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***Research article***

# A periodic boundary value problem of fractional differential equation involving $p(t)$-Laplacian operator

**Tingting Xue**[1,*], **Xiaolin Fan**[1], **Hong Cao**[2] **and Lina Fu**[1]

[1] School of Mathematics and Physics, Xinjiang Institute of Engineering, Urumqi, Xinjiang, China
[2] College of Mathematics and System Science, Xinjiang University, Urumqi, Xinjiang, China

* **Correspondence:** Email:xuett@cumt.edu.cn.

**Abstract:** The purpose of this article is to research the existence of solutions for fractional periodic boundary value problems with $p(t)$-Laplacian operator. In this regard, the article needs to establish a continuation theorem corresponding to the above problem. By applying the continuation theorem, a new existence result for the problem is obtained, which enriches existing literature. In addition, we provide an example to verify the main result.

**Keywords:** fractional differential equation; boundary value problem; $p(t)$-Laplacian operator; continuation theorem

## 1. Introduction

Fractional differential equations have attracted the attention of scholars in many fields at home and abroad because of their wide application background (see [1–8]). For example, in [9], in order to study the decrease in the height of granular material in the silo over time, the authors proposed the following fractional mathematical model:

$$ {}^{C}D_{T^-}^{\alpha} D_{a^+}^{\alpha} h^*(t) + \beta h^*(t) = 0,\ \alpha \in (0,1),\ t \in [0,T], $$

where ${}^{C}D_{T^-}^{\alpha}$ and $D_{a^+}^{\alpha}$ are right Caputo type and left Riemann-Liouville fractional derivatives of order $\alpha$, respectively. In recent years, fractional periodic boundary value problems (FPBVP for short) have attracted many people's attention (see [10–17]). There are also some researches on FPBVP with $p$-Laplacian operators. For more details, see [18-21]. For example, Hu et al. [22] discussed the FPBVP as follows:

$$ \begin{cases} {}^{C}D_{0+}^{\alpha} x(t) = f(t, x(t), x'(t)),\ 1 < \alpha \le 2,\ t \in [0,1], \\ x(0) = x(1), x'(0) = x'(1). \end{cases} $$

The authors mainly used the degree theory to consider the above FPBVP.

The differential equation of variable index $p(t)$-Laplacian has become a new research direction in recent years, which is generated in elasticity [23], image restoration [24] and electrorheological fluids [25], and has important application background. The so-called $p(t)$-Laplacian operator is written as

$$\phi_{p(t)}(x) = |x|^{p(t)-2}x, x \neq 0, t \in [0,T], p(t) \in C([0,T],\mathbb{R}), p(t) > 1, \phi_{p(t)}(0) = 0.$$

It is a generalization of the $p$-Laplacian operator. Many scholars have studied this kind of problem and obtained some valuable results, see [26–31]. For example, Shen and Liu [32] considered the solvability of fractional $p(t)$-Laplacian problems:

$$\begin{cases} D_{0+}^{\beta}\phi_{p(t)}\left(D_{0+}^{\alpha}x(t)\right) + f(t,x(t)) = 0, \ t \in (0,1), \\ x(0) = 0, \ D_{0+}^{\alpha-1}x(1) = \gamma I_{0+}^{\alpha-1}x(\eta), \ D_{0+}^{\alpha}x(0) = 0, \end{cases}$$

where $1 < \alpha \leq 2, \ 0 < \beta \leq 1, \ \gamma > 1, \ 0 < \eta < 1$. Through reading the literature [31,32], we find that the condition $D_{0+}^{\alpha}x(0) = 0$ is critical for studying the solvability of the fractional $p(t)$-Laplacian problem. Using this condition, one can transform the fractional $p(t)$-Laplacian operator equation into the linear differential operator equation with appropriate transformations, and then use the continuation theorem of Mawhin to deal with this type of problem. However, the periodic boundary conditions lack this key condition, $D_{0+}^{\alpha}x(0) = 0$, so the continuation theorem cannot be directly applied. Therefore, there is not much research in this area. Based on this, this paper studies the following FPBVP with $p(t)$-Laplacian operator:

$$\begin{cases} {}^{C}D_{0+}^{\beta}\phi_{p(t)}\left(D_{0+}^{\alpha}x(t)\right) = f\left(t,x(t),D_{0+}^{\alpha}x(t)\right), \quad t \in (0,T], \\ \lim\limits_{t\to 0^+} t^{1-\alpha}x(t) = \lim\limits_{t\to T} t^{1-\alpha}x(t), \ \lim\limits_{t\to 0^+}\phi_{p(t)}\left(D_{0+}^{\alpha}x(t)\right) = \lim\limits_{t\to T}\phi_{p(t)}\left(D_{0+}^{\alpha}x(t)\right), \end{cases} \tag{1.1}$$

where $0 < \alpha,\beta \leq 1$, $1 < \alpha+\beta \leq 2$, $D_{0+}^{\alpha}$ is a Riemann-Liouville fractional derivative, ${}^{C}D_{0+}^{\beta}$ is a Caputo fractional derivative, and $f \in C([0,T]\times\mathbb{R}^2,\mathbb{R})$, $p(t) \in C^1([0,T],\mathbb{R})$, $p(t) > 1$, $\min_{t\in[0,T]}p(t) = P_m$, $\max_{t\in[0,T]}p(t) = P_M$. Note that, the nonlinear $p(t)$-Laplacian operator will be reduced to a famous $p$-Laplacian operator when $p(t) = p$. Therefore, this paper will further enrich and extend the existing results. In addition, because the periodic boundary conditions lack this key condition, $D_{0+}^{\alpha}x(0) = 0$, the above-mentioned nonlinear operator equation can not be transformed into a linear operator equation in this paper, so the continuation theorem can not be applied directly. In the following sections, we first establish a new continuation theorem corresponding to the above FPBVP Eq (1.1). By applying the continuation theorem, a new existence result for the problem is obtained. Obviously, this method is different from the method used in [31,32]. As far as we know, the fractional $p(t)$-Laplacian differential equations with periodic boundary conditions have not been considered so far.

## 2. Preliminaries

For basic concepts and lemmas of fractional derivatives and integrals, please see [33,34]. Here, we give some important lemmas and definitions.

**Definition 1.** *([33]). The Riemann-Liouville fractional derivative of order $\alpha > 0$ for a function $x$ : $(0,+\infty) \to \mathbb{R}$: is given by*

$$D_{0+}^{\alpha}x(t) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_0^t (t-s)^{n-\alpha-1}x(s)ds,$$

*where* $n = [\alpha] + 1$, *provided that the right-hand side integral is defined on* $(0, +\infty)$.

**Definition 2.** *([33]). The Caputo fractional derivative of order* $\beta > 0$ *for the function* $x : (0, +\infty) \to \mathbb{R}$*: is defined as*

$$ {}^{C}D_{0+}^{\beta} x(t) = \frac{1}{\Gamma(n-\beta)} \int_0^t (t-s)^{n-\beta-1} x^{(n)}(s) ds, $$

*where* $n = [\beta] + 1$, *provided that the right-hand side integral is defined on* $(0, +\infty)$.

**Lemma 1.** *([30]). The function* $\phi_{p(t)}(x)$ *is an homeomorphism from* $\mathbb{R}$ *to* $\mathbb{R}$ *and strictly monotone increasing with respect to x for any fixed t. Its inverse operator* $\phi_{p(t)}^{-1}(\cdot)$ *is defined by*

$$ \begin{cases} \phi_{p(t)}^{-1}(x) = |x|^{\frac{2-p(t)}{p(t)-1}} x, \ x \in \mathbb{R} \backslash \{0\}, \ t \in [0, T], \\ \phi_{p(t)}^{-1}(0) = 0, \ x = 0, \end{cases} $$

*which is continuous and sends bounded sets into bounded sets.*

**Lemma 2.** *([30]). Function* $\phi_{p(t)}(x)$ *has the following properties:*

(i) *For any* $x_1, x_2 \in \mathbb{R}$, $x_1 \neq x_2$, *for all* $t \in [0, T]$, *one has*

$$ \left\langle \phi_{p(t)}(x_1) - \phi_{p(t)}(x_2), x_1 - x_2 \right\rangle > 0; $$

(ii) *There exists a function* $\varphi : [0, +\infty) \to [0, +\infty)$, $\varphi(s) \to +\infty$ *as* $s \to +\infty$, *such that*

$$ \left\langle \phi_{p(t)}(x), x \right\rangle \geq \varphi(|x|)|x|, \quad \text{for all } x \in \mathbb{R}. $$

Let $X$ and $Y$ be two real Banach spaces, and let $L : \mathrm{dom}L \subset X \to Y$ and $P : X \to X$, $Q : Y \to Y$ be projectors such that $\mathrm{Im}P = \mathrm{Ker}L$, $\mathrm{Ker}Q = \mathrm{Im}L$, $X = \mathrm{Ker}L \oplus \mathrm{Ker}P$, $Y = \mathrm{Im}L \oplus \mathrm{Im}Q$. Then $L|_{\mathrm{dom}L \cap \mathrm{Ker}P}$: $\mathrm{dom}L \cap \mathrm{Ker}P \to \mathrm{Im}L$ is invertible. In this paper, we set $Y = C([0, T], \mathbb{R})$ endowed with the norm $\|y\|_\infty = \max_{t\in[0,T]} |y(t)|$, and set

$$ X = \left\{ x \left| t^{1-\alpha} x, D_{0+}^{\alpha} x \in Y, \lim_{t\to 0^+} \phi_{p(t)}(D_{0+}^{\alpha} x(t)) \text{ and } \lim_{t\to T} \phi_{p(t)}(D_{0+}^{\alpha} x(t)) \text{ exist} \right. \right\}, $$

$$ X_T = \left\{ x \in X \left| \lim_{t\to 0^+} t^{1-\alpha} x(t) = \lim_{t\to T} t^{1-\alpha} x(t), \ \lim_{t\to 0^+} \phi_{p(t)}(D_{0+}^{\alpha} x(t)) = \lim_{t\to T} \phi_{p(t)}(D_{0+}^{\alpha} x(t)) \right. \right\} $$

endowed with the norm $\|x\|_X = \max\left\{ \left\| t^{1-\alpha} x \right\|_\infty, \left\| D_{0+}^{\alpha} x \right\|_\infty \right\}$. Obviously, $X$ and $X_T$ are two Banach spaces. The operator $L : \mathrm{dom}L \subset X \to Y$ is defined as follows:

$$ Lx = {}^{C}D_{0+}^{\beta} \phi_{p(t)}(D_{0+}^{\alpha} x), \tag{2.1} $$

where $\mathrm{dom}L = \left\{ x \in X_T \left| {}^{C}D_{0+}^{\beta} \phi_{p(t)}\left(D_{0+}^{\alpha} x\right) \in Y \right. \right\}$. Let $N_f : X \to Y$ be the Nemytskii operator defined by

$$ N_f x(t) = f(t, x(t), D_{0+}^{\alpha} x(t)). \tag{2.2} $$

It is clear that FPBVP Eq (1.1) can be converted to the following operator equation

$$ Lx = N_f x, \ x \in \mathrm{dom}L. $$

## 3. Forced fractional $p(t)$-Laplacian equation with periodic boundary conditions

In order to establish the continuation theorem for FPBVP Eq (1.1), this section begins with FPBVP as below:

$$\begin{cases} {}^{C}D_{0+}^{\beta}\phi_{p(t)}\left(D_{0+}^{\alpha}x\right) = h(t), \quad t \in (0, T], \\ \lim\limits_{t\to 0^+} t^{1-\alpha}x(t) = \lim\limits_{t\to T} t^{1-\alpha}x(t), \ \lim\limits_{t\to 0^+} \phi_{p(t)}\left(D_{0+}^{\alpha}x\right) = \lim\limits_{t\to T} \phi_{p(t)}\left(D_{0+}^{\alpha}x\right), \end{cases} \tag{3.1}$$

where $h \in Y$ satisfies

$$\bar{h} := \frac{\beta}{T^{\beta}} \int_0^T (T-s)^{\beta-1} h(s)\, ds = 0,$$

and let $x$ be a solution of FPBVP Eq (3.1). Based on the definition of Caputo fractional integral, we get

$$\phi_{p(t)}\left(D_{0+}^{\alpha}x\right) = a + I_{0+}^{\beta}h(t) = a + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} h(s)\, ds, \ \forall a \in \mathbb{R}. \tag{3.2}$$

Furthermore, by $\lim\limits_{t\to 0^+} t^{1-\alpha}x(t) = \lim\limits_{t\to T} t^{1-\alpha}x(t)$, we can obtain

$$\int_0^T (T-s)^{\alpha-1} \phi_{p(s)}^{-1}\left(a + I_{0+}^{\beta}h(s)\right) ds = 0.$$

For any fixed $l \in Y$, the function is defined here

$$G_l(a) = \frac{\alpha}{T^{\alpha}} \int_0^T (T-s)^{\alpha-1} \phi_{p(s)}^{-1}(a + l(s))\, ds. \tag{3.3}$$

**Lemma 3.** *The function $G_l$ has two characteristics, as follows:*

*(1) For $\forall l \in Y$, the equation*

$$G_l(a) = 0 \tag{3.4}$$

*has one unique solution $\tilde{a}(l)$.*

*(2) The function $\tilde{a} : Y \to \mathbb{R}$ is continuous and maps a bounded set to a bounded set.*

**Proof.** (1) It follows from Lemma 2 that

$$\langle G_l(a_1) - G_l(a_2), a_1 - a_2 \rangle > 0, \quad \text{for } a_1 \neq a_2.$$

It is clear that if Eq (3.4) has one solution, then it is unique. Next, it will be proved that when $|a|$ is large enough, $\langle G_l(a), a \rangle > 0$. Then, one has

$$\begin{aligned} \langle G_l(a), a \rangle &= \frac{\alpha}{T^{\alpha}} \int_0^T (T-s)^{\alpha-1} \left\langle \phi_{p(s)}^{-1}(a + l(s)), a \right\rangle ds \\ &= \frac{\alpha}{T^{\alpha}} \int_0^T (T-s)^{\alpha-1} \left\langle \phi_{p(s)}^{-1}(a + l(s)), a + l(s) \right\rangle ds \\ &\quad - \frac{\alpha}{T^{\alpha}} \int_0^T (T-s)^{\alpha-1} \left\langle \phi_{p(s)}^{-1}(a + l(s)), l(s) \right\rangle ds, \end{aligned}$$

thus

$$\begin{aligned}\left\langle G_l(a), a\right\rangle \geq & \frac{\alpha}{T^\alpha} \int_0^T (T-s)^{\alpha-1}\left\langle \phi_{p(s)}^{-1}(a+l(s)), a+l(s)\right\rangle ds \\ & -\frac{\alpha}{T^\alpha}\|l\|_\infty \int_0^T (T-s)^{\alpha-1}\left|\phi_{p(s)}^{-1}(a+l(s))\right| ds.\end{aligned} \tag{3.5}$$

It follows from Lemma 2 for any $y \in \mathbb{R}$ that

$$\left\langle \phi_{p(t)}^{-1}(y), y\right\rangle \geq \varphi\left(\left|\phi_{p(t)}^{-1}(y)\right|\right)\left|\phi_{p(t)}^{-1}(y)\right|. \tag{3.6}$$

By Eqs (3.5) and (3.6), one has

$$\left\langle G_l(a), a\right\rangle \geq \frac{\alpha}{T^\alpha} \int_0^T (T-s)^{\alpha-1}\left[\varphi\left(\left|\phi_{p(s)}^{-1}(a+l(s))\right|\right)-\|l\|_\infty\right]\left|\phi_{p(s)}^{-1}(a+l(s))\right| ds. \tag{3.7}$$

Because $|a| \to \infty$ means that $\phi_{p(s)}^{-1}(a+l(s)) \to \infty$, uniformly for $t \in [0, T]$, so by Eq (3.7), we get that there is $r > 0$ such that

$$\left\langle G_l(a), a\right\rangle > 0 \quad \text{for all } a \in \mathbb{R} \text{ with } |a| = r.$$

Next, according to an elementary topological degree argument, for each $l \in Y$, there is a solution to the equation $Gl(a) = 0$ and the preceding analysis implies this solution is unique. Therefore, the following defines a function $\tilde{a} : Y \to \mathbb{R}$, which satisfies

$$\frac{\alpha}{T^\alpha} \int_0^T (T-s)^{\alpha-1} \phi_{p(s)}^{-1}(\tilde{a}(l)+l(s)) ds = 0, \quad \text{for any } l \in Y. \tag{3.8}$$

(2) Here, let $\Lambda$ be a bounded subset of $Y$ and $l \in \Lambda$. By Eq (3.8), one has

$$\frac{\alpha}{T^\alpha} \int_0^T (T-s)^{\alpha-1}\left\langle \phi_{p(s)}^{-1}(\tilde{a}(l)+l(s)), \tilde{a}(l)\right\rangle ds = 0,$$

thus

$$\begin{aligned}& \frac{\alpha}{T^\alpha} \int_0^T (T-s)^{\alpha-1}\left\langle \phi_{p(s)}^{-1}(\tilde{a}(l)+l(s)), \tilde{a}(l)+l(s)\right\rangle ds \\ = & \frac{\alpha}{T^\alpha} \int_0^T (T-s)^{\alpha-1}\left\langle \phi_{p(s)}^{-1}(\tilde{a}(l)+l(s)), l(s)\right\rangle ds.\end{aligned} \tag{3.9}$$

Let's say that $\{\tilde{a}(l), l \in \Lambda\}$ is not bounded. For any constant $A > 0$, there exists $l \in \Lambda$ such that $\|l\|_\infty$ is large enough, so that the following inequality relationship holds

$$A \leq \varphi\left(\left|\phi_{p(t)}^{-1}(\tilde{a}(l)+l(t))\right|\right), \ t \in [0, T].$$

By Eqs (3.6) and (3.9), we obtain

$$A \int_0^T (T-s)^{\alpha-1}\left|\phi_{p(s)}^{-1}(\tilde{a}(l)+l(s))\right| ds \leq \|l\|_\infty \int_0^T (T-s)^{\alpha-1}\left|\phi_{p(s)}^{-1}(\tilde{a}(l)+l(s))\right| ds.$$

Therefore, $A \leq \|l\|_\infty$, which is a contradiction. Hence $\tilde{a}$ sends bounded sets in $Y$ into bounded sets in $\mathbb{R}$.

Last, it will prove that $\tilde{a}$ is continuous. Let $\{l_n\}$ be a convergent sequence in $Y$, then $l_n \to l\,(n \to \infty)$. Because $\{\tilde{a}\,(l_n)\}$ is a bounded sequence, any of its subsequences contain a convergent subsequence, which may be expressed as $\left\{\tilde{a}\left(l_{n_j}\right)\right\}$. Make $\tilde{a}\left(l_{n_j}\right) \to \hat{a}\ (j \to \infty)$ is established. By making $j \to \infty$ in

$$\frac{\alpha}{T^{\alpha}} \int_0^T (T-s)^{\alpha-1} \phi_{p(s)}^{-1} \left(\tilde{a}\left(l_{n_j}\right) + l_{n_j}(s)\right) ds = 0,$$

then

$$\frac{\alpha}{T^{\alpha}} \int_0^T (T-s)^{\alpha-1} \phi_{p(s)}^{-1} (\hat{a} + l(s))\, ds = 0.$$

So $\tilde{a}\,(l) = \hat{a}$, which indicates the continuity of $\tilde{a}$.

Define $a : Y \to \mathbb{R}$ by

$$a\,(h) = \tilde{a}\left(I_{0+}^{\beta} h\right).$$

Then, by applying Lemma 3, you can conclude that $a$ is a completely continuous mapping. By Eq (3.2), one has

$$x\,(t) = \left[\lim_{t \to 0^+} t^{1-\alpha} x\,(t)\right] t^{\alpha-1} + I_{0+}^{\alpha} \phi_{p(t)}^{-1} \left(a\,(h) + I_{0+}^{\beta} h\right)(t)\,. \tag{3.10}$$

**Lemma 4.** *If the definition of L is shown in Eq (2.1), then the following conclusions hold*

$$\mathrm{Ker}L = \left\{x \in X \,\middle|\, x\,(t) = ct^{\alpha-1},\ c \in \mathbb{R}\right\}, \tag{3.11}$$

$$\mathrm{Im}L = \left\{y \in Y \,\middle|\, \int_0^T (T-s)^{\beta-1} y\,(s)\, ds = 0\right\}. \tag{3.12}$$

*The following projection operators $P : X \to X$ and $Q : Y \to Y$:*

$$Px\,(t) = \left[\lim_{t \to 0^+} t^{1-\alpha} x\,(t)\right] t^{\alpha-1}, \quad Qy\,(t) = \frac{\beta}{T^{\beta}} \int_0^T (T-s)^{\beta-1} y\,(s)\, ds. \tag{3.13}$$

*It's easy to show that ImP = KerL, KerQ = ImL, X =* $\mathrm{Ker}L \oplus \mathrm{Ker}P$*, Y =* $\mathrm{Im}L \oplus \mathrm{Im}Q$*, then $L\,|_{domL \cap KerP}$: domL ∩ KerP → ImL is invertible. So, if $x \in X_T$ is one solution of Eq (3.1), then x satisfies the abstract equation*

$$x = Px + Qh + \mathcal{K}h, \tag{3.14}$$

*where the operator $\mathcal{K} : Y \to X_T$ is expressed as follows*

$$\mathcal{K}h\,(t) = I_{0+}^{\alpha} \phi_{p(t)}^{-1} \left[a\,((I-Q)\,h) + I_{0+}^{\beta}\,(I-Q)\,h\right](t)\,. \tag{3.15}$$

*Then again, by definition of the mapping a, we have*

$$I_{0+}^{\alpha} \phi_{p(t)}^{-1} \left[a\,((I-Q)\,h) + I_{0+}^{\beta}\,(I-Q)\,h\right](T) = 0,$$

*it is clear that if x satisfies Eq (3.14), then x is one solution of FPBVP Eq (3.1). Notice that $a(0) = \tilde{a}(0) = 0$, by Eqs (3.15) and (3.8), we have $\mathcal{K}(0) = 0$.*

**Lemma 5.** *Operator $\mathcal{K}$ is completely continuous.*

**Proof.** As defined by $\mathcal{K}$, one has

$$D_{0+}^{\alpha}\mathcal{K}h(t) = \phi_{p(t)}^{-1}\left[a\left((I-Q)h\right) + I_{0+}^{\beta}(I-Q)h\right](t).$$

Obviously, we find that operators $t^{1-\alpha}\mathcal{K}$ and $D_{0+}^{\alpha}\mathcal{K}$ are also a combination of continuous operators. Hence, $t^{1-\alpha}\mathcal{K},\ D_{0+}^{\alpha}\mathcal{K}$ are continuous in $Y$. In other words, the operator $\mathcal{K}$ is continuous. According to Lemma 3, there is $M > 0$ such that

$$\left|\left[a\left((I-Q)h\right) + I_{0+}^{\beta}(I-Q)h\right](t)\right| \le M,\ \forall h \in \bar{\Omega},\ t \in (0,T].$$

Thus, we have $\left\|D_{0+}^{\alpha}\mathcal{K}h\right\|_{\infty} \le M^{\frac{1}{p_m-1}}$ and

$$\begin{aligned}
\left\|t^{1-\alpha}\mathcal{K}h\right\|_{\infty} &= \left|\frac{1}{\Gamma(\alpha)}\int_0^t t^{1-\alpha}(t-s)^{\alpha-1}\phi_{p(s)}^{-1}\left[a\left((I-Q)h\right) + I_{0+}^{\beta}(I-Q)h\right]ds\right| \\
&\le \left|\frac{t^{1-\alpha}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\phi_{p(s)}^{-1}(M)\,ds\right| \\
&\le \frac{TM^{\frac{1}{p_m-1}}}{\Gamma(\alpha+1)}.
\end{aligned}$$

Suppose the set $\Omega \subset Y$ is open and bounded, then $t^{1-\alpha}\mathcal{K}(\bar{\Omega})$ and $D_{0+}^{\alpha}\mathcal{K}(\bar{\Omega})$ are bounded. Combined with Arzelà-Ascoli theorem, let's just verify that $\mathcal{K}(\bar{\Omega}) \subset X_T$ is equicontinuous. For $0 < t_1 < t_2 < T,\ h \in \bar{\Omega}$, one has

$$\begin{aligned}
&\left|t_2^{1-\alpha}\mathcal{K}h(t_2) - t_1^{1-\alpha}\mathcal{K}h(t_1)\right| \\
&= \frac{1}{\Gamma(\alpha)}\left|\int_0^{t_2} t_2^{1-\alpha}(t_2-s)^{\alpha-1}\phi_{p(s)}^{-1}\left[a\left((I-Q)h\right) + I_{0+}^{\beta}(I-Q)h\right]ds\right. \\
&\left. - \int_0^{t_1} t_1^{1-\alpha}(t_1-s)^{\alpha-1}\phi_{p(s)}^{-1}\left[a\left((I-Q)h\right) + I_{0+}^{\beta}(I-Q)h\right]ds\right| \\
&\le \frac{M^{\frac{1}{p_m-1}}}{\Gamma(\alpha)}\left|\int_0^{t_2} t_2^{1-\alpha}(t_2-s)^{\alpha-1}ds - \int_0^{t_1} t_1^{1-\alpha}(t_1-s)^{\alpha-1}ds\right| \\
&= \frac{M^{\frac{1}{p_m-1}}}{\Gamma(\alpha+1)}(t_2-t_1).
\end{aligned}$$

Because on $[0,T]$, $t$ is uniformly continuous, so $t^{1-\alpha}\mathcal{K}(\bar{\Omega}) \subset Y$ is equicontinuous. And by the same token, $\left[a(I-Q) + I_{0+}^{\beta}(I-Q)\right](\bar{\Omega}) \subset Y$ is equicontinuous. This, combined on $[-M,M]$, $\phi_{p(s)}^{-1}(\cdot)$ is continuous, you get $D_{0+}^{\alpha}\mathcal{K}(\bar{\Omega}) \subset Y$ is also equicontinuous. Hence, $\mathcal{K} : Y \to X_T$ is compact.

## 4. A new extension theorem for periodic boundary value problems of fractional $p(t)$-Laplacian equation

On the basis of the results in Section 3, this section establishes a new theorem for fractional $p(t)$-Laplacian equation, which is a generalization of the problems related to linear differential operators.

**Theorem 1.** *If $f \in C([0,T] \times \mathbb{R}^2, \mathbb{R})$. Define $L$, $N_f$, $Q$ respectively by Eqs (2.1), (2.2) and (3.13), and assume that the set $\Omega$ is an open bounded subset of $X_T$ that satisfies* $\mathrm{dom}L \cap \bar{\Omega} \neq \emptyset$ *and also satisfies the following three conditions.*

$(C_1)$ *For each $\lambda \in (0,1)$, the equation*

$$Lx = \lambda N_f x \tag{4.1}$$

*has no solution on $(domL \setminus KerL) \cap \partial\Omega$.*

$(C_2)$ *The equation*

$$QN_f x = 0$$

*has no solution on $KerL \cap \partial\Omega$.*

$(C_3)$ *The Brouwer degree*

$$deg(QN_f \mid_{KerL}, \Omega \cap KerL, 0) \neq 0.$$

*Then the abstract equation $Lx = N_f x$ has at least a solution in* $\mathrm{dom}L \cap \bar{\Omega}$.

**Proof.** First, we give the following homotopic equation of $Lx = N_f x$

$$Lx = \lambda N_f x + (1-\lambda) QN_f x, \quad x \in \mathrm{dom}L, \tag{4.2}$$

i.e.,

$$\begin{cases} {}^C D_{0+}^{\beta} \phi_{p(t)} \left( D_{0+}^{\alpha} x(t) \right) = \lambda f\left(t, x(t), D_{0+}^{\alpha} x(t)\right) + (1-\lambda) \dfrac{\beta}{T^{\beta}} \displaystyle\int_0^T (T-s)^{\beta-1} f\left(s, x(s), D_{0+}^{\alpha} x(s)\right) ds, \\ \lim\limits_{t \to 0^+} t^{1-\alpha} x(t) = \lim\limits_{t \to T} t^{1-\alpha} x(t), \ \lim\limits_{t \to 0^+} \phi_{p(t)} \left( D_{0+}^{\alpha} x(t) \right) = \lim\limits_{t \to T} \phi_{p(t)} \left( D_{0+}^{\alpha} x(t) \right). \end{cases}$$

Obviously, for $\lambda \in (0,1]$, if $x$ is one solution of Eq (4.1) or (4.2), the following necessary condition can be obtained

$$QN_f x(t) = \frac{\beta}{T^{\beta}} \int_0^T (T-s)^{\beta-1} f\left(s, x(s), D_{0+}^{\alpha} x(s)\right) ds = 0.$$

Therefore, Eqs (4.1) and (4.2) have the same solution. In addition, Eq (4.2) is equivalent to the following form:

$$x = \mathcal{G}_f(x, \lambda), \tag{4.3}$$

where $\mathcal{G}_f : X_T \times [0,1] \to X_T$ is denoted by

$$\mathcal{G}_f(x,\lambda) = Px + QN_f x + [\mathcal{K} \circ (\lambda N_f + (1-\lambda) QN_f)]x.$$

It follows from Lemma 5 and the continuity of $f$ that the operator $\mathcal{G}_f$ is completely continuous.

If $\lambda = 1$, then Eq (4.3) has no solution on $\partial\Omega$, if not, Theorem 1 is verified. For $(x, \lambda) \in \partial\Omega \times (0,1]$, the condition $(C_1)$ implies Eq (4.3) has no solution. For $\lambda = 0$, Eq (4.2) is equivalent to the problem as below

$$\begin{cases} {}^C D_{0+}^{\beta} \phi_{p(t)} \left( D_{0+}^{\alpha} x(t) \right) = \dfrac{\beta}{T^{\beta}} \displaystyle\int_0^T (T-s)^{\beta-1} f\left(s, x(s), D_{0+}^{\alpha} x(s)\right) ds, \\ \lim\limits_{t \to 0^+} t^{1-\alpha} x(t) = \lim\limits_{t \to T} t^{1-\alpha} x(t), \ \lim\limits_{t \to 0^+} \phi_{p(t)} \left( D_{0+}^{\alpha} x(t) \right) = \lim\limits_{t \to T} \phi_{p(t)} \left( D_{0+}^{\alpha} x(t) \right). \end{cases}$$

So, if $x$ is one solution to this problem, one has

$$\frac{\beta}{T^{\beta}}\int_0^T (T-s)^{\beta-1} f\left(s, x(s), D_{0+}^{\alpha} x(s)\right) ds = 0.$$

From Eq (3.11), we have $x(t) = ct^{\alpha-1} \in \mathrm{Ker}L,\ \forall c \in \mathbb{R}$. Thus,

$$\left(QN_f|_{\mathrm{Ker}L}\right) x(t) = \frac{\beta}{T^{\beta}}\int_0^T (T-s)^{\beta-1} f\left(s, cs^{\alpha-1}, 0\right) ds = 0.$$

This, combined with $(C_2)$ implies that $x = ct^{\alpha-1} \notin \partial\Omega$. Hence, Eq (4.3) has no solution, for $(x,\lambda) \in \partial\Omega \times [0,1]$. According to the homotopy property of degree, one has

$$\deg\left(I - G_f(\cdot,1), \Omega, 0\right) = \deg\left(I - G_f(\cdot,0), \Omega, 0\right). \tag{4.4}$$

Clearly, equation $x = \mathcal{G}_f(x,1)$ is equivalent to $Lx = N_f x$. From Eq (4.4) we can obtain that it will have a solution if $\deg\left(I - G_f(\cdot,0), \Omega, 0\right) \neq 0$. As defined by $\mathcal{G}_f$,

$$\mathcal{G}_f(x,0) = Px + QN_f x + \mathcal{K}(0) = Px + QN_f x,$$

then

$$x - \mathcal{G}_f(x,0) = x - Px - QN_f x.$$

Thus,

$$\deg\left(I - G_f(\cdot,0), \Omega, 0\right) = -\deg(QN_f\ |_{\mathrm{Ker}L}, \Omega \cap \mathrm{Ker}L, 0).$$

From $(C_3)$, we have the last degree is not zero. Hence, $Lx = N_f x$ has at least a solution.

## 5. Application of Theorem 1

This section studies the solvability of FPBVP Eq (1.1) and a new existing result is given and proved.

**Theorem 2.** *Let the function* $f \in C([0,T] \times \mathbb{R}^2, \mathbb{R})$. *Suppose that the conditions* $(\mathrm{H}_1)$ *and* $(\mathrm{H}_2)$ *hold.*

$(\mathrm{H}_1)$ *There are three non-negative functions* $a, b, c \in Y$, *which satisfy the following relationship*

$$|f(t,u,v)| \leq a(t) + b(t)\left|t^{1-\alpha}u\right|^{\theta-1} + c(t)|v|^{\theta-1},\ \forall t \in [0,T],\ (u,v) \in \mathbb{R}^2,\ 1 < \theta \leq P_m.$$

$(\mathrm{H}_2)$ *There is* $\mathrm{A} > 0$, *so one of the following is true*

$$uf(t,u,v) > 0,\ \forall t \in [0,T],\ v \in \mathbb{R},\ |u| > \mathrm{A}; \tag{5.1}$$

$$uf(t,u,v) < 0,\ \forall t \in [0,T],\ v \in \mathbb{R},\ |u| > \mathrm{A}. \tag{5.2}$$

*Then FPBVP Eq (1.1) has at least a solution, provided with*

$$\frac{4T^{\beta}}{\Gamma(\beta+1)}\left[\|c\|_{\infty} + \|b\|_{\infty}\left(\frac{4T}{\Gamma(\alpha+1)}\right)^{\theta-1}\right] < 1. \tag{5.3}$$

**Proof.** The certification process is mainly divided into three steps.

**Step 1.** Make $\Omega_1 = \left\{x \in \text{dom}L \setminus \text{Ker}L \,\middle|\, Lx = \lambda N_f x\,,\ \lambda \in (0,1)\right\}$. For $x \in \Omega_1$, you get $N_f x \in \text{Im}L$. Using Eq (3.12) we see that

$$\int_0^T (T-s)^{\beta-1} f\left(s, x(s), D_{0+}^{\alpha} x(s)\right) ds = 0.$$

Moreover, according to the integral mean value theorem, we can get that there is $\xi \in (0,T)$, which satisfies $f\left(\xi, x(\xi), D_{0+}^{\alpha} x(\xi)\right) = 0$. Thus, by (H$_2$), we have $|x(\xi)| \le A$. Furthermore, it follows from $x(t) = I_{0+}^{\alpha} D_{0+}^{\alpha} x(t) + c_1 t^{\alpha-1}$ that

$$\left|c_1 t^{\alpha-1}\right| \le |x(t)| + \left|I_{0+}^{\alpha} D_{0+}^{\alpha} x(t)\right|.$$

Thus,

$$\begin{aligned} |c_1| &\le \frac{1}{\xi^{\alpha-1}}\left[|x(\xi)| + \frac{1}{\Gamma(\alpha)}\int_0^{\xi} (\xi-s)^{\alpha-1}\left|D_{0+}^{\alpha} x(s)\right| ds\right] \\ &\le \frac{A}{\xi^{\alpha-1}} + \frac{\xi}{\Gamma(\alpha+1)}\left\|D_{0+}^{\alpha} x\right\|_\infty, \end{aligned}$$

and

$$\begin{aligned} \left|t^{1-\alpha} x(t)\right| &\le \frac{t^{1-\alpha}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left|D_{0+}^{\alpha} x\right| ds + |c_1| \\ &\le \frac{t}{\Gamma(\alpha+1)}\left\|D_{0+}^{\alpha} x\right\|_\infty + \frac{A}{\xi^{\alpha-1}} + \frac{\xi}{\Gamma(\alpha+1)}\left\|D_{0+}^{\alpha} x\right\|_\infty \\ &\le \frac{A}{T^{\alpha-1}} + \frac{2T}{\Gamma(\alpha+1)}\left\|D_{0+}^{\alpha} x\right\|_\infty. \end{aligned}$$

That is,

$$\left\|t^{1-\alpha} x\right\|_\infty \le \frac{A}{T^{\alpha-1}} + \frac{2T}{\Gamma(\alpha+1)}\left\|D_{0+}^{\alpha} x\right\|_\infty. \tag{5.4}$$

Then, by (H$_1$), Eq (5.4), we get

$$\begin{aligned} &\left|I_{0+}^{\beta} N_f x(t)\right| = \frac{1}{\Gamma(\beta)}\left|\int_0^t (t-s)^{\beta-1} f\left(s, x(s), D_{0+}^{\alpha} x(s)\right) ds\right| \\ &\le \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}\left[a(s) + b(s)\left|s^{1-\alpha} x(s)\right|^{\theta-1} + c(s)\left|D_{0+}^{\alpha} x(s)\right|^{\theta-1}\right] ds \\ &\le \frac{1}{\Gamma(\beta)}\left(\|a\|_\infty + \|b\|_\infty\left\|t^{1-\alpha} x\right\|_\infty^{\theta-1} + \|c\|_\infty\left\|D_{0+}^{\alpha} x\right\|_\infty^{\theta-1}\right)\cdot\frac{1}{\beta}t^{\beta} \\ &\le \frac{T^{\beta}}{\Gamma(\beta+1)}\left[\|a\|_\infty + \|b\|_\infty\left(\frac{A}{T^{\alpha-1}} + \frac{2T}{\Gamma(\alpha+1)}\left\|D_{0+}^{\alpha} x\right\|_\infty\right)^{\theta-1} + \|c\|_\infty\left\|D_{0+}^{\alpha} x\right\|_\infty^{\theta-1}\right]. \end{aligned} \tag{5.5}$$

By $Lx = \lambda N_f x$, one has

$$x(t) = d_2 + I_{0+}^{\alpha}\phi_{p(t)}^{-1}\left(d_1 + \lambda I_{0+}^{\beta} N_f x\right)(t), \quad \forall d_1,\ d_2 \in \mathbb{R}.$$

Combined with the boundary value condition $\lim_{t\to 0^+} t^{1-\alpha} x(t) = \lim_{t\to T} t^{1-\alpha} x(t)$, you get

$$\frac{1}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}\phi_{p(s)}^{-1}\left(d_1 + \lambda I_{0+}^{\beta} N_f x(s)\right) ds = 0.$$

Therefore, there is $\eta \in (0,T)$ satisfying $\phi_{p(t)}^{-1}\left(d_1 + \lambda I_{0+}^{\beta} N_f x(\eta)\right) = 0$, which means $d_1 = -\lambda I_{0+}^{\beta} N_f x(\eta)$. So, we get

$$\phi_{p(t)}\left(D_{0+}^{\alpha} x\right) = -\lambda I_{0+}^{\beta} N_f x(\eta) + \lambda I_{0+}^{\beta} N_f x(t).$$

According to $\left|\phi_{p(t)}\left(D_{0+}^{\alpha} x(t)\right)\right| = \left|D_{0+}^{\alpha} x(t)\right|^{p(t)-1}$ and Eq (5.5), one has

$$\left\|D_{0+}^{\alpha} x\right\|_{\infty}^{p(t)-1} \leq \frac{2T^{\beta}}{\Gamma(\beta+1)}\left[\|a\|_{\infty} + \|b\|_{\infty}\left(\frac{A}{T^{\alpha-1}} + \frac{2T}{\Gamma(\alpha+1)}\left\|D_{0+}^{\alpha} x\right\|_{\infty}\right)^{\theta-1} + \|c\|_{\infty}\left\|D_{0+}^{\alpha} x\right\|_{\infty}^{\theta-1}\right]. \quad (5.6)$$

Through the inequality $(|x|+|y|)^p \leq 2^p\left(|x|^p + |y|^p\right),\ p > 0$, the following formula can be obtained

$$\left\|D_{0+}^{\alpha} x\right\|_{\infty}^{p(t)-1} \leq \Lambda_1 + \Lambda_2\left\|D_{0+}^{\alpha} x\right\|_{\infty}^{\theta-1},$$

where

$$\Lambda_1 = \frac{2T^{\beta}}{\Gamma(\beta+1)}\left[\|a\|_{\infty} + \|b\|_{\infty}\left(\frac{2A}{T^{\alpha-1}}\right)^{\theta-1}\right], \Lambda_2 = \frac{2T^{\beta}}{\Gamma(\beta+1)}\left[\|b\|_{\infty}\left(\frac{4T}{\Gamma(\alpha+1)}\right)^{\theta-1} + \|c\|_{\infty}\right].$$

Thus, we get

$$\left\|D_{0+}^{\alpha} x\right\|_{\infty} \leq 2^{\frac{1}{p(t)-1}}\left(\Lambda_1^{\frac{1}{p(t)-1}} + \Lambda_2^{\frac{1}{p(t)-1}}\left\|D_{0+}^{\alpha} x\right\|_{\infty}^{\frac{\theta-1}{p(t)-1}}\right).$$

Because $\frac{\theta-1}{p(t)-1} \in (0,1]$ and $x^k \leq x+1,\ x>0,\ k\in(0,1]$, it means

$$\left\|D_{0+}^{\alpha} x\right\|_{\infty} \leq (2\Lambda_1)^{\frac{1}{p(t)-1}} + (2\Lambda_2)^{\frac{1}{p(t)-1}}\left(\left\|D_{0+}^{\alpha} x\right\|_{\infty} + 1\right).$$

From Eq (5.3), we get that there exists a constant $\mathrm{M}_1 > 0$ such that

$$\left\|D_{0+}^{\alpha} x\right\|_{\infty} \leq \mathrm{M}_1. \quad (5.7)$$

Thus, by Eq (5.4), one gets

$$\left\|t^{1-\alpha} x\right\|_{\infty} \leq \frac{A}{T^{\alpha-1}} + \frac{2TM_1}{\Gamma(\alpha+1)}. \quad (5.8)$$

Therefore, by Eqs (5.7) and (5.8), we obtain

$$\|x\|_X = \max\left\{\left\|t^{1-\alpha} x\right\|_{\infty},\ \left\|D_{0+}^{\alpha} x\right\|_{\infty}\right\} \leq \max\left\{\frac{A}{T^{\alpha-1}} + \frac{2TM_1}{\Gamma(\alpha+1)},\ M_1\right\} := M.$$

Hence, $\Omega_1$ is bounded.

**Step 2.** Make $\Omega_2 = \left\{x \in \mathrm{Ker}L \mid QN_f x = 0\right\}$. For $x \in \Omega_2$, one has $x(t) = ct^{\alpha-1}$, $c \in \mathbb{R}$. Then

$$\int_0^T (T-s)^{\beta-1} f\left(s, cs^{\alpha-1}, 0\right) ds = 0.$$

Combined with condition $(H_2)$, we get $|c| \leq \frac{A}{T^{\alpha-1}}$. Therefore,

$$\|x\|_X \leq \max\left\{\frac{A}{T^{\alpha-1}}, 0\right\} = \frac{A}{T^{\alpha-1}}.$$

Hence, $\Omega_2$ is bounded.

**Step 3.** If Eq (5.1) holds, let

$$\Omega_3 = \left\{x \in \mathrm{Ker}L | \lambda I x + (1-\lambda) Q N_f x = 0,\ \lambda \in [0,1]\right\}.$$

For $x \in \Omega_3$, one has $x(t) = ct^{\alpha-1}$, $\forall c \in \mathbb{R}$ and the following formula is established

$$\lambda c s^{\alpha-1} + (1-\lambda)\frac{\beta}{T^\beta}\int_0^T (T-s)^{\beta-1} f\left(s, c s^{\alpha-1}, 0\right) ds = 0. \tag{5.9}$$

If $\lambda = 0$, by Eq (5.1), one has $|c| \le \frac{A}{T^{\alpha-1}}$. If $\lambda \in (0,1]$, you get $|c| \le \frac{A}{T^{\alpha-1}}$. Otherwise, if $|c| > \frac{A}{T^{\alpha-1}}$, by Eq (5.1), we get

$$\lambda\left(c s^{\alpha-1}\right)^2 + (1-\lambda)\frac{\beta}{T^\beta}\int_0^T (T-s)^{\beta-1} c s^{\alpha-1} f\left(s, c s^{\alpha-1}, 0\right) ds > 0,$$

which is contradictory to Eq (5.9). Hence, $\Omega_3$ is bounded. Besides, if Eq (5.2) is true, then let

$$\Omega'_3 = \left\{x \in \mathrm{Ker}L | -\lambda I x + (1-\lambda) Q N_f x = 0,\ \lambda \in [0,1]\right\}.$$

Analogously, in the same way, we can verify that $\Omega'_3$ is also bounded.

Finally, we aim to verify that all the conditions of Theorem 1 are true. Let

$$\Omega = \left\{x \in X_T \middle| \|x\|_X < \max\left\{\mathrm{M},\ \frac{A}{T^{\alpha-1}}\right\} + 1\right\}.$$

It is clear that $(\Omega_1 \cup \Omega_2 \cup \Omega_3) \subset \Omega$ (or $(\Omega_1 \cup \Omega_2 \cup \Omega'_3) \subset \Omega$). By Step 1, 2, we find that the conditions $(C_1)$ and $(C_2)$ of Theorem 1 are satisfied. Next, let's verify condition $(C_3)$ of Theorem 1. Define the homotopy

$$H(x,\lambda) = \pm\lambda I x + (1-\lambda) Q N_f x.$$

From Step 3, we get $H(x,\lambda) \ne 0$, $\forall x \in \mathrm{Ker}L \cap \partial\Omega$, then

$$\begin{aligned}\deg\left(QN_f|_{\mathrm{Ker}L},\ \Omega \cap \mathrm{Ker}L,\ 0\right) &= \deg\left(H(\cdot,0),\ \Omega \cap \mathrm{Ker}L,\ 0\right)\\ &= \deg\left(H(\cdot,1),\ \Omega \cap \mathrm{Ker}L,\ 0\right)\\ &= \deg\left(\pm I,\ \Omega \cap \mathrm{Ker}L,\ 0\right) \ne 0.\end{aligned}$$

Thus, condition $(C_3)$ of Theorem 1 is met. Applying Theorem 1, $Lx = N_f x$ has at least a fixed point in $\mathrm{dom}L \cap \overline{\Omega}$. Therefore, FPBVP Eq (1.1) has at least a solution in $X_T$.

**Example 1.** *Consider the following problem:*

$$\begin{cases} {}^C D_{0+}^{\frac{3}{4}} \phi_{(t^2+2)}\left(D_{0+}^{\frac{1}{2}} x(t)\right) = \frac{1}{20}\left|t^{\frac{1}{2}} x(t)\right| - 2t^{\frac{1}{2}} + t e^{-\left(D_{0+}^{\frac{1}{2}} x(t)\right)^2}, \quad t \in (0,1], \\ \lim\limits_{t\to 0^+} t^{\frac{1}{2}} x(t) = x(1),\ \lim\limits_{t\to 0^+} \phi_{(t^2+2)}\left(D_{0+}^{\frac{1}{2}} x(t)\right) = \lim\limits_{t\to 1} \phi_{(t^2+2)}\left(D_{0+}^{\frac{1}{2}} x(t)\right). \end{cases} \tag{5.10}$$

Corresponding to FPBVP Eq (1.1), we have $p(t) = t^2 + 2,\ \alpha = \frac{1}{2},\ \beta = \frac{3}{4}, T = 1, \theta = 2$ and

$$f(t,u,v) = \frac{1}{20}\left|t^{\frac{1}{2}}u\right| - 2t^{\frac{1}{2}} + te^{-v^2}.$$

Take $a(t) = 2$, $b(t) = \frac{1}{20}$, $c(t) = 0$. Obviously, $\|b\|_\infty = \frac{1}{20}$, $\|c\|_\infty = 0$. Obviously, $(H_1)$ of Theorem 2 is satisfied. Let $A = 40$, then

$$uf(t,u,v) = u\left[\frac{1}{20}t^{\frac{1}{2}}(|u| - 40) + te^{-v^2}\right] > 0\ , \quad \forall t \in [0,1],\ v \in \mathbb{R},\ u > 40,$$

$$uf(t,u,v) = u\left[\frac{1}{20}t^{\frac{1}{2}}(|u| - 40) + te^{-v^2}\right] < 0\ , \quad \forall t \in [0,1],\ v \in \mathbb{R},\ u < -40,$$

$$\frac{4T^\beta}{\Gamma(\beta+1)}\left[\|c\|_\infty + \|b\|_\infty\left(\frac{4T}{\Gamma(\alpha+1)}\right)^{\theta-1}\right] = \frac{4}{5\Gamma\left(\frac{3}{4}+1\right)\Gamma\left(\frac{1}{2}+1\right)} < 1.$$

Thus, problem Eq (5.10) satisfies all conditions of Theorem 2. Hence, there is at least one solution to problem Eq (5.10).

## 6. Conclusions

This paper deals with FPBVP with $p(t)$-Laplacian operator. Since the periodic boundary value condition lacks this key condition $D^\alpha_{0+}x(0) = 0$, the method used in [31,32] is not applicable to FPBVP Eq (1.1). To this end, we establish a continuation theorem (see Theorem 1). By applying the continuation theorem, a new existence result for the problem is obtained (see Theorem 2). In addition, when $p(t) = p$, the $p(t)$-Laplacian operator is reduced to the well-known $p$-Laplacian operator, so our paper will further enrich and extend the existing results. This theory can provide a solid foundation for studying similar periodic boundary value problems of fractional differential equations. For example, one can consider the solvability of periodic boundary value problems for fractional differential equations with impulse effects. In addition, the proposed theory can also be used to study the existence of solutions to the periodic boundary value problems of fractional differential equations and their corresponding coupling systems in high-dimensional kernel spaces.

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## Conflict of interest

The authors declare there is no conflict of interest.

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