Mathematical Biosciences and Engineering

http://www.aimspress.com/journal/mbe

MBE, 20(11): 19929–19940.
DOI: 10.3934/mbe.2023882
Received: 09 October 2023
Revised: 25 October 2023
Accepted: 30 October 2023
Published: 02 November 2023

Research article

Weighted Hermite–Hadamard integral inequalities for general convex functions

Péter Kórus[1,*] **Juan Eduardo Nápoles Valdés**[2,3] **and Bahtiyar Bayraktar**[4]**,**

[1] Institute of Applied Pedagogy, Juhász Gyula Faculty of Education, University of Szeged, Hattyas utca 10, H-6725 Szeged, Hungary

[2] UNNE, FaCENA, Ave. Libertad 5450, Corrientes 3400, Argentina

[3] UTN-FRRE, French 414, Resistencia, Chaco 3500, Argentina

[4] Faculty of Education, Bursa Uludag University, Gorukle Campus, 16059, Bursa, Turkey

* **Correspondence:** Email: korus.peter@szte.hu.

Abstract: In this article, starting with an equation for weighted integrals, we obtained several extensions of the well-known Hermite–Hadamard inequality. We used generalized weighted integral operators, which contain the Riemann–Liouville and the k-Riemann–Liouville fractional integral operators. The functions for which the operators were considered satisfy various conditions such as the h-convexity, modified h-convexity and s-convexity.

Keywords: weighted integral operators; convex functions; general convex functions; Hermite–Hadamard inequality

1. Introduction

Discussing convexity in the field of mathematics is immersed in the realm of interdisciplinary relations. Convexity serves as a foundational concept not only within the domains of geometry and analysis, but also finds extensive applications across various diverse fields. The utility of convexity techniques permeates numerous branches of mathematics, encompassing both pure and applied disciplines. These include optimization, the theory of inequalities, functional analysis, mathematical programming, game theory, number theory, variational calculus and their intricate interplay. This continual fusion of convexity concepts with other mathematical realms consistently yields fertile ground for the inception of novel research endeavors and the development of practical applications. Thus, it becomes evident that the concept of convex functions plays an indispensable and central role in the landscape of contemporary mathematics.

A real function $\Phi : [\varrho_1, \varrho_2] \to \mathbb{R}$ is said to be convex if

$$\Phi(\xi y + (1-\xi)x) \leq \xi\Phi(y) + (1-\xi)\Phi(x)$$

holds for all $y, x \in [\varrho_1, \varrho_2]$ and $0 \leq \xi \leq 1$. If the above inequality is reversed, then function Φ is said to be concave on $[\varrho_1, \varrho_2]$.

Convex functions have been studied extensively. Moreover generalizations have been considered, including the n-convex, r-convex, m-convex, s-convex, modified h-convex and (h, m)-convex functions and numerous others (a much more general overview of the different notions of convexity can be found in [1]).

The following definitions favor the reading of our work.

Definition 1.1. *[2] If a function $\Phi : I \to \mathbb{R}$ satisfies the conditions of being nonnegative and for all $x, y \in I$ and $0 \leq \xi \leq 1$, the inequality $\Phi(\xi x + (1-\xi)y) \leq \Phi(x) + \Phi(y)$ holds, then it can be classified as a member of the set $P(I)$.*

Definition 1.2. *[3] For some fixed s from the interval $(0, 1]$, a real function Φ given in $[0, \infty)$ is defined as s-convex in the second sense if the condition $\Phi(\xi x + (1-\xi)y) \leq \xi^s\Phi(x) + (1-\xi)^s\Phi(y)$ is satisfied for all x, y in the domain $[0, \infty)$ and for any ξ in the interval $(0, 1)$.*

Definition 1.3. *[4] Consider a nonnegative function $h : J \to \mathbb{R}$, where h is not equal to zero. We designate a function $\Phi : I \to \mathbb{R}$ as an h-convex function, or state that Φ belongs to the class $SX(h, I)$, under the condition that Φ is nonnegative and for all x, y in the domain I, and ξ in the open interval $(0, 1)$, the following inequality holds:*

$$\Phi(\xi x + (1-\xi)y) \leq h(\xi)\Phi(x) + h(1-\xi)\Phi(y). \tag{1.1}$$

If inequality (1.1) is reversed, then Φ is said to be h-concave. Clearly, if $h(\xi) = \xi$, then we have the classic convex, if $h(\xi) = 1$, then we have the P-functions and if $h(\xi) = \xi^s$, where $s \in (0, 1]$, then we obtain the s-convex functions of second sense.

Definition 1.4. *[5] A function $\Phi : [0, \varrho_2] \to \mathbb{R}$ is called (α, m)-convex, with (α, m) belonging to the closed interval $[0, 1]^2$. If it satisfies the condition that for all $x, y \in [0, \varrho_2]$ and $0 \leq \xi \leq 1$, the inequality below holds:*

$$\Phi(\xi x + m(1-\xi)y) \leq \xi^\alpha\Phi(x) + m(1-\xi^\alpha)\Phi(y).$$

In our work, we will use the following definition.

Definition 1.5. *[6] Let $\Phi, h : [\varrho_1, \varrho_2] \subseteq \mathbb{R} \to [0, \infty)$. Function Φ is called modified h-convex if*

$$\Phi(\xi x + (1-\xi)y) \leq h(\xi)\Phi(x) + (1 - h(\xi))\Phi(y)$$

holds for any $x, y \in [\varrho_1, \varrho_2]$ and $0 \leq \xi \leq 1$. Note that for $h(\xi) = \xi$, we have the convex functions, while for $h(\xi) = \xi^\alpha$, $\alpha \in [0, 1]$, and we have the so-called $(\alpha, 1)$-convex functions.

One of the most significant inequalities associated with the concept of convexity, which has garnered the attention of inequality experts over recent decades, is the renowned Hermite–Hadamard inequality:

$$\Phi\left(\frac{\varrho_1 + \varrho_2}{2}\right) \leq \frac{1}{\varrho_2 - \varrho_1}\int_{\varrho_1}^{\varrho_2} \Phi(x)\, dx \leq \frac{\Phi(\varrho_1) + \Phi(\varrho_2)}{2}. \tag{1.2}$$

This inequality applies to any function f that is convex over the interval $[a, b]$. Hermite first published this inequality in 1883, and, independently, Hadamard presented it in 1893. The Hermite–Hadamard inequality not only offers an estimation for the average value of a convex function, but also serves as an refinement to the Jensen inequality. If the reader wishes to delve deeper into this topic and explore further extensions of the Hermite–Hadamard inequality, we recommend the cited sources [7–16] and the references contained therein.

Below, in some definitions we use the functions $\Gamma(z)$ (see [17]) and $\Gamma_k(z)$ (see [9]):

$$\Gamma(z) = \int_0^\infty \xi^{z-1} e^{-\xi}\, d\xi, \quad \Re(z) > 0,$$
$$\Gamma_k(z) = \int_0^\infty \xi^{z-1} e^{-\xi^k/k}\, d\xi, \quad k > 0.$$

Clearly, if $k \to 1$, then we have $\Gamma_k(z) \to \Gamma(z)$. Furthermore, $\Gamma_k(z) = k^{\frac{z}{k}-1}\Gamma\left(\frac{z}{k}\right)$ and $\Gamma_k(z+k) = z\Gamma_k(z)$.

To make the subject easier to understand, let us present the definitions of a fractional integral in the case of $[\varrho_1, \varrho_2] \subseteq [0, \infty)$. These are the classical fractional Riemann–Liouville integrals (see [18]).

Definition 1.6. *Let $\Phi \in L_1(\varrho_1, \varrho_2)$, then the Riemann–Liouville fractional integrals of order $\alpha \in \mathbb{C}$, $\Re(\alpha) > 0$ are defined by (left and right, respectively):*

$${}^{\alpha}I_{\varrho_1^+}\Phi(\eth) = \frac{1}{\Gamma(\alpha)} \int_{\varrho_1}^{\eth} (\eth - \xi)^{\alpha-1} \Phi(\xi)\, d\xi, \quad \eth > \varrho_1,$$
$${}^{\alpha}I_{\varrho_2^-}\Phi(\eth) = \frac{1}{\Gamma(\alpha)} \int_{\eth}^{\varrho_2} (\xi - \eth)^{\alpha-1} \Phi(\xi)\, d\xi, \quad \eth < \varrho_2.$$

One can also define the k-Riemann–Liouville fractional integrals (see [19]).

Definition 1.7. *If $k > 0$, let $\Phi \in L_1(\varrho_1, \varrho_2)$, then the left and right k-Riemann–Liouville fractional integrals of order $\alpha > 0$ are defined by*

$${}^{\alpha,k}I_{\varrho_1^+}\Phi(\eth) = \frac{1}{k\Gamma_k(\alpha)} \int_{\varrho_1}^{\eth} (\eth - \xi)^{\frac{\alpha}{k}-1} \Phi(\xi)\, d\xi, \quad \eth > \varrho_1,$$
$${}^{\alpha,k}I_{\varrho_2^-}\Phi(\eth) = \frac{1}{k\Gamma_k(\alpha)} \int_{\eth}^{\varrho_2} (\xi - \eth)^{\frac{\alpha}{k}-1} \Phi(\xi)\, d\xi, \quad \eth < \varrho_2.$$

Now, we introduce the integral operators with weights, which will serve as the foundation of our research.

Definition 1.8. *Let $\Phi \in L_1(\varrho_1, \varrho_2)$ and let $w : [0, \infty) \to [0, \infty)$, $w \in C[0, \infty)$, w' and $w'' \in L_1[0, \infty)$ with $w(0) = 0$. Then, the weighted fractional integrals are defined by (left and right, respectively):*

$${}^{w}I_{\varrho_1+}\Phi(\eth) = \int_{\varrho_1}^{\eth} w''\left(\frac{\eth - \xi}{\eth - \varrho_1}\right) \Phi(\xi)\, d\xi, \quad \eth > \varrho_1,$$
$${}^{w}I_{\varrho_2-}\Phi(\eth) = \int_{\eth}^{\varrho_2} w''\left(\frac{\xi - \eth}{\varrho_2 - \eth}\right) \Phi(\xi)\, d\xi, \quad \eth < \varrho_2.$$

Remark 1.1. *Putting $w''(\xi) \equiv 1$, we obtain the classical Riemann integral. If $w''(\xi) = \frac{(x-\varrho_1)^{\alpha-1}\xi^{\alpha-1}}{\Gamma(\alpha)}$ or $w''(\xi) = \frac{(\varrho_2-x)^{\alpha-1}\xi^{\alpha-1}}{\Gamma(\alpha)}$, then we obtain the Riemann–Liouville fractional integrals of order $\alpha > 0$, left and right, respectively. Other fractional integral operators, such as k-Riemann–Liouville fractional integrals, can be easily obtained with the proper selection of $w''(\xi)$.*

In this study, we introduce several variants of inequality (1.2) within the context of the weighted integral operators as defined in Definition 1.8.

2. Results

The following Lemma is a basic result of our work.

Lemma 2.1. *Let $\Phi : [\varrho_1, \varrho_2] \to \mathbb{R}$ and $\Phi \in C^2(\varrho_1, \varrho_2)$. If $\Phi'' \in L_1(\varrho_1, \varrho_2)$, then we have the following equation:*

$$\begin{aligned}
&-\frac{8}{(\varrho_2-\varrho_1)^2} w'(1)\Phi\left(\frac{\varrho_1+\varrho_2}{2}\right) + \frac{8}{(\varrho_2-\varrho_1)^2} w'(0)\left(\frac{\Phi(\varrho_1)+\Phi(\varrho_2)}{2}\right) \\
&+\frac{8}{(\varrho_2-\varrho_1)^3}\left[{}^{w}I_{\frac{\varrho_1+\varrho_2}{2}+}\Phi(\varrho_2) + {}^{w}I_{\frac{\varrho_1+\varrho_2}{2}-}\Phi(\varrho_1)\right] \qquad (2.1)\\
&= \int_0^1 w(\xi)\left(\Phi''\left(\frac{\xi}{2}\varrho_1 + \frac{2-\xi}{2}\varrho_2\right) + \Phi''\left(\frac{2-\xi}{2}\varrho_1 + \frac{\xi}{2}\varrho_2\right)\right)d\xi.
\end{aligned}$$

Proof. First note that

$$\begin{aligned}
&\int_0^1 w(\xi)\left(\Phi''\left(\frac{\xi}{2}\varrho_1 + \frac{2-\xi}{2}\varrho_2\right) + \Phi''\left(\frac{2-\xi}{2}\varrho_1 + \frac{\xi}{2}\varrho_2\right)\right)d\xi \\
&= \int_0^1 w(\xi)\Phi''\left(\frac{\xi}{2}\varrho_1 + \frac{2-\xi}{2}\varrho_2\right)d\xi + \int_0^1 w(\xi)\Phi''\left(\frac{2-\xi}{2}\varrho_1 + \frac{\xi}{2}\varrho_2\right)d\xi = I_1 + I_2.
\end{aligned}$$

By integrating each of these integrals two times using integration by parts, we achieve

$$\begin{aligned}
I_1 =& -\frac{2}{\varrho_2-\varrho_1}\Phi'\left(\frac{\varrho_1+\varrho_2}{2}\right)w(1) - \frac{4}{(\varrho_2-\varrho_1)^2}\left(\Phi\left(\frac{\varrho_1+\varrho_2}{2}\right)w'(1) - \Phi(\varrho_2)w'(0)\right) \\
&+ \frac{4}{(\varrho_2-\varrho_1)^2}\int_0^1 w''(\xi)\Phi\left(\frac{\xi}{2}\varrho_1 + \frac{2-\xi}{2}\varrho_2\right)d\xi, \\
I_2 =& \frac{2}{\varrho_2-\varrho_1}\Phi'\left(\frac{\varrho_1+\varrho_2}{2}\right)w(1) - \frac{4}{(\varrho_2-\varrho_1)^2}\left(\Phi\left(\frac{\varrho_1+\varrho_2}{2}\right)w'(1) - \Phi(\varrho_1)w'(0)\right) \\
&+ \frac{4}{(\varrho_2-\varrho_1)^2}\int_0^1 w''(\xi)\Phi\left(\frac{2-\xi}{2}\varrho_1 + \frac{\xi}{2}\varrho_2\right)d\xi,
\end{aligned}$$

hence, we get

$$I_1 + I_2 = -\frac{8}{(\varrho_2-\varrho_1)^2}w'(1)\Phi\left(\frac{\varrho_1+\varrho_2}{2}\right) + \frac{8}{(\varrho_2-\varrho_1)^2}w'(0)\left(\frac{\Phi(\varrho_1)+\Phi(\varrho_2)}{2}\right) +$$

$$+\frac{4}{(\varrho_2-\varrho_1)^2}\left[\int_0^1 w''(\xi)\Phi\left(\frac{\xi}{2}\varrho_1+\frac{2-\xi}{2}\varrho_2\right)d\xi+\int_0^1 w''(\xi)\Phi\left(\frac{2-\xi}{2}\varrho_1+\frac{\xi}{2}\varrho_2\right)d\xi\right].$$

From this last equality, with changes of variables $x=\frac{\xi}{2}\varrho_1+\frac{2-\xi}{2}\varrho_2$ and $x=\frac{2-\xi}{2}\varrho_1+\frac{\xi}{2}\varrho_2$, respectively and after some algebraic technicality, the desired result is obtained. □

Remark 2.1. *Lemma 2.1 extends [20, Lemma 2.1]. If we consider* $w(\xi)=\frac{\xi^{\alpha+1}}{\Gamma(\alpha+2)}$ *and* $w(\xi)=\frac{\xi^{\alpha/k+1}}{k^{\alpha/k}\Gamma(\alpha/k+2)}$*, Lemma 2.1 becomes [21, Lemma 1] and [20, Corollary 2.1], respectively.*

Our first two fundamental results are the following.

Theorem 2.1. *Given the conditions outlined in Lemma 2.1, in the event that* $|\Phi''|$ *is modified h-convex over the interval* $[\varrho_1,\varrho_2]$*, the subsequent inequality holds:*

$$\left|A+\frac{8}{(\varrho_2-\varrho_1)^3}\left[{}^wI_{\frac{\varrho_1+\varrho_2}{2}+}\Phi(\varrho_2)+{}^w I_{\frac{\varrho_1+\varrho_2}{2}-}\Phi(\varrho_1)\right]\right|\le B\Big(|\Phi''(\varrho_1)|+|\Phi''(\varrho_2)|\Big) \tag{2.2}$$

with $A=-\frac{8}{(\varrho_2-\varrho_1)^2}w'(1)\Phi\left(\frac{\varrho_1+\varrho_2}{2}\right)+\frac{8}{(\varrho_2-\varrho_1)^2}w'(0)\left(\frac{\Phi(\varrho_1)+\Phi(\varrho_2)}{2}\right)$ *and* $B=\int_0^1 w(\xi)\,d\xi$*.*

Proof. Using the modified h-convexity of $|\Phi''|$, we get

$$\int_0^1 w(\xi)\left|\Phi''\left(\frac{\xi}{2}\varrho_1+\frac{2-\xi}{2}\varrho_2\right)\right|d\xi\le|\Phi''(\varrho_1)|\int_0^1 h\left(\frac{\xi}{2}\right)w(\xi)\,d\xi+|\Phi''(\varrho_2)|\int_0^1\left(1-h\left(\frac{\xi}{2}\right)\right)w(\xi)\,d\xi,$$
$$\int_0^1 w(\xi)\left|\Phi''\left(\frac{2-\xi}{2}\varrho_1+\frac{\xi}{2}\varrho_2\right)\right|d\xi\le|\Phi''(\varrho_1)|\int_0^1\left(1-h\left(\frac{\xi}{2}\right)\right)w(\xi)\,d\xi+|\Phi''(\varrho_2)|\int_0^1 h\left(\frac{\xi}{2}\right)w(\xi)\,d\xi.$$

By adding both inequalities and denoting $B=\int_0^1 w(\xi)\,d\xi$, (2.1) yields (2.2) after taking absolute values. □

Remark 2.2. *If we consider* $w''(\xi)\equiv 1$*,* $h(\xi)=\xi$*, from the previous result, we obtain the following estimate of the left hand side of the Hermite–Hadamard inequality* (1.2)*:*

$$\left|-\Phi\left(\frac{\varrho_1+\varrho_2}{2}\right)+\frac{1}{\varrho_2-\varrho_1}\int_{\varrho_1}^{\varrho_2}\Phi(x)\,dx\right|\le(\varrho_2-\varrho_1)^2\left(\frac{|\Phi''(\varrho_1)|+|\Phi''(\varrho_2)|}{48}\right).$$

This inequality was obtained in several previous papers, for example in [22, inequality (15)] and in [23, Proposition 1].

Theorem 2.2. *Given the conditions outlined in Lemma 2.1, in the event that* $|\Phi''|$ *is s-convex over the interval* $[\varrho_1,\varrho_2]$ *with* $s\in(0,1]$*, the subsequent inequality holds:*

$$\left|A+\frac{8}{(\varrho_2-\varrho_1)^3}\left[{}^wI_{\frac{\varrho_1+\varrho_2}{2}+}\Phi(\varrho_2)+{}^w I_{\frac{\varrho_1+\varrho_2}{2}-}\Phi(\varrho_1)\right]\right|\le 2^{1-s}B\Big(|\Phi''(\varrho_1)|+|\Phi''(\varrho_2)|\Big) \tag{2.3}$$

with $A=-\frac{8}{(\varrho_2-\varrho_1)^2}w'(1)\Phi\left(\frac{\varrho_1+\varrho_2}{2}\right)+\frac{8}{(\varrho_2-\varrho_1)^2}w'(0)\left(\frac{\Phi(\varrho_1)+\Phi(\varrho_2)}{2}\right)$ *and* $B=\int_0^1 w(\xi)\,d\xi$*.*

Proof. Considering the s-convexity of $|\Phi''|$, we get

$$\int_0^1 w(\xi)\left|\Phi''\left(\frac{\xi}{2}\varrho_1+\frac{2-\xi}{2}\varrho_2\right)\right|d\xi \le |\Phi''(\varrho_1)|\int_0^1 \left(\frac{\xi}{2}\right)^s w(\xi)\,d\xi + |\Phi''(\varrho_2)|\int_0^1 \left(\frac{2-\xi}{2}\right)^s w(\xi)\,d\xi,$$
$$\int_0^1 w(\xi)\left|\Phi''\left(\frac{2-\xi}{2}\varrho_1+\frac{\xi}{2}\varrho_2\right)\right|d\xi \le |\Phi''(\varrho_1)|\int_0^1 \left(\frac{2-\xi}{2}\right)^s w(\xi)\,d\xi + |\Phi''(\varrho_2)|\int_0^1 \left(\frac{\xi}{2}\right)^s w(\xi)\,d\xi.$$

By the concavity of $h(\xi)=\xi^s$ ($s\in(0,1]$), we have $\left(\frac{\xi}{2}\right)^s+\left(\frac{2-\xi}{2}\right)^s \le 2\left(\frac{1}{2}\right)^s$. Therefore, by adding both inequalities above, (2.3) is obtained from (2.1). □

Remark 2.3. *Theorems 2.1 and 2.2 extend [20, Theorem 2.1]. If in Theorem 2.1 or 2.2 we set $h(\xi)=\xi$ or $s=1$, respectively, and put $w(\xi)=\frac{\xi^{\alpha+1}}{\Gamma(\alpha+2)}$, we get [21, Theorem 3].*

Further improvements to the preceding outcomes can be attained by introducing fresh supplementary conditions, specifically by considering the general convexity of $|\Phi''|^q$.

Theorem 2.3. *Given the conditions outlined in Lemma 2.1, in the event that $|\Phi''|^q$ is modified h-convex over the interval $[\varrho_1,\varrho_2]$, for $q>1$ the subsequent inequality holds:*

$$\begin{aligned}
&\left|A+\frac{8}{(\varrho_2-\varrho_1)^3}\left[{}^wI_{\frac{\varrho_1+\varrho_2}{2}+}\Phi(\varrho_2)+{}^wI_{\frac{\varrho_1+\varrho_2}{2}-}\Phi(\varrho_1)\right]\right|\\
&\le B_p\left[\left(H_1|\Phi''(\varrho_1)|^q+H_2|\Phi''(\varrho_2)|^q\right)^{\frac{1}{q}}+\left(H_2|\Phi''(\varrho_1)|^q+H_1|\Phi''(\varrho_2)|^q\right)^{\frac{1}{q}}\right] \qquad (2.4)\\
&\le 2^{\frac{1}{p}}B_p\Big(|\Phi''(\varrho_1)|+|\Phi''(\varrho_2)|\Big)
\end{aligned}$$

with $\frac{1}{p}+\frac{1}{q}=1$,

$$A=-\frac{8}{(\varrho_2-\varrho_1)^2}w'(1)\Phi\left(\frac{\varrho_1+\varrho_2}{2}\right)+\frac{8}{(\varrho_2-\varrho_1)^2}w'(0)\left(\frac{\Phi(\varrho_1)+\Phi(\varrho_2)}{2}\right),$$
$$B_p=\left(\int_0^1 w^p(\xi)\,d\xi\right)^{\frac{1}{p}},\ H_1=2\int_0^{1/2}h(\xi)\,d\xi\ \text{ and }\ H_2=1-H_1.$$

Proof. From Hölder's inequality considered for $w(\xi)$ and $|\Phi''|$, we obtain

$$\left|\int_0^1 w(\xi)\Phi''\left(\frac{\xi}{2}\varrho_1+\frac{2-\xi}{2}\varrho_2\right)d\xi\right| \le \left[\int_0^1 w^p(\xi)\,d\xi\right]^{\frac{1}{p}}\left[\int_0^1\left|\Phi''\left(\frac{\xi}{2}\varrho_1+\frac{2-\xi}{2}\varrho_2\right)\right|^q d\xi\right]^{\frac{1}{q}} \qquad (2.5)$$

and

$$\left|\int_0^1 w(\xi)\Phi''\left(\frac{2-\xi}{2}\varrho_1+\frac{\xi}{2}\varrho_2\right)d\xi\right| \le \left[\int_0^1 w^p(\xi)\,d\xi\right]^{\frac{1}{p}}\left[\int_0^1\left|\Phi''\left(\frac{2-\xi}{2}\varrho_1+\frac{\xi}{2}\varrho_2\right)\right|^q d\xi\right]^{\frac{1}{q}} \qquad (2.6)$$

for $\frac{1}{p}+\frac{1}{q}=1$. Denoting, for brevity, $B_p=\left(\int_0^1 w^p(\xi)\,d\xi\right)^{\frac{1}{p}}$ and using the modified h-convexity of $|\Phi''|^q$, we have

$$\left|\Phi''\left(\frac{\xi}{2}\varrho_1+\frac{2-\xi}{2}\varrho_2\right)\right|^q \le h\left(\frac{\xi}{2}\right)|\Phi''(\varrho_1)|^q+\left(1-h\left(\frac{\xi}{2}\right)\right)|\Phi''(\varrho_2)|^q, \qquad (2.7)$$

$$\left|\Phi''\left(\frac{2-\xi}{2}\varrho_1+\frac{\xi}{2}\varrho_2\right)\right|^q \le \left(1-h\left(\frac{\xi}{2}\right)\right)|\Phi''(\varrho_1)|^q + h\left(\frac{\xi}{2}\right)|\Phi''(\varrho_2)|^q. \tag{2.8}$$

Therefore, after integrating on $[0,1]$, noting $\int_0^1 h\left(\frac{\xi}{2}\right)d\xi = 2\int_0^{1/2} h(\xi)\,d\xi = H_1$ and $\int_0^1 1-h\left(\frac{\xi}{2}\right)d\xi = 1-H_1 = H_2$, we get

$$\left|\int_0^1 w(\xi)\Phi''\left(\frac{\xi}{2}\varrho_1+\frac{2-\xi}{2}\varrho_2\right)d\xi\right| \le B_p\left(H_1|\Phi''(\varrho_1)|^q + H_2|\Phi''(\varrho_2)|^q\right)^{\frac{1}{q}}, \tag{2.9}$$

$$\left|\int_0^1 w(\xi)\Phi''\left(\frac{2-\xi}{2}\varrho_1+\frac{\xi}{2}\varrho_2\right)d\xi\right| \le B_p\left(H_2|\Phi''(\varrho_1)|^q + H_1|\Phi''(\varrho_2)|^q\right)^{\frac{1}{q}}. \tag{2.10}$$

Adding up (2.9)-(2.10) gives us the first inequality in (2.4). Moreover, using the fact that $(\varrho_1+\varrho_2)^{\frac{1}{q}} \le \varrho_1^{\frac{1}{q}} + \varrho_2^{\frac{1}{q}}$ for $0<\frac{1}{q}<1$, we have

$$\left(H_1|\Phi''(\varrho_1)|^q + H_2|\Phi''(\varrho_2)|^q\right)^{\frac{1}{q}} \le H_1^{\frac{1}{q}}|\Phi''(\varrho_1)| + H_2^{\frac{1}{q}}|\Phi''(\varrho_2)|,$$
$$\left(H_2|\Phi''(\varrho_1)|^q + H_1|\Phi''(\varrho_2)|^q\right)^{\frac{1}{q}} \le H_2^{\frac{1}{q}}|\Phi''(\varrho_1)| + H_1^{\frac{1}{q}}|\Phi''(\varrho_2)|.$$

Combining these inequalities with (2.9) and (2.10), while keeping in mind

$$H_1^{\frac{1}{q}} + H_2^{\frac{1}{q}} = H_1^{\frac{1}{q}} + (1-H_1)^{\frac{1}{q}} \le 2\left(\frac{1}{2}\right)^{\frac{1}{q}},$$

allows us to obtain the second inequality in (2.4). □

Theorem 2.4. *Given the conditions outlined in Lemma 2.1, in the event that* $|\Phi''|^q$ *is s-convex over the interval* $[\varrho_1,\varrho_2]$*, for* $q>1$ *the subsequent inequality holds:*

$$\begin{aligned}
&\left|A + \frac{8}{(\varrho_2-\varrho_1)^3}\left[{}^{w}I_{\frac{\varrho_1+\varrho_2}{2}+}\Phi(\varrho_2) + {}^{w}I_{\frac{\varrho_1+\varrho_2}{2}-}\Phi(\varrho_1)\right]\right| \\
&\le B_p\left[\left(S_1|\Phi''(\varrho_1)|^q + S_2|\Phi''(\varrho_2)|^q\right)^{\frac{1}{q}} + \left(S_2|\Phi''(\varrho_1)|^q + S_1|\Phi''(\varrho_2)|^q\right)^{\frac{1}{q}}\right] \\
&\le \frac{2B_p}{(s+1)^{\frac{1}{q}}}\left(|\Phi''(\varrho_1)| + |\Phi''(\varrho_2)|\right)
\end{aligned} \tag{2.11}$$

with $\frac{1}{p}+\frac{1}{q}=1$,

$$A = -\frac{8}{(\varrho_2-\varrho_1)^2}w'(1)\Phi\left(\frac{\varrho_1+\varrho_2}{2}\right) + \frac{8}{(\varrho_2-\varrho_1)^2}w'(0)\left(\frac{\Phi(\varrho_1)+\Phi(\varrho_2)}{2}\right),$$
$$B_p = \left(\int_0^1 w^p(\xi)\,d\xi\right)^{\frac{1}{p}},\ S_1 = \frac{2^{-s}}{s+1}\ \textit{and}\ S_2 = \frac{2-2^{-s}}{s+1}.$$

Proof. First note that inequalities (2.5)-(2.6) hold. Moreover, the s-convexity of $|\Phi''|^q$ implies

$$\left|\Phi''\left(\frac{\xi}{2}\varrho_1+\frac{2-\xi}{2}\varrho_2\right)\right|^q \le \left(\frac{\xi}{2}\right)^s|\Phi''(\varrho_1)|^q + \left(1-\frac{\xi}{2}\right)^s|\Phi''(\varrho_2)|^q,$$

$$\left|\Phi''\left(\frac{2-\xi}{2}\varrho_1+\frac{\xi}{2}\varrho_2\right)\right|^q \le \left(1-\frac{\xi}{2}\right)^s |\Phi''(\varrho_1)|^q + \left(\frac{\xi}{2}\right)^s |\Phi''(\varrho_2)|^q .$$

After integrating on $[0,1]$, noting

$$\int_0^1 \left(\frac{\xi}{2}\right)^s d\xi = \frac{2^{-s}}{s+1} = S_1 \quad \text{and} \quad \int_0^1 \left(1-\frac{\xi}{2}\right)^s d\xi = \frac{2-2^{-s}}{s+1} = S_2,$$

we have

$$\left|\int_0^1 w(\xi)\Phi''\left(\frac{\xi}{2}\varrho_1+\frac{2-\xi}{2}\varrho_2\right)d\xi\right| \le B_p\left(S_1|\Phi''(\varrho_1)|^q + S_2|\Phi''(\varrho_2)|^q\right)^{\frac{1}{q}}, \tag{2.12}$$

$$\left|\int_0^1 w(\xi)\Phi''\left(\frac{2-\xi}{2}\varrho_1+\frac{\xi}{2}\varrho_2\right)d\xi\right| \le B_p\left(S_2|\Phi''(\varrho_1)|^q + S_1|\Phi''(\varrho_2)|^q\right)^{\frac{1}{q}}. \tag{2.13}$$

The first inequality in (2.11) is obtained from adding up (2.12)-(2.13). Moreover, using again $(\varrho_1+\varrho_2)^{\frac{1}{q}} \le \varrho_1^{\frac{1}{q}} + \varrho_2^{\frac{1}{q}}$, we get

$$\left(S_1|\Phi''(\varrho_1)|^q + S_2|\Phi''(\varrho_2)|^q\right)^{\frac{1}{q}} \le S_1^{\frac{1}{q}}|\Phi''(\varrho_1)| + S_2^{\frac{1}{q}}|\Phi''(\varrho_2)|,$$

$$\left(S_2|\Phi''(\varrho_1)|^q + S_1|\Phi''(\varrho_2)|^q\right)^{\frac{1}{q}} \le S_2^{\frac{1}{q}}|\Phi''(\varrho_1)| + S_1^{\frac{1}{q}}|\Phi''(\varrho_2)|.$$

Combining these inequalities with (2.12) and (2.13) and considering

$$S_1^{\frac{1}{q}} + S_2^{\frac{1}{q}} = \left(\frac{2^{-s}}{s+1}\right)^{\frac{1}{q}} + \left(\frac{2-2^{-s}}{s+1}\right)^{\frac{1}{q}} \le 2\left(\frac{1}{s+1}\right)^{\frac{1}{q}}$$

yields the second inequality in (2.11). □

Remark 2.4. *[20, Theorem 2.2] can be obtained from either Theorem 2.3 or 2.4 by setting $h(\xi)=\xi$ or $s=1$, respectively.*

Variants of Theorems 2.3 and 2.4 are given in the following results.

Theorem 2.5. *Given the conditions outlined in Lemma 2.1, in the event that $|\Phi''|^q$ is modified h-convex over the interval $[\varrho_1,\varrho_2]$, for $q>1$ the subsequent inequality holds:*

$$\begin{aligned}&\left|A+\frac{8}{(\varrho_2-\varrho_1)^3}\left[{}^{w}I_{\frac{\varrho_1+\varrho_2}{2}+}\Phi(\varrho_2)+{}^{w}I_{\frac{\varrho_1+\varrho_2}{2}-}\Phi(\varrho_1)\right]\right|\\ &\le B^{\frac{1}{p}}\left[\left(C_1|\Phi''(\varrho_1)|^q+C_2|\Phi''(\varrho_2)|^q\right)^{\frac{1}{q}}+\left(C_2|\Phi''(\varrho_1)|^q+C_1|\Phi''(\varrho_2)|^q\right)^{\frac{1}{q}}\right]\\ &\le 2^{\frac{1}{p}}\left(|\Phi''(\varrho_1)|+|\Phi''(\varrho_2)|\right)\end{aligned} \tag{2.14}$$

with $\frac{1}{p}+\frac{1}{q}=1$,

$$A=-\frac{8}{(\varrho_2-\varrho_1)^2}w'(1)\Phi\left(\frac{\varrho_1+\varrho_2}{2}\right)+\frac{8}{(\varrho_2-\varrho_1)^2}w'(0)\left(\frac{\Phi(\varrho_1)+\Phi(\varrho_2)}{2}\right),$$

$$B=\int_0^1 w(\xi)\,d\xi,\quad C_1=\int_0^1 w(\xi)h\left(\frac{\xi}{2}\right)d\xi \ \textit{and}\ C_2=B-C_1.$$

Proof. Taking into account Hölder's inequality for $w(\xi)^{\frac{1}{p}}$ and $w(\xi)^{\frac{1}{q}}|\Phi''|$, we have

$$\left|\int_0^1 w(\xi)\Phi''\left(\frac{\xi}{2}\varrho_1+\frac{2-\xi}{2}\varrho_2\right)d\xi\right| \le \left[\int_0^1 w(\xi)\,d\xi\right]^{\frac{1}{p}}\left[\int_0^1 w(\xi)\left|\Phi''\left(\frac{\xi}{2}\varrho_1+\frac{2-\xi}{2}\varrho_2\right)\right|^q d\xi\right]^{\frac{1}{q}} \tag{2.15}$$

and

$$\left|\int_0^1 w(\xi)\Phi''\left(\frac{2-\xi}{2}\varrho_1+\frac{\xi}{2}\varrho_2\right)d\xi\right| \le \left[\int_0^1 w(\xi)\,d\xi\right]^{\frac{1}{p}}\left[\int_0^1 w(\xi)\left|\Phi''\left(\frac{2-\xi}{2}\varrho_1+\frac{\xi}{2}\varrho_2\right)\right|^q d\xi\right]^{\frac{1}{q}} \tag{2.16}$$

for $\frac{1}{p}+\frac{1}{q}=1$. By the modified h-convexity of $|\Phi''|^q$, (2.7)-(2.8) hold, and therefore

$$\int_0^1 w(\xi)\left|\Phi''\left(\frac{\xi}{2}\varrho_1+\frac{2-\xi}{2}\varrho_2\right)\right|^q d\xi \le |\Phi''(\varrho_1)|^q\int_0^1 w(\xi)h\left(\frac{\xi}{2}\right)d\xi+|\Phi''(\varrho_2)|^q\int_0^1 w(\xi)\left(1-h\left(\frac{\xi}{2}\right)\right)d\xi,$$
$$\int_0^1 w(\xi)\left|\Phi''\left(\frac{2-\xi}{2}\varrho_1+\frac{\xi}{2}\varrho_2\right)\right|^q d\xi \le |\Phi''(\varrho_1)|^q\int_0^1 w(\xi)\left(1-h\left(\frac{\xi}{2}\right)\right)d\xi+|\Phi''(\varrho_2)|^q\int_0^1 w(\xi)h\left(\frac{\xi}{2}\right)d\xi.$$

These inequalities together with (2.15)-(2.16) imply the first inequality in (2.14). The second is obtained by noting

$$\begin{aligned}(C_1\,|\Phi''(\varrho_1)|^q+C_2\,|\Phi''(\varrho_2)|^q)^{\frac{1}{q}} &\le\ C_1{}^{\frac{1}{q}}\,|\Phi''(\varrho_1)|+\ C_2{}^{\frac{1}{q}}\,|\Phi''(\varrho_2)|\,,\\ (C_2\,|\Phi''(\varrho_1)|^q+C_1\,|\Phi''(\varrho_2)|^q)^{\frac{1}{q}} &\le\ C_2{}^{\frac{1}{q}}\,|\Phi''(\varrho_1)|+\ C_1{}^{\frac{1}{q}}\,|\Phi''(\varrho_2)|\end{aligned}$$

and

$$C_1^{\frac{1}{q}}+C_2^{\frac{1}{q}}=C_1^{\frac{1}{q}}+(B-C_1)^{\frac{1}{q}}\le 2\left(\frac{B}{2}\right)^{\frac{1}{q}}.$$ □

Theorem 2.6. *Given the conditions outlined in Lemma 2.1, in the event that $|\Phi''|^q$ is h-convex over the interval $[\varrho_1,\varrho_2]$, for $q>1$ the subsequent inequality holds:*

$$\begin{aligned}&\left|A+\frac{8}{(\varrho_2-\varrho_1)^3}\left[{}^wI_{\frac{\varrho_1+\varrho_2}{2}+}\Phi(\varrho_2)+{}^wI_{\frac{\varrho_1+\varrho_2}{2}-}\Phi(\varrho_1)\right]\right|\\ &\le B^{\frac{1}{p}}\left[\left(D_1\,|\Phi''(\varrho_1)|^q+D_2\,|\Phi''(\varrho_2)|^q\right)^{\frac{1}{q}}+\left(D_2\,|\Phi''(\varrho_1)|^q+D_1\,|\Phi''(\varrho_2)|^q\right)^{\frac{1}{q}}\right]\\ &\le B^{\frac{1}{p}}\left(D_1^{\frac{1}{q}}+D_2^{\frac{1}{q}}\right)(|\Phi''(\varrho_1)|+|\Phi''(\varrho_2)|)\end{aligned}$$

with $\frac{1}{p}+\frac{1}{q}=1$*,*

$$\begin{aligned}A&=-\frac{8}{(\varrho_2-\varrho_1)^2}w'(1)\Phi\left(\frac{\varrho_1+\varrho_2}{2}\right)+\frac{8}{(\varrho_2-\varrho_1)^2}w'(0)\left(\frac{\Phi(\varrho_1)+\Phi(\varrho_2)}{2}\right)\\ B&=\int_0^1 w(\xi)\,d\xi,\ D_1=\int_0^1 w(\xi)h\left(\frac{\xi}{2}\right)d\xi,\ D_2=\int_0^1 w(\xi)h\left(1-\frac{\xi}{2}\right)d\xi.\end{aligned}$$

Proof. The proof is analogous to that of Theorem 2.5. We omit the details. □

3. Conclusions

In this work we have obtained some inequalities using a certain weighted integral, which contained several already published results. Apart from the remarks made, we can point out the strength of our approach due to the fact that we considered general convex functions such as modified h-convex, s-convex or h-convex functions. Consider the following example.

Example 3.1. *Let* $\Phi : [0,1] \to \mathbb{R}$, $\Phi(x) = x^{2+s}$, $s \in (0,1)$ *and let* $w : [0,\infty) \to [0,\infty)$ *and* $w \in C[0,\infty)$ *with* w' *and* w'' *piecewise continuous on* $[0,\infty)$ *with* $w(0) = 0$. *Then,* $\Phi'' \in L_1(0,1)$, $|\Phi''|$ *is* s*-convex on* $[0,1]$, *although* $|\Phi''|$ *is not convex. Therefore, Theorem 2.2 implies*

$$\left|-\frac{2^{1-s}}{(\varrho_2-\varrho_1)^2}w'(1) + \frac{4}{(\varrho_2-\varrho_1)^2}w'(0) + \frac{8}{(\varrho_2-\varrho_1)^3}\left[\int_0^x w''(x-\xi)\xi^{2+s}\,d\xi + \int_x^1 w''(\xi-x)\xi^{2+s}\,d\xi\right]\right|$$
$$\leq 2^{1-s}(2+s)(1+s)\int_0^1 w(\xi)\,d\xi.$$

Moreover, we can cover some known results other than the above remarks. The following is sufficient as an example. Consider the continuous function $w : [0,1] \to [0,\infty)$ with first and second order derivatives piecewise continuous on $[0,1]$ so that $w(0) = w(1) = 0$, then we can formulate the following result that can be proved very similarly to Lemma 2.1.

Proposition 3.1. *Let function* w *be as above,* $\Phi : [\varrho_1,\varrho_2] \to \mathbb{R}$ *and* $\Phi \in C^2(\varrho_1,\varrho_2)$. *Assuming that* $\Phi'' \in L_1(\varrho_1,\varrho_2)$, *we can establish the following equations:*

$$w'(0)\Phi(\varrho_2) - w'(1)\Phi(\varrho_1) + \frac{1}{\varrho_2-\varrho_1}{}^{w}I_{\varrho_1+}\Phi(\varrho_2) = (\varrho_2-\varrho_1)^2\int_0^1 w(\xi)\Phi''(\xi\varrho_1 + (1-\xi)\varrho_2)\,d\xi,$$
$$w'(0)\Phi(\varrho_1) - w'(1)\Phi(\varrho_2) + \frac{1}{\varrho_2-\varrho_1}{}^{w}I_{\varrho_2-}\Phi(\varrho_1) = (\varrho_2-\varrho_1)^2\int_0^1 w(\xi)\Phi''((1-\xi)\varrho_1 + \xi\varrho_2)\,d\xi.$$

This outcome includes a specific instance from [24, Lemma 1] (also discussed in [9]) by setting $w(\xi) = \xi(1-\xi)$.

We can derive alternative formulations of our findings by pursuing two directions: first, by introducing supplementary constraints on the function w, and second, by exploring alternative concepts of convexity.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare there is no conflict of interest.

References

1. J. E. Nápoles Valdés, F. Rabossi, A. D. Samaniego, Convex functions: Ariadne's thread or Charlotte's Spiderweb?, *Adv. Math. Models Appl.*, **5** (2020), 176–191. Available from: `http://jomardpublishing.com/UploadFiles/Files/journals/AMMAV1N1/V5N2/Napoles%20Valdes.pdf`
2. S. S. Dragomir, J. Pečarić, L.-E. Persson, Some inequalities of Hadamard type, *Soochow J. Math.*, **21** (1995), 335–341.
3. W. W. Breckner, Stetigkeitsaussagen für eine klasse verallgemeinerter konvexer funktionen in topologischen linearen Räumen, *Pub. Inst. Math.*, **23** (1978), 13–20. Available from: `http://elib.mi.sanu.ac.rs/files/journals/publ/43/3.pdf`
4. G. Toader, Some generalizations of the convexity, *Proc. Colloq. Approx. Optim. Cluj-Naploca (Romania)* (1984), 329–338.
5. V. G. Mihesan, A generalization of convexity, *Proceedings of the Seminar on Functional Equations, Approximation and Convexity*, Cluj-Napoca, Romania, 1993.
6. S. Varošanec, On h-convexity, *J. Math. Anal. Appl.*, **326** (2007), 303–311. https://doi.org/10.1016/j.jmaa.2006.02.086
7. M. Bessenyei, Z. Páles, On generalized higher-order convexity and Hermite–Hadamard-type inequalities, *Acta Sci. Math. (Szeged)*, **70** (2004), 13–24.
8. S. S. Dragomir, Inequalities of Hermite–Hadamard type for GA-convex functions, *Ann. Math. Sil.*, **32** (2018), 145–168. https://doi.org/10.2478/amsil-2018-0001
9. S. S. Dragomir, C. E. M. Pearce, *Selected Topics on Hermite-Hadamard Inequalities and Applications*, RGMIA Monographs, Victoria University, 2000. Available from: `https://rgmia.org/papers/monographs/Master.pdf`
10. T. S. Du, C. Y. Luo, Z. J. Cao, On the Bullen-type inequalities via generalized fractional integrals and their applications, *Fractals*, **29** (2021), no. 7, 2150188. https://doi.org/10.1142/S0218348X21501887
11. T. S. Du, Y. Peng, Hermite–Hadamard type inequalities for multiplicative Riemann–Liouville fractional integrals, *J. Comput. Appl. Math.* (2023), 115582. https://doi.org/10.1016/j.cam.2023.115582
12. T. S. Du, T. C. Zhou, On the fractional double integral inclusion relations having exponential kernels via interval-valued co-ordinated convex mappings, *Chaos Solitons Fractals*, **156** (2022), 111846. https://doi.org/10.1016/j.chaos.2022.111846
13. M. Klaričić, E. Neuman, J. Pečarić, V. Šimić, Hermite-Hadamard's inequalities for multivariate g-convex functions, *Math. Inequal. Appl.*, **8** (2005), no. 2, 305–316. https://doi.org/10.7153/mia-08-28
14. M. S. Moslehian, Matrix Hermite–Hadamard type inequalities, *Houston J. Math.*, **39** (2013), no. 1, 177–189. Available from: `arXiv:1203.5300`
15. J. E. Nápoles Valdés, J. M. Rodríguez, J. M. Sigarreta, New Hermite–Hadamard Type Inequalities Involving Non-Conformable Integral Operators, *Symmetry*, **11** (2019), 1108. https://doi.org/10.3390/sym11091108

16. T. C. Zhou, Z. R. Yuan, T. S. Du, On the fractional integral inclusions having exponential kernels for interval-valued convex functions, *Math. Sci.*, **17** (2023), 107–120. https://doi.org/10.1007/s40096-021-00445-x

17. E. D. Rainville, *Special Functions*, Macmillan Co., New York, 1960.

18. I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, 1999.

19. S. Mubeen, G. M. Habibullah, k-Fractional Integrals and Application, *Int. J. Contemp. Math. Sci.*, **7** (2012), no. 2, 89–94. Available from: `http://m-hikari.com/ijcms/ijcms-2012/1-4-2012/mubeenIJCMS1-4-2012-1.pdf`

20. P. O. Mohammed, M. Z. Sarıkaya, On generalized fractional integral inequalities for twice differentiable convex functions, *Int. J. Comput. Appl. Math.*, **372** (2020), 112740. https://doi.org/10.1016/j.cam.2020.112740

21. M. Tomar, E. Set, M. Z. Sarıkaya, Hermite-Hadamard type Riemann-Liouville fractional integral inequalities for convex functions, *AIP Conf. Proc.*, **1726** (2016), 020035. https://doi.org/10.1063/1.4945861

22. B. Bayraktar, Some new generalizations of Hadamard–type Midpoint inequalities involving fractional integrals, *Probl. Anal. Issues Anal.*, **9** (27), no. 3, (2020), 66–82. https://doi.org/10.15393/j3.art.2020.8270

23. M. Z. Sarıkaya, N. Aktan, On the generalization of some integral inequalities and their applications, *Math. Comput. Modelling*, **54** (2011), 2175 –2182. https://doi.org/10.1016/j.mcm.2011.05.026

24. M. Alomari, M. Darus, S. S. Dragomir, New inequalities of Hermite-Hadamard type for functions whose second derivatives absolute values are quasi-convex, *RGMIA Res. Rep. Coll.*, **12** (2009), Supplement, Article 17. Available from: `https://rgmia.org/papers/v12e/Correctedmdssd.pdf`

AIMS Press