}$, *whenever it is applicable. Unless otherwise specified,* C *will denote a generic positive constant which is independent of time. The value of the constant may vary line by line according to the context.*

Theorem 2.2. *Consider the initial-boundary value problem* (1.3), *where the parameters satisfy* $\alpha > 0$, $\tau \geq 0$, $d > 0$, $A_0 > 0$, $a > 0$, $b > 0$, *and* $\beta \in \mathbb{R}$. *Suppose the initial data satisfy* $\rho_0(x) > 0$ *for all* $x \in [0,1]$, $(\rho_0, u_0) \in [H^2((0,1))]^2$, $\Phi_0 \in H^4((0,1))$, *and are compatible with the boundary conditions. Assume further that there is a small constant* $\varepsilon_0 > 0$, *such that* $\|(u_0, \rho_0 - \widehat{\rho})\|_{H^2}^2 + \|\Phi_0 - \widehat{\Phi}\|_{H^4}^2 \leq \varepsilon_0$, *and there is a large constant* $d_0 > 0$, *such that the diffusion coefficient* $d \geq d_0$. *Then there exists a unique solution to* (1.3), *which satisfies*

$$\begin{aligned}
&\|(\widetilde{\rho}, u)(t)\|_2^2 + \sum_{k=0}^{2} \|(\partial_t^k \widetilde{\Phi})(t)\|_{H^{4-2k}}^2 \\
&\quad + \int_0^t \left(\|(\widetilde{\rho}, u)(\tau)\|_2^2 + \|\widetilde{\Phi}(\tau)\|_{H^4}^2 + \|\widetilde{\Phi}_t(\tau)\|_{H^3}^2 + \|\widetilde{\Phi}_{tt}(\tau)\|_{H^1}^2\right) d\tau \leq C, \quad \forall\, t > 0,
\end{aligned}$$

where $\widetilde{\rho} = \rho - \widehat{\rho}$, $\widetilde{\Phi} = \Phi - \widehat{\Phi}$, *and the constant* $C > 0$ *is independent of* $t > 0$. *Moreover, there are positive constants* η_1 *and* η_2 *which are independent of* $t > 0$, *such that*

$$\|(\widetilde{\rho}, u)(t)\|_2^2 + \sum_{k=0}^{2} \|(\partial_t^k \widetilde{\Phi})(t)\|_{H^{4-2k}}^2 \leq \eta_1 e^{-\eta_2 t}, \quad \forall\, t > 0.$$

We prove Theorem 2.2 by applying L^2-based energy methods. First of all, it should be mentioned that the local well-posedness of classical solutions to (3.3) can be established by using some classical approaches in the literature, see e.g., [22, 23], and we omit the details to simplify the presentation. The bulk of this paper is devoted to deriving the *a priori* estimates of the local solution, in order to extend it to a global one. We begin the proof with reformulating the first and second equations in (1.3) by using the sound speed transformation to obtain a symmetric hyperbolic system. Note that the boundary data of the spatial derivatives of the solution are unknown. Hence, the direct energy method (differentiating with respect to x, then performing L^2-type estimates) is not accessible for the problem under consideration. One of the key steps in the proof is to reduce the estimate of the total (spatial and temporal) derivatives to the temporal ones only, using an iteration scheme based on the structure of the equations. Moreover, note that in the hyperbolic portion of the system, only the dissipation of u appears on the right-hand side of the second equation. We recover the dissipation mechanism of ρ by essentially working a wave-type equation of the function.

3. Proof of Theorem 2.2

In this section, we give a proof of Theorem 2.2. The proof consists of three major steps: 1) apply the sound speed transformation to symmetrize the first two equations in (1.3); 2) reduce the estimate of the total (spatial and temporal) derivatives of the solution to the temporal ones only; 3) perform L^2-based energy estimates. We first present the symmetrization process.

3.1. Reformulation

Since the principle part of the first two equations in (1.3) is hyperbolic, one needs to introduce an appropriate new variable to symmetrize these two equations, after which one can carry out L^2-based energy estimates. For this purpose, we let $\sigma = 2\sqrt{2A_0\rho}$ be the sound speed. Then the initial-boundary value problem (1.3) can be written in terms of σ, in the regime of classical solutions, as

$$\begin{cases} 2\sigma_t + 2u\sigma_x + \sigma u_x = 0, & x \in I,\ t > 0, \\ 2u_t + 2uu_x + \sigma\sigma_x = -2\alpha u + 2\beta\Phi_x, & x \in I,\ t > 0, \\ 8\tau A_0\Phi_t = 8dA_0\Phi_{xx} - 8aA_0\Phi + b\sigma^2, & x \in I,\ t > 0; \\ (\sigma, u, \Phi)(x,0) = (2\sqrt{2A_0\rho_0}, u_0, \Phi_0)(x), & x \in I; \\ u|_{x=0,\,x=1} = 0, \quad \Phi|_{x=0,\,x=1} = 0, & t > 0. \end{cases} \tag{3.1}$$

To perform asymptotic analysis, leading to the global dynamics of the solution to (3.1), we need to write the system of equations in (3.1) in terms of the perturbed variables around the stationary solution. Since the stationary solution satisfies (2.1), letting $\widehat{\sigma} = 2\sqrt{2A_0\widehat{\rho}}$, we can show that

$$\begin{cases} \widehat{\sigma}\,\widehat{\sigma}_x = 2\beta\widehat{\Phi}_x, \\ 8dA_0\widehat{\Phi}_{xx} - 8aA_0\widehat{\Phi} + b\widehat{\sigma}^2 = 0. \end{cases} \tag{3.2}$$

Letting $\widetilde{\sigma} = \sigma - \widehat{\sigma}$ and $\widetilde{\Phi} = \Phi - \widehat{\Phi}$, we update (3.1) by using (3.2) as

$$\begin{cases} 2\widetilde{\sigma}_t + 2u\widetilde{\sigma}_x + \widetilde{\sigma}u_x + 2u\widehat{\sigma}_x + \widehat{\sigma}u_x = 0, & x \in I,\ t > 0, \\ 2u_t + 2uu_x + \widetilde{\sigma}\,\widetilde{\sigma}_x + \widetilde{\sigma}\,\widehat{\sigma}_x + \widehat{\sigma}\,\widetilde{\sigma}_x = -2\alpha u + 2\beta\widetilde{\Phi}_x, & x \in I,\ t > 0, \\ 8\tau A_0\widetilde{\Phi}_t = 8dA_0\widetilde{\Phi}_{xx} - 8aA_0\widetilde{\Phi} + b(\widetilde{\sigma} + 2\widehat{\sigma})\widetilde{\sigma}, & x \in I,\ t > 0; \\ (\widetilde{\sigma}, u, \widetilde{\Phi})(x,0) = (2\sqrt{2A_0\rho_0} - 2\sqrt{2A_0\widehat{\rho}}, u_0, \Phi_0 - \widehat{\Phi})(x), & x \in I; \\ u|_{x=0,\,x=1} = 0, \quad \widetilde{\Phi}|_{x=0,\,x=1} = 0, & t > 0. \end{cases} \tag{3.3}$$

The energy estimates derived in the rest of this section are based on the *a priori* assumption:

$$\operatorname*{ess\,sup}_{t\in[0,T]} X(t) \equiv \operatorname*{ess\,sup}_{t\in[0,T]} \|(\widetilde{\sigma}, u, \widetilde{\Phi})(t)\|_2^2 \le \varepsilon^2, \tag{3.4}$$

where $T > 0$ denotes the lifespan of the local solution and $\varepsilon > 0$ is a small number to be determined later. Note the smallness of ε can be realized by the smallness assumption of the initial perturbation in Theorem 2.2 and the local well-posedness theory. We will focus on deriving the time-independent *a priori* estimates of the local solution under (3.4), which, when combined with standard continuation argument, will generate the global well-posedness and long-time behavior of the solution in one stroke.

The rest of the proof consists of two major steps which are contained in two subsections. As was discussed in §2.1, the stationary solution takes on different forms, depending on the sign of $b\beta - 2aA_0$. In the analysis presented below, we shall focus on the case when $b\beta - 2aA_0 > 0$, in which the stationary solution is given by (2.8)–(2.9). The other case, i.e., $b\beta - 2aA_0 \leq 0$, can be proved in exactly the same fashion, and we omit the details for brevity.

We first deal with the case of $\tau > 0$ in §3.2 and §3.3. The proof of the case of $\tau = 0$ will be sketched in §3.4. We begin with the reduction of the total derivatives of the solution to (3.3).

3.2. Reduction of total derivatives

Lemma 3.1. *Let* $(\widetilde{\sigma}, u, \widetilde{\Phi})$ *be the local solution to the IBVP* (3.3) *with* $\tau > 0$ *up to some finite time* $T > 0$*. Assume* (3.4) *holds for some small* $\varepsilon > 0$*. Then, under the conditions of Theorem 2.2, there exists a constant* $D_0 > 1$*, which is independent of t, such that*

$$X(t) \leq D_0 X_1(t) := D_0 \|(\widetilde{\sigma}_t, \widetilde{\sigma}_{tt}, u, u_t, u_{tt}, \widetilde{\Phi}_x, \widetilde{\Phi}_{xt}, \widetilde{\Phi}_{tt})(t)\|_{L^2}^2. \tag{3.5}$$

Proof. **Step 1.** We first derive a Poincaré-type inequality for $\widetilde{\sigma}$. From the discussions in §2.1 we infer that when the diffusion coefficient is sufficiently large, the stationary solution $\widehat{\rho}$ satisfies (2.11). Denote the spatial integral of ρ_0 by $\overline{\rho}$ (which is positive by the assumptions of Theorem 2.2). Then we have

$$\frac{1}{2}\overline{\rho} < \widehat{\rho} < 2\overline{\rho}. \tag{3.6}$$

According to the definition of $\widehat{\sigma}$, we know

$$2\sqrt{A_0\overline{\rho}} < \widehat{\sigma} < 4\sqrt{A_0\overline{\rho}}. \tag{3.7}$$

Note that by definition,

$$\widetilde{\sigma} = \sigma - \widehat{\sigma} = 2\sqrt{2A_0}(\sqrt{\rho} - \sqrt{\widehat{\rho}}) = 2\sqrt{2A_0}\,\frac{\rho - \widehat{\rho}}{\sqrt{\rho} + \sqrt{\widehat{\rho}}}. \tag{3.8}$$

Since ρ_0 is sufficiently close to $\widehat{\rho}$ (by assumptions of Theorem 2.2) and $\widehat{\rho} > \frac{1}{2}\overline{\rho} > 0$, from the local well-posedness theory we know $\rho(x, t)$ is positive within the lifespan of the local solution. Using such information, we deduce from (3.8) and (3.6) that

$$|\widetilde{\sigma}| \leq \frac{2\sqrt{2A_0}}{\sqrt{\widehat{\rho}}}\,|\rho - \widehat{\rho}| \leq \frac{4\sqrt{A_0}}{\sqrt{\overline{\rho}}}\,|\rho - \widehat{\rho}|,$$

which implies

$$\|\widetilde{\sigma}\|_{L^2} \leq \frac{4\sqrt{A_0}}{\sqrt{\overline{\rho}}}\,\|\rho - \widehat{\rho}\|_{L^2}. \tag{3.9}$$

Since $\rho - \widehat{\rho}$ is mean-free, it can be shown that

$$\|\rho - \widehat{\rho}\|_{L^2} \leq \|(\rho - \widehat{\rho})_x\|_{L^2}. \tag{3.10}$$

Since

$$\rho - \widehat{\rho} = \frac{\sigma^2 - \widehat{\sigma}^2}{8A_0} = \frac{\widetilde{\sigma}(\widetilde{\sigma} + 2\widehat{\sigma})}{8A_0},$$

we have

$$(\rho - \widehat{\rho})_x = \frac{\widetilde{\sigma}_x(\widetilde{\sigma} + 2\widehat{\sigma})}{8A_0} + \frac{\widetilde{\sigma}(\widetilde{\sigma}_x + 2\widehat{\sigma}_x)}{8A_0},$$

which implies

$$\|(\rho - \widehat{\rho})_x\|_{L^2} \leq \frac{(\|\widetilde{\sigma}\|_{L^\infty} + 2\|\widehat{\sigma}\|_{L^\infty})\|\widetilde{\sigma}_x\|_{L^2}}{8A_0} + \frac{(\|\widetilde{\sigma}_x\|_{L^\infty} + 2\|\widehat{\sigma}_x\|_{L^\infty})\|\widetilde{\sigma}\|_{L^2}}{8A_0}. \tag{3.11}$$

Using (3.11), we update (3.10) as

$$\|\rho - \widehat{\rho}\|_{L^2} \leq \frac{(\|\widetilde{\sigma}\|_{L^\infty} + 2\|\widehat{\sigma}\|_{L^\infty})\|\widetilde{\sigma}_x\|_{L^2}}{8A_0} + \frac{(\|\widetilde{\sigma}_x\|_{L^\infty} + 2\|\widehat{\sigma}_x\|_{L^\infty})\|\widetilde{\sigma}\|_{L^2}}{8A_0}. \tag{3.12}$$

Substituting (3.12) into (3.9), we arrive at

$$\begin{aligned}
\|\widetilde{\sigma}\|_{L^2} &\leq \frac{1}{2\sqrt{A_0\overline{\rho}}}\left[(\|\widetilde{\sigma}\|_{L^\infty} + 2\|\widehat{\sigma}\|_{L^\infty})\|\widetilde{\sigma}_x\|_{L^2} + (\|\widetilde{\sigma}_x\|_{L^\infty} + 2\|\widehat{\sigma}_x\|_{L^\infty})\|\widetilde{\sigma}\|_{L^2}\right] \\
&\leq \frac{1}{2\sqrt{A_0\overline{\rho}}}\left[(\sqrt{2}\,\|\widetilde{\sigma}\|_{H^1} + 8\sqrt{A_0\overline{\rho}})\|\widetilde{\sigma}_x\|_{L^2} + (\sqrt{2}\,\|\widetilde{\sigma}_x\|_{H^1} + 2\|\widehat{\sigma}_x\|_{L^\infty})\|\widetilde{\sigma}\|_{L^2}\right] \\
&\leq \frac{1}{2\sqrt{A_0\overline{\rho}}}\left[(\sqrt{2}\,\varepsilon + 8\sqrt{A_0\overline{\rho}})\|\widetilde{\sigma}_x\|_{L^2} + (\sqrt{2}\,\varepsilon + 2\|\widehat{\sigma}_x\|_{L^\infty})\|\widetilde{\sigma}\|_{L^2}\right],
\end{aligned} \tag{3.13}$$

where we used the 1D Sobolev inequality: $\|f\|_{L^\infty} \leq \sqrt{2}\,\|f\|_{H^1}$, (3.4) and (3.7). Since

$$\widehat{\sigma}_x = \frac{\sqrt{2A_0}}{\sqrt{\widehat{\rho}}}\widehat{\rho}_x,$$

using (3.6), we can show that

$$\|\widehat{\sigma}_x\|_{L^\infty} \leq \frac{\sqrt{2A_0}}{\sqrt{\underline{\rho}}}\|\widehat{\rho}_x\|_{L^\infty}. \tag{3.14}$$

From the discussions in §2.1 we know when d is large enough, $\|\widehat{\rho}_x\|_{L^\infty}$ is sufficiently small. In this case, we denote (3.14) by

$$\|\widehat{\sigma}_x\|_{L^\infty} \leq \delta, \tag{3.15}$$

where the constant δ decreases as d increases. Using (3.15), we update (3.13) as

$$\|\widetilde{\sigma}\|_{L^2} \leq \frac{1}{2\sqrt{A_0\overline{\rho}}}[(\sqrt{2}\,\varepsilon + 8\sqrt{A_0\overline{\rho}})\|\widetilde{\sigma}_x\|_{L^2} + (\sqrt{2}\,\varepsilon + 2\delta)\|\widetilde{\sigma}\|_{L^2}].$$

This implies when ε and δ are sufficiently small, such that

$$(\sqrt{2}\,\varepsilon + 2\delta) \leq \sqrt{A_0\overline{\rho}}, \tag{3.16}$$

it holds that

$$\|\widetilde{\sigma}\|_{L^2} \leq \frac{9}{2}\|\widetilde{\sigma}_x\|_{L^2} + \frac{1}{2}\|\widetilde{\sigma}\|_{L^2}.$$

Hence,

$$\|\widetilde{\sigma}\|_{L^2} \leq 9\|\widetilde{\sigma}_x\|_{L^2}. \tag{3.17}$$

Step 2. From the first equation of (3.3) we see that

$$u_x = -\frac{2}{\widetilde{\sigma} + \widehat{\sigma}}(\widetilde{\sigma}_t + u\widetilde{\sigma}_x + u\widehat{\sigma}_x). \tag{3.18}$$

Using (3.7), Sobolev embedding, (3.4), and (3.16), we deduce that

$$\|\widetilde{\sigma} + \widehat{\sigma}\|_{L^\infty} \geq \|\widehat{\sigma}\|_{L^\infty} - \|\widetilde{\sigma}\|_{L^\infty} \geq 2\sqrt{A_0\overline{\rho}} - \sqrt{2}\,\|\widetilde{\sigma}\|_{H^1} \geq 2\sqrt{A_0\overline{\rho}} - \sqrt{2}\,\varepsilon \geq \sqrt{A_0\overline{\rho}}. \tag{3.19}$$

Using (3.19), we deduce from (3.18) that

$$\|u_x\|_{L^2}^2 \leq \frac{12}{A_0\overline{\rho}}(\|\widetilde{\sigma}_t\|_{L^2}^2 + \|u\|_{L^\infty}^2\|\widetilde{\sigma}_x\|_{L^2}^2 + \|\widehat{\sigma}_x\|_{L^\infty}^2\|u\|_{L^2}^2). \tag{3.20}$$

Since u satisfies the zero boundary condition, it can be shown that

$$\|u\|_{L^2} \leq \|u_x\|_{L^2}. \tag{3.21}$$

Using Sobolev embedding, (3.4), (3.15) and (3.21), we update (3.20) as

$$\|u_x\|_{L^2}^2 \leq \frac{12}{A_0\overline{\rho}}(\|\widetilde{\sigma}_t\|_{L^2}^2 + 2\varepsilon^2\|\widetilde{\sigma}_x\|_{L^2}^2 + \delta^2\|u_x\|_{L^2}^2). \tag{3.22}$$

Now, from the second equation of (3.3) we see that

$$\widetilde{\sigma}_x = -\frac{1}{\widetilde{\sigma} + \widehat{\sigma}}(2u_t + 2uu_x + \widetilde{\sigma}\widehat{\sigma}_x + 2\alpha u - 2\beta\widetilde{\Phi}_x). \tag{3.23}$$

Using (3.19), we can show that

$$\|\widetilde{\sigma}_x\|_{L^2}^2 \leq \frac{5}{A_0\overline{\rho}}(4\|u_t\|_{L^2}^2 + 4\|u\|_{L^\infty}^2\|u_x\|_{L^2}^2 + \|\widehat{\sigma}_x\|_{L^\infty}^2\|\widetilde{\sigma}\|_{L^2}^2 + 4\alpha^2\|u\|_{L^2}^2 + 4\beta^2\|\widetilde{\Phi}_x\|_{L^2}^2). \tag{3.24}$$

Using (3.17), we update (3.24) as

$$\|\widetilde{\sigma}_x\|_{L^2}^2 \leq \frac{5}{A_0\overline{\rho}}(4\|u_t\|_{L^2}^2 + 8\varepsilon^2\|u_x\|_{L^2}^2 + 81\delta^2\|\widetilde{\sigma}_x\|_{L^2}^2 + 4\alpha^2\|u\|_{L^2}^2 + 4\beta^2\|\widetilde{\Phi}_x\|_{L^2}^2). \tag{3.25}$$

Taking the sum of (3.22) and (3.25) gives us

$$\begin{aligned}\|u_x\|_{L^2}^2 + \|\widetilde{\sigma}_x\|_{L^2}^2 \leq \frac{1}{A_0\overline{\rho}}[&20\|u_t\|_{L^2}^2 + 12\|\widetilde{\sigma}_t\|_{L^2}^2 + 20\alpha^2\|u\|_{L^2}^2 + 20\beta^2\|\widetilde{\Phi}_x\|_{L^2}^2\\ &+ (40\varepsilon^2 + 12\delta^2)\|u_x\|_{L^2}^2 + (24\varepsilon^2 + 405\delta^2)\|\widetilde{\sigma}_x\|_{L^2}^2].\end{aligned}$$

When ε and δ are sufficiently small, we conclude that

$$\|u_x\|_{L^2}^2 + \|\widetilde{\sigma}_x\|_{L^2}^2 \leq C\|(u_t, \widetilde{\sigma}_t, u, \widetilde{\Phi}_x)\|_{L^2}^2. \tag{3.26}$$

Step 3. Taking ∂_t to (3.18), we obtain

$$u_{xt} = -\frac{2}{\widetilde{\sigma} + \widehat{\sigma}}(\widetilde{\sigma}_{tt} + u_t\widetilde{\sigma}_x + u\widetilde{\sigma}_{xt} + u_t\widehat{\sigma}_x) + \frac{2\widetilde{\sigma}_t}{(\widetilde{\sigma} + \widehat{\sigma})^2}(\widetilde{\sigma}_t + u\widetilde{\sigma}_x + u\widehat{\sigma}_x).$$

Using similar arguments as in **Step 1**, we can derive the following estimate:

$$\begin{aligned} \|u_{xt}\|_{L^2}^2 \leq & \frac{32}{A_0\overline{\rho}}[\|\widetilde{\sigma}_{tt}\|_{L^2}^2 + (2\varepsilon^2 + \delta^2)\|u_t\|_{L^2}^2 + 2\varepsilon^2\|\widetilde{\sigma}_{xt}\|_{L^2}^2] \\ & + \frac{48}{A_0^2\overline{\rho}^2}(2\varepsilon^4 + \varepsilon^2 + \varepsilon^2\delta^2)\|\widetilde{\sigma}_t\|_{L^2}^2. \end{aligned} \tag{3.27}$$

Taking ∂_t to (3.23), we can show that

$$\begin{aligned} \|\widetilde{\sigma}_{xt}\|_{L^2}^2 \leq & \frac{48}{A_0\overline{\rho}}[4\|u_{tt}\|_{L^2}^2 + 8\varepsilon^2\|u_t\|_{L^2}^2 + 8\varepsilon^2\|u_{xt}\|_{L^2}^2 + \delta^2\|\widetilde{\sigma}_t\|_{L^2}^2 + 4\alpha^2\|u_t\|_{L^2}^2 + 4\beta^2\|\widetilde{\Phi}_{xt}\|_{L^2}^2] \\ & + \frac{80}{A_0^2\overline{\rho}^2}(8\varepsilon^4 + 4\varepsilon^2 + \varepsilon^2\delta^2 + 4\alpha^2\varepsilon^2 + 4\beta^2\varepsilon^2)\|\widetilde{\sigma}_t\|_{L^2}^2. \end{aligned} \tag{3.28}$$

Taking the sum of (3.27) and (3.28), we have

$$\begin{aligned} \|u_{xt}\|_{L^2}^2 + \|\widetilde{\sigma}_{xt}\|_{L^2}^2 \leq & \frac{16}{A_0\overline{\rho}}[2\|\widetilde{\sigma}_{tt}\|_{L^2}^2 + 12\|u_{tt}\|_{L^2}^2 + (28\varepsilon^2 + 2\delta^2 + 12\alpha^2)\|u_t\|_{L^2}^2 \\ & + 3\delta^2\|\widetilde{\sigma}_t\|_{L^2}^2 + 12\beta^2\|\widetilde{\Phi}_{xt}\|_{L^2}^2 + 24\varepsilon^2\|u_{xt}\|_{L^2}^2 + 4\varepsilon^2\|\widetilde{\sigma}_{xt}\|_{L^2}^2] \\ & + \frac{16}{A_0^2\overline{\rho}^2}(46\varepsilon^4 + 23\varepsilon^2 + 8\varepsilon^2\delta^2 + 20\alpha^2\varepsilon^2 + 20\beta^2\varepsilon^2)\|\widetilde{\sigma}_t\|_{L^2}^2. \end{aligned}$$

When ε and δ are sufficiently small, there holds that

$$\|u_{xt}\|_{L^2}^2 + \|\widetilde{\sigma}_{xt}\|_{L^2}^2 \leq C\|(u_{tt}, \widetilde{\sigma}_{tt}, u_t, \widetilde{\sigma}_t, \widetilde{\Phi}_{xt})\|_{L^2}^2. \tag{3.29}$$

Step 4. Taking ∂_x to (3.18) and using Poincaré inequality for u, we can derive the following estimate:

$$\begin{aligned} \|u_{xx}\|_{L^2}^2 \leq & \frac{20}{A_0\overline{\rho}}(\|\widetilde{\sigma}_{xt}\|_{L^2}^2 + (2\varepsilon^2 + \delta^2 + \|\widehat{\sigma}_{xx}\|_{L^\infty}^2)\|u_x\|_{L^2}^2 + 2\varepsilon^2\|\widetilde{\sigma}_{xx}\|_{L^2}^2) \\ & + \frac{24}{A_0^2\overline{\rho}^2}(2\varepsilon^2 + \delta^2)(\|\widetilde{\sigma}_t\|_{L^2}^2 + (2\varepsilon^2 + \delta^2)\|u_x\|_{L^2}^2). \end{aligned} \tag{3.30}$$

Note that

$$\widehat{\sigma}_{xx} = \frac{\sqrt{2A_0}}{\sqrt{\widehat{\rho}}}\widehat{\rho}_{xx} - \frac{\sqrt{2A_0}}{2\widehat{\rho}\sqrt{\widehat{\rho}}}(\widehat{\rho}_x)^2.$$

From the discussions in §2.1 we know that $\widehat{\rho}_x$ and $\widehat{\rho}_{xx}$ are small when d is large. Hence, as in (3.15), we may assume $\|\widetilde{\sigma}_{xx}\|_{L^\infty} \leq \delta$, as well. Then, we update (3.30) as

$$\|u_{xx}\|_{L^2}^2 \leq C(\|\widetilde{\sigma}_{xt}\|_{L^2}^2 + \|\widetilde{\sigma}_t\|_{L^2}^2 + \|u_x\|_{L^2}^2 + \varepsilon^2\|\widetilde{\sigma}_{xx}\|_{L^2}^2). \tag{3.31}$$

Next, taking ∂_x to (3.23) and using (3.17), we can show that

$$\|\widetilde{\sigma}_{xx}\|_{L^2}^2 \leq C(\|u_{xt}\|_{L^2}^2 + \|u_t\|_{L^2}^2 + \|u_x\|_{L^2}^2 + \|\widetilde{\sigma}_x\|_{L^2}^2 + \|\widetilde{\Phi}_{xx}\|_{L^2}^2 + \|\widetilde{\Phi}_x\|_{L^2}^2 + \varepsilon^2\|u_{xx}\|_{L^2}^2). \tag{3.32}$$

Taking the sum of (3.31) and (3.32) gives us

$$\begin{aligned}\|u_{xx}\|_{L^2}^2 + \|\widetilde{\sigma}_{xx}\|_{L^2}^2 \leq C(&\|u_{xt}\|_{L^2}^2 + \|\widetilde{\sigma}_{xt}\|_{L^2}^2 + \|u_t\|_{L^2}^2 + \|\widetilde{\sigma}_t\|_{L^2}^2 + \|u_x\|_{L^2}^2 \\ &+ \|\widetilde{\sigma}_x\|_{L^2}^2 + \|\widetilde{\Phi}_{xx}\|_{L^2}^2 + \|\widetilde{\Phi}_x\|_{L^2}^2 + \varepsilon^2(\|\widetilde{\sigma}_{xx}\|_{L^2}^2 + \|u_{xx}\|_{L^2}^2)).\end{aligned} \tag{3.33}$$

It is clear that when ε is small, we can update (3.33) as

$$\begin{aligned}\|u_{xx}\|_{L^2}^2 + \|\widetilde{\sigma}_{xx}\|_{L^2}^2 \leq C(&\|u_{xt}\|_{L^2}^2 + \|\widetilde{\sigma}_{xt}\|_{L^2}^2 + (\|u_x\|_{L^2}^2 + \|\widetilde{\sigma}_x\|_{L^2}^2) \\ &+ \|u_t\|_{L^2}^2 + \|\widetilde{\sigma}_t\|_{L^2}^2 + \|\widetilde{\Phi}_{xx}\|_{L^2}^2 + \|\widetilde{\Phi}_x\|_{L^2}^2).\end{aligned} \tag{3.34}$$

By (3.26) and (3.29), we further update (3.34) as

$$\begin{aligned}\|u_{xx}\|_{L^2}^2 + \|\widetilde{\sigma}_{xx}\|_{L^2}^2 \leq C(&\|u_{tt}\|_{L^2}^2 + \|\widetilde{\sigma}_{tt}\|_{L^2}^2 + \|u_t\|_{L^2}^2 + \|\widetilde{\sigma}_t\|_{L^2}^2 \\ &+ \|u\|_{L^2}^2 + \|\widetilde{\Phi}_{xt}\|_{L^2}^2 + \|\widetilde{\Phi}_x\|_{L^2}^2 + \|\widetilde{\Phi}_{xx}\|_{L^2}^2).\end{aligned} \tag{3.35}$$

Step 5. Since $\widetilde{\Phi}$ and $\widetilde{\Phi}_t$ satisfy the zero boundary condition, it follows from Poincaré inequality that

$$\|\widetilde{\Phi}\|_{L^2}^2 \leq \|\widetilde{\Phi}_x\|_{L^2}^2 \quad \text{and} \quad \|\widetilde{\Phi}_t\|_{L^2}^2 \leq \|\widetilde{\Phi}_{xt}\|_{L^2}^2. \tag{3.36}$$

Moreover, using (3.36), (3.17) and (3.26), we can deduce from the third equation of (3.3) that

$$\|\widetilde{\Phi}_{xx}\|_{L^2}^2 \leq C\|(\widetilde{\Phi}_{xt}, \widetilde{\Phi}_x, u_t, \widetilde{\sigma}_t, u)\|_{L^2}^2. \tag{3.37}$$

Substituting (3.37) into (3.35), we get

$$\|u_{xx}\|_{L^2}^2 + \|\widetilde{\sigma}_{xx}\|_{L^2}^2 \leq C\|(u_{tt}, \widetilde{\sigma}_{tt}, u_t, \widetilde{\sigma}_t, u, \widetilde{\Phi}_{xt}, \widetilde{\Phi}_x)\|_{L^2}^2. \tag{3.38}$$

Combining (3.17), (3.26), (3.29), (3.36), (3.37) and (3.38), we arrive at (3.5). This completes the proof of the lemma. □

3.3. Energy estimates

In this subsection, we examine the quantity $X_1(t)$ defined in (3.5) and derive the desired energy estimates, along with the exponential decaying of the perturbed solution.

Lemma 3.2. *Let* $(\widetilde{\sigma}, u, \widetilde{\Phi})$ *be the local solution to the IBVP* (3.3) *with* $\tau > 0$ *up to some finite time* $T > 0$. *Then under the conditions of Theorem 2.2, the quantity*

$$\|(\widetilde{\sigma}, u)(t)\|_2^2 + \sum_{k=0}^{2} \|(\partial_t^k \widetilde{\Phi})(t)\|_{H^{4-2k}}^2$$

$$+\int_0^t (\|(\widetilde{\sigma},u)(\tau)\|_2^2 + \|\widetilde{\Phi}(\tau)\|_{H^4}^2 + \|\widetilde{\Phi}_t(\tau)\|_{H^3}^2 + \|\widetilde{\Phi}_{tt}(\tau)\|_{H^1}^2)\mathrm{d}\tau$$

is uniformly bounded with respect to $t > 0$*, and* $\|(\widetilde{\sigma},u)(t)\|_2^2 + \sum_{k=0}^{2}\|(\partial_t^k\widetilde{\Phi})(t)\|_{H^{4-2k}}^2$ *decays exponentially rapidly to zero as* $t \to \infty$*.*

Proof. **Step 1.** Taking L^2 inner product of the first equation in (3.3) with $\widetilde{\sigma}$, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|\widetilde{\sigma}\|_{L^2}^2 &= -2\int_0^1 u\widetilde{\sigma}_x\widetilde{\sigma}\mathrm{d}x - \int_0^1 \widetilde{\sigma}^2u_x\mathrm{d}x - 2\int_0^1 u\widehat{\sigma}_x\widetilde{\sigma}\mathrm{d}x - \int_0^1 \widehat{\sigma}u_x\widetilde{\sigma}\mathrm{d}x\\ &= -2\int_0^1 u\widehat{\sigma}_x\widetilde{\sigma}\mathrm{d}x - \int_0^1 \widehat{\sigma}u_x\widetilde{\sigma}\mathrm{d}x,\end{aligned} \tag{3.39}$$

where we used the zero boundary condition for u. Taking L^2 inner product of the second equation in (3.3) with u gives us

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_{L^2}^2 + 2\alpha\|u\|_{L^2}^2 &= -2\int_0^1 u^2u_x\mathrm{d}x - \int_0^1 \widetilde{\sigma}\widetilde{\sigma}_xu\mathrm{d}x - \int_0^1 \widetilde{\sigma}\widehat{\sigma}_xu\mathrm{d}x - \int_0^1 \widehat{\sigma}\widetilde{\sigma}_xu\mathrm{d}x + 2\beta\int_0^1 \widetilde{\Phi}_xu\mathrm{d}x\\ &= -\int_0^1 \widetilde{\sigma}\widetilde{\sigma}_xu\mathrm{d}x - \int_0^1 \widetilde{\sigma}\widehat{\sigma}_xu\mathrm{d}x - \int_0^1 \widehat{\sigma}\widetilde{\sigma}_xu\mathrm{d}x + 2\beta\int_0^1 \widetilde{\Phi}_xu\mathrm{d}x.\end{aligned} \tag{3.40}$$

Taking the sum of (3.39) and (3.40), we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(\|\widetilde{\sigma}\|_{L^2}^2 + \|u\|_{L^2}^2) + 2\alpha\|u\|_{L^2}^2 &= -2\int_0^1 u\widehat{\sigma}_x\widetilde{\sigma}\mathrm{d}x - \int_0^1 \widetilde{\sigma}\widetilde{\sigma}_xu\mathrm{d}x + 2\beta\int_0^1 \widetilde{\Phi}_xu\mathrm{d}x\\ &\le (2\|\widehat{\sigma}_x\|_{L^\infty} + \|\widetilde{\sigma}_x\|_{L^\infty})\|u\|_{L^2}\|\widetilde{\sigma}\|_{L^2} + 2\beta\|\widetilde{\Phi}_x\|_{L^2}\|u\|_{L^2}.\end{aligned} \tag{3.41}$$

Since, by Young's inequality,

$$2\beta\|\widetilde{\Phi}_x\|_{L^2}\|u\|_{L^2} \le \alpha\|u\|_{L^2}^2 + \alpha^{-1}\beta^2\|\widetilde{\Phi}_x\|_{L^2}^2,$$

we update (3.41) as

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|\widetilde{\sigma}\|_{L^2}^2 + \|u\|_{L^2}^2) + \alpha\|u\|_{L^2}^2 \le (2\|\widehat{\sigma}_x\|_{L^\infty} + \|\widetilde{\sigma}_x\|_{L^\infty})\|u\|_{L^2}\|\widetilde{\sigma}\|_{L^2} + \alpha^{-1}\beta^2\|\widetilde{\Phi}_x\|_{L^2}^2.$$

By Sobolev embedding and Cauchy-Schwarz inequality, we can show that

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|\widetilde{\sigma}\|_{L^2}^2 + \|u\|_{L^2}^2) + \alpha\|u\|_{L^2}^2 \le (\delta + 2^{-\frac{1}{2}}\varepsilon)(\|u\|_{L^2}^2 + \|\widetilde{\sigma}\|_{L^2}^2) + \alpha^{-1}\beta^2\|\widetilde{\Phi}_x\|_{L^2}^2, \tag{3.42}$$

where we used (3.15) and (3.4).

Step 2. Taking ∂_t to the three equations in (3.3), we have

$$\begin{cases}2\widetilde{\sigma}_{tt} + 2u_t\widetilde{\sigma}_x + 2u\widetilde{\sigma}_{xt} + \widetilde{\sigma}_tu_x + \widetilde{\sigma}u_{xt} + 2u_t\widehat{\sigma}_x + \widehat{\sigma}u_{xt} = 0,\\ 2u_{tt} + 2u_tu_x + 2uu_{xt} + \widetilde{\sigma}_t\widetilde{\sigma}_x + \widetilde{\sigma}\,\widetilde{\sigma}_{xt} + \widetilde{\sigma}_t\widehat{\sigma}_x + \widehat{\sigma}\,\widetilde{\sigma}_{xt} = -2\alpha u_t + 2\beta\widetilde{\Phi}_{xt},\\ 8\tau A_0\widetilde{\Phi}_{tt} = 8dA_0\widetilde{\Phi}_{xxt} - 8aA_0\widetilde{\Phi}_t + 2b(\widetilde{\sigma} + \widehat{\sigma})\widetilde{\sigma}_t.\end{cases} \tag{3.43}$$

Taking L^2 inner product of the first equation in (3.43) with $\widetilde{\sigma}_t$, we have

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\|\widetilde{\sigma}_t\|_{L^2}^2 = &-2\int_0^1 u_t\widetilde{\sigma}_x\widetilde{\sigma}_t\mathrm{d}x - 2\int_0^1 u\widetilde{\sigma}_{xt}\widetilde{\sigma}_t\mathrm{d}x - \int_0^1 \widetilde{\sigma}_t^2 u_x\mathrm{d}x - \int_0^1 \widetilde{\sigma}u_{xt}\widetilde{\sigma}_t\mathrm{d}x \\
&- 2\int_0^1 u_t\widehat{\sigma}_x\widetilde{\sigma}_t\mathrm{d}x - \int_0^1 \widehat{\sigma}u_{xt}\widetilde{\sigma}_t\mathrm{d}x \\
= &-\int_0^1 \widetilde{\sigma}_x u_t\widetilde{\sigma}_t\mathrm{d}x + \int_0^1 \widetilde{\sigma}u_t\widetilde{\sigma}_{xt}\mathrm{d}x - \int_0^1 u_t\widehat{\sigma}_x\widetilde{\sigma}_t\mathrm{d}x + \int_0^1 \widehat{\sigma}u_t\widetilde{\sigma}_{xt}\mathrm{d}x. \qquad (3.44)
\end{aligned}
$$

Taking L^2 inner product of the second equation in (3.43) with u_t, we obtain

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\|u_t\|_{L^2}^2 + 2\alpha\|u_t\|_{L^2}^2 = &-2\int_0^1 u_x u_t^2\mathrm{d}x - 2\int_0^1 uu_{xt}u_t\mathrm{d}x - \int_0^1 \widetilde{\sigma}_t\widetilde{\sigma}_x u_t\mathrm{d}x - \int_0^1 \widetilde{\sigma}\widetilde{\sigma}_{xt}u_t\mathrm{d}x \\
&- \int_0^1 \widetilde{\sigma}_t\widehat{\sigma}_x u_t\mathrm{d}x - \int_0^1 \widehat{\sigma}\widetilde{\sigma}_{xt}u_t\mathrm{d}x + 2\beta\int_0^1 \widetilde{\Phi}_{xt}u_t\mathrm{d}x \\
= &-\int_0^1 u_x u_t^2\mathrm{d}x - \int_0^1 \widetilde{\sigma}_t\widetilde{\sigma}_x u_t\mathrm{d}x - \int_0^1 \widetilde{\sigma}\widetilde{\sigma}_{xt}u_t\mathrm{d}x \\
&- \int_0^1 \widetilde{\sigma}_t\widehat{\sigma}_x u_t\mathrm{d}x - \int_0^1 \widehat{\sigma}\widetilde{\sigma}_{xt}u_t\mathrm{d}x + 2\beta\int_0^1 \widetilde{\Phi}_{xt}u_t\mathrm{d}x. \qquad (3.45)
\end{aligned}
$$

Taking the sum of (3.44) and (3.45), we arrive at

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}(\|\widetilde{\sigma}_t\|_{L^2}^2 + \|u_t\|_{L^2}^2) + 2\alpha\|u_t\|_{L^2}^2 \\
= &-2\int_0^1 (\widetilde{\sigma}_x + \widehat{\sigma}_x)u_t\widetilde{\sigma}_t\mathrm{d}x - \int_0^1 u_x u_t^2\mathrm{d}x + 2\beta\int_0^1 \widetilde{\Phi}_{xt}u_t\mathrm{d}x \\
\le &\, 2(\|\widetilde{\sigma}_x\|_{L^\infty} + \|\widehat{\sigma}_x\|_{L^\infty})\|u_t\|_{L^2}\|\widetilde{\sigma}_t\|_{L^2} + \|u_x\|_{L^\infty}\|u_t\|_{L^2}^2 + 2\beta\|\widetilde{\Phi}_{xt}\|_{L^2}\|u_t\|_{L^2}.
\end{aligned}
$$

Similar to (3.42), it can be shown that

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}(\|\widetilde{\sigma}_t\|_{L^2}^2 + \|u_t\|_{L^2}^2) + \alpha\|u_t\|_{L^2}^2 \le &\,(\delta + 2^{-\frac{1}{2}}\varepsilon)(\|u_t\|_{L^2}^2 + \|\widetilde{\sigma}_t\|_{L^2}^2) + \sqrt{2}\,\varepsilon\|u_t\|_{L^2}^2 \\
&+ \alpha^{-1}\beta^2\|\widetilde{\Phi}_{xt}\|_{L^2}^2. \qquad (3.46)
\end{aligned}
$$

In completely the same fashion, we can show that

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}(\|\widetilde{\sigma}_{tt}\|_{L^2}^2 + \|u_{tt}\|_{L^2}^2) + \alpha\|u_{tt}\|_{L^2}^2 \\
\le &\,(\delta + \sqrt{2}\,\varepsilon)(\|u_{tt}\|_{L^2}^2 + \|\widetilde{\sigma}_{tt}\|_{L^2}^2) + 3\sqrt{2}\,\varepsilon(\|u_{tt}\|_{L^2}^2 + \|u_{xt}\|_{L^2}^2 + \|\widetilde{\sigma}_{tt}\|_{L^2}^2 + \|\widetilde{\sigma}_{xt}\|_{L^2}^2) \\
&+ \alpha^{-1}\beta^2\|\widetilde{\Phi}_{xtt}\|_{L^2}^2. \qquad (3.47)
\end{aligned}
$$

Step 3. Taking L^2 inner product of the third equation in (3.3) with $-\widetilde{\Phi}_{xx}$, we have

$$
\frac{\mathrm{d}}{\mathrm{d}t}(4\tau A_0\|\widetilde{\Phi}_x\|_{L^2}^2) + 8dA_0\|\widetilde{\Phi}_{xx}\|_{L^2}^2 + 8aA_0\|\widetilde{\Phi}_x\|_{L^2}^2 = -b\int_0^1 (\widetilde{\sigma} + 2\widehat{\sigma})\widetilde{\sigma}\widetilde{\Phi}_{xx}\mathrm{d}x
$$

$$\leq 2b(\|\widetilde{\sigma}\|_{L^\infty} + \|\widehat{\sigma}\|_{L^\infty})\|\widetilde{\sigma}\|_{L^2}\|\widetilde{\Phi}_{xx}\|_{L^2}. \tag{3.48}$$

By Cauchy-Schwarz inequality, we update (3.48) as

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}(4\tau A_0\|\widetilde{\Phi}_x\|_{L^2}^2) + 8dA_0\|\widetilde{\Phi}_{xx}\|_{L^2}^2 + 8aA_0\|\widetilde{\Phi}_x\|_{L^2}^2 \\
&\leq \frac{b^2}{dA_0}(\|\widetilde{\sigma}\|_{L^\infty} + \|\widehat{\sigma}\|_{L^\infty})^2\|\widetilde{\sigma}\|_{L^2}^2 + dA_0\|\widetilde{\Phi}_{xx}\|_{L^2}^2 \\
&\leq \frac{2b^2}{dA_0}(\|\widetilde{\sigma}\|_{L^\infty}^2 + \|\widehat{\sigma}\|_{L^\infty}^2)\|\widetilde{\sigma}\|_{L^2}^2 + dA_0\|\widetilde{\Phi}_{xx}\|_{L^2}^2,
\end{aligned}$$

which implies, by (3.7),

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|\widetilde{\Phi}_x\|_{L^2}^2) + \frac{7d}{4\tau}\|\widetilde{\Phi}_{xx}\|_{L^2}^2 + \frac{2a}{\tau}\|\widetilde{\Phi}_x\|_{L^2}^2 \leq \frac{b^2}{\tau dA_0^2}(8A_0\bar{\rho} + \varepsilon^2)\|\widetilde{\sigma}\|_{L^2}^2. \tag{3.49}$$

Similarly, by taking L^2 inner product of the third equation in (3.43) with $-\widetilde{\Phi}_{xxt}$, we can show that

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|\widetilde{\Phi}_{xt}\|_{L^2}^2) + \frac{7d}{4\tau}\|\widetilde{\Phi}_{xxt}\|_{L^2}^2 + \frac{2a}{\tau}\|\widetilde{\Phi}_{xt}\|_{L^2}^2 \leq \frac{b^2}{\tau dA_0^2}(8A_0\bar{\rho} + \varepsilon^2)\|\widetilde{\sigma}_t\|_{L^2}^2. \tag{3.50}$$

Moreover, taking ∂_t to the third equation in (3.43), then taking L^2 inner product of the resulting equation with $\widetilde{\Phi}_{tt}$, it can be shown that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}(\|\widetilde{\Phi}_{tt}\|_{L^2}^2) + \frac{7d}{4\tau}\|\widetilde{\Phi}_{xtt}\|_{L^2}^2 + \frac{2a}{\tau}\|\widetilde{\Phi}_{tt}\|_{L^2}^2 &\leq \frac{b}{\sqrt{2}\,\tau A_0}\|\widetilde{\sigma}_t\|_{H^1}\|\widetilde{\sigma}_t\|_{L^2}\|\widetilde{\Phi}_{tt}\|_{L^2} \\
&\quad + \frac{b^2}{\tau dA_0^2}(8A_0\bar{\rho} + \varepsilon^2)\|\widetilde{\sigma}_{tt}\|_{L^2}^2.
\end{aligned} \tag{3.51}$$

For the first term on the right-hand side of (3.51), we have

$$\begin{aligned}
\frac{b}{\sqrt{2}\,\tau A_0}\|\widetilde{\sigma}_t\|_{H^1}\|\widetilde{\sigma}_t\|_{L^2}\|\widetilde{\Phi}_{tt}\|_{L^2} &\leq \frac{b}{\sqrt{2}\,\tau A_0}\|\widetilde{\sigma}_t\|_{H^1}\|\widetilde{\sigma}_t\|_{L^2}\|\widetilde{\Phi}_{xtt}\|_{L^2} \\
&\leq \frac{b^2}{2\tau dA_0^2}\|\widetilde{\sigma}_t\|_{H^1}^2\|\widetilde{\sigma}_t\|_{L^2}^2 + \frac{d}{4\tau}\|\widetilde{\Phi}_{xtt}\|_{L^2}^2,
\end{aligned}$$

where we applied Poincaré inequality. Then we update (3.51) as

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}(\|\widetilde{\Phi}_{tt}\|_{L^2}^2) + \frac{3d}{2\tau}\|\widetilde{\Phi}_{xtt}\|_{L^2}^2 + \frac{2a}{\tau}\|\widetilde{\Phi}_{tt}\|_{L^2}^2 &\leq \frac{b^2}{2\tau dA_0^2}\|\widetilde{\sigma}_t\|_{H^1}^2\|\widetilde{\sigma}_t\|_{L^2}^2 \\
&\quad + \frac{b^2}{\tau dA_0^2}(8A_0\bar{\rho} + \varepsilon^2)\|\widetilde{\sigma}_{tt}\|_{L^2}^2.
\end{aligned} \tag{3.52}$$

Step 4. Taking the sum of (3.42), (3.46), and (3.47) gives us

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|(\widetilde{\sigma}, \widetilde{\sigma}_t, \widetilde{\sigma}_{tt}, u, u_t, u_{tt})\|_{L^2}^2) + \alpha\|(u, u_t, u_{tt})\|_{L^2}^2$$

$$\leq (\delta + 4\sqrt{2}\,\varepsilon)X(t) + \alpha^{-1}\beta^2\|(\widetilde{\Phi}_x, \widetilde{\Phi}_{xt}, \widetilde{\Phi}_{xtt})\|_{L^2}^2. \tag{3.53}$$

Taking the sum of (3.49), (3.50) and (3.52), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|(\widetilde{\Phi}_x, \widetilde{\Phi}_{xt}, \widetilde{\Phi}_{tt})\|_{L^2}^2) + \frac{3d}{2\tau}\|(\widetilde{\Phi}_{xx}, \widetilde{\Phi}_{xxt}, \widetilde{\Phi}_{xtt})\|_{L^2}^2 \leq \frac{b^2}{\tau d A_0^2}(8A_0\bar{\rho} + \varepsilon^2)X(t), \tag{3.54}$$

where we threw away the non-negative terms involving a. Taking the sum of (3.53) and (3.54), and using the definition of $X_1(t)$ (c.f. (3.5)), we obtain

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}(\|\widetilde{\sigma}(t)\|_{L^2}^2 + X_1(t)) + \alpha\|(u, u_t, u_{tt})\|_{L^2}^2 + \frac{3d}{2\tau}\|(\widetilde{\Phi}_{xx}, \widetilde{\Phi}_{xxt}, \widetilde{\Phi}_{xtt})\|_{L^2}^2 \\ &\leq (\delta + 4\sqrt{2}\,\varepsilon)X(t) + \frac{b^2}{\tau d A_0^2}(8A_0\bar{\rho} + \varepsilon^2)X(t) + \alpha^{-1}\beta^2\|(\widetilde{\Phi}_x, \widetilde{\Phi}_{xt}, \widetilde{\Phi}_{xtt})\|_{L^2}.\end{aligned} \tag{3.55}$$

Note that by Poincaré inequality, we have $\|\widetilde{\Phi}_x\|_{L^2} \leq \|\widetilde{\Phi}_{xx}\|_{L^2}$ and $\|\widetilde{\Phi}_{xt}\|_{L^2} \leq \|\widetilde{\Phi}_{xxt}\|_{L^2}$. Hence, using the assumption that $d > 0$ is sufficiently large, we update (3.55) as

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}(\|\widetilde{\sigma}(t)\|_{L^2}^2 + X_1(t)) + \alpha\|(u, u_t, u_{tt})\|_{L^2}^2 + \frac{d}{\tau}\|(\widetilde{\Phi}_{xx}, \widetilde{\Phi}_{xxt}, \widetilde{\Phi}_{xtt})\|_{L^2}^2 \\ &\leq (\delta + 4\sqrt{2}\,\varepsilon)X(t) + \frac{b^2}{\tau d A_0^2}(8A_0\bar{\rho} + \varepsilon^2)X(t).\end{aligned} \tag{3.56}$$

Again, by Poincaré inequality, we deduce from (3.56) that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}(\|\widetilde{\sigma}(t)\|_{L^2}^2 + X_1(t)) + \alpha\|(u, u_t, u_{tt})\|_{L^2}^2 + \frac{d}{2\tau}\|(\widetilde{\Phi}_x, \widetilde{\Phi}_{xt}, \widetilde{\Phi}_{tt}, \widetilde{\Phi}_{xx}, \widetilde{\Phi}_{xxt}, \widetilde{\Phi}_{xtt})\|_{L^2}^2 \\ &\leq (\delta + 4\sqrt{2}\,\varepsilon)X(t) + \frac{b^2}{\tau d A_0^2}(8A_0\bar{\rho} + \varepsilon^2)X(t).\end{aligned} \tag{3.57}$$

Step 5. Taking L^2 inner product of the first equation in (3.43) with $-\widetilde{\sigma}$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\int_0^1 -\widetilde{\sigma}\widetilde{\sigma}_t\mathrm{d}x\Big) + \|\widetilde{\sigma}_t\|_{L^2}^2 = \frac{1}{2}\int_0^1 (2u_t\widetilde{\sigma}_x + 2u\widetilde{\sigma}_{xt} + \widetilde{\sigma}_t u_x + \widetilde{\sigma}u_{xt} + 2u_t\widehat{\sigma}_x + \widehat{\sigma}u_{xt})\widetilde{\sigma}\mathrm{d}x. \tag{3.58}$$

For the integral involving the first five integrands on the right-hand side of (3.58), we can show that

$$\begin{aligned}\Big|\frac{1}{2}\int_0^1 (2u_t\widetilde{\sigma}_x + 2u\widetilde{\sigma}_{xt} + \widetilde{\sigma}_t u_x + \widetilde{\sigma}u_{xt} + 2u_t\widehat{\sigma}_x)\widetilde{\sigma}\mathrm{d}x\Big| &\leq \frac{1}{2}(\|\widetilde{\sigma}\|_{L^\infty} + \|\widehat{\sigma}_x\|_{L^\infty})X(t) \\ &\leq \frac{1}{2}(\sqrt{2}\,\varepsilon + \delta)X(t).\end{aligned} \tag{3.59}$$

For the integral of the last integrand, using integration by parts, we have

$$\begin{aligned}\Big|\frac{1}{2}\int_0^1 \widehat{\sigma}u_{xt}\widetilde{\sigma}\mathrm{d}x\Big| &= \frac{1}{2}\Big|\int_0^1 (\widehat{\sigma}_x u_t\widetilde{\sigma} + \widehat{\sigma}u_t\widetilde{\sigma}_x)\mathrm{d}x\Big| \\ &\leq \frac{\delta}{4}X(t) + \sqrt{A_0\bar{\rho}}(\|u_t\|_{L^2}^2 + \|\widetilde{\sigma}_x\|_{L^2}^2).\end{aligned} \tag{3.60}$$

Similar to (3.25), we can derive the following estimate:

$$\|\widetilde{\sigma}_x\|_{L^2}^2 \le \frac{5}{A_0\overline{\rho}}(4\|u_t\|_{L^2}^2 + 8\varepsilon^2\|u\|_{L^2}^2 + 81\delta^2\|\widetilde{\sigma}_x\|_{L^2}^2 + 4\alpha^2\|u\|_{L^2}^2 + 4\beta^2\|\widetilde{\Phi}_x\|_{L^2}^2). \tag{3.61}$$

Since δ is small, we update (3.61) as

$$\|\widetilde{\sigma}_x\|_{L^2}^2 \le C(\|u_t\|_{L^2}^2 + \|u\|_{L^2}^2 + \|\widetilde{\Phi}_x\|_{L^2}^2). \tag{3.62}$$

Substituting (3.62) into (3.60), we obtain

$$\Big|\frac{1}{2}\int_0^1 \widehat{\sigma}u_{xt}\widetilde{\sigma}\mathrm{d}x\Big| \le \frac{\delta}{4}X(t) + D_1\|(u_t, u, \widetilde{\Phi}_x)\|_{L^2}^2, \tag{3.63}$$

where the constant D_1 depends only on $A_0, \overline{\rho}$. Substituting (3.59) and (3.63) into (3.58) gives us

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\int_0^1 -\widetilde{\sigma}\widetilde{\sigma}_t\mathrm{d}x\Big) + \|\widetilde{\sigma}_t\|_{L^2}^2 \le \Big(\frac{\sqrt{2}}{2}\varepsilon + \frac{3}{4}\delta\Big)X(t) + D_1\|(u_t, u, \widetilde{\Phi}_x)\|_{L^2}^2.$$

Next, taking ∂_t to the first equation in (3.43), then taking L^2 inner product of the resulting equation with $-\widetilde{\sigma}_t$, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\int_0^1 -\widetilde{\sigma}_t\widetilde{\sigma}_{tt}\mathrm{d}x\Big) + \|\widetilde{\sigma}_{tt}\|_{L^2}^2 = \frac{1}{2}\int_0^1 (2u_t\widetilde{\sigma}_x + 2u\widetilde{\sigma}_{xt} + \widetilde{\sigma}_t u_x + \widetilde{\sigma}u_{xt} + 2u_t\widehat{\sigma}_x + \widehat{\sigma}u_{xt})_t\widetilde{\sigma}_t\mathrm{d}x. \tag{3.64}$$

For the integral involving the first five integrands on the right-hand side of (3.64), using integration by parts, we can show that

$$\begin{aligned}
&\frac{1}{2}\int_0^1 (2u_t\widetilde{\sigma}_x + 2u\widetilde{\sigma}_{xt} + \widetilde{\sigma}_t u_x + \widetilde{\sigma}u_{xt} + 2u_t\widehat{\sigma}_x)_t\widetilde{\sigma}_t\mathrm{d}x\\
&= \frac{1}{2}\int_0^1 (2\widetilde{\sigma}_x u_{tt} + 4u_t\widetilde{\sigma}_{xt} + u_x\widetilde{\sigma}_{tt} + 2\widetilde{\sigma}_t u_{xt} + 2\widehat{\sigma}_x u_{tt})\widetilde{\sigma}_t\mathrm{d}x + \frac{1}{2}\int_0^1 (2u\widetilde{\sigma}_{xtt} + \widetilde{\sigma}u_{xtt})\widetilde{\sigma}_t\mathrm{d}x\\
&= \frac{1}{2}\int_0^1 (\widetilde{\sigma}_x u_{tt} + 4u_t\widetilde{\sigma}_{xt} - u_x\widetilde{\sigma}_{tt} + 2\widetilde{\sigma}_t u_{xt} + 2\widehat{\sigma}_x u_{tt})\widetilde{\sigma}_t\mathrm{d}x - \frac{1}{2}\int_0^1 (2u\widetilde{\sigma}_{tt} + \widetilde{\sigma}u_{tt})\widetilde{\sigma}_{xt}\mathrm{d}x.
\end{aligned}$$

Similar to (3.59), we have

$$\begin{aligned}
&\Big|\frac{1}{2}\int_0^1 (2u_t\widetilde{\sigma}_x + 2u\widetilde{\sigma}_{xt} + \widetilde{\sigma}_t u_x + \widetilde{\sigma}u_{xt} + 2u_t\widehat{\sigma}_x)_t\widetilde{\sigma}_t\mathrm{d}x\Big|\\
&\le (\|\widetilde{\sigma}_x\|_{L^\infty} + \|u_x\|_{L^\infty} + \|\widetilde{\sigma}_t\|_{L^\infty} + \|u_t\|_{L^\infty} + \|\widetilde{\sigma}\|_{L^\infty} + \|u\|_{L^\infty} + \|\widehat{\sigma}_x\|_{L^\infty})X(t)\\
&\le (2\sqrt{3}\,\varepsilon + \delta)X(t).
\end{aligned} \tag{3.65}$$

For the integral of the last integrand on the right-hand side of (3.64), we deduce that

$$\begin{aligned}
\Big|\frac{1}{2}\int_0^1 \widehat{\sigma}u_{xtt}\widetilde{\sigma}_t\mathrm{d}x\Big| &= \frac{1}{2}\Big|\int_0^1 (\widehat{\sigma}_x u_{tt}\widetilde{\sigma}_t + \widehat{\sigma}u_{tt}\widetilde{\sigma}_{xt})\mathrm{d}x\Big|\\
&\le \frac{\delta}{4}X(t) + \sqrt{A_0\overline{\rho}}(\|u_{tt}\|_{L^2}^2 + \|\widetilde{\sigma}_{xt}\|_{L^2}^2).
\end{aligned}$$

According to (3.27) and (3.28), we know

$$\|\widetilde{\sigma}_{xt}\|_{L^2}^2 \leq C\|(u_{tt}, u_t, \widetilde{\Phi}_{xt})\|_{L^2}^2 + C\varepsilon\|u_{xt}\|_{L^2}^2 + C(\varepsilon+\delta)\|\widetilde{\sigma}_t\|_{L^2}^2 \tag{3.66}$$

and

$$\|u_{xt}\|_{L^2}^2 \leq C\|\widetilde{\sigma}_{tt}\|_{L^2}^2 + C(\varepsilon+\delta)\|(u_t, \widetilde{\sigma}_t)\|_{L^2}^2 + C\varepsilon\|\widetilde{\sigma}_{xt}\|_{L^2}^2. \tag{3.67}$$

Substituting (3.67) into (3.66), we obtain

$$\|\widetilde{\sigma}_{xt}\|_{L^2}^2 \leq C\|(u_{tt}, u_t, \widetilde{\Phi}_{xt})\|_{L^2}^2 + C\varepsilon\|\widetilde{\sigma}_{tt}\|_{L^2}^2 + C(\varepsilon+\delta)\|\widetilde{\sigma}_t\|_{L^2}^2 + C\varepsilon\|\widetilde{\sigma}_{xt}\|_{L^2}^2. \tag{3.68}$$

When ε is sufficiently small, we update (3.68) as

$$\|\widetilde{\sigma}_{xt}\|_{L^2}^2 \leq C\|(u_{tt}, u_t, \widetilde{\Phi}_{xt})\|_{L^2}^2 + C\varepsilon\|\widetilde{\sigma}_{tt}\|_{L^2}^2 + C(\varepsilon+\delta)\|\widetilde{\sigma}_t\|_{L^2}^2. \tag{3.69}$$

Substituting (3.65) and (3.69) into (3.64), we arrive at

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Big(\int_0^1 -\widetilde{\sigma}_t\widetilde{\sigma}_{tt}\mathrm{d}x\Big) + \|\widetilde{\sigma}_{tt}\|_{L^2}^2 \leq \Big(2\sqrt{3}\,\varepsilon + \frac{5}{4}\delta\Big)X(t) + C\|(u_{tt}, u_t, \widetilde{\Phi}_{xt})\|_{L^2}^2 \\ + C\varepsilon\|\widetilde{\sigma}_{tt}\|_{L^2}^2 + C(\varepsilon+\delta)\|\widetilde{\sigma}_t\|_{L^2}^2.\end{aligned} \tag{3.70}$$

When ε and δ are sufficiently small, we update (3.70) as

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Big(\int_0^1 -(\widetilde{\sigma}\widetilde{\sigma}_t + \widetilde{\sigma}_t\widetilde{\sigma}_{tt})\mathrm{d}x\Big) + \frac{1}{2}\|(\widetilde{\sigma}_t, \widetilde{\sigma}_{tt})\|_{L^2}^2 \leq \Big(2\sqrt{3}\,\varepsilon + \frac{5}{4}\delta\Big)X(t) \\ + D_2\|(u_{tt}, u_t, u, \widetilde{\Phi}_x, \widetilde{\Phi}_{xt})\|_{L^2}^2.\end{aligned} \tag{3.71}$$

We observe from (3.27), (3.28), and (3.62) that when ε and δ are sufficiently small, the constant D_2 depends only on A_0, $\overline{\rho}$, α, and β.

Step 6. Note that the dissipations in (3.57) and (3.71) contain a quantity that is equivalent to $X_1(t)$ defined in (3.5). Hence, we shall make a coupling of (3.57) and (3.71) to close the overall energy estimates to capture the global dynamics of the perturbed solution. However, direct summation of (3.57) and (3.71) is problematic, as some leading terms are standing on the right-hand side of (3.71) and the summation of the terms inside the time derivatives does not cover the total H^2-norm of $\widetilde{\sigma}$. To overcome such a technical difficulty, we shall require $d > 0$ to be large enough, such that

$$4\tau D_2 d^{-1} \leq 2, \tag{3.72}$$

and let

$$\chi = \max\{2,\ 2D_2\alpha^{-1}\}. \tag{3.73}$$

Dividing (3.71) by χ, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\int_0^1 \frac{-(\widetilde{\sigma}\widetilde{\sigma}_t + \widetilde{\sigma}_t\widetilde{\sigma}_{tt})}{\chi}\mathrm{d}x\Big) + \frac{1}{2\chi}\|(\widetilde{\sigma}_t, \widetilde{\sigma}_{tt})\|_{L^2}^2 \leq \frac{1}{\chi}\Big(2\sqrt{3}\,\varepsilon + \frac{5}{4}\delta\Big)X(t)$$

$$+\frac{D_2}{\chi}\|(u_{tt},u_t,u,\widetilde{\Phi}_x,\widetilde{\Phi}_{xt})\|_{L^2}^2. \tag{3.74}$$

Taking the sum of (3.57) and (3.74) gives us

$$\frac{\mathrm{d}}{\mathrm{d}t}(V(t))+W(t)\le\theta X(t), \tag{3.75}$$

where

$$\begin{aligned}
V(t)&\equiv\|\widetilde{\sigma}(t)\|_{L^2}^2+X_1(t)-\int_0^1\frac{(\widetilde{\sigma}\widetilde{\sigma}_t+\widetilde{\sigma}_t\widetilde{\sigma}_{tt})}{\chi}\mathrm{d}x,\\
W(t)&\equiv\alpha\|(u,u_t,u_{tt})\|_{L^2}^2+\frac{d}{2\tau}\|(\widetilde{\Phi}_x,\widetilde{\Phi}_{xt},\widetilde{\Phi}_{tt},\widetilde{\Phi}_{xx},\widetilde{\Phi}_{xxt},\widetilde{\Phi}_{xtt})\|_{L^2}^2+\frac{1}{2\chi}\|(\widetilde{\sigma}_t,\widetilde{\sigma}_{tt})\|_{L^2}^2,\\
&\quad-\frac{D_2}{\chi}\|(u_{tt},u_t,u,\widetilde{\Phi}_x,\widetilde{\Phi}_{xt})\|_{L^2}^2,\\
\theta&\equiv\delta+4\sqrt{2}\,\varepsilon+\frac{b^2}{\tau dA_0^2}(8A_0\bar{\rho}+\varepsilon^2)X(t)+\frac{1}{\chi}\Big(2\sqrt{3}\,\varepsilon+\frac{5}{4}\delta\Big).
\end{aligned}$$

Note that under (3.72) and (3.73),

$$\begin{aligned}
\frac{D_2}{\chi}\|(u_{tt},u_t,u,\widetilde{\Phi}_x,\widetilde{\Phi}_{xt})\|_{L^2}^2&=\frac{D_2}{\chi}\|(u_{tt},u_t,u)\|_{L^2}^2+\frac{D_2}{\chi}\|(\widetilde{\Phi}_x,\widetilde{\Phi}_{xt})\|_{L^2}^2\\
&\le\frac{\alpha}{2}\|(u_{tt},u_t,u)\|_{L^2}^2+\frac{d}{4\tau}\|(\widetilde{\Phi}_x,\widetilde{\Phi}_{xt})\|_{L^2}^2.
\end{aligned} \tag{3.76}$$

Hence, it follows from the definition of $X_1(t)$ that

$$\begin{aligned}
W(t)&\ge\frac{\alpha}{2}\|(u,u_t,u_{tt})\|_{L^2}^2+\frac{d}{4\tau}\|(\widetilde{\Phi}_x,\widetilde{\Phi}_{xt},\widetilde{\Phi}_{tt})\|_{L^2}^2+\frac{1}{2\chi}\|(\widetilde{\sigma}_t,\widetilde{\sigma}_{tt})\|_{L^2}^2\\
&\quad+\frac{d}{2\tau}\|(\widetilde{\Phi}_{xx},\widetilde{\Phi}_{xxt},\widetilde{\Phi}_{xtt})\|_{L^2}^2\\
&\ge D_3X_1(t)+\frac{d}{2\tau}\|(\widetilde{\Phi}_{xx},\widetilde{\Phi}_{xxt},\widetilde{\Phi}_{xtt})\|_{L^2}^2,
\end{aligned} \tag{3.77}$$

where

$$D_3=\min\Big\{\frac{\alpha}{2},\ \frac{d}{4\tau},\ \frac{1}{2\chi}\Big\}.$$

Since $\chi\ge2$, from the definition of $V(t)$ we see that

$$\begin{aligned}
V(t)&\ge\|(\widetilde{\sigma},\widetilde{\sigma}_t,\widetilde{\sigma}_{tt})\|_{L^2}^2-\frac{1}{4}(\|\widetilde{\sigma}\|_{L^2}^2+2\|\widetilde{\sigma}_t\|_{L^2}^2+\|\widetilde{\sigma}_{tt}\|_{L^2}^2)+\|(u,u_t,u_{tt},\widetilde{\Phi}_x,\widetilde{\Phi}_{xt},\widetilde{\Phi}_{tt})\|_{L^2}^2\\
&=\frac{1}{4}(3\|\widetilde{\sigma}\|_{L^2}^2+2\|\widetilde{\sigma}_t\|_{L^2}^2+3\|\widetilde{\sigma}_{tt}\|_{L^2}^2)+\|(u,u_t,u_{tt},\widetilde{\Phi}_x,\widetilde{\Phi}_{xt},\widetilde{\Phi}_{tt})\|_{L^2}^2\\
&\ge\frac{1}{2}\|(\widetilde{\sigma},\widetilde{\sigma}_t,\widetilde{\sigma}_{tt},u,u_t,u_{tt},\widetilde{\Phi}_x,\widetilde{\Phi}_{xt},\widetilde{\Phi}_{tt})\|_{L^2}^2=\frac{1}{2}\|\widetilde{\sigma}\|_{L^2}^2+\frac{1}{2}X_1(t).
\end{aligned} \tag{3.78}$$

Using Lemma 3.1 and (3.77), we update (3.75) as

$$\frac{\mathrm{d}}{\mathrm{d}t}(V(t))+D_3X_1(t)+\frac{d}{2\tau}\|(\widetilde{\Phi}_{xx},\widetilde{\Phi}_{xxt},\widetilde{\Phi}_{xtt})\|_{L^2}^2\le\theta D_0X_1(t),$$

Since $\delta \to 0$ as $d \to \infty$, from the definition of θ we see that $\theta \to 0$ as $\varepsilon \to 0$ and $d \to \infty$. Hence, when ε is sufficiently small and d is sufficiently large, it holds that

$$\frac{\mathrm{d}}{\mathrm{d}t}(V(t)) + \frac{D_3}{2}X_1(t) + \frac{d}{2\tau}\|(\widetilde{\Phi}_{xx}, \widetilde{\Phi}_{xxt}, \widetilde{\Phi}_{xtt})\|_{L^2}^2 \leq 0. \tag{3.79}$$

Integrating (3.79) with respect to t, we obtain

$$V(t) + \int_0^t \Big(\frac{D_3}{2}X_1(t) + \frac{d}{2\tau}\|(\widetilde{\Phi}_{xx}, \widetilde{\Phi}_{xxt}, \widetilde{\Phi}_{xtt})(t)\|_{L^2}^2\Big) \leq V(0). \tag{3.80}$$

Since, according to (3.78) and Lemma 3.1, $\frac{1}{2}X_1(t) \leq V(t) \leq X(t) \leq D_0X_1(t)$, the estimate (3.80) yields

$$\|(\widetilde{\sigma}, u, \widetilde{\Phi})(t)\|_2^2 + \int_0^t (\|(\widetilde{\sigma}, u, \widetilde{\Phi})(\tau)\|_2^2 + \|(\widetilde{\Phi}_{xxt}, \widetilde{\Phi}_{xtt})(\tau)\|_{L^2}^2)\mathrm{d}\tau \leq D_4, \quad \forall\, t > 0, \tag{3.81}$$

where the constant D_4 is independent of t. Moreover, using the third equation in (3.3) and Poincaré inequality, we can show that

$$\|\widetilde{\Phi}_{xxx}\|_{L^2}^2 \lesssim \|(\widetilde{\Phi}_{xt}, \widetilde{\Phi}_x, \widetilde{\sigma}_x, \widetilde{\sigma})\|_{L^2}^2, \tag{3.82}$$

$$\begin{aligned} &\|\widetilde{\Phi}_{xxxx}\|_{L^2}^2 \lesssim \|(\widetilde{\Phi}_{xxt}, \widetilde{\Phi}_{xx}, \widetilde{\sigma}_{xx}, \widetilde{\sigma}_x, \widetilde{\sigma})\|_{L^2}^2 \lesssim \|(\widetilde{\Phi}_{tt}, \widetilde{\Phi}_t, \widetilde{\sigma}_t)\|_{L^2}^2, \\ &\|\widetilde{\Phi}_{xxxt}\|_{L^2}^2 \lesssim \|(\widetilde{\Phi}_{xtt}, \widetilde{\Phi}_{xt}, \widetilde{\sigma}_{xt}, \widetilde{\sigma}_t)\|_{L^2}^2. \end{aligned} \tag{3.83}$$

Hence, it follows from (3.81), (3.82), and (3.83) that

$$\begin{aligned} &\|(\widetilde{\sigma}, u)(t)\|_2^2 + \sum_{k=0}^{2} \|(\partial_t^k\widetilde{\Phi})(t)\|_{H^{4-2k}}^2 \\ &\quad + \int_0^t (\|(\widetilde{\sigma}, u)(\tau)\|_2^2 + \|\widetilde{\Phi}(\tau)\|_{H^4}^2 + \|\widetilde{\Phi}_t(\tau)\|_{H^3}^2 + \|\widetilde{\Phi}_{tt}(\tau)\|_{H^1}^2)\mathrm{d}\tau \leq D_5, \quad \forall\, t > 0, \end{aligned}$$

for some constant D_5 which is independent of t.

To derive the exponential decaying of the perturbation, we note that by dropping the non-negative term $\frac{d}{2\tau}\|(\widetilde{\Phi}_{xx}, \widetilde{\Phi}_{xxt}, \widetilde{\Phi}_{xtt})(t)\|_{L^2}^2$ from the left-hand side of (3.79), and using the equivalency of $V(t)$ and $X_1(t)$, it holds that

$$\frac{\mathrm{d}}{\mathrm{d}t}(V(t)) + \frac{D_3}{2D_0}V(t) \leq 0,$$

which yields the exponential decaying of $V(t)$, and hence of $X(t)$. Moreover, the exponential decaying of $\sum_{k=0}^{2} \|(\partial_t^k\widetilde{\Phi})(t)\|_{H^{4-2k}}^2$ follows from the decaying of $X(t)$ and (3.82)–(3.83). This completes the proof of Lemma 3.2. □

3.4. Global Dynamics when $\tau = 0$

In this subsection, we mainly consider the case of $\tau = 0$ in (3.3):

$$\begin{cases} 2\widetilde{\sigma}_t + 2u\widetilde{\sigma}_x + \widetilde{\sigma}u_x + 2u\widehat{\sigma}_x + \widehat{\sigma}u_x = 0, & x \in I,\ t > 0, \\ 2u_t + 2uu_x + \widetilde{\sigma}\,\widetilde{\sigma}_x + \widetilde{\sigma}\,\widehat{\sigma}_x + \widehat{\sigma}\,\widetilde{\sigma}_x = -2\alpha u + 2\beta\widetilde{\Phi}_x, & x \in I,\ t > 0, \\ 8dA_0\widetilde{\Phi}_{xx} - 8aA_0\widetilde{\Phi} + b(\widetilde{\sigma} + 2\widehat{\sigma})\widetilde{\sigma} = 0, & x \in I,\ t > 0; \\ (\widetilde{\sigma}, u)(x, 0) = (2\sqrt{2A_0\rho_0} - 2\sqrt{2A_0\widehat{\rho}}, u_0)(x), & x \in I; \\ u|_{x=0,\,x=1} = 0, \quad \widetilde{\Phi}|_{x=0,\,x=1} = 0, & t > 0. \end{cases} \tag{3.84}$$

In this case, instead of $X(t)$ defined by (3.4), we let

$$Y(t) \equiv \|(\widetilde{\sigma}, u)(t)\|_2^2,$$

and derive the *a priori* estimates based on the assumptions that (1) $Y(t)$ is sufficiently small within the lifespan of the local solution, and (2) the diffusion coefficient d is sufficiently large.

First, by using the third equation in (3.84), we can modify the proof of Lemma 3.1 to get the qualitative equivalency of $Y(t)$ and $\|(\widetilde{\sigma}_t, \widetilde{\sigma}_{tt}, u, u_t, u_{tt})(t)\|_{L^2}^2$. Indeed, using Sobolev embedding, (3.7), (3.16) and Poincaré inequality, it can be shown that

$$\begin{aligned} 8dA_0\|\widetilde{\Phi}_x\|_{L^2}^2 + 8aA_0\|\widetilde{\Phi}\|_{L^2}^2 &\le b\|\widetilde{\sigma} + \widehat{\sigma}\|_{L^\infty}\|\widetilde{\sigma}\|_{L^2}\|\widetilde{\Phi}\|_{L^2} \\ &\le 5b\sqrt{A_0\widehat{\rho}}\,\|\widetilde{\sigma}\|_{L^2}\|\widetilde{\Phi}_x\|_{L^2} \le \frac{C}{d}\|\widetilde{\sigma}\|_{L^2}^2 + 4dA_0\|\widetilde{\Phi}_x\|_{L^2}^2, \end{aligned}$$

which implies

$$\|\widetilde{\Phi}_x\|_{L^2}^2 \le \frac{C}{d^2}\|\widetilde{\sigma}\|_{L^2}^2 \le \frac{C}{d^2}\|\widetilde{\sigma}_x\|_{L^2}^2, \tag{3.85}$$

where we also used (3.17). Substituting (3.85) into (3.25), we obtain

$$\|\widetilde{\sigma}_x\|_{L^2}^2 \le C(\|u_t\|_{L^2}^2 + \varepsilon^2\|u_x\|_{L^2}^2 + \delta^2\|\widetilde{\sigma}_x\|_{L^2}^2 + \alpha^2\|u\|_{L^2}^2 + d^{-2}\|\widetilde{\sigma}_x\|_{L^2}^2). \tag{3.86}$$

Taking the sum of (3.86) and (3.22) gives us

$$\begin{aligned} \|\widetilde{\sigma}_x\|_{L^2}^2 + \|u_x\|_{L^2}^2 \le C[&\|u_t\|_{L^2}^2 + \|u\|_{L^2}^2 + \|\widetilde{\sigma}_t\|_{L^2}^2 + (\varepsilon^2 + \delta^2)\|u_x\|_{L^2}^2 \\ &+ (\varepsilon^2 + \delta^2 + d^{-2})\|\widetilde{\sigma}_x\|_{L^2}^2]. \end{aligned}$$

Hence, when ε and δ are sufficiently small and d is sufficiently large, such that the coefficients in front of $\|u_x\|_{L^2}^2$ and $\|\widetilde{\sigma}_x\|_{L^2}^2$ are smaller than $\frac{1}{2}$, there holds that

$$\|\widetilde{\sigma}_x\|_{L^2}^2 + \|u_x\|_{L^2}^2 \le C(\|u_t\|_{L^2}^2 + \|u\|_{L^2}^2 + \|\widetilde{\sigma}_t\|_{L^2}^2). \tag{3.87}$$

Similarly, it follows from the elliptic equation that

$$\|\widetilde{\Phi}_{xt}\|_{L^2}^2 \le \frac{C}{d^2}\|\widetilde{\sigma}_t\|_{L^2}^2 \quad \text{and} \quad \|\widetilde{\Phi}_{xx}\|_{L^2}^2 \le \frac{C}{d^2}\|\widetilde{\sigma}_x\|_{L^2}^2, \tag{3.88}$$

by using which we can show that

$$\|u_{xt}\|_{L^2}^2 + \|\widetilde{\sigma}_{xt}\|_{L^2}^2 \le C\|(u_{tt}, \widetilde{\sigma}_{tt}, u_t, \widetilde{\sigma}_t)\|_{L^2}^2,$$

and

$$\|u_{xx}\|_{L^2}^2 + \|\widetilde{\sigma}_{xx}\|_{L^2}^2 \le C\|(u_{tt}, \widetilde{\sigma}_{tt}, u_t, \widetilde{\sigma}_t, u)\|_{L^2}^2.$$

Hence,

$$Y(t) \cong Y_1(t) \equiv \|(\widetilde{\sigma}_t, \widetilde{\sigma}_{tt}, u, u_t, u_{tt})(t)\|_{L^2}^2.$$

Regarding Lemma 3.2, it follows from the elliptic equation that

$$\|\widetilde{\Phi}_{xtt}\|_{L^2} \le \frac{\varepsilon^2}{d^2}\|\widetilde{\sigma}_t\|_{L^2}^2 + \frac{1}{d^2}\|\widetilde{\sigma}_{tt}\|_{L^2}^2. \tag{3.89}$$

Similar to (3.53), by using (3.85), (3.87), (3.88), and (3.89), we can derive the following estimate:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|(\widetilde{\sigma}, \widetilde{\sigma}_t, \widetilde{\sigma}_{tt}, u, u_t, u_{tt})\|_{L^2}^2) + \alpha\|(u, u_t, u_{tt})\|_{L^2}^2 \lesssim O(\varepsilon, \delta, d^{-1})Y(t). \tag{3.90}$$

Similar to (3.71) and using the modified estimates in this section, it can be shown that

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\int_0^1 -(\widetilde{\sigma}\widetilde{\sigma}_t + \widetilde{\sigma}_t\widetilde{\sigma}_{tt})\mathrm{d}x\Big) + \frac{1}{2}\|(\widetilde{\sigma}_t, \widetilde{\sigma}_{tt})\|_{L^2}^2 \le O(\varepsilon, \delta, d^{-1})Y(t) + O(1)\|(u_{tt}, u_t, u)\|_{L^2}^2. \tag{3.91}$$

By coupling (3.90) and (3.91) together, and using the smallness of ε, δ and the largeness of d, we can derive the exponential decaying of $Y_1(t)$, and hence equivalently of $Y(t)$. Moreover, it follows from the elliptic equation that

$$\sum_{k=0}^{2} \|(\partial_t^k \widetilde{\Phi})(t)\|_{H^{4-k}}^2 \lesssim \|\widetilde{\sigma}(t)\|_2^2,$$

which yields the exponential decaying of $\widetilde{\Phi}$ in the corresponding topology. Thus, the proof of Theorem 2.2 is completed.

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Conflict of interest

The authors declare there is no conflict of interest.

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