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Research article

Existence results of fractional differential equations with nonlocal double-integral boundary conditions

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Abstract: This article presents the existence outcomes concerning a family of singular nonlinear differential equations containing Caputo's fractional derivatives with nonlocal double integral boundary conditions. According to the nature of Caputo's fractional calculus, the problem is converted into an equivalent integral equation, while two standard fixed theorems are employed to prove its uniqueness and existence results. An example is presented at the end of this paper to illustrate our obtained results.

Keywords: singular non-linear fractional differential equations; nonlocal double integral boundary conditions; uniqueness and existence of solutions; fixed-point theorems

1. Introduction

Due to its extensive applications in several fields like science and engineering, fractional calculus (FC) has acquired remarkable generality and significance, especially within the last few decades. FC is widely used to describe such practical problems as viscoelastic bodies, continuous media with memory, transformation of temperature, etc. Compared with the traditional integer-order models, the fractional order models can accurately reflect the properties and laws of related phenomena. Recently, there has been a lot of literature on FC. Some of them focus on the basic theory of FC, and the others focus their research on the solvability of initial problems or boundary problems in term of special functions, readers can refer to references [1–7] for details. Researchers have made great advancement in the study of qualitative and quantitative properties of solutions for fractional differential equations (FDEs), including existence, uniqueness, boundedness, continuous dependence on initial data and so on [8–15]. The methods used for analysis include fixed point theorems, the comparison principle, chaos control, nonlinear alternatives of the Leray-Schauder type, upper and lower solutions and numerical calculation. For various studies performed on FC, we refer the reader to more literature [16–23] and the references therein.

In recent years, the issues related to singular FDEs (SFDEs) have been verified. The positive solutions regarding a category of SFDEs were verified in [21] by

$$\begin{cases} D^{\alpha}_{0^+}u(t) = f(t, u(t)), 2 < \alpha \leq 3, 0 < t < 1, \\ u(0) = u'(0) = u''(0) = 0, \end{cases}$$

where $f : (0, 1] \times [0, +\infty)$ and $\lim_{t \to 0^+} f(t, x(t)) = \infty$. They employed the fixed-point theorem and the Leray-Schauder type with nonlinear form in a cone to obtain two results for this problem.

Other works related to this kind of problem have been presented in [24–31].

FDEs have been investigated in various studies when integral boundary conditions (BCs) are under consideration. This type of problems arose from many research areas such as heat conduction, chemical engineering, underground water flow, population dynamics, and so forth. For further information about FDEs with integral BCs, we refer the reader to the [32–39] and the references therein. For instance, Ahmad and Agarwal [39] investigated both the existence and uniqueness of solutions (EUS) for fractional boundary value problems (FBVPs) with some novel versions regarding slit-strips conditions. One of the problems that they considered is as follows:

$$\begin{cases} {}^cD^{q}_{0^+}x(t) = f(t, x(t)), n-1 < q \leq n, t \in (0, 1) \\ x(0) = x'(0) = x''(0) = \cdots = x^{n-2}(0) = 0, \\ x(\zeta) = a\int_0^{\eta} x(s)ds + b\int_{\xi}^{1} x(s)ds, 0 < \eta < \zeta < \xi < 1, \end{cases}$$

where ${}^cD^q$ stands for a special derivative with order q called the fractional derivative of Caputo type and a continuous mapping expressed by $f(t, x(t))$ in $([0, 1] \times R)$ is considered. They obtained the EUS conditions for the mentioned problems by applying fixed principles.

Researchers are also interested in singular nonlinear FDEs with integral BCs [40–44]. Yan [44] investigated just such a problem. Specifically, the upcoming problem was studied:

$$D^{\alpha}_{0^+}x(t) = f(t, x(t)), 0 < t < 1,$$

subject to conditions: $x(0) = 0 = x'(0)$ and $x(1) = \int_{\gamma}^{1} x(\tau)d\tau, 0 < \gamma < 1$. Both $t = 0$ and $t = 1$ lead to the singular non-linear mapping $f(t, x(t))$.

Inspired by the mentioned studies, the current study discusses the following singular nonlinear FDE containing nonlocal double integral BCs:

$$\begin{cases} {}^cD^{\delta}_{0^+}x(t) = f(t, x(t)), 0 < t < 1, \\ x(0) = \int_0^{\eta} x(\tau)d\tau, \\ x'(0) = x''(0) = \cdots = x^{(n-2)}(0) = 0, \\ x(1) = \int_{\gamma}^{1} x(\tau)d\tau \end{cases} \tag{1.1}$$

where ${}^cD^{\delta}_{0^+}$ is Caputo's differentiation of order δ; δ, η and γ are real numbers satisfying $1 \leq n-1 < \delta \leq n < +\infty$ and $0 < \eta < \gamma < 1$, and $n = [\delta] + 1$ is an integer number, the nonlinear term $f(t, x(t)) \in ((0, 1) \times R, R)$ becomes singular when both $t = 0$ and $t = 1$, namely, $\lim_{t \to 0^+} f(t, \cdot) = \infty$ and $\lim_{t \to 1^-} f(t, \cdot) = \infty$. For the physical meaning of the integral BCs in (1.1), $x(t)$ can be interpreted as the distribution of heat on a linear body, and the integral condition $x(0) = \int_0^{\eta} x(\tau)d\tau$ states that the heat absorbed or emitted by the body at $t = 0$ is equal to the variable of its heat over $[0, \eta]$. The other integral condition has a similar explanatory and physical meaning. The current study aims to demonstrate the EUS to the problem (1.1). The generalized Hölder's inequality and fixed-point theories are applied in this paper, while the use of the generalized Hölder's inequality is the highlight of this article. This category of problems discussed in this article and the methods used make a contribution to the existing literature.

This paper consists of a total of five parts. In the first part, the related situation of FDEs is introduced. The second part mainly introduces some basic knowledge of FC, such as definitions and related lemmas, which will be employed in the following content. The third part is the core of the manuscript, including the key conclusions and their proofs. The fourth part includes an example, which aims to use the results of this paper to solve the relevant problems. The last part is the summary of this paper.

2. Preliminaries

The characteristics of FC, the lemmas to be used, and pertinent principles are presented in the current subsection.

Definition 2.1([3]) Consider that $\Omega = [a, b](-\infty < a < b < +\infty)$ is a limited range in R. The fractional integrals denoted by $I^{\alpha}_{a^+} f$ and $I^{\alpha}_{b^-} f$ of order $\alpha \in \mathbb{C}(\Re(\alpha) > 0)$ called the Riemann-Liouville type can be represented by

$$(I^{\alpha}_{a^+} f)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(t)}{(x-t)^{1-\alpha}} dt (x > a; \Re(\alpha) > 0)$$

and

$$(I^{\alpha}_{b^-} f)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \frac{f(t)}{(t-x)^{1-\alpha}} dt (x < b; \Re(\alpha) > 0),$$

respectively. In the above relations, $\Gamma(\cdot)$ stands for the gamma function.

Definition 2.2([3]) Consider $y(x) \in AC^n[a, b]$. Now, the derivatives $({}^cD^{\alpha}_{a^+} y)(x)$ and $({}^cD^{\alpha}_{b^-} y)(x)$, called the Caputo's, can subsist nearly on the whole interval $[a, b]$.
(1) If $\alpha \notin N_0$, $({}^cD^{\alpha}_{a^+} y)(x)$ and $({}^cD^{\alpha}_{b^-} y)(x)$ are defined as follows:

$$({}^cD^{\alpha}_{a^+} y)(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x \frac{y^{(n)}(t)}{(x-t)^{\alpha-n+1}} dt$$

and

$$({}^cD^{\alpha}_{b^-} y)(x) = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_x^b \frac{y^{(n)}(t)}{(t-x)^{\alpha-n+1}} dt,$$

respectively, where D describes the derivative operator and $n = [\Re(\alpha)] + 1$, $\alpha \in \mathbb{C}$, $\Re(\alpha) \geq 0$.
(2) If $\alpha \in N_0$, then

$$({}^cD^{n}_{a^+} y)(x) = y^{(n)}(x), ({}^cD^{n}_{b^-} y)(x) = (-1)^{(n)} y^{(n)}(x), n \in N_0.$$

Lemma 2.1([3]) The FDE's public solutions denoted by $({}^cD^{\alpha}_{a^+} y)(x) = 0$ is derived as

$$y(x) = \sum_{k=0}^{n-1} \frac{y^{(i)}(a)}{i!} (x-a)^i.$$

Especially, for $a = 0$, this result can be presented as

$$y(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_{n-1} x^{n-1},$$

where $c_i = \frac{y^{(i)}(0)}{i!} (i = 0, 1, \cdots n-1)$ denotes certain constants.

Lemma 2.2 Suppose $x(t)$ fulfills the subsequent BVP:

$$\begin{cases} {}^c D_{0^+}^{\delta} x(t) = h(t), 0 < t < 1, \\ x(0) = \int_0^{\eta} x(\tau) d\tau, \\ x'(0) = x''(0) = \cdots = x^{(n-2)}(0) = 0, \\ x(1) = \int_{\gamma}^1 x(\tau) d\tau. \end{cases} \tag{2.1}$$

Now, BVP (2.1) possesses the following unique solution for a certain function $h(t) \in C[0, 1]$

$$x(t) = \frac{1}{\Gamma(\delta)} \int_0^t (t-\tau)^{\delta-1} h(\tau) d\tau + \frac{a + bt^{n-1}}{\Gamma(\delta)} \int_0^1 (1-\tau)^{\delta-1} h(\tau) d\tau + \frac{c + dt^{n-1}}{\Gamma(\delta+1)} \int_0^1 (1-\tau)^{\delta} h(\tau) d\tau$$

$$+ \frac{e + ft^{n-1}}{\Gamma(\delta+1)} \int_0^{\eta} (\eta-\tau)^{\delta} h(\tau) d\tau + \frac{a + bt^{n-1}}{\Gamma(\delta+1)} \int_0^{\gamma} (\gamma-\tau)^{\delta} h(\tau) d\tau \tag{2.2}$$

where ${}^c D_{0^+}^{\delta}$ stands for the Caputo's differentiation of order δ; δ, η, γ and n are defined as in problem (1.1), and $a = -\frac{\eta^n}{\Delta_1}, b = -\frac{n(1-\eta)}{\Delta_1}, c = \frac{\eta}{\Delta_1}, d = \frac{n+\eta^n-(n+1)\eta}{\Delta_1}, e = \frac{n+\gamma^n-1}{\Delta_1}, f = -\frac{n\gamma}{\Delta_1}$ and $\Delta_1 = [(n-1) + \gamma^n](1 - \eta) + \gamma\eta^n$.

Proof According to Lemma 2.1, one can gain

$$x(t) = \frac{1}{\Gamma(\delta)} \int_0^t (t-\tau)^{\delta-1} h(\tau) d\tau + c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1} \tag{2.3}$$

for some $c_0, c_1, c_2 \cdots c_{n-1} \in R$. From the condition $x(0) = \int_0^{\eta} x(\tau) d\tau$, we get

$$c_0 = \int_0^{\eta} x(\tau) d\tau \tag{2.4}$$

By differentiating $x(t)$ based on the expression in (2.3), the following relations are obtained

$$x'(t) = \frac{1}{\Gamma(\delta-1)} \int_0^t (t-\tau)^{\delta-2} h(\tau) d\tau + c_1 + 2c_2 t + 3c_3 t^2 \cdots + (n-1) c_{n-1} t^{n-2},$$

$$x''(t) = \frac{1}{\Gamma(\delta-2)} \int_0^t (t-\tau)^{\delta-3} h(\tau) d\tau + 2c_2 + 3 \cdot 2 c_3 t \cdots + (n-1)(n-2) c_{n-1} t^{n-3},$$

$$\vdots$$

$$x^{(n-2)}(t) = \frac{1}{\Gamma(\delta-n+2)} \int_0^t (t-\tau)^{\delta-n+1} h(\tau) d\tau + (n-2)(n-3) \cdots 2 \cdot 1 c_{n-2} + (n-1)(n-2) \cdots 2 \cdot 1 c_{n-1} t.$$

From the BCs $x'(0) = x''(0) = \cdots = x^{(n-2)}(0) = 0, x(1) = \int_{\gamma}^1 x(\tau) d\tau$ in (2.1), we have

$$c_1 = \cdots = c_{n-2} = 0 \tag{2.5}$$

and

$$c_{n-1} = \int_{\gamma}^{1} x(\tau)d\tau - \frac{1}{\Gamma(\delta)} \int_{0}^{1} (1-s)^{\delta-1} h(\tau)d\tau - \int_{0}^{\eta} x(\tau)d\tau \tag{2.6}$$

Combining (2.3)–(2.6) gives

$$x(t) = \frac{1}{\Gamma(\delta)} \int_{0}^{t} (t-\tau)^{\delta-1} h(\tau)d\tau - \frac{t^{n-1}}{\Gamma(\delta)} \int_{0}^{1} (1-\tau)^{\delta-1} h(\tau)d\tau + (1-t^{n-1}) \int_{0}^{\eta} x(\tau)d\tau + t^{n-1} \int_{\gamma}^{1} x(\tau)d\tau \tag{2.7}$$

Both sides' integration of (2.7) regarding the lower and upper bounds of 0 and η, respectively is denoted by

$$\int_{0}^{\eta} x(t)dt = \frac{1}{\Gamma(\delta+1)} \int_{0}^{\eta} (\eta-\tau)^{\delta} h(\tau)d\tau - \frac{\eta^n}{n\Gamma(\delta)} \int_{0}^{1} (1-\tau)^{\delta-1} h(\tau)d\tau + (\eta - \frac{1}{n}\eta^n) \int_{0}^{\eta} x(\tau)d\tau + \frac{1}{n}\eta^n \int_{\gamma}^{1} x(\tau)d\tau.$$

By transposing and rearranging, we can get the following from the above formula

$$(n+\eta^n - n\eta) \int_{0}^{\eta} x(t)dt - \eta^n \int_{\gamma}^{1} x(\tau)d\tau = \frac{n}{\Gamma(\delta+1)} \int_{0}^{\eta} (\eta-\tau)^{\delta} h(\tau)d\tau - \frac{\eta^n}{\Gamma(\delta)} \int_{0}^{1} (1-\tau)^{\delta-1} h(\tau)d\tau \tag{2.8}$$

Both sides' integration of (2.7) by using the lower and upper bounds γ and 1, respectively is represented by

$$\int_{\gamma}^{1} x(t)dt = \frac{1}{\Gamma(\delta+1)} \int_{0}^{1} (1-\tau)^{\delta} h(\tau)d\tau - \frac{1}{\Gamma(\delta+1)} \int_{0}^{\gamma} (\gamma-\tau)^{\delta} h(\tau)d\tau - \frac{1-\gamma^n}{n\Gamma(\delta)} \int_{0}^{1} (1-\tau)^{\delta-1} h(\tau)d\tau$$
$$+ \frac{n+\gamma^n - n\gamma - 1}{n} \int_{0}^{\eta} x(\tau)d\tau + \frac{1-\gamma^n}{n} \int_{\gamma}^{1} x(\tau)d\tau.$$

By transposing and rearranging, we can get the following from the above formula

$$(n+\gamma^n - n\gamma - 1) \int_{0}^{\eta} x(t)dt - (n+\gamma^n - 1) \int_{\gamma}^{1} x(\tau)d\tau = \frac{n}{\Gamma(\delta+1)} \int_{0}^{\gamma} (\gamma-\tau)^{\delta} h(\tau)d\tau$$
$$+ \frac{1-\gamma^n}{\Gamma(\delta)} \int_{0}^{1} (1-\tau)^{\delta-1} h(\tau)d\tau - \frac{n}{\Gamma(\delta+1)} \int_{0}^{1} (1-\tau)^{\delta} h(\tau)d\tau \tag{2.9}$$

Equations (2.8) and (2.9) constitute a system with $\int_0^{\eta} x(\tau)d\tau$ and $\int_{\gamma}^{1} x(\tau)d\tau$ as the unknown elements, and the coefficients of this system are represented by

$$\Delta = \begin{vmatrix} n+\eta^n - n\eta & -\eta^n \\ n+\gamma^n - n\gamma - 1 & -(n+\gamma^n - 1) \end{vmatrix} = -n[(n-1)+\gamma^n](1-\eta) - n\gamma\eta^n < 0.$$

So, using the Cramer's rule, we can get

$$\int_{0}^{\eta} x(t)dt = \frac{n+\gamma^n - 1}{\Delta_1 \Gamma(\delta+1)} \int_{0}^{\eta} (\eta-\tau)^{\delta} h(\tau)d\tau + \frac{\eta}{\Delta_1 \Gamma(\delta+1)} \int_{0}^{1} (1-\tau)^{\delta} h(\tau)d\tau$$
$$- \frac{\eta^n}{\Delta_1 \Gamma(\delta)} \int_{0}^{1} (1-\tau)^{\delta-1} h(\tau)d\tau - \frac{\eta^n}{\Delta_1 \Gamma(\delta+1)} \int_{0}^{\gamma} (1-\tau)^{\delta} h(\tau)d\tau \tag{2.10}$$

and

$$\int_\gamma^1 x(t)dt = -\frac{n+\eta^n-n\eta}{\Delta_1\Gamma(\delta+1)}\int_0^\gamma(\gamma-\tau)^\delta h(\tau)d\tau - \frac{(1-\eta)(1-\gamma^n)+(1-\gamma)\eta^n}{\Delta_1\Gamma(\delta)}\int_0^1(1-\tau)^{\delta-1}h(\tau)d\tau$$

$$+\frac{n+\eta^n-n\eta}{\Delta_1\Gamma(\delta+1)}\int_0^1(1-\tau)^\delta h(\tau)d\tau + \frac{n+\gamma^n-n\gamma-1}{\Delta_1\Gamma(\delta+1)}\int_0^\eta(\eta-\tau)^\delta h(\tau)d\tau \quad (2.11)$$

where $\Delta_1 = [(n-1)+\gamma^n](1-\eta)+\gamma\eta^n > 0$.

The result can be derived after substituting Eqs (2.10) and (2.11) into Eq (2.7). This finishes the proof.

Banach's fixed point theorem and its subsequent theorem help to attain the main outcomes of the current article.

Lemma 2.3([45]) (The fixed point theorem by Krasnoselskii) Suppose that M is defined as a non-empty subset of a Banach space X with properties of closedness, boundedness and convexity. Moreover, consider that A and B stand for the operators meeting the subsequent requirements (a) $Ax + By \in M$, for $x, y \in M$; (b) both compactness and continuity of A exist; (c) a contraction mapping is represented by B. Now, $z \in M$ exists such that $z = Az + Bz$.

This part ends with showing some fundamental understanding of the L^p space and introducing an inequality and its corresponding extended format called the Hölder's inequality [46].

Consider that an open (or measurable) set is denoted by $V \subset R^n$ and a measurable mapping of real numbers defined on V is denoted by $g(x)$. $|g(x)|^p$ turns out to be measurable on V for $1 \le p < \infty$ and $\int_V |g(x)|^p dx$ is meaningful. Now, we introduce a function space $L^p(V)$ as follows:

$$L^p(V) = \{g(x)|g(x) \text{ is measurable on } V, \int_V |g(x)|^p dx < \infty\}.$$

For $g \in L^p(V)$, the upcoming norm is defined

$$\| g \|_p = (\int_V |g(x)|^p dx)^{1/p}.$$

$1 < p_1$ and $p_2 < \infty$ are called conjugate exponentials of each other if $\frac{1}{p_1} + \frac{1}{p_2} = 1$.

Lemma 2.4([46]) (Hölder's inequality) Consider that $V \subset R^n$ is an open set, p_1 and p_2 are conjugate exponentials, $g(x) \in L^{p_1}(V), h(x) \in L^{p_2}(V)$ and $g(x)h(x)$ is integrable on V, while the following equality holds

$$\int_V | g(x)h(x) | dx \le \| g \|_{p_1} \| g \|_{p_2} .$$

The mentioned result is extended as

$$\int_V | g_1(x)\cdots g_n(x) | dx \le \| g_1 \|_{p_1} \cdots \| g_n \|_{p_n},$$

where $g_i(x) \in L^{p_i}(V)$ and $\sum_{i=1}^n \frac{1}{p_i} = 1$. The above expression is just called the generalized Hölder's inequality.

3. Main results

Suppose that $E = C([0,1], R)$ encompasses continuous function space on interval $[0,1]$. Now, a Banach space is denoted by $X = (E, \|\cdot\|)$, where $\|\cdot\|$ is the maximum norm $\| x \| = \max_{t\in[0,1]} | x(t) |$ with $x(t) \in E$.

Define an operator $\phi : X \to X$ as

$$(\phi x)(t) = \frac{1}{\Gamma(\delta)}\int_0^t (t-\tau)^{\delta-1} f(\tau, x(\tau))d\tau + \frac{a+bt^{n-1}}{\Gamma(\delta)}\int_0^1 (1-\tau)^{\delta-1} f(\tau, x(\tau))d\tau$$
$$+\frac{c+dt^{n-1}}{\Gamma(\delta+1)}\int_0^1 (1-\tau)^{\delta} f(\tau, x(\tau))d\tau$$
$$+\frac{e+ft^{n-1}}{\Gamma(\delta+1)}\int_0^\eta (\eta-\tau)^{\delta} f(\tau, x(\tau))d\tau + \frac{a+bt^{n-1}}{\Gamma(\delta+1)}\int_0^\gamma (\gamma-\tau)^{\delta} f(\tau, x(\tau))d\tau \tag{3.1}$$

There exists equality between the solutions of the problem (1.1) and the fixed points regarding the operator ϕ. This paper presents the following assumptions that are put on $f(t, x(t))$ that appears in (1.1) in the sequel.

(H1) Both $t = 0$ and $t = 1$ lead to a singular $f(t, x(t))$ which satisfies

$$\lim_{t\to 0^+} f(t,\cdot) = \infty, \lim_{t\to 1^-} f(t,\cdot) = \infty.$$

Besides, there are two constants $\sigma_1 > 0$ and $\sigma_2 > 0$, where $t^{\sigma_1}(1-t)^{\sigma_2} f(t, x(t))$ is a continuous function in $[0,1]$.

By the assumption of (H1), it can be deduced that a number N_0 exists and meets

$$|t^{\sigma_1}(1-t)^{\sigma_2} f(t, x(t))| \le N_0 \tag{3.2}$$

where $t \in [0,1]$ and $x(t) \in E$. Throughout the rest of this article, we always employ s, s_1 and s_2 to represent any set of real numbers that meet the following conditions

(H2) (i) $s > 1, s_1 > 1, s_2 > 1$; (ii) $\frac{1}{s} + \frac{1}{s_1} + \frac{1}{s_2} = 1$; (iii) $0 < s_1\sigma_1 < 1, 0 < s_2\sigma_2 < 1$.

Accordingly, avoiding excessive conjugate exponent notations is possible while using the generalized Hölder's inequality in different contexts.

Lemma 3.1 Assume that $1 \le n-1 < \delta \le n$, and s, s_1, s_2, σ_1 and σ_2 are positive constants satisfying (H2). Define an operator $K_l(t)$ for some real number $l \ge 1$ as

$$K_l(t) = \int_0^t (t-\tau)^l \tau^{-\theta_1}(1-\tau)^{-\theta_2} d\tau, t \in [0,1].$$

Then, the following results are valid:

(1) $\lim_{t\to 0^+} K_l(t) = 0$;

(2) $K_l(t) \le \frac{1}{\sqrt[s]{1+sl}} \frac{1}{\sqrt[s_1]{1-s_1\sigma_1}} \frac{1}{\sqrt[s_2]{1+s_2\sigma_2}}$ for any $t \in [0,1]$;

(3) For any $t_1, t_2 \in [0,1]$, $| K_l(t_1) - K_l(t_2) | < \frac{l}{\sqrt[s]{1+s(l-1)}} \frac{1}{\sqrt[s_1]{1-s_1\sigma_1}} \frac{1}{\sqrt[s_2]{1+s_2\sigma_2}} | t_1 - t_2 |$.

Proof (1) Recall Lemma 3.2 in [44].

(2) According to the generalized Hölder's inequality, one obtains

$$\begin{aligned}K_l(t) &= \int_0^t (t-\tau)^l \tau^{-\theta_1}(1-\tau)^{-\theta_2} d\tau \\ &\le [\int_0^t (t-\tau)^{sl} d\tau]^{1/s} [\int_0^t \tau^{-s_1\sigma_1} d\tau]^{1/s_1} [\int_0^t (1-\tau)^{-s_2\sigma_2} d\tau]^{1/s_2} \\ &\le \frac{1}{\sqrt[s]{1+sl}} \sqrt[s]{t^{1+sl}} \frac{1}{\sqrt[s_1]{1-s_1\sigma_1}} \sqrt[s_1]{t^{1-s_1\sigma_1}} \frac{1}{\sqrt[s_2]{1-s_2\sigma_2}} \sqrt[s_2]{1-(1-t)^{1-s_2\sigma_2}} \\ &\le \frac{1}{\sqrt[s]{1+sl}} \frac{1}{\sqrt[s_1]{1-s_1\sigma_1}} \frac{1}{\sqrt[s_2]{1-s_2\sigma_2}}.\end{aligned}$$

(3) Deriving the function $K_l(t)$ and using the generalized Hölder's inequality, one can obtain

$$\begin{aligned}K_l'(t) &= l \int_0^t (t-\tau)^{l-1} \tau^{-\theta_1}(1-\tau)^{-\theta_2} d\tau \\ &\le l[\int_0^t (t-\tau)^{s(l-1)} d\tau]^{1/s} [\int_0^t \tau^{-s_1\sigma_1} d\tau]^{1/s_1} [\int_0^t (1-\tau)^{-s_2\sigma_2} d\tau]^{1/s_2} \\ &\le \frac{l}{\sqrt[s]{1+s(l-1)}} \sqrt[s]{t^{1+s(l-1)}} \frac{1}{\sqrt[s_1]{1-s_1\sigma_1}} \sqrt[s_1]{t^{1-s_1\sigma_1}} \frac{1}{\sqrt[s_2]{1-s_2\sigma_2}} \sqrt[s_2]{1-(1-t)^{1-s_2\sigma_2}} \\ &\le \frac{l}{\sqrt[s]{1+s(l-1)}} \frac{1}{\sqrt[s_1]{1-s_1\sigma_1}} \frac{1}{\sqrt[s_2]{1-s_2\sigma_2}}.\end{aligned}$$

By the mean value theorem, we have

$$\mid K_l(t_1) - K_l(t_2) \mid \le K_l'(\xi) \mid t_1 - t_2 \mid \le \frac{l}{\sqrt[s]{1+s(l-1)}} \frac{1}{\sqrt[s_1]{1-s_1\sigma_1}} \frac{1}{\sqrt[s_2]{1+s_2\sigma_2}} \mid t_1 - t_2 \mid,$$

where ξ is a number between t_1 and t_2.

Lemma 3.2 Assume that $1 \le n-1 < \delta \le n$ and a function $h(t) : (0,1) \to R$ is continuous and satisfying $\lim_{t\to 0^+} h(t) = \infty$ and $\lim_{t\to 1^-} h(t) = \infty$. A new function $H(t)$ is defined as

$$\begin{aligned}H(t) =& \frac{1}{\Gamma(\delta)} \int_0^t (t-\tau)^{\delta-1} h(\tau) d\tau + \frac{a+bt^{n-1}}{\Gamma(\delta)} \int_0^(1-\tau)^{\delta-1} h(\tau) d\tau + \frac{c+dt^{n-1}}{\Gamma(\delta+1)} \int_0^1 (1-\tau)^{\delta} h(\tau) d\tau \\ &+ \frac{e+ft^{n-1}}{\Gamma(\delta+1)} \int_0^\eta (\eta-\tau)^{\delta} h(\tau) d\tau + \frac{a+bt^{n-1}}{\Gamma(\delta+1)} \int_0^\gamma (\gamma-\tau)^{\delta} h(\tau) d\tau.\end{aligned}$$

Then the continuity of $t^{\sigma_1}(1-t)^{\sigma_2} h(t)$ on $[0,1]$ leads $H(t)$ to be continuous in $[0,1]$.

Proof Since $t^{\sigma_1}(1-t)^{\sigma_2} h(t)$ is a continuous mapping in $[0,1]$, there is a positive constant N_1 that satisfies $|t^{\sigma_1}(1-t)^{\sigma_2} h(t)| \le N_1$.

According to the definition of $H(t)$, we have

$$\begin{aligned}H(0) =& \frac{a}{\Gamma(\delta)} \int_0^1 (1-\tau)^{\delta-1} h(\tau) d\tau + \frac{c}{\Gamma(\delta+1)} \int_0^1 (1-\tau)^{\delta} h(\tau) d\tau \\ &+ \frac{e}{\Gamma(\delta+1)} \int_0^\eta (\eta-\tau)^{\delta} h(\tau) d\tau + \frac{a}{\Gamma(\delta+1)} \int_0^\gamma (\gamma-\tau)^{\delta} h(\tau) d\tau.\end{aligned}$$

For any $t' \in [0,1], H(t)$ is continuous and t' will be proven.
(I) For $t' = 0$ and $t \in [0,1]$, the following equality is attained.

$$\begin{aligned}
\mid H(t) - H(0) \mid =& \mid \frac{1}{\Gamma(\delta)} \int_0^t (t-\tau)^{\delta-1} h(\tau) d\tau + \frac{bt^{n-1}}{\Gamma(\delta)} \int_0^1 (1-\tau)^{\delta-1} h(\tau) d\tau + \frac{dt^{n-1}}{\Gamma(\delta+1)} \int_0^1 (1-\tau)^{\delta} h(\tau) d\tau \\
&+ \frac{ft^{n-1}}{\Gamma(\delta+1)} \int_0^\eta (\eta-\tau)^{\delta} h(\tau) d\tau + \frac{bt^{n-1}}{\Gamma(\delta+1)} \int_0^\gamma (\gamma-\tau)^{\delta} h(\tau) d\tau \mid \\
&\le \frac{N_1}{\Gamma(\delta)} \int_0^t (t-\tau)^{\delta-1} \tau^{-\sigma_1} (1-\tau)^{-\sigma_2} d\tau + \frac{|b| N_1 t^{n-1}}{\Gamma(\delta)} \int_0^1 (1-\tau)^{\delta-1} \tau^{-\sigma_1} (1-\tau)^{-\sigma_2} d\tau \\
&+ \frac{|d| N_1 t^{n-1}}{\Gamma(\delta+1)} \int_0^1 (1-\tau)^{\delta} \tau^{-\sigma_1} (1-\tau)^{-\sigma_2} d\tau + \frac{|f| N_1 t^{n-1}}{\Gamma(\delta+1)} \int_0^\eta (\eta-\tau)^{\delta} \tau^{-\sigma_1} (1-\tau)^{-\sigma_2} d\tau \\
&+ \frac{|b| N_1 t^{n-1}}{\Gamma(\delta+1)} \int_0^\gamma (\gamma-\tau)^{\delta} \tau^{-\sigma_1} (1-\tau)^{-\sigma_2} d\tau \\
&\le \frac{N_1}{\Gamma(\delta)} K_{\delta-1}(t) + N_1 [\frac{|b| K_{\delta-1}(1)}{\Gamma(\delta)} + \frac{|d| K_\delta(1)}{\Gamma(\delta+1)} + \frac{|f| K_\delta(\eta)}{\Gamma(\delta+1)} + \frac{|b| K_\delta(\gamma)}{\Gamma(\delta+1)}] t^{n-1}.
\end{aligned}$$

Thus, by the results (1) and (2) in Lemma 3.1, we have $\mid H(t) - H(0) \mid \to 0$ as $t \to t' = 0$ that is

$$\lim_{t \to 0^+} H(t) = H(0).$$

(II) For each $t' \in (0,1]$ and $t \in [0,1], t' \neq t$, one can obtain

$$\begin{aligned}
\mid H(t) - H(t') \mid =& \mid \frac{1}{\Gamma(\delta)} [\int_0^t (t-\tau)^{\delta-1} h(\tau) d\tau - \int_0^{t'} (t'-\tau)^{\delta-1} h(\tau) d\tau] + \frac{b(t^{n-1} - t'^{n-1})}{\Gamma(\delta)} \int_0^1 (1-\tau)^{\delta-1} h(\tau) d\tau \\
&+ \frac{d(t^{n-1} - t'^{n-1})}{\Gamma(\delta+1)} \int_0^1 (1-\tau)^{\delta} h(\tau) d\tau + \frac{f(t^{n-1} - t'^{n-1})}{\Gamma(\delta)} \int_0^\eta (\eta-\tau)^{\delta} h(\tau) d\tau \\
&+ \frac{b(t^{n-1} - t'^{n-1})}{\Gamma(\delta+1)} \int_0^\gamma (\gamma-\tau)^{\delta} h(\tau) d\tau \mid \\
&\le \frac{1}{\Gamma(\delta)} \mid \int_0^{t'} (t'-\tau)^{\delta-1} h(\tau) d\tau - \int_0^t (t-\tau)^{\delta-1} h(\tau) d\tau \mid \\
&+ \frac{|b| N_1 |t^{n-1} - t'^{n-1}|}{\Gamma(\delta)} \int_0^1 (1-\tau)^{\delta-1} \tau^{-\sigma_1} (1-\tau)^{-\sigma_2} d\tau \\
&+ \frac{|d| N_1 |t^{n-1} - t'^{n-1}|}{\Gamma(\delta+1)} \int_0^1 (1-\tau)^{\delta} \tau^{-\sigma_1} (1-\tau)^{-\sigma_2} d\tau \\
&+ \frac{|f| N_1 |t^{n-1} - t'^{n-1}|}{\Gamma(\delta+1)} \int_0^\eta (\eta-\tau)^{\delta} \tau^{-\sigma_1} (1-\tau)^{-\sigma_2} d\tau \\
&+ \frac{|b| N_1 |t^{n-1} - t'^{n-1}|}{\Gamma(\delta+1)} \int_0^\gamma (\gamma-\tau)^{\delta} \tau^{-\sigma_1} (1-\tau)^{-\sigma_2} d\tau \\
&\le \frac{1}{\Gamma(\delta)} \mid \int_0^t [(t'-\tau)^{\delta-1} - (t-\tau)^{\delta-1}] h(\tau) d\tau + \int_t^{t'} (t'-\tau)^{\delta-1} h(\tau) d\tau \mid \\
&+ N_1 [\frac{|b| K_{\delta-1}(1)}{\Gamma(\delta)} + \frac{|d| K_\delta(1) + |f| K_\delta(\eta) + |b| K_\delta(\gamma)}{\Gamma(\delta+1)}] |t^{n-1} - t'^{n-1}|
\end{aligned}$$

$$\leq \frac{N_1}{\Gamma(\delta)} \mid \int_0^t [(t'-\tau)^{\delta-1} - (t-\tau)^{\delta-1}]\tau^{-\sigma_1}(1-\tau)^{-\sigma_2}d\tau + \int_t^{t'} (t'-\tau)^{\delta-1}\tau^{-\sigma_1}(1-\tau)^{-\sigma_2}d\tau \mid$$
$$+ N_1[\frac{|b|K_{\delta-1}(1)}{\Gamma(\delta)} + \frac{|d|K_\delta(1) + |f|K_\delta(\eta) + |b|K_\delta(\gamma)}{\Gamma(\delta+1)}]|t^{n-1} - t'^{n-1}|$$
$$= \frac{N_1}{\Gamma(\delta)} \mid \int_0^{t'} (t'-\tau)^{\delta-1}\tau^{-\sigma_1}(1-\tau)^{-\sigma_2}d\tau - \int_0^t (t-\tau)^{\delta-1}\tau^{-\sigma_1}(1-\tau)^{-\sigma_2}d\tau \mid$$
$$+ N_1[\frac{|b|K_{\delta-1}(1)}{\Gamma(\delta)} + \frac{|d|K_\delta(1) + |f|K_\delta(\eta) + |b|K_\delta(\gamma)}{\Gamma(\delta+1)}]|t^{n-1} - t'^{n-1}|$$
$$= \frac{N_1}{\Gamma(\delta)} \mid K_{\delta-1}(t') - K_{\delta-1}(t) \mid + N_1[\frac{|b|K_{\delta-1}(1)}{\Gamma(\delta)} + \frac{|d|K_\delta(1) + |f|K_\delta(\eta) + |b|K_\delta(\gamma)}{\Gamma(\delta+1)}]|t^{n-1} - t'^{n-1}|.$$

By the results (2) and (3) in Lemma 3.1, one can obtain

$$\mid H(t) - H(t') \mid \leq \frac{N_1}{\Gamma(\delta)} \frac{l}{\sqrt[s]{1+s(\delta-1)}} \frac{1}{\sqrt[s_1]{1-s_1\sigma_1}} \frac{1}{\sqrt[s_2]{1-s_2\sigma_2}}|t'-t|$$
$$+ N_1[\frac{|b|K_{\delta-1}(1)}{\Gamma(\delta)} + \frac{|d|K_\delta(1) + |f|K_\delta(\eta) + |b|K_\delta(\gamma)}{\Gamma(\delta+1)}]|t^{n-1} - t'^{n-1}| \to 0$$

when $t \to t'$, and this means that

$$\lim_{t\to t'} H(t) = H(t').$$

Since t' is any point in $[0, 1]$, we prove the assertion of Lemma 3.2.

Theorem 3.1 Assume that $1 \leq n-1 < \delta \leq n, \sigma_1 > 0$ and $\sigma_2 > 0$ are constants; (H1) and the subsequent assumption are satisfied by $f(t, x(t))$:

(H3) $m(t) \in C([0, 1], R)$ is a mapping such that

$$t^{\sigma_1}(1-t)^{\sigma_2} \mid f(t, x(t)) - f(t, y(t)) \mid \leq \mid m(t) \parallel x(t) - y(t) \mid .$$

Assume that the condition (H2) and the subsequent inequality are fulfilled

$$\parallel m \parallel \{\frac{1+\mid a\mid+\mid b\mid}{\Gamma(\delta)} \frac{1}{\sqrt[s]{1+s(\delta-1)}} \frac{1}{\sqrt[s_1]{1-s_1\sigma_1}} \frac{1}{\sqrt[s_2]{1-s_2\sigma_2}} + \frac{1}{\Gamma(\delta+1)} \frac{1}{\sqrt[s]{1+s\delta}} \frac{1}{\sqrt[s_1]{1-s_1\sigma_1}} \frac{1}{\sqrt[s_2]{1-s_2\sigma_2}}$$
$$\cdot[\mid c\mid + \mid d\mid + (\mid e\mid + \mid f\mid)\sqrt[s]{\eta^{1+s\delta}} \sqrt[s_1]{\eta^{1-s_1\sigma_1}} \sqrt[s_2]{1-(1-\eta)^{1-s_2\sigma_2}}$$
$$+(\mid a\mid + \mid b\mid)\sqrt[s]{\gamma^{1+s\delta}} \sqrt[s_1]{\gamma^{1-s_1\sigma_1}} \sqrt[s_2]{1-(1-\gamma)^{1-s_2\sigma_2}}]\} < 1 \tag{3.3}$$

Now, a unique solution regarding the $([0, 1], R)$ space is possessed for the problem (1.1).

Proof For every $x, y \in X = (C[0, 1], R)$, the second assertion of Lemma 3.1 and the generalized Hölder's inequality can be deduced by (H3)

$$\mid (\phi x)(t) - (\phi y)(t) \mid \leq \frac{1}{\Gamma(\delta)} \int_0^t (t-\tau)^{\delta-1} \mid f(\tau, x(\tau)) - f(\tau, y(\tau)) \mid d\tau$$

$$+\frac{\mid a\mid + \mid b\mid}{\Gamma(\delta)} \int_0^1 (1-\tau)^{\delta-1} \mid f(\tau, x(\tau)) - f(\tau, y(\tau)) \mid d\tau + \frac{\mid c\mid + \mid d\mid}{\Gamma(\delta+1)} \int_0^1 (1-\tau)^{\delta} \mid f(\tau, x(\tau)) - f(\tau, y(\tau)) \mid d\tau$$

$$
\begin{aligned}
&+\frac{|e|+|f|}{\Gamma(\delta+1)}\int_0^\eta(\eta-\tau)^\delta \mid f(\tau,x(\tau))-f(\tau,y(\tau))\mid d\tau\\
&+\frac{|a|+|b|}{\Gamma(\delta+1)}\int_0^\gamma(\gamma-\tau)^\delta \mid f(\tau,x(\tau))-f(\tau,y(\tau))\mid d\tau\\
&\le \frac{1}{\Gamma(\delta)}\int_0^t(t-\tau)^{\delta-1}\tau^{-\sigma_1}(1-\tau)^{-\sigma_2}\mid m(\tau)\mid\mid x(\tau)-y(\tau)\mid d\tau\\
&+\frac{|a|+|b|}{\Gamma(\delta)}\int_0^1(1-\tau)^{\delta-1}\tau^{-\sigma_1}(1-\tau)^{-\sigma_2}\mid m(\tau)\mid\mid x(\tau)-y(\tau)\mid d\tau\\
&+\frac{|c|+|d|}{\Gamma(\delta+1)}\int_0^1(1-\tau)^{\delta}\tau^{-\sigma_1}(1-\tau)^{-\sigma_2}\mid m(\tau)\mid\mid x(\tau)-y(\tau)\mid d\tau\\
&+\frac{|e|+|f|}{\Gamma(\delta+1)}\int_0^\eta(\eta-\tau)^{\delta}\tau^{-\sigma_1}(1-\tau)^{-\sigma_2}\mid m(\tau)\mid\mid x(\tau)-y(\tau)\mid d\tau\\
&+\frac{|a|+|b|}{\Gamma(\delta+1)}\int_0^\gamma(\gamma-\tau)^{\delta}\tau^{-\sigma_1}(1-\tau)^{-\sigma_2}\mid m(\tau)\mid\mid x(\tau)-y(\tau)\mid d\tau\\
&\le \|m(\tau)\|\|x(\tau)-y(\tau)\|\{\frac{1}{\Gamma(\delta)}K_{\delta-1}(t)+\frac{|a|+|b|}{\Gamma(\delta)}K_{\delta-1}(1)+\frac{|c|+|d|}{\Gamma(\delta+1)}K_\delta(1)\\
&+\frac{|e|+|f|}{\Gamma(\delta+1)}[\int_0^\eta(\eta-\tau)^{s\delta}d\tau]^{1/s}[\int_0^\eta\tau^{-s_1\sigma_1}d\tau]^{1/s_1}[\int_0^\eta(1-\tau)^{-s_2\sigma_2}d\tau]^{1/s_2}\\
&+\frac{|a|+|b|}{\Gamma(\delta+1)}[\int_0^\gamma(\gamma-\tau)^{s\delta}d\tau]^{1/s}[\int_0^\gamma\tau^{-s_1\sigma_1}d\tau]^{1/s_1}[\int_0^\gamma(1-\tau)^{-s_2\sigma_2}d\tau]^{1/s_2}\}\\
&\le \|m(\tau)\|\|x(\tau)-y(\tau)\|\{\frac{1}{\Gamma(\delta)}\frac{1}{\sqrt[s]{1+s(\delta-1)}}\frac{1}{\sqrt[s_1]{1-s_1\sigma_1}}\frac{1}{\sqrt[s_2]{1-s_2\sigma_2}}\\
&+\frac{|a|+|b|}{\Gamma(\delta)}\frac{1}{\sqrt[s]{1+s(\delta-1)}}\frac{1}{\sqrt[s_1]{1-s_1\sigma_1}}\frac{1}{\sqrt[s_2]{1-s_2\sigma_2}}\\
&+\frac{|c|+|d|}{\Gamma(\delta+1)}\frac{1}{\sqrt[s]{1+s(\delta)}}\frac{1}{\sqrt[s_1]{1-s_1\sigma_1}}\frac{1}{\sqrt[s_2]{1-s_2\sigma_2}}\\
&+\frac{|e|+|f|}{\Gamma(\delta+1)}\frac{1}{\sqrt[s]{1+s\delta}}\sqrt[s]{\eta^{1+s\delta}}\frac{1}{\sqrt[s_1]{1-s_1\sigma_1}}\sqrt[s_1]{\eta^{1-s_1\sigma_1}}\frac{1}{\sqrt[s_2]{1-s_2\sigma_2}}\sqrt[s_2]{1-(1-\eta)^{1-s_2\sigma_2}}\\
&+\frac{|a|+|b|}{\Gamma(\delta+1)}\frac{1}{\sqrt[s]{1+s\delta}}\sqrt[s]{\gamma^{1+s\delta}}\frac{1}{\sqrt[s_1]{1-s_1\sigma_1}}\sqrt[s_1]{\gamma^{1-s_1\sigma_1}}\frac{1}{\sqrt[s_2]{1-s_2\sigma_2}}\sqrt[s_2]{1-(1-\gamma)^{1-s_2\sigma_2}}\}\\
&\le\| m \|\{\frac{1+|a|+|b|}{\Gamma(\delta)}\frac{1}{\sqrt[s]{1+s(\delta-1)}}\frac{1}{\sqrt[s_1]{1-s_1\sigma_1}}\frac{1}{\sqrt[s_2]{1-s_2\sigma_2}}\\
&+\frac{1}{\Gamma(\delta+1)}\frac{1}{\sqrt[s]{1+s\delta}}\frac{1}{\sqrt[s_1]{1-s_1\sigma_1}}\frac{1}{\sqrt[s_2]{1-s_2\sigma_2}}\\
&\cdot[|c|+|d|+(|e|+|f|)\sqrt[s]{\eta^{1+s\delta}}\ \sqrt[s_1]{\eta^{1-s_1\sigma_1}}\ \sqrt[s_2]{1-(1-\eta)^{1-s_2\sigma_2}}\\
&+(|a|+|b|)\sqrt[s]{\gamma^{1+s\delta}}\ \sqrt[s_1]{\gamma^{1-s_1\sigma_1}}\ \sqrt[s_2]{1-(1-\gamma)^{1-s_2\sigma_2}}]\}\cdot\|x(\tau)-y(\tau)\|.
\end{aligned}
$$

The condition (3.3) ensures that the operator ϕ is a contractive mapping. Accordingly, Banach's fixed-point theorem indicates that ϕ possesses a unique fixed-point that is equal to the problem (1.1) unique solution.

Theorem 3.2 Suppose that $1 \le n-1 < \delta \le n, \sigma_1 > 0$ and $\sigma_2 > 0$ are constants; both (H1) and (H3) are satisfied by $f(t, x(t))$. Assume that (H2) and the subsequent inequality are true

$$\| m \| \{\frac{|a|+|b|}{\Gamma(\delta)} \frac{1}{\sqrt[s]{1+s(\delta-1)}} \frac{1}{\sqrt[s_1]{1-s_1\sigma_1}} \frac{1}{\sqrt[s_2]{1-s_2\sigma_2}} + \frac{1}{\Gamma(\delta+1)} \frac{1}{\sqrt[s]{1+s\delta}} \frac{1}{\sqrt[s_1]{1-s_1\sigma_1}} \frac{1}{\sqrt[s_2]{1-s_2\sigma_2}}$$
$$\cdot[|c|+|d|+(|e|+|f|)\sqrt[s]{\eta^{1+s\delta}}\sqrt[s_1]{\eta^{1-s_1\sigma_1}}\sqrt[s_2]{1-(1-\eta)^{1-s_2\sigma_2}}$$
$$+(|a|+|b|)\sqrt[s]{\gamma^{1+s\delta}}\sqrt[s_1]{\gamma^{1-s_1\sigma_1}}\sqrt[s_2]{1-(1-\gamma)^{1-s_2\sigma_2}}]\} < 1 \tag{3.4}$$

Now, the interval $([0, 1], R)$ contains at least one solution for the problem (1.1).

Proof Take a constant L satisfying

$$N_0\{\frac{1+|a|+|b|}{\Gamma(\delta)} \frac{1}{\sqrt[s]{1+s(\delta-1)}} \frac{1}{\sqrt[s_1]{1-s_1\sigma_1}} \frac{1}{\sqrt[s_2]{1-s_2\sigma_2}} + \frac{1}{\Gamma(\delta+1)} \frac{1}{\sqrt[s]{1+s\delta}} \frac{1}{\sqrt[s_1]{1-s_1\sigma_1}} \frac{1}{\sqrt[s_2]{1-s_2\sigma_2}}$$
$$\cdot[|c|+|d|+(|e|+|f|)\sqrt[s]{\eta^{1+s\delta}}\sqrt[s_1]{\eta^{1-s_1\sigma_1}}\sqrt[s_2]{1-(1-\eta)^{1-s_2\sigma_2}}$$
$$+(|a|+|b|)\sqrt[s]{\gamma^{1+s\delta}}\sqrt[s_1]{\gamma^{1-s_1\sigma_1}}\sqrt[s_2]{1-(1-\gamma)^{1-s_2\sigma_2}}]\} \le L.$$

The number N_0 is defined in (3.2).
The set $B_L = \{x \in X = C([0, 1], \||\, x \| \le L\}$ is a ball in X. Define two operators ϕ_1 and ϕ_2 on B_L as

$$(\phi_1 x)(t) = \frac{1}{\Gamma(\delta)} \int_0^t (t-\tau)^{\delta-1} f(\tau, x(\tau))d\tau,$$
$$(\phi_2 x)(t) = \frac{a+bt^{n-1}}{\Gamma(\delta)} \int_0^1 (1-\tau)^{\delta-1} f(\tau, x(\tau))d\tau + \frac{c+dt^{n-1}}{\Gamma(\delta+1)} \int_0^1 (1-\tau)^{\delta} f(\tau, x(\tau))d\tau$$
$$+ \frac{e+ft^{n-1}}{\Gamma(\delta+1)} \int_0^\eta (\eta-\tau)^{\delta} f(\tau, x(\tau))d\tau + \frac{a+bt^{n-1}}{\Gamma(\delta+1)} \int_0^\gamma (\gamma-\tau)^{\delta} f(\tau, x(\tau))d\tau.$$

For any $x, y \in B_L$, the following relation can be obtained by taking a process similar to Theorem 3.1:

$$\| \phi_1 x + \phi_2 \| \le \frac{N_0}{\Gamma(\delta)} \int_0^t (t-\tau)^{\delta-1}\tau^{-\sigma_1}(1-\tau)^{-\sigma_2}d\tau + \frac{(|a|+|b|)N_0}{\Gamma(\delta)} \int_0^1 (1-\tau)^{\delta-1}\tau^{-\sigma_1}(1-\tau)^{-\sigma_2}d\tau$$
$$+ \frac{(|c|+|d|)N_0}{\Gamma(\delta+1)} \int_0^1 (1-\tau)^{\delta}\tau^{-\sigma_1}(1-\tau)^{-\sigma_2}d\tau$$
$$+ \frac{(|e|+|f|)N_0}{\Gamma(\delta+1)} \int_0^\eta (\eta-\tau)^{\delta}\tau^{-\sigma_1}(1-\tau)^{-\sigma_2}d\tau$$
$$+ \frac{(|a|+|b|)N_0}{\Gamma(\delta+1)} \int_0^\gamma (\gamma-\tau)^{\delta}\tau^{-\sigma_1}(1-\tau)^{-\sigma_2}d\tau$$
$$\le N_0\{\frac{1+|a|+|b|}{\Gamma(\delta)} \frac{1}{\sqrt[s]{1+s(\delta-1)}} \frac{1}{\sqrt[s_1]{1-s_1\sigma_1}} \frac{1}{\sqrt[s_2]{1-s_2\sigma_2}}$$
$$+ \frac{1}{\Gamma(\delta+1)} \frac{1}{\sqrt[s]{1+s\delta}} \frac{1}{\sqrt[s_1]{1-s_1\sigma_1}} \frac{1}{\sqrt[s_2]{1-s_2\sigma_2}}$$
$$\cdot [|c|+|d|+(|e|+|f|)\sqrt[s]{\eta^{1+s\delta}}\sqrt[s_1]{\eta^{1-s_1\sigma_1}}\sqrt[s_2]{1-(1-\eta)^{1-s_2\sigma_2}}$$
$$+ (|a|+|b|)\sqrt[s]{\gamma^{1+s\delta}}\sqrt[s_1]{\gamma^{1-s_1\sigma_1}}\sqrt[s_2]{1-(1-\gamma)^{1-s_2\sigma_2}}]\} \le L.$$

This means that $\phi_1 x + \phi_2 y \in B_L$.

The operator ϕ_2 is a contractive mapping deduced from (H1)–(H3) and (3.4) with a process similar to Theorem 3.1.

The operator ϕ_1 is continuous in B_L by (H1) and Lemma 3.2. ϕ_1 is uniformly bounded on B_L since the following inequality is true

$$\| \phi_1 x \| \leq \frac{N_0}{\Gamma(\delta)} K_{\delta-1}(t) \leq \frac{N_0}{\Gamma(\delta)} \frac{l}{\sqrt[s]{1+s(\delta-1)}} \frac{1}{\sqrt[s_1]{1-s_1\sigma_1}} \frac{l}{\sqrt[s_2]{1-s_2\sigma_2}},$$

where $x \in B_L$.

For every $t_1, t_2 \in [0,1]$ and $t_1 < t_2$, one can obtain

$$\begin{aligned} | (\phi_1 x)(t_2) - (\phi_2 x)(t_1) | &= \frac{1}{\Gamma(\delta)} | \int_0^{t_2} (t_2-\tau)^{\delta-1} f(\tau, x(\tau)) d\tau - \int_0^{t_1} (t_1-\tau)^{\delta-1} f(\tau, x(\tau)) d\tau | \\ &\leq \frac{N_0}{\Gamma(\delta)} \{ \int_0^{t_1} [(t_2-\tau)^{\delta-1} - (t_1-\tau)^{\delta-1}] \tau^{-\sigma_1} (1-\tau)^{-\sigma_2} d\tau \\ &+ \int_{t_1}^{t_2} (t_2-\tau)^{\delta-1} \tau^{-\sigma_1} (1-\tau)^{-\sigma_2} d\tau \} \\ &\leq \frac{N_0}{\Gamma(\delta)} [K_{\delta-1}(t_2) - K_{\delta-1}(t_1)]. \end{aligned}$$

By the third assertion in Lemma 3.1, we have

$$| (\phi_1 x)(t_2) - (\phi_2 x)(t_1) | \leq \frac{N_0}{\Gamma(\delta)} \frac{l}{\sqrt[s]{1+s(\delta-1)}} \frac{1}{\sqrt[s_1]{1-s_1\sigma_1}} \frac{l}{\sqrt[s_2]{1-s_2\sigma_2}} | t_2 - t_1 | .$$

Thus, both the equicontinuity and relative compactness of ϕ_1 on B_L are attained. The Arzelà-Ascoli theorem ensures that the operator ϕ_1 is compact in B_L. Therefore, the existence of a solution in $([0,1], R)$ is ensured by Lemma 2.3.

4. Numerical example

The current section introduces an example to verify the efficiency of the fundamental outcomes in the current article.

Example 4.1 Assume the upcoming fractional BVP:

$$\begin{cases} {}^cD_{0^+}^{2.5} x(t) = \frac{\sin^r x}{\sqrt[12]{t} \cdot \sqrt[20]{1-t}}, 0 < t < 1, \\ x(0) = \int_0^{\frac{1}{3}} x(\tau) d\tau, \\ x'(0) = 0, \\ x(1) = \int_{0.5}^1 x(\tau) d\tau. \end{cases} \tag{4.1}$$

In this BVP, $f(t,x) = \frac{\sin^r x}{\sqrt[12]{t} \cdot \sqrt[20]{1-t}}$, $r \geq 1$ is a real number, $\delta = 2.5$, $n = 3, \eta = \frac{1}{3}, \gamma = \frac{1}{2}$. Take $\sigma_1 = \sigma_2 = \frac{1}{10}$; we have

$$t^{\frac{1}{10}} (1-t)^{\frac{1}{10}} | f(t, x(t)) - f(t, y(t)) | = t^{\frac{1}{60}} (1-t)^{\frac{1}{20}} | \sin^r x - \sin^r y | \leq r t^{\frac{1}{60}} (1-t)^{\frac{1}{20}} | x - y | .$$

Thus, $m(t) = rt^{\frac{1}{60}}(1-t)^{\frac{1}{20}}, t \in [0,1]$ and $\| m \| = max_{0\leq t\leq 1} \mid m(t) \mid = r \sqrt[60]{\frac{1}{4}(\frac{3}{4})^3} \approx 0.9627r$.
Now, the following values can be obtained:

$$\Delta_1 = \frac{261}{108}, a = -\frac{4}{261}, b = -\frac{216}{261}, c = \frac{36}{261}, d = \frac{184}{261}, e = \frac{459}{522}, f = -\frac{54}{87},$$
$$\Gamma(2.5) \approx 1.3293, \Gamma(1+2.5) \approx 3.3233.$$

Take conjugate exponentials s, s_1 and s_2 when $s = \frac{4}{3}$ and $s_1 = s_2 = 8$. Then, we can calculate

$$\begin{aligned}\| m \| \{ &\frac{1+\mid a\mid+\mid b\mid}{\Gamma(\delta)} \frac{1}{\sqrt[s]{1+s(\delta-1)}} \frac{1}{\sqrt[s_1]{1-s_1\sigma_1}} \frac{1}{\sqrt[s_2]{1-s_2\sigma_2}} + \frac{1}{\Gamma(\delta+1)} \frac{1}{\sqrt[s]{1+s\delta}} \frac{1}{\sqrt[s_1]{1-s_1\sigma_1}} \frac{1}{\sqrt[s_2]{1-s_2\sigma_2}} \\ &\cdot[\mid c\mid+\mid d\mid+(\mid e\mid+\mid f\mid)\sqrt[s]{\eta^{1+s\delta}} \sqrt[s_1]{\eta^{1-s_1\sigma_1}} \sqrt[s_2]{1-(1-\eta)^{1-s_2\sigma_2}} \\ &+(\mid a\mid+\mid b\mid)\sqrt[s]{\gamma^{1+s\delta}} \sqrt[s_1]{\gamma^{1-s_1\sigma_1}} \sqrt[s_2]{1-(1-\gamma)^{1-s_2\sigma_2}}]\} \\ &\approx 0.9627r \times 0.4793 + 0.0636 \times [0.8589 + 2.0455] \approx 0.6393r.\end{aligned}$$

And

$$\begin{aligned}\| m \| \{ &\frac{\mid a\mid+\mid b\mid}{\Gamma(\delta)} \frac{1}{\sqrt[s]{1+s(\delta-1)}} \frac{1}{\sqrt[s_1]{1-s_1\sigma_1}} \frac{1}{\sqrt[s_2]{1-s_2\sigma_2}} + \frac{1}{\Gamma(\delta+1)} \frac{1}{\sqrt[s]{1+s\delta}} \frac{1}{\sqrt[s_1]{1-s_1\sigma_1}} \frac{1}{\sqrt[s_2]{1-s_2\sigma_2}} \\ &\cdot[\mid c\mid+\mid d\mid+(\mid e\mid+\mid f\mid)\sqrt[s]{\eta^{1+s\delta}} \sqrt[s_1]{\eta^{1-s_1\sigma_1}} \sqrt[s_2]{1-(1-\eta)^{1-s_2\sigma_2}} \\ &+(\mid a\mid+\mid b\mid)\sqrt[s]{\gamma^{1+s\delta}} \sqrt[s_1]{\gamma^{1-s_1\sigma_1}} \sqrt[s_2]{1-(1-\gamma)^{1-s_2\sigma_2}}]\} \\ &\approx 0.9627r \times 0.2102 + 0.0636 \times [0.8589 + 2.0455] \approx 0.3802r.\end{aligned}$$

Thus, we can deduce the following:

(a1) When $0.6393r < 1$, or $1 \leq r < 1.5642$, a unique solution in $[0,1]$ for the problem (4.1) is possessed by Theorem 3.1.

(a2) When $0.3802r < 1$, or $1 \leq r < 2.63019$, Theorem 3.2 guarantees that the problem (4.1) attains solutions on the interval $[0,1]$.

5. Conclusions

This work dealt with the existence results for a category of singular nonlinear FDEs with nonlocal double integral BCs. The results we obtained depended on the parameters that appeared in the integral BCs, this is due to the use of the generalized Hölder's inequality. So the type of problems, conclusions and methods discussed in this paper complemented the existing literature.

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Conflict of interest

The author declares that there is no conflict of interest.

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