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Research article

Stationary distribution and extinction of a stochastic influenza virus model with disease resistance

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Abstract: Influenza is a respiratory infection caused influenza virus. To evaluate the effect of environment noise on the transmission of influenza, our study focuses on a stochastic influenza virus model with disease resistance. We first prove the existence and uniqueness of the global solution to the model. Then we obtain the existence of a stationary distribution to the positive solutions by stochastic Lyapunov function method. Moreover, certain sufficient conditions are provided for the extinction of the influenza virus flu. Finally, several numerical simulations are revealed to illustrate our theoretical results. Conclusively, according to the results of numerical models, increasing disease resistance is favorable to disease control. Furthermore, a simple example demonstrates that white noise is favorable to the disease's extinction.

Keywords: stochastic influenza virus model; disease resistance; stationary distribution; extinction; ergodic property

1. Introduction

Influenza, usually known as the flu, is a virus-borne illness that mostly affects the throat, nose and lungs. It can be transferred from person to person by sneezes, contaminated air, or coughing, as well as through direct contact with infected people. Many authors have considered the mathematical models of the influenza virus [1–5]. In real life, there is an exposed time following infection transfer from susceptible to possibly infective person. However, these potentially infective people without any acquired symptoms are capable of spreading the flu. In 2016, Khanh [6] proposed a novel human virus transmission SEIR model with disease resistance. Briefly, the model, reports the recovery of an infected group without any prior therapy. Depending on the treatment, the other groups can be returned to the susceptible group. It describes the susceptible (S), the exposed (E), the infected (I) and

the recovered (R) as four unique epidemiological subgroups of persons in the overall population. A set of ordinary differential equations provides the model as

$$\begin{aligned}
&\tfrac{dS}{dt} = \Lambda - \tfrac{\beta S(E+I)}{N} + cE + bI + \alpha R - \mu S,\\
&\tfrac{dE}{dt} = \tfrac{\beta S(E+I)}{N} - (c+\epsilon+\mu)E,\\
&\tfrac{dI}{dt} = \epsilon E - (\gamma+b+\mu)I,\\
&\tfrac{dR}{dt} = \gamma I - (\alpha+\mu)R,\\
&N = S + E + I + R,
\end{aligned} \tag{1.1}$$

where Λ is the susceptible recruitment rate; α is the susceptibility constant rate for recovered individuals; the contact transmission rates of the virus is denoted by β; the susceptibility rate of individual exposed and infected people is expressed as c and b; γ is the persistent pace of recovery; ϵ is the continuous infectious proportion of the exposed population and μ is the population's natural mortality rate. Positive values are expected for all parameters. In System (1.1), $R_0 = \frac{\beta(\gamma+b+\mu+\epsilon)}{(\gamma+b+\mu)(c+\epsilon+\mu)}$ represents the basic reproduction number. Importantly, infection persists in the population when $R_0 > 1$ but finally dies off when $R_0 \leq 1$.

Epidemic models are powerful tools to understand the dynamics of the tramission of an infectious disease. There is a large number of researchers who pay attention to the study of epidemic models. For example, Allegretti et al. [7] derived an SIR model to describe the evolution in time of the infectious disease by Sars-Cov-2; Kumar and Erturk [8] considered a cholera epidemic model from the perspective of generalized Liouville-Caputo fractional derivatives; Özkše and Yavuz [9] established a fractional-order pandemic model to discuss interactions between COVID-19 and diabetes by using real data from Türkiye; Naik et al. [10] proposed and analyzed a fractional-order epidemic model with a classic Caputo operator and the Atangana-Balenu-Caputo operator for the transmisson during the COVID-19 epidemic; Yavuz and Özdemir [11] used an SIR model to simulate the transmission dynamics of diseases where individuals acquire permanent immunity. Besides, there are also several works devoted to population systems, such as [12–14] and their references.

The population system for the actual world is unavoidably influenced by ambient white noise. The population size may be changed greatly in a short time due to fuctuations in the environment, such as earthquakes and tsunamis. Hence, the parameters of the system may not be absolute constant, and they may fluctuate around some constants. Thus, incorporating environmental noise into the epidemic models seems to be a good way to describe these phenomena. Additionally, in population dynamics, stochastic differential equation model is one of the significant types models because they offer a more realistic description than deterministic models. Biological and epidemiological stochastic models have been examined by many researchers [15–19]. Therefore, in the current research, Imhof and Walcher's technique is followed under the assumption that ambient white noise is a function of $S(t)$, $R(t)$, $E(t)$ and $I(t)$. For sufficiently small Δt, one can model $X = (S, E, I, R)^T$ as a Markov process with the following specifications

$$\mathbb{E}[S(t+\Delta t) - S(t)|X = x] \approx \Big[\Lambda - \frac{\beta S(E+I)}{N} + cE + bI + \alpha R - \mu S\Big]\Delta t,$$

$$\mathbb{E}[E(t+\Delta t) - E(t)|X = x] \approx \Big[\frac{\beta S(E+I)}{N} - (c+\epsilon+\mu)E\Big]\Delta t,$$

$$\mathbb{E}[I(t+\Delta t) - I(t)|X = x] \approx [\epsilon E - (\gamma+b+\mu)I]\Delta t,$$

$$\mathbb{E}[R(t+\Delta t) - R(t)|X = x] \approx [\gamma I - (\alpha + \mu)R]\Delta t,$$

and

$$Var[S(t+\Delta t) - S(t)|X = x] \approx \sigma_1^2 S^2 \Delta t, Var[E(t+\Delta t) - E(t)|X = x] \approx \sigma_2^2 E^2 \Delta t,$$

$$Var[I(t+\Delta t) - I(t)|X = x] \approx \sigma_3^2 I^2 \Delta t, Var[R(t+\Delta t) - R(t)|X = x] \approx \sigma_4^2 R^2 \Delta t.$$

Following is the outline of the model,

$$\begin{aligned}
dS &= \left[\Lambda - \tfrac{\beta S(E+I)}{N} + cE + bI + \alpha R - \mu S\right]dt + \sigma_1 S\, dB_1(t),\\
dE &= \left[\tfrac{\beta S(E+I)}{N} - (c + \epsilon + \mu)E\right]dt + \sigma_2 E dB_2(t),\\
dI &= [\epsilon E - (\gamma + b + \mu)I]dt + \sigma_3 I dB_3(t),\\
dR &= [\gamma I - (\alpha + \mu)R]dt + \sigma_4 R dB_4(t),\\
N &= S + E + I + R,
\end{aligned} \tag{1.2}$$

where $\sigma_i^2 > 0 (i = 1, 2, 3, 4)$ signifies the intensity of the white noise. Brownian movements $B_i(t) (i = 1, 2, 3, 4)$ are defined on a complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ that meets the standard criteria of mutually independent i.e., while $\mathcal{F}_0$ includes all P-null sets, it is increasing and right continuous. Comparatively, the parameters of Systems (1.1) and (1.2) imply the same meaning. And $\mathbb{R}_+^4 = \{x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4; x_i > 0, i = 1, 2, 3, 4\}$ in this paper. We also need to assume that these variables $\Lambda, \beta, c, b, \alpha, \mu, \epsilon$ and γ should be positive constants.

We should ponit out that Ikram et al. [20] considered a stochastic epidemic model consisting of four human classes, i.e., susceptible, infected, vaccinated and removed. They established the global dynamcis of the model in terms of the stochastic basic reproduction number R_0^s and used a stochastic Runge-Kutta method to implement the numerical simulations. In real life, there is an exposed period after the transmission of infection from susceptible to potentially infective members but before these potential infectives develop symptoms and can transmit infection. Due to this reason, the method in [20] cannot be applied to System (1.2). Thus, we need to construct a new auxiliary function to consider this system. On the other hand, a stochastic model may not have the positive equilibrium state. Therefore, we hope to study the existence and stability of "stochastic positive equilibrium" and the existence of a stationary distribution, for this stochastic influenza virus model. As we all know, there are no results about this system. Since the model described by System (1.2) consists of four equations, it is very difficult to construct the Lyapunov function if we consider the existence of an ergodic distribution. Thus, we need to construct a new Lyapunov function and a rectangular set. We also should note that the conditions for extinction of the influenza virus have a close relationship with the basic reproduction ratio R_0 in System (1.1).

The structure of the paper is outlined below. In Section 2, we show that there exists a unique global positive solution of System (1.2) with initial value $(S_0, E_0, I_0, R_0) \in \mathbb{R}_+^4$. The existence of ergodic stationary distribution of System (1.2) is established in Section 3. These conclusions can be generated by constructing a suitable stochastic Lyapunov function and a rectangular set. Furthermore, the sufficient condition for the extinction of disease can be found in Section 4. In Section 5, numerical simulations involving Milstein's order method are introduced to illustrate our theoretical results. To wrap up the article, certain conclusions are offered.

2. Existence and uniqueness of the positive solution

Investigating an epidemic model's dynamic behavior begins with determining whether or not the solution is both positive and global. In this section, inspired by the methods presented in [21], we demonstrate that there exsits a solution for System (1.2) that is a unique, global positive one.

Focus on a stochastic differential equation with an initial value of $z(0) = z_0 \in \mathbb{R}^l$ in an l-dimensional, i.e.,

$$dz(t) = f(z(t), t)dt + g(z(t), t)dB(t) \text{ on } t \geq t_0, \tag{2.1}$$

where $f \in L^1(\mathbb{R}^l \times \mathbb{R}_+, \mathbb{R}^l)$ and $g \in L^1(\mathbb{R}^l \times \mathbb{R}_+, \mathbb{R}^{l\times m})$. Equation (2.1) is connected with a differential operator L, which is defined as follows

$$L = \frac{\partial}{\partial t} + \sum_{i=1}^{l} f_i(z,t)\frac{\partial}{\partial z_i} + \frac{1}{2}\sum_{i,j=1}^{l}[g^T(z,t)g(z,t)]_{ij}\frac{\partial^2}{\partial z_i \partial z_j}.$$

Theorem 2.1. *There exists a unique solution* $(I(t), E(t), R(t), S(t))$ *to System* (1.2) *for each given initial value* $(S_0, E_0, I_0, R_0) \in \mathbb{R}^4_+$*, and this solution will persist with a probablity of* 1*, i.e., the solution will remain in* $\mathbb{R}^4_+$ *almost surely (a.s.).*

Proof. Because the local Lipschitz condition is met by the coefficients of System (1.2), there exists a unique local solution $(S(t), E(t), I(t), R(t))$ on $t \in [0, \tau_e)$ for every initial value $(S_0, E_0, I_0, R_0) \in \mathbb{R}^4_+$, while τ_e is the explosion time. To show that this solution is global, we only need to verify $\tau_e = \infty$ a.s.. To begin, we demonstrate that $S(t)$, $E(t)$, $I(t)$ and $R(t)$ do not explode to infinity in a limited period. Assume that $S(0)$, $R(0)$, $E(0)$ and $I(0)$ all fall inside the interval $[\frac{1}{k_0}, k_0]$ when $k \geq k_0$. For each integer value $k \geq k_0$, we determine the stopping time to be

$$\tau_k = \inf\{t \in [0, \tau_e) : \min\{(S(t), E(t), I(t), R(t)\} \leq \frac{1}{k} \text{or} \max\{S(t), E(t), I(t), R(t)\} \geq k\}.$$

We have specified $\inf \emptyset = \infty$ throughout this work (as is customary, $\emptyset$ signifies an empty set). Clearly, as a result, when $k \to \infty$, τ_k is increasing. Denote $\tau_\infty = \lim_{k\to\infty} \tau_k$ a.s. There exist two constants $T > 0$ and $\epsilon \in (0, 1)$ such that if this assumption is false, $\mathbb{P}\{\tau_\infty \geq T\} > \epsilon$. Hence, there exists an integer $k_1 \geq k_0$ satisfying

$$\mathbb{P}\{\tau_k \leq T\} \geq \epsilon, \ \forall\, k \geq k_1. \tag{2.2}$$

Let a C^2-function $V : \mathbb{R}^4_+ \to \mathbb{R}_+$ be defined by

$$V(S, E, I, R) = (S - 1 - \ln S) + (E - 1 - \ln E) + (I - 1 - \ln I) + (R - 1 - \ln R).$$

For any $u > 0$, this function's nonnegativity may be determined by using $u - 1 - \ln u \geq 0$. It does not matter what number is in $k \geq k_0$ and $T > 0$. Itô's formula may be applied to $V(S, E, I, R)$; then

$$\begin{aligned} dV(S, E, I, R) = LV(S, E, I, R)dt + \sigma_1(S - 1)dB_1(t) + \sigma_2(E - 1)dB_2(t) \\ + \sigma_3(I - 1)dB_3(t) + \sigma_4(R - 1)dB_4(t), \end{aligned}$$

where $LV : \mathbb{R}^4_+ \to \mathbb{R}$ and

$$\begin{aligned} LV(S,E,I,R) =&(1-\frac{1}{S})[A-\frac{\beta S(E+I)}{N}+cE+bI+\alpha R-\mu S]+\frac{1}{2}\sigma_1^2\\ &+(1-\frac{1}{E})[\frac{\beta S(E+I)}{N}-(c+\epsilon+\mu)E]+\frac{1}{2}\sigma_2^2\\ &+(1-\frac{1}{I})[\epsilon E-(\gamma+b+\mu)I]+\frac{1}{2}\sigma_3^2\\ &+(1-\frac{1}{R})[\gamma I-(\alpha+\mu)R]+\frac{1}{2}\sigma_4^2\\ &=\Lambda-\frac{\Lambda}{S}-\mu S-\frac{cE}{S}-\frac{bI}{S}-\frac{\alpha aR}{S}+\beta\frac{S(E+I)}{N}-\mu E\\ &-\frac{\beta S}{NE}(E+I)-\mu I-\frac{\epsilon E}{I}-\mu R-\frac{\gamma I}{R}\\ &+c+\epsilon+\gamma+b+\alpha+4\mu+\frac{1}{2}\sigma_1^2+\frac{1}{2}\sigma_2^2+\frac{1}{2}\sigma_3^2+\frac{1}{2}\sigma_4^2. \end{aligned}$$

Then

$$LV(S,E,I,R)\le\Lambda+c+\epsilon+\gamma+b+\alpha+4\mu+\frac{1}{2}\sigma_1^2+\frac{1}{2}\sigma_2^2+\frac{1}{2}\sigma_3^2+\frac{1}{2}\sigma_4^2:=K,$$

where K is a positive constant in this equation. As a result, we can get

$$dV(S,E,I,R)\le Kdt+\sigma_1^2S\,dB_1(t)+\sigma_2^2E dB_2(t)+\sigma_3^2I dB_3(t)+\sigma_4^2R dB_4(t). \tag{2.3}$$

Integrating Eq (2.3) from 0 to $\tau_k\wedge T$, we apply the mathematical expectation

$$\mathbb{E}V(S(\tau_k\wedge T),E(\tau_k\wedge T),I(\tau_k\wedge T),R(\tau_k\wedge T))\le V(S_0,E_0,I_0,R_0)+K\mathbb{E}(\tau_k\wedge T).$$

Thus,

$$\mathbb{E}V(S(\tau_k\wedge T),E(\tau_k\wedge T),I(\tau_k\wedge T),R(\tau_k\wedge T))\le V(S_0,E_0,I_0,R_0)+KT. \tag{2.4}$$

Setting $\Omega_k=\{\tau_k\le T\}$ for $k\ge k_1$ and by virtue of Eq (2.2), we get $\mathbb{P}(\Omega_k)\ge\epsilon$. It is worth noting that for each $\omega\in\Omega_k$, there exists $S(\tau_k,\omega)$, $I(\tau_k,\omega)$, $E(\tau_k,\omega)$ or $R(\tau_k,\omega)$ that is equal to either $\frac{1}{k}$ or k. As a result, either $k-1-\ln k$ or $\frac{1}{k}-1-\ln\frac{1}{k}=\frac{1}{k}-1+\ln k$ is no more than $V(S(\tau_k,\omega),E(\tau_k,\omega),I(\tau_k,\omega),R(\tau_k,\omega))$. As a result, we may acquire

$$(S(\tau_k,\omega),E(\tau_k,\omega),I(\tau_k,\omega),R(\tau_k,\omega))\le[k-1-\ln k]\wedge[\frac{1}{k}-1+\ln k].$$

It then follows from Eq (2.4) that

$$\begin{aligned} V(S_0,E_0,I_0,R_0)+KT&\ge\mathbb{E}[I_{\Omega_k}(\omega)V(S(\tau_k,\omega),E(\tau_k,\omega),I(\tau_k,\omega),R(\tau_k,\omega))]\\ &\ge\epsilon[k-1-\ln k]\wedge[\frac{1}{k}-1+\ln k], \end{aligned}$$

where the indicator function of Ω_k , i.e., I_{Ω_k}, is inconsistent with what we have allowed. Letting $k\to\infty$,

$$\infty>V(S_0,E_0,I_0,R_0))+KT=\infty.$$

Thus, we have a contradiction. Finally, we have $\tau_\infty=\infty$, and $S(t)$, $I(t)$, $R(t)$ and $E(t)$ do not seem to be exploding in a limited period.

□

3. Stationary distribution

Eventually, it becomes important to know when an epidemic dynamical system will continue and propagate through a population. Using deterministic models, this issue may be resolved by considering evidence indicating that the model's global attractor or globally asymptotically stable endemic equilibrium exists. On the other hand, System (1.2) may not have an endemic equilibrium. Considering the hypothesis of Khasminskii [22], we demonstrate that the disease is likely to persist in the mean that there is a stationary distribution. We shall go over a theory introduced in [22] on stationary distributions. The following stochastic differential equation in D-dimensional Euclidean space defines $X(t)$ as a Markov homogeneous process in D-dimensional Euclidean space E_d (E_d denotes d-dimensional Euclidean space)

$$dX(t) = b(X)dt + \sum_{r=1}^{k} g_r(X)dB_r(t),$$

where $b \in L^1(\mathbb{R}^d, \mathbb{R}^d)$ and $g_r \in L^2(\mathbb{R}^d, \mathbb{R}^d)$, $r = 1, 2, \cdots, k$. And, following is the definition of the diffusion matrix,

$$A(x) = (a_{i,j}(x))_{n\times n},\ a_{i,j}(x) = \sum_{r=1}^{k} g_r^i(x)g_r^j(x).$$

Lemma 3.1. [22, Chapter 4.3] *The Markov process $X(t)$ has a unique ergodic stationary distribution μ if there is a bound open set $D \subset E^d$ that has a regular boundary Γ and the following conditions:*
(1) *there exists a positive number M s.t. $\sum_{i,j=1}^{d} a_{i,j}(x)\xi_i\xi_j \geq M\|\xi\|^2, x \in D, \xi \in R^d$.*
(2) *for any $E_d\backslash D$, there is a nonnegative C^2-function V s.t. LV is negative.*
Let f be an integrable function that is integrable with respect to the measure $\mu(\cdot)$; then,

$$\mathbb{P}_x\{\lim_{T\to+\infty} \frac{1}{T}\int_0^T f(X(t))dt = \int_{E_d} f(x)\mu(dx)\} = 1.$$

We will give a definition of the parameter $\hat{R}_0^s$,

$$\hat{R}_0^s = \frac{\beta\mu(\gamma + \mu + b + \epsilon + \frac{\sigma_3^2}{2})}{(\mu + \frac{\sigma_1^2}{2})(\gamma + \mu + b + \frac{\sigma_3^2}{2})(c + \epsilon + \mu + \frac{\sigma_2^2}{2})}.$$

Theorem 3.2. *Consider that $\hat{R}_0^s > 1$ and $(S_0, E_0, I_0, R_0) \in R_+^4$, there exists a unique stationary distribution $\mu(\cdot)$ for System* (1.2) *that has the ergodic property.*

Proof. It then follows from Theorem 2.1 that there exists a unique global solution $(S(t), E(t), I(t), R(t)) \in \mathbb{R}_+^4$ with the initial value of $(S_0, E_0, I_0, R_0) \in \mathbb{R}_+^4$. We simply need to confirm Conditions (1) and (2) in Lemma 3.1 to prove this theorem. Now, we shall verify the situation given by Condition (1). The diffusion matrix of System (1.2) is provided by

$$H_x = \begin{bmatrix} \sigma_1^2 S^2 & 0 & 0 & 0 \\ 0 & \sigma_2^2 E^2 & 0 & 0 \\ 0 & 0 & \sigma_3^2 I^2 & 0 \\ 0 & 0 & 0 & \sigma_4^2 R^2 \end{bmatrix}. \tag{3.1}$$

Selecting $M = \min\limits_{(S,E,I,R)\in D_k\subset\mathbb{R}^4_+}\{\sigma_1^2S^2, \sigma_2^2E^2, \sigma_3^2I^2, \sigma_4^2R^2\}$, we get

$$\sum_{i,j=1}^{4} a_{ij}(S,E,I,R)\xi_i\xi_j = \sigma_1^2S^2\xi_1^2 + \sigma_2^2E^2\xi_2^2 + \sigma_3^2I^2\xi_3^2 + \sigma_4^2R^2\xi_4^2 \geq M\|\xi\|^2,$$

$$\forall (S,E,I,R) \in D_\delta, \xi = (\xi_1,\xi_2,\xi_3,\xi_4) \in \mathbb{R}^4,$$

where $D_k = [k, \frac{1}{k}] \times [k, \frac{1}{k}] \times [k, \frac{1}{k}] \times [k, \frac{1}{k}]$. Then, Condition (1) of Lemma 3.1 is satisfied.

Create a C^2-function $Q : \mathbb{R}^4_+ \to \mathbb{R}$ as follows:

$$\begin{aligned} Q(S,E,I,R) = M[&(-c_1\ln S - c_2\ln E - c_3\ln I + c_4(S+E+I+R) - c_5\ln S \\ &+ c_6(S+E+I+R)] - \ln S - \ln E - \ln I - \ln R + (S+E+I+R) \\ &+ \frac{1}{\theta+1}(S+E+I+R)^{(\theta+1)}, \end{aligned}$$

where $0 < \theta < \frac{2\mu}{\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2\vee\sigma_4^2}$,

$$c_1 = \frac{\beta\epsilon\mu\Lambda}{(\mu+\frac{\sigma_1^2}{2})^2(\mu+\gamma+\epsilon+\frac{\sigma_2^2}{2})}, c_2 = \Lambda, c_3 = \frac{\beta\epsilon\mu\Lambda}{(\mu+\frac{\sigma_1^2}{2})(\mu+\gamma+\epsilon+\frac{\sigma_2^2}{2})^2},$$

$$c_4 = \frac{\beta\epsilon\mu}{(\mu+\frac{\sigma_1^2}{2})(\mu+\gamma+\epsilon+\frac{\sigma_2^2}{2})^2}, c_5 = \frac{\beta\mu\Lambda}{(\mu+\frac{\sigma_1^2}{2})^2}, c_6 = \frac{\beta\mu}{\mu+\frac{\sigma_1^2}{2}}.$$

Furthermore, selecting $M > 0$ fulfills the following requirement

$$-M\Lambda(\mu+\gamma+\epsilon+\frac{\sigma_3^2}{2})(R_0^s - 1) + G < -2, \tag{3.2}$$

where

$$G = 2\beta + 4\mu + c + \epsilon + b + \gamma + \alpha + \frac{\sigma_1^2}{2} + \frac{\sigma_2^2}{2} + \frac{\sigma_3^2}{2} + \frac{\sigma_4^2}{2} + B + \Lambda.$$

It is simple to verify this

$$\liminf_{k\to\infty,(S,E,I,R)\in\mathbb{R}^4_+\setminus U_k} Q(S,E,I,R) = \infty,$$

where $U_k = (\frac{1}{k},k)\times(\frac{1}{k},k)\times(\frac{1}{k},k)\times(\frac{1}{k},k)$. Additionally, $Q(S,E,I,R)$ is a continuous function. This means that just a minimum point (S_0,E_0,I_0,R_0) which is in the interior of $\mathbb{R}^4_+$ is essential for $Q(S,E,I,R)$. Thus a nonnegative C^2-function $V : \mathbb{R}^4_+ \to \mathbb{R}^4_+$ is defined by

$$V(S,E,I,R) = Q(S,E,I,R) - Q(S_0,E_0,I_0,R_0).$$

If $V_1(S,E,I,R) = -c_1 \ln S - c_2 \ln E - c_3 \ln I + c_4(S+E+I+R) - c_5 \ln S + c_6(S+E+I+R)$, it holds

$$\begin{aligned}
LV_1(S,E,I,R) &= -\frac{c_1\Lambda}{S} - c_2\frac{\beta S I}{NE} - c_3\epsilon\frac{E}{I} - c_4\mu N + c_1\Big(\mu + \frac{\sigma_1^2}{2}\Big) \\
&\quad + c_2\Big(\mu+\epsilon+c+\frac{\sigma_2^2}{2}\Big) + c_3\Big(\mu+\gamma+b+\frac{\sigma_3^2}{2}\Big) + c_4\Lambda - c_5\frac{\Lambda}{S} \\
&\quad - c_2\frac{\beta S}{N} - c_6\mu N + c_5\Big(\mu+\frac{\sigma_1^2}{2}\Big) + c_6\Lambda + \beta(c_1+c_5)\frac{(E+I)}{N} \\
&\quad - c_1 c\frac{E}{S} - c_1 b\frac{I}{S} - c_1\alpha\frac{R}{S} \\
&\le -4\Big(c_1\Lambda c_2\beta c_3 c_4\mu\epsilon\Big)^{\frac{1}{4}} - 3\Big(c_2c_5\Lambda\beta c_6\mu\Big)^{\frac{1}{3}} + c_1\Big(\mu+\frac{\sigma_1^2}{2}\Big) \\
&\quad + c_2\Big(\mu+\epsilon+c+\frac{\sigma_2^2}{2}\Big) + c_3\Big(\mu+\gamma+b+\frac{\sigma_3^2}{2}\Big) + c_4\Lambda + c_5\Big(\mu+\frac{\sigma_1^2}{2}\Big) \\
&\quad + c_6\Lambda + (c_1+c_5)\frac{\beta(E+I)}{N} \\
&= -\Lambda\Big(\mu+\gamma+b+\frac{\sigma_3^2}{2}\Big)(R_0^s - 1) + (c_1+c_5)\frac{\beta(E+I)}{N}.
\end{aligned}$$

If $V_2(S,E,I,R) = -\ln S - \ln E - \ln I - \ln R$, it holds that

$$\begin{aligned}
LV_2(S,E,I,R) &= -\frac{\Lambda}{S} + \beta\frac{E+I}{N} - c\frac{E}{S} - b\frac{I}{S} - \alpha\frac{R}{S} + \mu + \frac{\sigma_1^2}{2} - \frac{\beta S(E+I)}{EN} \\
&\quad + c + \mu + \epsilon + \frac{\sigma_2^2}{2} - \epsilon\frac{E}{I} + \gamma + b + \mu + \frac{\sigma_3^2}{2} - \gamma\frac{I}{R} + \alpha + \mu + \frac{\sigma_4^2}{2} \\
&\le -\frac{\Lambda}{S} - \frac{\beta S(E+I)}{EN} - \epsilon\frac{E}{I} - \gamma\frac{I}{R} + 2\beta + 4\mu + c + \epsilon + b + \gamma + \alpha \\
&\quad + \frac{\sigma_1^2}{2} + \frac{\sigma_2^2}{2} + \frac{\sigma_3^2}{2} + \frac{\sigma_4^2}{2}.
\end{aligned}$$

If $V_3(S,E,I,R) = S+E+I+R$, it holds that

$$LV_3(S,E,I,R) = \Lambda - \mu(S+E+I+R).$$

If $V_4(S,E,I,R) = \frac{1}{\theta+1}(S+E+I+R)^{\theta+1}$, it holds that

$$\begin{aligned}
LV_4(S,E,I,R) &= (S+E+I+R)^{\theta}[\Lambda - \mu(S+E+I+R)] \\
&\quad + \frac{1}{2}\theta(S+E+I+R)^{(\theta-1)} \times (\sigma_1^2 S^2 + \sigma_2^2 E^2 + \sigma_3^2 I^2 + \sigma_4^2 R^2) \\
&\le (S+E+I+R)^{\theta}[\Lambda - \mu(S+E+I+R)] \\
&\quad + \frac{1}{2}\theta(S+E+I+R)^{\theta+1}(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2) \\
&= \Lambda(S+E+I+R)^{\theta} - [\mu - \frac{1}{2}\theta(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2)](S+E+I+R)^{\theta+1} \\
&\le B - \frac{1}{2}[\mu - \frac{1}{2}\theta(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2)](S+E+I+R)^{\theta+1} \\
&\le B - \frac{1}{2}[\mu - \frac{1}{2}\theta(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2)](S^{\theta+1} + E^{\theta+1} + I^{\theta+1} + R^{\theta+1}),
\end{aligned}$$

where

$$B = \sup_{(S,E,I,R)\in R_+^4} \{\Lambda(S+E+I+R)^\theta - \frac{1}{2}[\mu - \frac{1}{2}\theta(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2)(S+E+I+R)^{\theta+1}]\} < \infty.$$

Hence, we have

$$\begin{aligned}
LV(S,E,I,R) &\le -M[\Lambda(\mu+\gamma+b+\frac{\sigma_3^2}{2})(R_0^s-1)] + M(c_1+c_5)\frac{\beta(E+I)}{N} \\
&- \frac{\Lambda}{S} - \frac{\beta S(E+I)}{EN} - \epsilon\frac{E}{I} - \gamma\frac{I}{R} + 2\beta + 4\mu + c + \epsilon + b + \gamma + \alpha \\
&+ \frac{\sigma_1^2}{2} + \frac{\sigma_2^2}{2} + \frac{\sigma_3^2}{2} + \frac{\sigma_4^2}{2} + \Lambda - \mu(S+E+I+R) + B \\
&- \frac{1}{2}[\mu - \frac{1}{2}\theta(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2)](S^{\theta+1} + E^{\theta+1} + I^{\theta+1} + R^{\theta+1}) \\
&\le -M\Lambda(\mu+d+r+\frac{\sigma_3^2}{2})(R_0^s-1) + \beta[M(c_1+c_5)+1]\frac{E+I}{N} \\
&- 2(\frac{\mu\beta S(E+I)}{E})^{\frac{1}{2}} - \frac{\Lambda}{S} - \epsilon\frac{E}{I} - \gamma\frac{I}{R} + 2\beta + 4\mu + c + \epsilon + b + \gamma \\
&+ \alpha + \frac{\sigma_1^2}{2} + \frac{\sigma_2^2}{2} + \frac{\sigma_3^2}{2} + \frac{\sigma_4^2}{2} + \Lambda + B \\
&- \frac{1}{2}[\mu - \frac{1}{2}\theta(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2)](S^{\theta+1} + E^{\theta+1} + I^{\theta+1} + R^{\theta+1}).
\end{aligned}$$

Now, we will prove the Condition (2) holds. Set

$$\begin{aligned}
F = \sup_{(S,E,I,R)\in R_+^4} &\{\beta[M(c_1+c_5)+1]\frac{I}{N} + \mu + \frac{\sigma_1^2}{2} + 3\mu + k + \Lambda + \frac{\sigma_2^2}{2} + \frac{\sigma_4^2}{2} + B \\
&- \frac{1}{2}[\mu - \frac{1}{2}\theta(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2)](S^{\theta+1} + E^{\theta+1} + I^{\theta+1} + R^{\theta+1})\} < \infty,
\end{aligned}$$

$$L = \sup_{(S,E,I,R)\in R_+^4} \{\beta[M(c_1+c_5)+1]\frac{I}{N} + \mu + \frac{\sigma_1^2}{2} + 3\mu + k + \Lambda + \frac{\sigma_2^2}{2} + \frac{\sigma_4^2}{2} + B\}.$$

Let

$$D = \{\epsilon_1 \le S \le \frac{1}{\epsilon_1},\ \epsilon_2 \le I \le \frac{1}{\epsilon_2},\ \epsilon_3 \le E \le \frac{1}{\epsilon_3},\ \epsilon_4 \le R \le \frac{1}{\epsilon_4}\},$$

To make it easier, we've divided $\mathbb{R}_+^4 \backslash D$ into eight domains

$$\begin{aligned}
&D_1 = \{(S,E,I,R) \in \mathbb{R}_+^4 : 0 < S < \epsilon_1\},\ D_2 = \{(S,E,I,R) \in \mathbb{R}_+^4 : 0 < I < \epsilon_2, S \ge \epsilon_1\}, \\
&D_3 = \{(S,E,I,R) \in \mathbb{R}_+^4 : S \ge \epsilon_1, I \ge \epsilon_2, 0 < E < \epsilon_3\}, \\
&D_4 = \{(S,E,I,R) \in \mathbb{R}_+^4 : 0 < R < \epsilon_4, I \ge \epsilon_2\}, \\
&D_5 = \{(S,E,I,R) \in \mathbb{R}_+^4 : S > \frac{1}{\epsilon_1}\},\ D_6 = \{(S,E,I,R) \in \mathbb{R}_+^4 : I > \frac{1}{\epsilon_2}\}, \\
&D_7 = \{(S,E,I,R) \in \mathbb{R}_+^4 : E > \frac{1}{\epsilon_3}\},\ D_8 = \{(S,E,I,R) \in \mathbb{R}_+^4 : R > \frac{1}{\epsilon_4}\},
\end{aligned}$$

where $\epsilon_i (i = 1, 2, 3, 4)$ are sufficiently small positive constant factors that fulfill the following conditions:

$$-\frac{\Lambda}{\epsilon_1} + F \leq -1, \tag{3.3}$$

$$\epsilon_2 = \epsilon_1^2, \frac{\beta[M(c_1 + c_3) + 1]}{\epsilon_1} - \frac{\epsilon}{\epsilon_2} < 0, \tag{3.4}$$

$$-M\Lambda(\mu + d + r + \frac{\sigma_3^2}{2})(R_0^s - 1) + \beta[M(c_1 + c_3) + 1]\epsilon_1 + G < -1, \tag{3.5}$$

$$\epsilon_3 = \epsilon_1^4, \ -2(\frac{\mu\beta}{\epsilon_1}) + F < -1, \tag{3.6}$$

$$\epsilon_4 = \epsilon_1^3, \ -\gamma\frac{1}{\epsilon_1} + F < -1, \tag{3.7}$$

$$-\frac{1}{4}[\mu - \frac{1}{2}\theta(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2)]\frac{1}{\epsilon_1^{\theta+1}} + L < -1, \tag{3.8}$$

$$-\frac{1}{4}[\mu - \frac{1}{2}\theta(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2)]\frac{1}{\epsilon_1^{2(\theta+1)}} + L < -1, \tag{3.9}$$

$$-\frac{1}{4}[\mu - \frac{1}{2}\theta(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2)]\frac{1}{\epsilon_1^{3(\theta+1)}} + L < -1, \tag{3.10}$$

$$-\frac{1}{4}[\mu - \frac{1}{2}\theta(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2)]\frac{1}{\epsilon_1^{4(\theta+1)}} + L < -1. \tag{3.11}$$

Then, we show that $\forall(S, E, I, R) \in \mathbb{R}_+^4 \backslash D$, $LV(S, E, I, R) \leq -1$, which is comparable to demonstrating it on the above eight domains.

Case 1. If $(S, E, I, R) \in D_1$, then one has

$$\begin{aligned} LV(S, E, I, R) \leq & -\frac{\Lambda}{S} + \beta[M(c_1 + c_5) + 1]\frac{E + I}{N} + 2\beta + 4\mu + c + \epsilon + b + \gamma \\ & + \alpha + \frac{\sigma_1^2}{2} + \frac{\sigma_2^2}{2} + \frac{\sigma_3^2}{2} + \frac{\sigma_4^2}{2} + \Lambda + B \\ & - \frac{1}{2}[\mu - \frac{1}{2}\theta(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2)](S^{\theta+1} + E^{\theta+1} + I^{\theta+1} + R^{\theta+1}) \\ & \leq -\frac{\Lambda}{S} + F \leq -\frac{\Lambda}{\epsilon_1} + F. \end{aligned}$$

From Eq (3.3), $\forall(S, E, I, R) \in D_1 and LV(S, E, I, R) \leq -1$.

Case 2. If $(S, E, I, R) \in D_2$, then one has

$$\begin{aligned} LV(S,E,I,R) &\le -M\Lambda(\mu + d + r + \frac{\sigma_3^2}{2})(R_0^s - 1) + \beta[M(c_1 + c_5) + 1]\frac{E+I}{N} + \Lambda \\ &\quad - \epsilon\frac{E}{I} + 2\beta + 4\mu + c + \epsilon + b + \gamma + \alpha + \frac{\sigma_1^2}{2} + \frac{\sigma_2^2}{2} + \frac{\sigma_3^2}{2} + \frac{\sigma_4^2}{2} + B \\ &\quad - \frac{1}{2}[\mu - \frac{1}{2}\theta(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2)](S^{\theta+1} + E^{\theta+1} + I^{\theta+1} + R^{\theta+1}) \\ &\le -M\Lambda(\mu + d + r + \frac{\sigma_3^2}{2})(R_0^s - 1) + \beta[M(c_1 + c_3) + 1]\frac{\epsilon_2}{\epsilon_1} + G \\ &\quad + \Big\{\frac{\beta[M(c_1 + c_3) + 1]}{\epsilon_1} - \frac{\epsilon}{\epsilon_2}\Big\}E. \end{aligned}$$

Based on Eqs (3.4) and (3.5), we have

$$LV(S,E,I,R) \le -M(R_0^s - 1) + \beta[M(c_1 + c_3) + 1]\epsilon_2 + G \le -1, \forall (S,E,I,R) \in D_2.$$

Case 3. If $(S, E, I, R) \in D_3$, then one has

$$\begin{aligned} LV(S,E,I,R) &\le -2\Big(\frac{\mu\beta S I}{E}\Big)^{\frac{1}{2}} + \beta[M(c_1 + c_5) + 1]\frac{E+I}{N} + \Lambda \\ &\quad + 2\beta + 4\mu + c + \epsilon + b + \gamma + \alpha + \frac{\sigma_1^2}{2} + \frac{\sigma_2^2}{2} + \frac{\sigma_3^2}{2} + \frac{\sigma_4^2}{2} + B \\ &\quad - \frac{1}{2}\Big[\mu - \frac{1}{2}\theta(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2)\Big](S^{\theta+1} + E^{\theta+1} + I^{\theta+1} + R^{\theta+1}) \\ &\le -2\Big(\frac{\mu\beta S I}{E}\Big)^{\frac{1}{2}} + F \le -2\Big(\frac{\mu\beta\epsilon_1\epsilon_2}{\epsilon_3}\Big)^{\frac{1}{2}} + F. \end{aligned}$$

From Eq (3.6), $LV(S,E,I,R) \le -2\Big(\frac{\mu\beta}{\epsilon_1}\Big)^{\frac{1}{2}} + F < -1, \forall (S,E,I,R) \in D_3$.
Case 4. If $(S, E, I, R) \in D_4$, then one has

$$\begin{aligned} LV(S,E,I,R) &\le -\gamma + \frac{I}{R} + \beta[M(c_1 + c_5) + 1]\frac{E+I}{N} - 2\Big(\frac{\mu\beta S I}{E}\Big)^{\frac{1}{2}} \\ &\quad - \frac{1}{2}\Big[\mu - \frac{1}{2}\theta(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2)\Big](S^{\theta+1} + E^{\theta+1} + I^{\theta+1} + R^{\theta+1}) \\ &\quad + \frac{\sigma_1^2}{2} + 3\mu + c + \epsilon + \Lambda + \alpha + \frac{\sigma_2^2}{2} + \frac{\sigma_4^2}{2} + B \\ &\le -\gamma\frac{I}{R} + F \le -\gamma\frac{\epsilon_2}{\epsilon_4} + F. \end{aligned}$$

It then follows from Eq (3.7) that $LV(S,E,I,R) \le -\frac{\gamma}{\epsilon_1} + F < -1, \forall (S,E,I,R) \in D_4$.

Case 5. If $(S, E, I, R) \in D_5$, then one has

$$\begin{aligned} LV_(S, E, I, R) &\leq \beta[M(c_1 + c_5) + 1]\frac{E+I}{N} + \Lambda + 2\beta + 4\mu + c + \epsilon + b + \gamma + \alpha \\ &+ \frac{\sigma_1^2}{2} + \frac{\sigma_2^2}{2} + \frac{\sigma_3^2}{2} + \frac{\sigma_4^2}{2} + B - \frac{1}{2}\Big[\mu - \frac{1}{2}\theta(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2)\Big]S^{\theta+1} \\ &\leq -\frac{1}{2}\Big[\mu - \frac{1}{2}\theta(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2)\Big]S^{\theta+1} + L \\ &\leq -\frac{1}{2}\Big[\mu - \frac{1}{2}\theta(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2)\Big]\frac{1}{\epsilon_1^{\theta+1}} + L. \end{aligned}$$

It then follows from Eq (3.8) that $LV(S, E, I, R) < -1, \forall (S, E, I, R) \in D_5$.
Case 6. If $(S, E, I, R) \in D_6$, then one has

$$\begin{aligned} LV(S, E, I, R) &\leq \beta[M(c_1 + c_5) + 1]\frac{E+I}{N} + \Lambda + 2\beta + 4\mu + c + \epsilon + b + \gamma + \alpha \\ &+ \frac{\sigma_1^2}{2} + \frac{\sigma_2^2}{2} + \frac{\sigma_3^2}{2} + \frac{\sigma_4^2}{2} + B - \frac{1}{2}\Big[\mu - \frac{1}{2}\theta(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2)\Big]E^{\theta+1} \\ &\leq -\frac{1}{2}\Big[\mu - \frac{1}{2}\theta(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2)\Big]E^{\theta+1} + L \\ &\leq -\frac{1}{2}\Big[\mu - \frac{1}{2}\theta(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2)\Big]\frac{1}{\epsilon_1^{2(\theta+1)}} + L. \end{aligned}$$

According to Eq (3.9), $LV(S, E, I, R) < -1, \forall (S, E, I, R) \in D_6$.
Case 7. If $(S, E, I, R) \in D_7$, then one has

$$\begin{aligned} LV(S, E, I, R) &\leq \beta[M(c_1 + c_5) + 1]\frac{E+I}{N} + \Lambda + 2\beta + 4\mu + c + \epsilon + b + \gamma + \alpha \\ &+ \frac{\sigma_1^2}{2} + \frac{\sigma_2^2}{2} + \frac{\sigma_3^2}{2} + \frac{\sigma_4^2}{2} + B - \frac{1}{2}\Big[\mu - \frac{1}{2}\theta(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2)\Big]I^{\theta+1} \\ &\leq -\frac{1}{2}\Big[\mu - \frac{1}{2}\theta(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2)\Big]I^{\theta+1} + L \\ &\leq -\frac{1}{2}\Big[\mu - \frac{1}{2}\theta(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2)\Big]\frac{1}{\epsilon_1^{4(\theta+1)}} + L. \end{aligned}$$

From Eq (3.11), $LV(S, E, I, R) < -1, \forall (S, E, I, R) \in D_7$.
Case 8. If $(S, E, I, R) \in D_8$, then one has

$$\begin{aligned} LV(S, E, I, R) &\leq \beta[M(c_1 + c_5) + 1]\frac{E+I}{N} + \Lambda + 2\beta + 4\mu + c + \epsilon + b + \gamma + \alpha \\ &+ \frac{\sigma_1^2}{2} + \frac{\sigma_2^2}{2} + \frac{\sigma_3^2}{2} + \frac{\sigma_4^2}{2} + B - \frac{1}{2}\Big[\mu - \frac{1}{2}\theta(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2)\Big]R^{\theta+1} \\ &\leq -\frac{1}{2}\Big[\mu - \frac{1}{2}\theta(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2)\Big]R^{\theta+1} + L \\ &\leq -\frac{1}{2}\Big[\mu - \frac{1}{2}\theta(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2)\Big]\frac{1}{\epsilon_1^{3(\theta+1)}} + L. \end{aligned}$$

From Eq (3.10), $LV(S,E,I,R) < -1, \forall (S,E,I,R) \in D_8$.

As a result, for sufficiently small values $\epsilon_i, i = 1,2,3,4$, we can demonstrate that

$$LV(S,E,I,R) \leq -1, \forall (S,E,I,R) \in \mathbb{R}^4_+ \backslash D.$$

And, as a consequence, the Condition (2) of Lemma 3.1 is satisfied. It follows from Lemma 3.1 that System (1.2) has a unique ergodic stationary distribution. □

Remark 3.3. Theorem 3.2 states that if $\hat{R}_0^s = \frac{\beta\mu(\gamma+\mu+b+\epsilon+\frac{\sigma_3^2}{2})}{(\mu+\frac{\sigma_1^2}{2})(\gamma+\mu+b+\frac{\sigma_3^2}{2})(c+\epsilon+\mu+\frac{\sigma_2^2}{2})} > 1$, then System (1.2) has a unique ergodic stationary distribution $\mu(\cdot)$. If we ignore the white noise, the expression $\hat{R}_0^s$ is equal to the threshold for the deterministic autonomous system System (1.1). This demonstrates that the deterministic system's results may be generalized.

4. Extinction of the disease

We shall focus on the disease extinction in this section. Defining the following notations for ease of use and simplicity in the following analysis, we have

$$\hat{R}_0 = \frac{\beta(\gamma+b+\mu+\epsilon)}{(c+\epsilon+\mu)(\gamma+b+\mu)} - \frac{1}{2\min\{\frac{c+\epsilon+\mu}{\gamma+b+\mu+\epsilon}, 1\}(\gamma+b+\mu)(\sigma_2^{-2}+\sigma_3^{-2})},$$

and

$$\langle f \rangle_t = \frac{1}{t}\int_0^t f(s)ds$$

where f is an integrable function on $[0,\infty)$.

Lemma 4.1. *If $(S(t),E(t),I(t),R(t))$ is the solution of System (1.2) satisfying any initial value $(S_0,E_0,I_0,R_0) \in \mathbb{R}^4_+$, then*

$$\lim_{t\to\infty}\frac{S(t)}{t} = 0, \lim_{t\to\infty}\frac{E(t)}{t} = 0, \lim_{t\to\infty}\frac{I(t)}{t} = 0, \lim_{t\to\infty}\frac{R(t)}{t} = 0 \ a.s.$$

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t S(u)dB_1(u) = 0, \lim_{t\to\infty}\frac{1}{t}\int_0^t E(u)dB_2(u) = 0 \ a.s.$$

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t I(u)dB_3(u) = 0, \lim_{t\to\infty}\frac{1}{t}\int_0^t R(u)dB_4(u) = 0 \ a.s.$$

The proof is omitted since it is comparable to [24, Lemma 3.1].

Lemma 4.2. [21, Theorem 1.3.4] *Denote $M = \{M_t\}_{t\leq 0}$ as a real-valued continuous local martingale vanishing at $t = 0$. It holds that*

$$\lim_{t\to\infty}\langle M,M\rangle_t = \infty \ a.s. \Rightarrow \lim_{t\to\infty}\frac{M_t}{\langle M,M\rangle_t} = 0 \ a.s.$$

and

$$\limsup\frac{\langle M,M\rangle_t}{t} < \infty \ a.s. \Rightarrow \lim_{t\to\infty}\frac{M_t}{t} = 0 \ a.s.$$

Theorem 4.3. *Assuming that System* (1.2) *has the solution* $(S(t), E(t), I(t), R(t))$ *with any initial value* $(S_0, E_0, I_0, R_0) \in \mathbb{R}^4_+$. *If* $\hat{R}_0 < 1$, *the solution* $(S(t), E(t), I(t), R(t))$ *of the System* (1.2) *satisfies*

$$\limsup_{t\to\infty} \frac{\ln(\frac{\gamma+b+\mu+\epsilon}{c+\epsilon+\mu}E(t) + I(t))}{t} \le \min\{\frac{c+\epsilon+\mu}{\gamma+b+\mu+\epsilon}, 1\}(\gamma+b+\mu)(\hat{R}_0 - 1) < 0 \ a.s.$$

$$\lim_{t\to\infty} R(t) = 0 \ a.s. \ , \ \lim_{t\to\infty}\langle S(t)\rangle = \frac{A}{\mu} \ a.s.$$

Proof. By System (1.2), we have

$$dQ(t) = \Lambda - \mu Q(t) + \sigma_1 S(t)dB_1(t) + \sigma_2 E(t)dB_2(t) + \sigma_3 I(t)dB_3(t) + \sigma_4 R(t)dB_4(t), \tag{4.1}$$

where $Q(t) = S(t) + E(t) + I(t) + R(t)$. Integrating both sides of Eq (4.1) from 0 to t and then dividing by t, it follows that

$$\begin{aligned}\frac{Q(t)}{t} - \frac{S_0 + E_0 + I_0 + R_0}{t} = \Lambda - \frac{\int_0^t \mu Q(s)ds}{t} + \frac{\int_0^t \sigma_1 S(s)dB_1(s)}{t} + \frac{\int_0^t \sigma_2 E(s)dB_2(s)}{t} \\ + \frac{\int_0^t \sigma_3 I(s)dB_3(s)}{t} + \frac{\int_0^t \sigma_4 R(s)dB_4(s)}{t}.\end{aligned}$$

It then follows from Lemma 4.1 that

$$\lim_{t\to\infty}\langle Q\rangle_t = \frac{\Lambda}{\mu} \ a.s. \tag{4.2}$$

Let $C(t) = \frac{\gamma+b+\mu+\epsilon}{c+\epsilon+\mu}E(t) + I(t)$. Note that

$$C^2 = \Big(\sigma_2 \cdot \frac{\gamma+b+\mu+\epsilon}{c+\epsilon+\mu}\frac{E}{\sigma_2} + \frac{I}{\sigma_3}\Big)^2 \le \Big(\sigma_2^2(\frac{\gamma+b+\mu+\epsilon}{c+\epsilon+\mu})^2E^2 + \sigma_3^2I^2\Big)\Big(\frac{1}{\sigma_2^2} + \frac{1}{\sigma_3^2}\Big). \tag{4.3}$$

Using Itô's formula, it holds that

$$\begin{aligned} d\ln C(t) &= \Big\{\frac{\beta \cdot \frac{\gamma+b+\mu+\epsilon}{c+\epsilon+\mu}\frac{S(E+I)}{N} - (\gamma+b+\mu)E - (\gamma+b+\mu)I}{C(t)} \\ &\quad + \frac{\sigma_2^2(\frac{\gamma+b+\mu+\epsilon}{c+\epsilon+\mu})^2E^2 + \sigma_3^2I^2}{2C^2(t)}\Big\}dt + \frac{\sigma_2\frac{\gamma+b+\mu+\epsilon}{c+\epsilon+\mu}E}{C(t)}dB_2(t) + \frac{\sigma_3 I}{C(t)}dB_3(t) \\ &\le \Big\{\min\{\frac{c+\epsilon+\mu}{\gamma+b+\mu+\epsilon}, 1\}\Big[\beta \cdot \frac{\gamma+b+\mu+\epsilon}{c+\epsilon+\mu} - (\gamma+b+\mu)\Big] \\ &\quad - (2(\sigma_2^{-2} + \sigma_3^{-2}))^{-1}\Big\}dt + \frac{\sigma_2\frac{\gamma+b+\mu+\epsilon}{c+\epsilon+\mu}E}{C(t)}dB_2(t) + \frac{\sigma_3 I}{C(t)}dB_3(t) \\ &\le \min\{\frac{c+\epsilon+\mu}{\gamma+b+\mu+\epsilon}, 1\}(\gamma+b+\mu)(\widehat{R}_0 - 1)dt \\ &\quad + \frac{\sigma_2\frac{\gamma+b+\mu+\epsilon}{c+\epsilon+\mu}E}{C(t)}dB_2(t) + \frac{\sigma_3 I}{C(t)}dB_3(t). \end{aligned} \tag{4.4}$$

Integrating both sides of Eq (4.4) from 0 to t and then dividing by t, we have

$$\begin{aligned}\frac{\ln C(t)}{t} - \frac{\ln(C_0)}{t} \leq \min\{\frac{c+\epsilon+\mu}{\gamma+b+\mu+\epsilon}, 1\}(\gamma+b+\mu)(\widehat{R}_0 - 1) \\ + \frac{1}{t}\int_0^t \frac{\sigma_2 \frac{\gamma+b+\mu+\epsilon}{c+\epsilon+\mu} E}{C(s)} dB_2(t) + \frac{1}{t}\int_0^t \frac{\sigma_3 I}{C(s)} dB_3(t).\end{aligned} \tag{4.5}$$

Let $M(t) = \int_0^t \frac{\frac{\gamma+b+\mu+\epsilon}{c+\epsilon+\mu}E(s)}{C(s)} dB_2(s)$. It is clear that,

$$\limsup_{t\to\infty} \frac{\langle M, M\rangle_t}{t} = \limsup_{t\to\infty} \frac{\int_0^t \frac{(\frac{\gamma+b+\mu+\epsilon}{c+\epsilon+\mu}E(s))^2}{(C(s))^2} ds}{t} \leq 1 \ a.s.$$

According to Lemma 4.2, we have

$$\lim_{t\to\infty} \frac{\int_0^t \frac{\sigma_2 \cdot \frac{\gamma+b+\mu+\epsilon}{c+\epsilon+\mu}E(s)}{C(s)} dB_2(s)}{t} = 0. \tag{4.6}$$

Similarly, we get

$$\lim_{t\to\infty} \frac{\int_0^t \frac{\sigma_3 I(s)}{E(s)+I(s)} dB_3(s)}{t} = 0. \tag{4.7}$$

Based on Eqs (4.5)–(4.7), it holds that

$$\limsup_{t\to\infty} \frac{\ln C(t)}{t} \leq \min\{\frac{c+\epsilon+\mu}{\gamma+b+\mu+\epsilon}, 1\}(\gamma+b+\mu)(\widehat{R}_0 - 1). \tag{4.8}$$

Since $\widehat{R}_0 < 1$, we can get

$$\lim_{t\to\infty} E(t) = 0 \ a.s. \ , \ \lim_{t\to\infty} I(t) = 0 \ a.s. \ ,$$

based on Eq (4.8).

From System (1.2), we conclude that $\lim_{t\to\infty} R(t) = 0$ a.s. when $\lim_{t\to\infty} I(t) = 0$ $a.s.$ According to Eq (4.2), $\lim_{t\to\infty}\langle S\rangle_t = \frac{A}{\mu}$ $a.s.$ This completes the proof. □

Remark 4.4. *If* $R_0 = \frac{\beta(\gamma+b+\mu+\epsilon)}{(c+\epsilon+\mu)(\gamma+b+\mu)} > 1$ *in the determined model, the infection will continue to exist. However, we can come to this conclusion based on the expression of* $\hat{R}_0$ *at the beginning of this section which indicates that if the density of the noise is sufficiently large, the disease may die out.*

5. Numerical simulations

Several examples are presented in this section to exemplify the theoretical findings. Our results are presented as a result of using Milstein's higher method suggested in [23]. Consider the following

discretization equation

$$
\begin{aligned}
S_{j+1} &= S_j + \Big[\Lambda - \beta\frac{S_j(E_j+I_j)}{S_j+E_j+I_i+R_j} + cE_j + bI_j + \alpha R_j - \mu S_j\Big]\Delta t \\
&\quad + \sigma_1 S_j\sqrt{\Delta t}\epsilon_{1,j} + \frac{\sigma_1^2 S_j}{2}(\epsilon_{1,j}^2 - 1)\Delta t, \\
E_{j+1} &= E_j + \Big[\beta\frac{S_j(E_j+I_j)}{S_j+E_j+I_i+R_j} - (c+\epsilon+\mu)E_j\Big]\Delta t \\
&\quad + \sigma_2 E_j\sqrt{\Delta t}\epsilon_{2,j} + \frac{\sigma_2^2 E_j}{2}(\epsilon_{2,j}^2 - 1)\Delta t, \\
I_{j+1} &= I_j + [\epsilon E_j - (\gamma+b+\mu)I_j]\Delta t + \sigma_3 I_j\sqrt{\Delta t}\epsilon_{3,j} + \frac{\sigma_3^2 I_j}{2}(\epsilon_{3,j}^2 - 1)\Delta t, \\
R_{j+1} &= R_j + [\gamma I_j - (\alpha+\mu)R_j]\Delta t + \sigma_4 R_j\sqrt{\Delta t}\epsilon_{4,j} + \frac{\sigma_4^2 R_j}{2}(\epsilon_{4,j}^2 - 1)\Delta t,
\end{aligned}
$$

where the intensities of white noise are represented by the time increment $\Delta t > 0, \sigma_i^2 > 0 (i = 1, 2, 3, 4)$ and Gaussian random variables $\epsilon_{i,j} \sim N(0, 1)(i = 1, 2, 3, 4)$. We subdivide the time interval into 1000 equidistant time steps. Moreover, $(S_j, E_j, I_j, R_j)^T$ is the value of the ith iteration of the discretized equation.

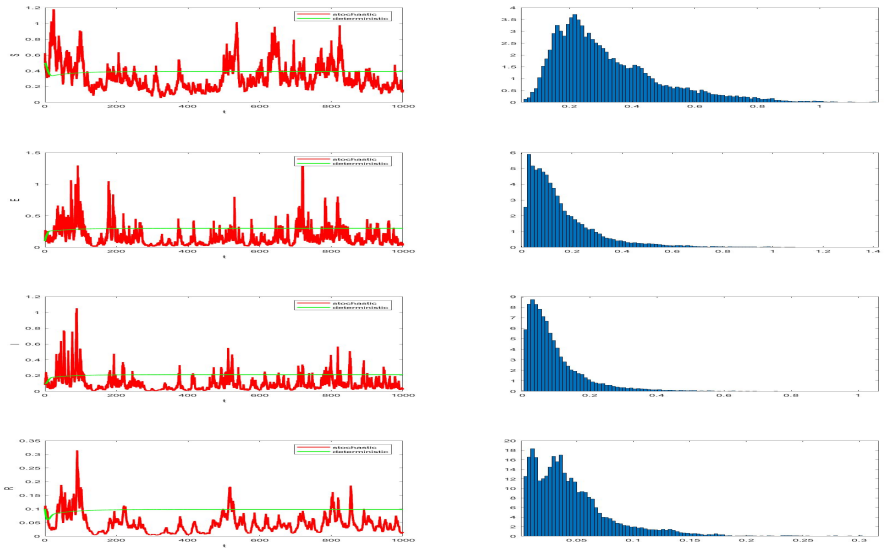

Figure 1. (Left) Red lines depict the solution of System (1.2 and green lines depict the solution of the undisturbed system System (1.1). (Right) Histograms of the probability density functions of S(t), E(t), I(t) and R(t).

For the parameters and initial values, refer to the data in [6]. We chose $\beta = 0.34, c = 0.25, b = 0.2, \epsilon = 0.15, \alpha = 0.2, \gamma = 0.1, \Lambda = 0.015, \mu = 0.01, \sigma_1^2 = 0.002$, $\sigma_2^2 = 0.02$, $\sigma_3^2 = 0.04$ and $\sigma_4^2 = 0.005$. In this example, numerical simulations were used to show the trajectory images and distribution function graphs of $E(t)$ and $I(t)$, as shown in Figure 1 . In the figure, one can find that the number of people making up the infectious population fluctuates around a nonzero value, which means

that both will persist in the community. According to Theorem 3.2, $\hat{R}_0^s = 1.05785$, and there exists a unique stationary distribution of System (1.2) in this circumstance. The results are supported by the results of numerical simulations.

Now, we will examine the influences of the contact transmission rate, susceptibility rate of exposed individuals and infected people.

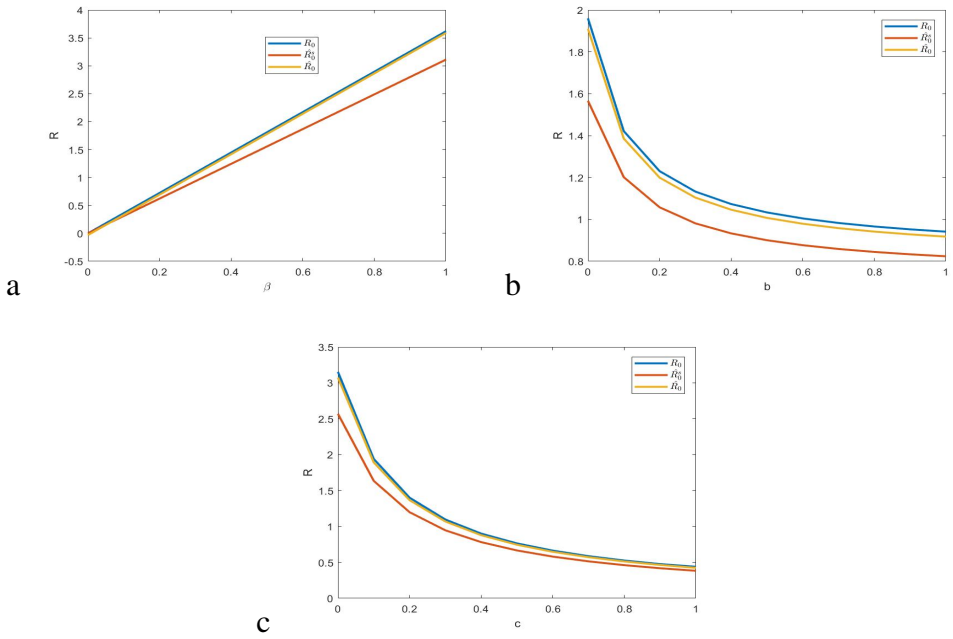

Figure 2. Effects of β, b and c.

Let β vary in $[0, 1]$ and keep the other parameters as shown in Figure 1. Figure 2(a) shows that $R_0, \hat{R}_0^s$ and $\hat{R}_0$ are increasing functions. If we lower the viral transmission contact rate β to 0.278 and keep the other parameters the same as in Figure 1, we get the following results. In this situation, the simulations using the initial value $(S_0, E_0, I_0, R_0) = (0.5, 0.1, 0.1, 0.1)$ show that $E(t)$ and $I(t)$ of the solutions of System (1.2) tend to 0 as $t \to +\infty$. This indicates that the disease is no longer present (see Figure 3). In this case, we may compute $\hat{R}_0 = 0.9751143 < 1$. The disease will die out with a probability of 1 based on Theorem 4.3. The results are supported by numerical simulations. We may deduce that the disease will be eradicated if the rate of infection is reduced.

Following that, let b vary in $[0, 1]$ and keep the other parameters as shown in Figure 1. Figure 2(b) shows that $R_0, \hat{R}_0^s$ and $\hat{R}_0$ are decreasing functions. We raise the amount of b to 0.55, as well as the other parameters as shown in Figure 1. The simulations with the initial value $(S_0, E_0, I_0, R_0) = (0.5, 0.1, 0.1, 0.1)$ show that when t approaches $+\infty$, $E(t)$ and $I(t)$ of solutions of System (1.2) tend to 0. This indicates that the disease is no longer present (see Figure 4). In this case, $\hat{R}_0 = 0.96 < 1$. Hence the disease will die out with a probability of 1 based on Theorem 4.3. The results are supported by the numerical simulations.

Let c vary in $[0, 1]$ and keep the other parameters as shown in Figure 1. Figure 2(c) shows that $R_0, \hat{R}_0^s$ and $\hat{R}_0$ are decreasing functions. Then, we increase the value of c to 0.3 and keep the other parameters the same as in Figure 1. In this situation, the simulations using the initial value $(S_0, E_0, I_0, R_0) = (0.5, 0.1, 0.1, 0.1)$ show that when t approaches $+\infty$, $E(t)$ and $I(t)$ of the solutions of System (1.2)

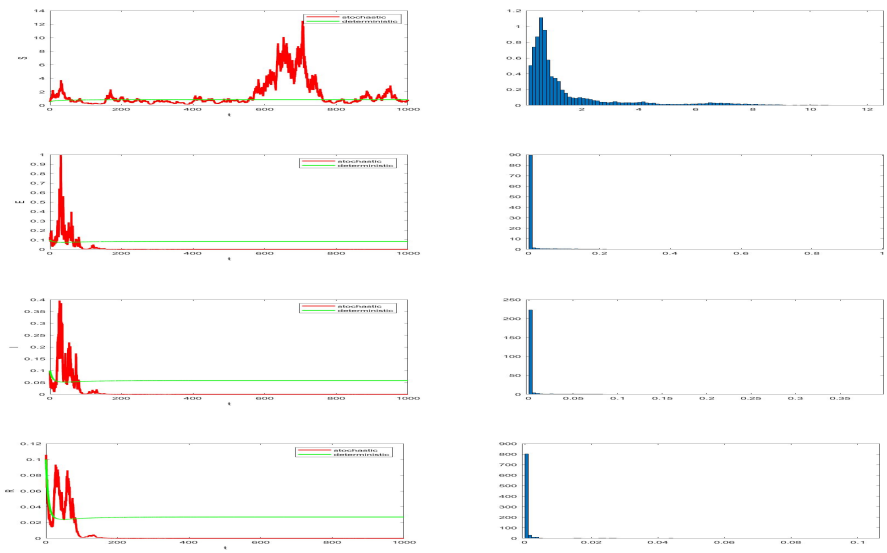

Figure 3. (Left) Red lines depict the solution of System (1.2) and green lines depict the solution of the undisturbed system System (1.1). (Right) Histograms of the probability density functions of S(t), E(t), I(t) and R(t).

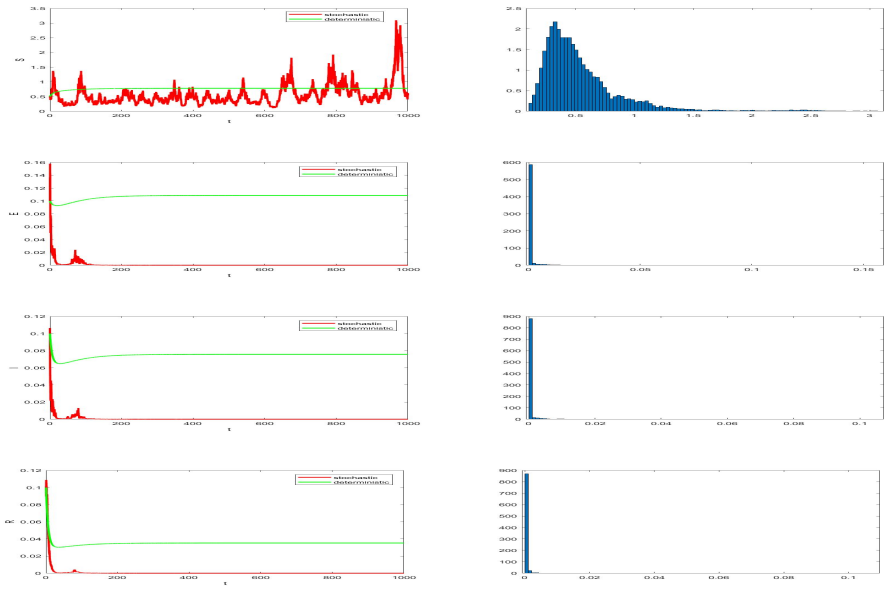

Figure 4. (Left) Red lines depict the solution of System(1.2 and green lines depict the solution of the undisturbed system System (1.1). (Right) Histograms of the probability density functions of S(t), E(t), I(t) and R(t).

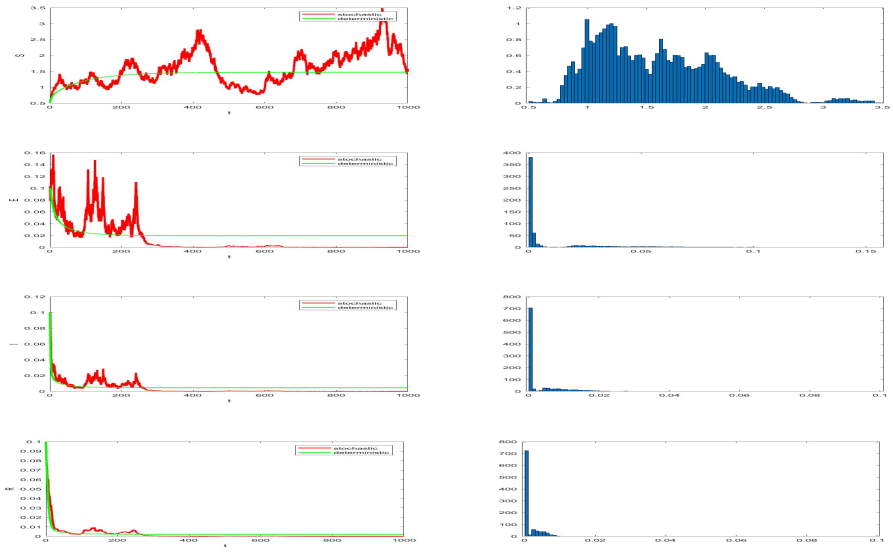

Figure 5. (Left) Red lines depict the solution of System(1.2 and green lines depict the solution of the undisturbed system System (1.1). (Right) Histograms of the probability density functions of S(t), E(t), I(t) and R(t).

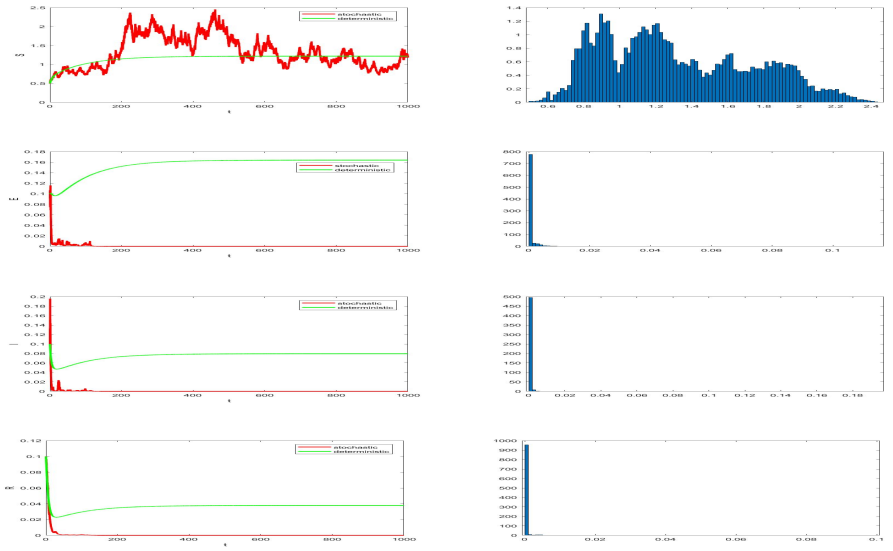

Figure 6. (Left) Red lines depict the solution of System(1.2 and green lines depict the solution of the undisturbed system System (1.1). (Right) Histograms of the probability density functions of S(t), E(t), I(t) and R(t).

tend 0 as $t \to +\infty$. This indicates that the disease is no longer present (see Figure 5). In this case, $\hat{R}_0 = 0.67 < 1$. According to Theorem 4.3, the disease will die out with a probability of 1. There is a good correlation between the numerical simulations and the results. In conclusion, the disease will die out by enhancing disease resistance.

From Figures 3–5, one can find that the components $E(t)$ and $I(t)$ of the stochastic epidemic model will approach 0 as $t \to +\infty$. We should also point out that the components $E(t)$ and $I(t)$ of the deterministic epidemic model will tend toward positive values as $t \to +\infty$. This fact shows that white noise is beneficial for disease extinction. On the other hand, Figure 2 shows that, no matter how we change the values of β, b and c, we have $R_0 > \hat{R}_0 > \hat{R}_0^s$. This result also shows that white noise is beneficial for disease extinction.

Consequently, we chose $\sigma_2^2 = 0.2, \sigma_3^2 = 0.3$, and the values shown in Figure 1 for the other variables. In this case, we observed that the $(S(t), E(t), I(t), R(t))$ of System (1.1) tended to $(1.2190, 0.1693, 0.0793, 0.0378)$ as $t \to +\infty$, whereas the $I(t)$ of System (1.2) gradually decreased to 0 as t approached ∞ (see Figure 6). Ultimately, it could be concluded that white noise is beneficial for disease extinction.

6. Conclusions

Briefly, the current study yielded a stochastic influenza virus model with disease resistance. According to mathematical approaches, in reality, the circumstances required for the disease to die out are rough to some extent. In this study, we focused on a stochastic influenza virus model with disease resistance. First, we obtained the existence of one unique global positive solution satisfying the initial value of $(S_0, E_0, I_0, R_0) \in \mathbb{R}_+^4$. Second, we get the existence of a stationary distribution for the positive solutions by using a stochastic Lyapunov function method. Third, we established the sufficient conditions for extinction of the disease. Finally, we conducted some numerical simulations. The results of the numerical simulations indicate that enhancing disease resistance is beneficial for the control of the disease. In addition, a simple example showed that the white noise is beneficial for the extinction of the disease.

Furthermore, there is a parametric region between the extinction conditions and a stationary distribution for positive solution values. As a result, it is only logical to wonder about the dynamics in this region. We shall address these incidents in future works.

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Conflict of interest

The authors declare that there is no conflict of interest.

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