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Research article

Global existence and blow up of solutions for a class of coupled parabolic systems with logarithmic nonlinearity

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Abstract: According to the difference of the initial energy, we consider three cases about the global existence and blow-up of the solutions for a class of coupled parabolic systems with logarithmic nonlinearity. The three cases are the low initial energy, critical initial energy and high initial energy, respectively. For the low initial energy and critical initial energy $J(u_0, v_0) \leq d$, we prove the existence of global solutions with $I(u_0, v_0) \geq 0$ and blow up of solutions at finite time $T < +\infty$ with $I(u_0, v_0) < 0$, where I is Nehari functional. On the other hand, we give sufficient conditions for global existence and blow up of solutions in the case of high initial energy $J(u_0, v_0) > d$.

Keywords: coupled parabolic systems; global existence; blow up; logarithmic nonlinearity

1. Introduction

In this paper, we consider the following initial-boundary value problem for a class of coupled parabolic systems with logarithmic nonlinearity.

$$\begin{cases} u_t - \Delta u = |v|^p |u|^{p-2} u \log(|uv|), & x \in \Omega,\ t > 0, \\ v_t - \Delta v = |u|^p |v|^{p-2} v \log(|uv|), & x \in \Omega,\ t > 0, \\ u(x,0) = u_0(x), & x \in \Omega, \\ v(x,0) = v_0(x), & x \in \Omega, \\ u(x,t) = v(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T], \end{cases} \tag{1.1}$$

where $(u_0, v_0) \in H_0^1(\Omega) \times H_0^1(\Omega)$, $T \in (0, +\infty)$, $\Omega \subset R^n (n \geq 2)$ is a bounded domain with smooth boundary $\partial\Omega$ and p satisfies the following assumptions:

$$2 < p < 2^* := \begin{cases} \infty, & \text{if } n = 2, \\ \frac{2n}{n-2}, & \text{if } n \geq 3. \end{cases} \tag{1.2}$$

Among the fields of mathematical physics, biosciences and engineering, problem (1.1) is one of the most important reaction-diffusion coupled systems with logarithmic nonlinearity. It can be used not only to predict the time evolution of various population density distributions, but also to describe the thermal propagation of a two-component combustible mixture [1–3]. In recent years, this kind of systematic research has attracted many mathematicians and has made remarkable progress [4–8]. In order to overcome the special difficulties brought by nonlinear terms, many new ideas and tools have been developed, which greatly enrich the theory of partial differential equations [9–14].

In the past years, many authors made efforts to the investigation of the existence and blow up of solutions for such kinds of systems. Galaktionov et al. [15, 16] investigated the following semilinear reaction-diffusion system

$$\begin{cases} u_t - \Delta u = v^p, \\ v_t - \Delta v = u^q. \end{cases} \tag{1.3}$$

They proved the local and global existence of solutions for the initial boundary value problem of (1.3). Subsequently, Escobedo and Herrero [17] considered the initial boundary value problem of (1.3) for a bounded open domain on R^n with smooth boundary. They obtained global solution under the condition $0 < pq \leq 1$, meanwhile global solution and blow up in finite time depending on sufficient small or large initial value and $pq > 1$. For more studies on problem (1.3) we refer the interested reader to [18–20] and references therein.

Recently, Xu et al. [21] considered the following nonlinear reaction-diffusion systems

$$\begin{cases} u_t - \Delta u = \left(|u|^{2p} + |v|^{p+1}|u|^{p-1}\right)u, & x \in \Omega, t > 0, \\ v_t - \Delta v = \left(|v|^{2p} + |u|^{p+1}|v|^{p-1}\right)v, & x \in \Omega, t > 0, \\ u(x,0) = u_0(x), & x \in \Omega, \\ v(x,0) = v_0(x), & x \in \Omega, \\ u(x,t) = v(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T]. \end{cases} \tag{1.4}$$

When initial energy $J(u_0, v_0) \leq d$, by virtue of Galerkin method [22] and concave function method [23], global existence and finite time blow-up of the solutions for the problem (1.4) were obtained. When initial energy $J(u_0, v_0) > d$, they discussed global existence, finite time blow-up of solutions and tried to find out the corresponding initial data with arbitrarily high initial energy. What's more, by using comparison principle and the ideas in [24,25], they described the structures of the initial data and gave some sufficient conditions of the initial data which ensured the finite time blow up and global existence of the solutions, respectively.

Inspired by the above works, we aim to use the Galerkin method, logarithmic inequalities [26], and concave function method to prove the global existence, decay, finite time blow-up of solutions for problem (1.1) with initial energy $J(u_0, v_0) \leq d$. When high initial energy $J(u_0, v_0) > d$, by constructing two sets Φ_α and Ψ_α defined as (5.1) and (5.2), we prove that the weak solution will blow up in finite or infinite time if the initial value belongs to Ψ_α, while the weak solution will exist globally and tends to zero as time $t \to +\infty$ when the initial value belongs to Φ_α.

The organization of the remaining part of this paper is as follows. In Section 2, we introduce some preliminaries and lemmas of this paper. In Sections 3–5, we will give our main results and the corresponding proofs.

2. Preliminary

Throughout this paper, we denote by $\|u\|_\gamma$ the norm of $L^\gamma(\Omega)$ for $1 \le \gamma \le +\infty$ and by $\|u\|_{H_0^1(\Omega)}$ the norm of $H_0^1(\Omega)$. For $u \in L^\gamma(\Omega)$,

$$\|u\|_\gamma = \begin{cases} \left(\int_\Omega |u(x)|^\gamma dx\right)^{\frac{1}{\gamma}}, & \text{if } 1 \le \gamma < +\infty, \\ \text{ess sup}_{x\in\Omega} |u(x)|, & \text{if } \gamma = +\infty, \end{cases}$$

and for $u \in H_0^1(\Omega)$,

$$\|u\|^2_{H_0^1(\Omega)} = \|u\|_2^2 + \|\nabla u\|_2^2.$$

By virtue of Poincaré inequality, we know that $\|u\|^2_{H_0^1(\Omega)}$ and $\|\nabla u\|_2^2$ are equivalent norms to each other, i.e., there exist C_1 and C_2 such that

$$C_1\|\nabla u\|_2^2 \le \|u\|^2_{H_0^1(\Omega)} \le C_2\|\nabla u\|_2^2,$$

which is denoted by $\|u\|^2_{H_0^1(\Omega)} \simeq \|\nabla u\|_2^2$. In addition, we denoted by $(\cdot,\cdot)$ the inner product in $L^2(\Omega)$ and c is an arbitrary positive number which may be different from line to line.

For $(u, v) \in H_0^1(\Omega) \times H_0^1(\Omega)$, we define the Nehari functional I and energy functional J as follows:

$$\begin{aligned} I(u,v) &= \int_\Omega |\nabla u|^2 dx + \int_\Omega |\nabla v|^2 dx - 2\int_\Omega |uv|^p |\log(|uv|) dx \\ &\simeq \|u\|^2_{H_0^1(\Omega)} + \|v\|^2_{H_0^1(\Omega)} - 2\int_\Omega |uv|^p \log(|uv|) dx, \end{aligned} \tag{2.1}$$

$$\begin{aligned} J(u,v) &= \frac{1}{2}\int_\Omega |\nabla u|^2 dx + \frac{1}{2}\int_\Omega |\nabla v|^2 dx + \frac{1}{p^2}\int_\Omega |uv|^p dx - \frac{1}{p}\int_\Omega |uv|^p \log(|uv|) dx \\ &\simeq \frac{1}{2}\|u\|^2_{H_0^1(\Omega)} + \frac{1}{2}\|v\|^2_{H_0^1(\Omega)} + \frac{1}{p^2}\|uv\|_p^p - \frac{1}{p}\int_\Omega |uv|^p \log(|uv|) dx. \end{aligned} \tag{2.2}$$

From (2.1) and (2.2), we have

$$J(u,v) \simeq \frac{1}{2p} I(u,v) + \frac{p-1}{2p}\left(\|u\|^2_{H_0^1(\Omega)} + \|v\|^2_{H_0^1(\Omega)}\right) + \frac{1}{p^2}\|uv\|_p^p. \tag{2.3}$$

Let

$$\mathcal{N} := \left\{(u,v) \in H_0^1(\Omega) \times H_0^1(\Omega) \setminus \{(0,0)\} \mid I(u,v) = 0\right\}$$

be the Nehari manifold. Furthermore, the potential well W and its corresponding set V are defined respectively by

$$W = \left\{(u,v) \in H_0^1(\Omega) \times H_0^1(\Omega) \mid I(u,v) > 0, J(u,v) < d\right\} \cup \{(0,0)\},$$

$$V = \left\{(u,v) \in H_0^1(\Omega) \times H_0^1(\Omega) \mid I(u,v) < 0, J(u,v) < d\right\},$$

where

$$d := \inf_{(u,v)\in H_0^1(\Omega)\times H_0^1(\Omega)\backslash\{(0,0)\}} \sup_{s_1,s_2>0} J(s_1u, s_2v) = \inf_{(u,v)\in\mathcal{N}} J(u, v) \tag{2.4}$$

is the depth of the potential well W.

To consider the weak solution with high energy level, we need to introduce some new notions.

$$J^\alpha = \{(u, v) \in H_0^1(\Omega) \times H_0^1(\Omega)|J(u, v) < \alpha\},$$

$$\mathcal{N}_\alpha = \mathcal{N} \cap J^\alpha = \Big\{(u, v) \in \mathcal{N}|\frac{p-1}{2p}\Big(\|u\|^2_{H_0^1(\Omega)} + \|v\|^2_{H_0^1(\Omega)}\Big) + \frac{1}{p^2}\|uv\|_p^p < \alpha\Big\},$$

and

$$\lambda_\alpha = \inf\Big\{\frac{1}{2}\Big(\|u\|_2^2 + \|v\|_2^2\Big)|(u, v) \in \mathcal{N}_\alpha\Big\},\ \Lambda_\alpha = \sup\Big\{\frac{1}{2}\Big(\|u\|_2^2 + \|v\|_2^2\Big)|(u, v) \in \mathcal{N}_\alpha\Big\} \text{ for all } \alpha > d,$$

Clearly, λ_α is nonincreasing, and Λ_α is nondecreasing with respect to α, respectively.

Now, we give the definitions of the weak solution, maximal existence time and finite time blow up of the problem (1.1) as follows.

Definition 1. *(Weak solution) We say that* $(u, v)=(u(x, t), v(x, t)) \in L^\infty([0, T), H_0^1(\Omega) \times H_0^1(\Omega))$ *with* $(u_t, v_t) \in L^2([0, T), L^2(\Omega) \times L^2(\Omega))$ *is a weak solution of problem (1.1) on* $\Omega \times [0, T)$*, if it satisfies the initial condition* $u(x, 0) = u_0(x), v(x, 0) = v_0(x)$ *in* $H_0^1(\Omega)$*,*

$$(u_t, w_1) + (\nabla u, \nabla w_1) = (|v|^p|u|^{p-2}u \log(|uv|), w_1)$$

and

$$(v_t, w_2) + (\nabla v, \nabla w_2) = (|u|^p|v|^{p-2}v \log(|uv|), w_2)$$

for all $w_1, w_2 \in H_0^1(\Omega)$ *and* $t \in (0, T)$*. Moreover, for all* $t \in (0, T)$*, we have*

$$\int_0^t \|u_\tau\|_2^2 + \|v_\tau\|_2^2 d\tau + J(u, v) \le J(u_0, v_0). \tag{2.5}$$

Remark 1. *For the global weak solution* $(u(t), v(t)) = (u(x, t), v(x, t))$ *of problem (1.1), we define the* ω*-limit set of* (u_0, v_0) *by*

$$\omega(u_0, v_0) := \bigcap_{t\ge 0} \overline{\{u(s), v(s) : s \ge t\}}.$$

Definition 2. *(Maximal existence time) Let* $(u, v) = (u(x, t), v(x, t))$ *be a weak solution of problem (1.1). We define the maximal existence time of* (u, v) *as follows*

(i) If (u, v) *exists for all* $t \in [0, +\infty)$*, then* $T = +\infty$*.*

(ii) If there exists a $t_0 \in (0, +\infty)$ *such that* (u, v) *exists for* $0 \le t < t_0$*, but it does not exist at* $t = t_0$*, then* $T = t_0$*.*

Definition 3. *(Finite time blow-up) Let* $(u(t), v(t)) = (u(x, t), v(x, t))$ *be a weak solution of problem (1.1). We say* $(u(t), v(t))$ *blows up in finite time if the maximal existence time* T *is finite and*

$$\lim_{t\to T} \|u(t)\|_2^2 + \|v(t)\|_2^2 = +\infty.$$

Lemma 1. *Let* $(u, v) \in H_0^1(\Omega) \times H_0^1(\Omega) \backslash \{(0, 0)\}$*, then the following hold*
(i) $\lim_{\lambda \to 0} J(\lambda u, \lambda v) = 0$*,* $\lim_{\lambda \to +\infty} J(\lambda u, \lambda v) = -\infty$.
(ii) There exists a unique $\lambda_* > 0$ *such that* $\frac{d}{d\lambda} J(\lambda u, \lambda v)\big|_{\lambda = \lambda_*} = 0$.
(iii) $J(\lambda u, \lambda v)$ *is increasing on* $(0, \lambda_*)$*, decreasing on* $(\lambda_*, +\infty)$*, and attains the maximum at* $\lambda = \lambda_*$.
(iv) $I(\lambda u, \lambda v) > 0$ *for* $0 < \lambda < \lambda_*$*,* $I(\lambda u, \lambda v) < 0$ *for* $\lambda_* < \lambda < +\infty$*, and* $I(\lambda_* u, \lambda_* v) = 0$.

Proof. (i) By definition of $J(u, v)$ and $\lambda > 0$, we have

$$J(\lambda u, \lambda v) = \frac{\lambda^2}{2} \|u\|^2_{H_0^1(\Omega)} + \frac{\lambda^2}{2} \|v\|^2_{H_0^1(\Omega)} + \frac{\lambda^{2p}}{p^2} \|uv\|_p^p - \frac{\lambda^{2p}}{p} log \lambda^2 \|uv\|_p^p - \frac{\lambda^{2p}}{p} \int_\Omega |uv|^p log(|uv|) dx.$$

Thus $\lim_{\lambda \to 0} J(\lambda u, \lambda v) = 0$, $\lim_{\lambda \to +\infty} J(\lambda u, \lambda v) = -\infty$.
(ii) Differentiating $J(\lambda u, \lambda v)$ with respect to λ, we get

$$\begin{aligned} \frac{d}{d\lambda} J(\lambda u, \lambda v) &= \lambda \|u\|^2_{H_0^1(\Omega)} + \lambda \|v\|^2_{H_0^1(\Omega)} - 2\lambda^{2p-1} \log \lambda^2 \|uv\|_p^p - 2\lambda^{2p-1} \int_\Omega |uv|^p \log(|uv|) dx \\ &= \lambda \left(\|u\|^2_{H_0^1(\Omega)} + \|v\|^2_{H_0^1(\Omega)} - 2\lambda^{2p-2} \log \lambda^2 \|uv\|_p^p - 2\lambda^{2p-2} \int_\Omega |uv|^p \log(|uv|) dx \right). \end{aligned}$$

Setting $g(\lambda) = \|u\|^2_{H_0^1(\Omega)} + \|v\|^2_{H_0^1(\Omega)} - 2\lambda^{2p-2} \log \lambda^2 \|uv\|_p^p - 2\lambda^{2p-2} \int_\Omega |uv|^p \log(|uv|) dx$, we have $\lim_{\lambda \to 0} g(\lambda) = \|u\|^2_{H_0^1(\Omega)} + \|v\|^2_{H_0^1(\Omega)} > 0$, $\lim_{\lambda \to +\infty} g(\lambda) = -\infty$, and

$$g'(\lambda) = -2(2p-2)\lambda^{2p-3} \log \lambda^2 \|uv\|_p^p - 4\lambda^{2p-3} \|uv\|_p^p - 2(2p-2)\lambda^{2p-3} \int_\Omega |uv|^p \log |uv| dx < 0.$$

Thus there exists a unique $\lambda_* > 0$ such that $g(\lambda_*) = 0$, i.e., $\frac{d}{d\lambda} J(\lambda u, \lambda v)\big|_{\lambda = \lambda_*} = 0$.
(iii) It is easy to find that $J(\lambda u, \lambda v)$ is strictly increasing on $(0, \lambda_*]$, strictly decreasing on $(\lambda_*, +\infty)$ and taking the maximum at $\lambda = \lambda_*$.
(iv) Since

$$I(\lambda u, \lambda v) = \|\lambda u\|^2_{H_0^1(\Omega)} + \|\lambda v\|^2_{H_0^1(\Omega)} - 2 \int_\Omega |\lambda u \lambda v|^p \log(|\lambda u \lambda v|) dx = \lambda \frac{d}{d\lambda} J(\lambda u, \lambda v),$$

then the conclusion follows immediately.

Lemma 2. *Assume (1.2) holds, let* $(u, v) \in H_0^1(\Omega) \times H_0^1(\Omega)$ *satisfy* $I(u, v) < 0$*, then*

$$I(u, v) < 2p(J(u, v) - d). \tag{2.6}$$

Proof. According to $I(u, v) < 0$ and Lemma 1, we have $(u, v) \neq (0, 0)$ and there exists a $\lambda_* \in (0, 1)$ such that $I(\lambda_* u, \lambda_* v) = 0$, i.e., $J(\lambda_* u, \lambda_* v) \geq d$. For $\lambda > 0$, set

$$h(\lambda) = 2pJ(\lambda u, \lambda v) - I(\lambda u, \lambda v) = (p-1)\lambda^2 \left(\|u\|^2_{H_0^1(\Omega)} + \|v\|^2_{H_0^1(\Omega)} \right) + \frac{2}{p} \lambda^{2p} \|uv\|_p^P,$$

then

$$h'(\lambda) = 2(p-1)\lambda \left(\|u\|^2_{H_0^1(\Omega)} + \|v\|^2_{H_0^1(\Omega)} \right) + 4\lambda^{2p-1} \|uv\|_p^P > 0.$$

Hence $h(\lambda)$ is strictly increasing for $\lambda > 0$. Together with $\lambda_* \in (0, 1)$, it follows that $h(1) > h(\lambda_*)$. i.e.,

$$2pJ(u, v) - I(u, v) > 2pJ(\lambda_* u, \lambda_* v) - I(\lambda_* u, \lambda_* v) = 2pJ(\lambda_* u, \lambda_* v) \geq 2pd.$$

Then (2.6) follows immediately.

Lemma 3. *Assume (1.2) holds, let* $(u_0, v_0) \in H_0^1(\Omega)\times H_0^1(\Omega)$ *and* $(u(t), v(t)) = (u(x,t), v(x,t))$ *be a weak solution of problem (1.1). If* $J(u_0, v_0) < d$ *and* $I(u_0, v_0) < 0$*, then* $(u(t), v(t)) \in V$ *for all* $0 \le t \le T$*, where* T *is the maximal existence time of* $(u(t), v(t))$*.*

Proof. We will show that $(u(t), v(t)) \in V$ for $0 \le t \le T$. Arguing by contradiction, suppose that $t_0 \in [0, T]$ be the smallest time for which $(u(t_0), v(t_0)) \notin V$, then by the continuity of the $(u(t), v(t))$, we get $(u(t_0), v(t_0)) \in \partial V$. Hence, it follows that

$$I(u(t_0), v(t_0)) = 0 \tag{2.7}$$

or

$$J(u(t_0), v(t_0)) = d. \tag{2.8}$$

If (2.7) is true, then $(u(t_0), v(t_0)) \in \mathcal{N}$, $J(u(t_0), v(t_0)) > d$,which contradicts with (2.5). While if (2.8) is true, it also contradicts with (2.5). Consequently, we have $(u(t), v(t)) \in V$ for all $0 \le t \le T$.

Lemma 4. *Let (u,v) be a weak solution of problem (1.1). Then for all* $t \in [0, T)$*,*

$$\frac{d}{dt}\left(\|u\|_2^2 + \|v\|_2^2\right) = -2I(u, v).$$

Proof. The proof of Lemma 4 directly follows by choosing $w_1 = u, w_2 = v$ in Definition 1.

3. Low initial energy $J(u_0, v_0) < d$

In this section, we prove global existence and finite time blow up of solutions for problem (1.1) with the initial energy $J(u_0, v_0) < d$.

Theorem 1. *Assume* $(u_0, v_0) \in H_0^1(\Omega) \times H_0^1(\Omega)$ *and (1.2) hold. If* $J(u_0, v_0) < d$ *and* $I(u_0, v_0) \ge 0$*, then the problem (1.1) has a global solution* $(u(t), v(t)) \in L^\infty((0,\infty); H_0^1(\Omega) \times H_0^1(\Omega))$ *with* $(u_t(t), v_t(t)) \in L^2((0,\infty); L^2(\Omega) \times L^2(\Omega))$ *and* $(u(t), v(t)) \in W$ *for* $0 \le t < \infty$*. Furthermore, if* $I(u_0, v_0) > 0$*, then there exists a* $c > 0$ *such that* $\|u\|_2^2 + \|v\|_2^2 \le (\|u_0\|_2^2 + \|v_0\|_2^2)e^{-2ct}$*.*

Proof. Since we know $J(u_0, v_0) < d$ and $I(u_0, v_0) \ge 0$, then it follows that
(i) If $0 < J(u_0, v_0) < d$ and $I(u_0, v_0) \ge 0$, then we have $I(u_0, v_0) > 0$. In fact, if $I(u_0, v_0) = 0$, then by the definition of d in (2.4), we have $J(u_0, v_0) \ge d$, which is a contradiction.
(ii) If $J(u_0, v_0) = 0$ and $I(u_0, v_0) \ge 0$, then we obtain $(u_0, v_0) = (0, 0)$. In fact, if $(u_0, v_0) \ne (0, 0)$, then by the (2.3) , we have $\frac{p-1}{2p}\left(\|u_0\|^2_{H_0^1(\Omega)} + \|v_0\|^2_{H_0^1(\Omega)}\right) + \frac{1}{p^2}\|u_0 v_0\|_p^p < 0$, which is also a contradiction.
(iii) If $J(u_0, v_0) < 0$ and $I(u_0, v_0) \ge 0$, then it is contradictive with (2.3).

From the discussions above, we consider the case $0 < J(u_0, v_0) < d$ and $I(u_0, v_0) > 0$. It is widely know that there is a basis $\{\omega_j(x)\}_{j=1}^\infty$ of $H_0^1(\Omega)$ such that ω_j is an eigenfunction of the Laplacian operator corresponding to the eigenvalue λ_j and

$$\begin{cases} -\Delta\omega_j = \lambda_j \omega_j, & x \in \Omega, \\ \omega_j = 0, & x \in \partial\Omega. \end{cases}$$

Hence, we choose $\{\omega_j(x)\}_{j=1}^\infty$ as the Galerkin basis for $-\Delta$ in $H_0^1(\Omega)$. Then we construct the Galerkin approximate solution $(u_m(x,t), v_m(x,t))$ of the problem (1.1),

$$\begin{cases} u_m(x,t) = \sum_{j=1}^m g_{jm}(t)\omega_j(x), \ m = 1, 2, \cdots, \\ v_m(x,t) = \sum_{j=1}^m h_{jm}(t)\omega_j(x), \ m = 1, 2, \cdots, \end{cases}$$

which satisfy, for $j = 1, 2, \cdots, m$,

$$(u_{mt}, \omega_j) + (\nabla u_m, \nabla \omega_j) = (|v_m|^p |u_m|^{p-2} u_m \log(|u_m v_m|), \omega_j) \tag{3.1}$$

and

$$(v_{mt}, \omega_j) + (\nabla v_m, \nabla \omega_j) = (|u_m|^p |v_m|^{p-2} v_m \log(|u_m v_m|), \omega_j), \tag{3.2}$$

with initial condition $u_m(x, 0) = u_{0m}$, $v_m(x, 0) = v_{0m}$, where u_{0m} and v_{0m} are chosen in span$\{\omega_1, \omega_2, \cdots, \omega_m\}$ so that

$$u_{0m} = \sum_{j=1}^{m} g_{jm}(0)\omega_j(x) \to u_0 \text{ in } H_0^1(\Omega), \text{ as } m \to +\infty \tag{3.3}$$

and

$$v_{0m} = \sum_{j=1}^{m} h_{jm}(0)\omega_j(x) \to v_0 \text{ in } H_0^1(\Omega), \text{ as } m \to +\infty. \tag{3.4}$$

According to the standard ordinary differential equation theory, the system (3.1)–(3.4) admit a solution

$$(g_{jm}(t), h_{jm}(t)) \in C^1[0, T_0) \times C^1[0, T_0),$$

where T_0 is the minimum of the existence time of $g_{jm}(t)$ and $h_{jm}(t)$ for each m. Thus $(u_m(x, t), v_m(x, t)) \in C^1\Big([0, T_0); H_0^1(\Omega) \times H_0^1(\Omega)\Big)$.

Next, multiplying (3.1) and (3.2) by $g'_{jm}(t)$ and $h'_{jm}(t)$, respectively, summing for j from 1 to m, integrating with respect to t from 0 to t and adding these two equations, we get

$$\int_0^t \|u_{m\tau}\|_2^2 + \|v_{m\tau}\|_2^2 \, d\tau + J(u_m, v_m) = J(u_{0m}, v_{0m}), \ 0 \le t < T_0. \tag{3.5}$$

From $J(u_0, v_0) < d$ and (3.3)–(3.4), we see that $J(u_{0m}, v_{0m}) < d$ for sufficiently large m. Then we get from (3.5) that

$$\int_0^t \|u_{m\tau}\|_2^2 + \|v_{m\tau}\|_2^2 \, d\tau + J(u_m, v_m) = J(u_{0m}, v_{0m}) < d, \ 0 \le t < T_0, \tag{3.6}$$

for sufficiently large m.

By (3.3) and (3.4) and $(u_0, v_0) \in W$, we know that $(u_{0m}, v_{0m}) \in W$ for large enough m. Next, we prove $(u_m(x, t), v(x, t)) \in W$ for large enough m and $0 \le t < T_0$. If it is false, then there exists $t_0 \in (0, T_0)$ such that $(u_m(x, t_0), v_m(x, t_0)) \in \partial W$, then $I(u_m(t_0), v_m(t_0)) = 0$ and $(u_m(t_0), v_m(t_0)) \neq (0, 0)$, or $J(u_m(t_0), v_m(t_0)) = d$.

By (3.6), $J(u_m(t_0), v_m(t_0)) = d$ is not true. On the other hand, if $I(u_m(t_0), v_m(t_0)) = 0$ and $(u_m(t_0), v_m(t_0)) \neq (0, 0)$, then by the definition of d, we have $J(u_m(t_0), v_m(t_0)) \ge d$, which is also contradiction with (3.6). So $(u_m(x, t), v_m(x, t)) \in W$ for large enough m and $0 \le t < T_0$.

From the fact $\Big(u_m(x, t), v_m(x, t)\Big) \in W$ for large enough m, (3.6) and

$$J(u_m(t), v_m(t)) = \frac{p-1}{2p}\Big(\|u_m(t)\|_{H_0^1(\Omega)}^2 + \|v_m(t)\|_{H_0^1(\Omega)}^2\Big) + \frac{1}{p^2}\|u_m(t)v_m(t)\|_p^p + \frac{1}{2p} I(u_m(t), v_m(t)),$$

we obtain

$$\int_0^t \|u_{m\tau}\|_2^2 + \|v_{m\tau}\|_2^2 d\tau + \frac{p-1}{2p}\Big(\|u_m(t)\|^2_{H_0^1(\Omega)} + \|v_m(t)\|^2_{H_0^1(\Omega)}\Big) + \frac{1}{p^2}\|u_m(t)v_m(t)\|_p^p < d,\ 0 \le t < T_0, \quad (3.7)$$

for sufficiently large m, which gives

$$\|u_m(t)\|^2_{H_0^1(\Omega)} < \frac{2p}{p-1}d, \quad (3.8)$$

$$\|v_m(t)\|^2_{H_0^1(\Omega)} < \frac{2p}{p-1}d, \quad (3.9)$$

$$\int_0^t \|u_{m\tau}\|_2^2 + \|v_{m\tau}\|_2^2 d\tau < d. \quad (3.10)$$

By (3.10), we know that $T_0 = +\infty$. Then by (3.8)–(3.10), there exist u, v with theirs subsequences of $\{u_m\}_{j=1}^{+\infty}$ and $\{v_m\}_{j=1}^{+\infty}$, such that, as $m \to +\infty$,

$$u_m \to u \text{ weakly star in } L^\infty\left(0, +\infty; H_0^1(\Omega)\right), \quad (3.11)$$

$$v_m \to v \text{ weakly star in } L^\infty\left(0, +\infty; H_0^1(\Omega)\right), \quad (3.12)$$

$$u_{mt} \to u_t \text{ weakly in } L^2\left(0, +\infty; L^2(\Omega)\right), \quad (3.13)$$

$$v_{mt} \longrightarrow v_t \text{ weakly in } L^2\left(0, +\infty; L^2(\Omega)\right). \quad (3.14)$$

Then it follows Aubin-lions compactness theorem [27] that

$$u_m \to u \text{ strongly in } C\left([0, +\infty); L^2(\Omega)\right),$$

$$v_m \to v \text{ strongly in } C\left([0, +\infty); L^2(\Omega)\right).$$

Clearly, this implies that

$$u_m \to u \text{ a.e. in } \Omega \times [0, +\infty),$$

$$v_m \to v \text{ a.e. in } \Omega \times [0, +\infty).$$

Furthermore, we get

$$|v_m|^p|u_m|^{p-2}u_m \log(|u_m v_m|) \to |v|^p|u|^{p-2}u \log(|uv|) \text{ a.e. in } \Omega \times [0, +\infty), \quad (3.15)$$

$$|u_m|^p|v_m|^{p-2}v_m \log(|u_m v_m|) \to |u|^p|v|^{p-2}v \log(|uv|) \text{ a.e. in } \Omega \times [0, +\infty). \quad (3.16)$$

On the other hand,

$$\begin{aligned}
&\int_\Omega \left||v_m|^p|u_m|^{p-2}u_m \log(|u_m v_m|)\right|^{\frac{p}{p-1}} dx = \int_\Omega \Big(|v_m|^p|u_m|^{p-1} \log(|u_m v_m|)\Big)^{\frac{p}{p-1}} dx \\
&= \int_{\{x\in\Omega: |u_m(x)v_m(x)|\le 1\}} \Big(|v_m|^p|u_m|^{p-1} \log(|u_m v_m|)\Big)^{\frac{p}{p-1}} dx + \int_{\{x\in\Omega: |u_m(x)v_m(x)|> 1\}} \Big(|v_m|^p|u_m|^{p-1} \log(|u_m v_m|)\Big)^{\frac{p}{p-1}} dx \\
&\le \int_\Omega (e(p-1)^{-1}|v_m|)^{\frac{p}{p-1}} dx + \int_\Omega |u_m|^{\frac{(p-1+r)p}{p-1}} |v_m|^{\frac{(p+r)p}{p-1}} dx \\
&\le e(p-1)^{-\frac{p}{p-1}} \|v_m\|^{\frac{p}{p-1}}_{\frac{p}{p-1}} + \|u_m\|^e_{2e}\|v_m\|^\sigma_{2\sigma} \\
&\le c\|v_m\|^{\frac{p}{p-1}}_{H_0^1(\Omega)} + c\|u_m\|^e_{H_0^1(\Omega)} \|v_m\|^\sigma_{H_0^1(\Omega)} \le c,
\end{aligned} \quad (3.17)$$

where $\sigma = \frac{(p+r)p}{p-1}$, $e = \frac{(p-1+r)p}{p-1}$, and since $|x^{q-1}\log x| \le (e(q-1))^{-1}$ for $0 < x < 1$ while $x^{-\mu}\log x \le (e\mu)^{-1}$ for $x \ge 1, \mu > 0$. Choosing a positive real number r so that $0 < 2e < 2^*$ and $0 < 2\sigma < 2^*$, and similar to the proof (3.17), we have

$$\int_\Omega \left||u_m|^p |v_m|^{p-2} v_m \log(|u_m v_m|)\right|^{\frac{p}{p-1}} dx \le c. \tag{3.18}$$

Hence, from (3.15)–(3.18) and Lion's Lemma(see [27], Lemma 1.3, p.12), we have

$$|v_m|^p |u_m|^{p-2} u_m \log(|u_m v_m|) \to |v|^p |u|^{p-2} u \log(|uv|) \text{ weakly star in } L^\infty\left(0,+\infty; L^{\frac{p}{p-1}}(\Omega)\right), \tag{3.19}$$

$$|u_m|^p |v_m|^{p-2} v_m \log(|u_m v_m|) \to |u|^p |v|^{p-2} v \log(|uv|) \text{ weakly star in } L^\infty\left(0,+\infty; L^{\frac{p}{p-1}}(\Omega)\right). \tag{3.20}$$

In view of (3.11)–(3.14) and (3.19),(3.20), for j fixed, we can pass to the limit in (3.1) and (3.2) to get

$$(u_t, \omega_j) + (\nabla u, \nabla \omega_j) = (|v|^p |u|^{p-2} u \log(|uv|), \omega_j)$$

and

$$(v_t, \omega_j) + (\nabla v, \nabla \omega_j) = (|u|^p |v|^{p-2} v \log(|uv|), \omega_j)$$

for a.e., $t \in (0,+\infty)$. Since $\{\omega_j(x)\}_{j=1}^\infty$ is the basis in $H_0^1(\Omega)$, we have

$$(u_t, w_1) + (\nabla u, \nabla w_1) = \left(|v|^p |u|^{p-2} u \log(|uv|), w_1\right) \tag{3.21}$$

and

$$(v_t, w_2) + (\nabla v, \nabla w_2) = \left(|u|^p |v|^{p-2} v \log(|uv|), w_2\right) \tag{3.22}$$

for any $w_1, w_2 \in H_0^1(\Omega)$ and a.e., $t \in (0,+\infty)$.

Fixing any $t \in (0,+\infty)$ and integrating (3.21) and (3.22) from 0 to t, we get

$$(u, w_1) + \int_0^t (\nabla u, \nabla w_1)\, d\tau = \int_0^t \left(|v|^p |u|^{p-2} u \log(|uv|), w_1\right) d\tau + (u(0), w_1),\ \forall w_1 \in H_0^1(\Omega), \tag{3.23}$$

and

$$(v, w_2) + \int_0^t (\nabla v, \nabla w_2)\, d\tau = \int_0^t \left(|u|^p |v|^{p-2} v \log(|uv|), w_2\right) d\tau + (v(0), w_2),\ \forall w_2 \in H_0^1(\Omega). \tag{3.24}$$

Similarly, integrating (3.1) and (3.2) from 0 to t, and passing to the limit, we get

$$(u, w_1) + \int_0^t (\nabla u_m, \nabla w_1)\, d\tau = \int_0^t \left(|v_m|^p |u_m|^{p-2} u_m \log(|u_m v_m|), w_1\right) d\tau + (u_0, w_1),\ \forall w_1 \in H_0^1(\Omega), \tag{3.25}$$

and

$$(v, w_2) + \int_0^t (\nabla v_m, \nabla w_2)\, d\tau = \int_0^t \left(|u_m|^p |v_m|^{p-2} v_m \log(|u_m v_m|), w_2\right) d\tau + (v_0, w_2),\ \forall w_2 \in H_0^1(\Omega). \tag{3.26}$$

From (3.23)–(3.26), we get $u(0) = u_0$ and $v(0) = v_0$.

According to (3.3), (3.4), (3.7), (3.11)–(3.14), (3.19), (3.20) and since the norm is weakly lower semicontinuous, we know that the energy inequality (2.5) holds. Then from Definition 2, $(u(t), v(t))$ is a global weak solution and $(u(t), v(t)) \in W$.

Next, we will prove the algebraic decay of the global solution $u(x, t)$. Combining (2.3), (2.5) and $(u(t), v(t)) \in W$, we have

$$\frac{p-1}{2p}\Big(\|u\|^2_{H^1_0(\Omega)} + \|v\|^2_{H^1_0(\Omega)}\Big) + \frac{1}{p^2}\|uv\|^p_p \le J(u(t)) \le J(u_0). \tag{3.27}$$

As $I(u, v) < 0$, then there exists a $\lambda_* \in (0, 1)$ such that $I(\lambda_* u, \lambda_* v) = 0$. Furthermore, we get

$$\lambda^p_*\left(\frac{p-1}{2p}\Big(\|u\|^2_{H^1_0(\Omega)} + \|v\|^2_{H^1_0(\Omega)}\Big)\right) + \frac{1}{p^2}\|uv\|^p_p \ge J(\lambda_* u(t)) \ge d. \tag{3.28}$$

It follows from (3.27) and (3.28) that

$$\lambda_* \ge \left(\frac{d}{J(u_0, v_0)}\right)^{\frac{1}{p}}. \tag{3.29}$$

Due to the $I(\lambda_* u, \lambda_* v) = 0$, we have

$$\begin{aligned} I(\lambda_* u, \lambda_* v) &= \lambda^2_*\Big(\|u\|^2_{H^1_0(\Omega)} + \|v\|^2_{H^1_0(\Omega)}\Big) - 2\lambda^{2p}_* \int_\Omega |uv|^p \log|uv| dx - 2\lambda^{2p}_* \log \lambda^2_* \|uv\|^p_p \\ &= (\lambda^2_* - 2\lambda^{2p}_*)\Big(\|u\|^2_{H^1_0(\Omega)} + \|v\|^2_{H^1_0(\Omega)}\Big) + 2\lambda^{2p}_* I(u, v) - 2\lambda^{2p}_* \log \lambda^2_* \|uv\|^p_p \\ &= 0, \end{aligned}$$

i.e.,

$$I(u, v) \ge \Big(1 - \frac{1}{2\lambda^{2(p-1)}_*}\Big)(\|u\|^2_{H^1_0(\Omega)} + \|v\|^2_{H^1_0(\Omega)}). \tag{3.30}$$

Combining (3.29) with (3.30), we have

$$I(u, v) \ge \left(1 - \frac{1}{2}\big(\frac{d}{J(u_0, v_0)}\big)^{2(\frac{1}{p}-1)}\right)(\|u\|^2_{H^1_0(\Omega)} + \|v\|^2_{H^1_0(\Omega)}).$$

By the emdedding $H^1_0(\Omega) \hookrightarrow L^2(\Omega)$, we have

$$I(u, v) \ge c(\|u\|^2_{L^2(\Omega)} + \|v\|^2_{L^2(\Omega)}). \tag{3.31}$$

On the other hand, by Lemma 4, we know

$$\frac{1}{2}\frac{d}{dt}(\|u\|^2_2 + \|v\|^2_2) + I(u, v) = 0,\ 0 \le t < \infty.$$

Combining this equality with (3.31), we get

$$\frac{1}{2}\frac{d}{dt}(\|u\|^2_2 + \|v\|^2_2) + c(\|u\|^2_{L^2(\Omega)} + \|v\|^2_{L^2(\Omega)}) \le 0,\ 0 \le t < \infty.$$

By Grönwall's inequality, we have

$$\|u\|^2_2 + \|v\|^2_2 \le (\|u_0\|^2_2 + \|v_0\|^2_2)e^{-2ct},\ 0 \le t < \infty.$$

The proof of Theorem 1 is complete.

Theorem 2. *Assume* $(u_0, v_0) \in H_0^1(\Omega) \times H_0^1(\Omega)$ *and (1.2) hold. If* $J(u_0, v_0) < d$ *and* $I(u_0, v_0) < 0$*, then the weak solution* $(u(x,t), v(x,t))$ *of the problem (1.1) blows up in finite time, i.e., there exists a* $T > 0$ *such that*

$$\lim_{t\to T} \int_0^t \|u\|_2^2 + \|v\|_2^2 d\tau = +\infty.$$

Proof. Step 1: Blow-up in finite time

By contradiction, we suppose that $(u(t), v(t))$ is global weak solution of problem (1.1), then $T_{max} = +\infty$. Let

$$G(t) = \int_0^t \|u\|_2^2 + \|v\|_2^2 d\tau,$$

then

$$G'(t) = \|u\|_2^2 + \|v\|_2^2$$

and

$$G''(t) = 2\left((u, u_t) + (v, v_t)\right) = -2\left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2\right) + 4\int_\Omega |u|^p|v|^p \log(|uv|)dx = -2I(u, v). \tag{3.32}$$

From (3.32) and energy inequality (2.5), it follows that

$$\begin{aligned} G''(t) &\simeq 2(p-1)\Big(\|u\|_{H_0^1(\Omega)}^2 + \|v\|_{H_0^1(\Omega)}^2\Big) + \frac{4}{p}\|uv\|_p^p - 4pJ(u, v) \\ &\geq 2(p-1)\Big(\|u\|_{H_0^1(\Omega)}^2 + \|v\|_{H_0^1(\Omega)}^2\Big) + \frac{4}{p}\|uv\|_p^p - 4pJ(u_0, v_0) + 4p\int_0^t \|u_\tau\|_2^2 + \|v_\tau\|_2^2 d\tau \\ &\geq 4p\int_0^t \|u_\tau\|_2^2 + \|v_\tau\|_2^2 d\tau + 2(p-1)cG'(t) - 4pJ(u_0, v_0), \end{aligned} \tag{3.33}$$

where the constant c is from the Poincaré inequality $\|u\|_2^2 \leq c\|\nabla u\|_2^2$.

Note that

$$\begin{aligned} &\left(\int_0^t (u_\tau, u) + (v_\tau, v)d\tau\right)^2 \\ &= \left(\frac{1}{2}\int_0^t \frac{d}{d\tau}(\|u\|_2^2 + \|v\|_2^2)d\tau\right)^2 \\ &= \left(\frac{1}{2}(\|u\|_2^2 + \|v\|_2^2 - \|u_0\|_2^2 - \|v_0\|_2^2)\right)^2 \\ &= \frac{1}{4}[(\|u\|_2^2 + \|v\|_2^2)^2 - 2(\|u_0\|_2^2 + \|v_0\|_2^2)(\|u\|_2^2 + \|v\|_2^2) + (\|u_0\|_2^2 + \|v_0\|_2^2)^2] \\ &= \frac{1}{4}[(G'(t))^2 - 2G'(t)(\|u_0\|_2^2 + \|v_0\|_2^2) + (\|u_0\|_2^2 + \|v_0\|_2^2)^2], \end{aligned}$$

then

$$G'(t) = 4\left(\int_0^t (u_\tau, u) + (v_\tau, v)d\tau\right)^2 + 2G'(t)(\|u_0\|_2^2 + \|v_0\|_2^2) - (\|u_0\|_2^2 + \|v_0\|_2^2)^2. \tag{3.34}$$

Hence by (3.33) and (3.34) we know that

$$G(t)G''(t) - p(G'(t))^2 \geqslant 4p \int_0^t \|u_\tau\|_2^2 + \|v_\tau\|_2^2 d\tau \int_0^t \|u\|_2^2 + \|v\|_2^2 d\tau - 4p\left(\int_0^t (u_\tau, u) + (v_\tau, v) d\tau\right)^2$$
$$+ 2(p-1)cG(t)G'(t) - 4pG(t)J(u_0, v_0) - 2pG'(t)(\|u_0\|_2^2 + \|v_0\|_2^2) + p(\|u_0\|_2^2 + \|v_0\|_2^2)^2.$$

By Schwartz inequality, we have

$$\int_0^t \|u_\tau\|_2^2 + \|v_\tau\|_2^2 d\tau \int_0^t \|u\|_2^2 + \|v\|_2^2 d\tau - \left(\int_0^t (u_\tau, u) + (v_\tau, v) d\tau\right)^2$$
$$\geq \int_0^t \|u_\tau\|_2^2 + \|v\tau_2^2 d\tau \int_0^t \|u\|_2^2 + \|v\|_2^2 d\tau - \left(\int_0^t \|u\|_2 \|u_\tau\|_2 + \|v\|_2 \|v\tau\|_2 d\tau\right)^2$$
$$\geq \int_0^t \|u_\tau\|_2^2 + \|v_\tau\|_2^2 d\tau \int_0^t \|u\|_2^2 + \|v\|_2^2 d\tau - \left(\int_0^t \sqrt{\|u_\tau\|_2^2 + \|v_\tau\|_2^2} \sqrt{\|u\|_2^2 + \|v\|_2^2} d\tau\right)^2$$
$$\geq 0.$$

It implies that

$$G(t)G''(t) - p(G'(t))^2 \geq 2(p-1)cG'(t)G(t) - 4pJ(u_0, v_0)G(t) - 2pG'(t)(\|u_0\|_2^2 + \|v_0\|_2^2). \tag{3.35}$$

From Lemma 3 we have $I(u(t), v(t)) < 0$ for $0 \leq t < +\infty$. Thus from Lemma 2 one has

$$-2I(u(t), v(t)) > 4p(d - J(u(t), v(t))),\ 0 \leq t < +\infty. \tag{3.36}$$

Combing (3.36) and (2.5) we get

$$G''(t) = -2I(u, v) > 4p(d - J(u, v)) \geq 4p(d - J(u_0, v_0)) := C_1 > 0,\ 0 \leq t < +\infty$$

and

$$G'(t) \geq C_1 t + G'(0) = C_1 t,\ 0 \leq t < +\infty,$$
$$G(t) \geq \frac{1}{2} C_1 t^2 + G(0) = \frac{1}{2} C_1 t^2,\ 0 \leq t < +\infty.$$

Hence for sufficiently large t, we have

$$(p-1)cG(t) > 2p(\|u_0\|_2^2 + \|v_0\|_2^2) \text{ and } (p-1)cG'(t) > 4pJ(u_0, v_0). \tag{3.37}$$

Combining (3.35) with (3.37), we obtain

$$G(t)G''(t) - p(G'(t))^2 \geq \left((p-1)cG(t) - 2p(\|u_0\|_2^2 + \|v_0\|_2^2)\right) G'(t)$$
$$+ \left((p-1)cG'(t) - 4pJ(u_0, v_0)\right) G(t) > 0,$$

for sufficiently large t. Note that

$$\left(G^{-(p-1)}(t)\right)'' = \frac{-(p-1)}{G^{p+1}(t)} \left(G(t)G''(t) - p(G'(t))^2\right) < 0.$$

It follows that there exists a finite time $T > 0$ such that $\lim_{t\to T} G^{-(p-1)}(t) = 0$, i.e., $\lim_{t\to T} \int_0^t \|u\|_2^2 + \|v\|_2^2 d\tau = +\infty$.

Step 2: Upper bound estimation of the blow-up time.

We next give an upper bound estimate of T. Suppose $(u(t), v(t))$ be a solution of problem (1.1) with initial value (u_0, v_0) satisfying $I(u_0, v_0) < 0$ and $J(u_0, v_0) < d$. By Step 1, the maximal existence time $T < \infty$. By Lemma 3, we get $(u(t), v(t)) \in V$,$\forall t \in [0, T)$, i.e., $I(u(t), v(t)) < 0, t \in [0, T)$. For $T_1 \in (0, T)$, we define the auxiliary functional $M : [0, T_1] \to R$ which is defined by

$$M(t) := \int_0^t \|u\|_2^2 + \|v\|_2^2 d\tau + (T - t)(\|u_0\|_2^2 + \|v_0\|_2^2) + \beta(t + \gamma)^2, \tag{3.38}$$

with $\beta > 0$ and $\gamma > 0$ specified later. Through a direct calculation, we have

$$\begin{aligned} M'(t) &= \|u(t)\|_2^2 + \|v(t)\|_2^2 - (\|u_0\|_2^2 + \|v_0\|_2^2) + 2\beta(t + \gamma) \\ &= 2\int_0^t (u_\tau, u) + (v_\tau, v) d\tau + 2\beta(t + \gamma) \end{aligned} \tag{3.39}$$

and

$$M''(t) = 2((u_t, u) + (v_t, v)) + 2\beta = 2\beta - 2I(u, v).$$

It follows from Lemma 2 and (2.5) that

$$M''(t) > 4p(d - J(u, v)) + 2\beta \geq 4p(d - J(u_0, v_0)) + 4p\int_0^t \|u_\tau\|_2^2 + \|v_\tau\|_2^2 d\tau + 2\beta. \tag{3.40}$$

From (3.39) and Hölder inequality, we have

$$\begin{aligned} (M'(t))^2 &= 4\left[\int_0^t (u_\tau, u) + (v_\tau, v) d\tau + 2\beta(t + \gamma)\right]^2 \\ &\leq 4\left[\int_0^t \|u_\tau\|_2\|u\|_2 + \|v_\tau\|_2\|v\|_2 d\tau + 2\beta(t + \gamma)\right]^2. \end{aligned} \tag{3.41}$$

According to the inequality

$$xz + yw \leq (x^2 + y^2)^{\frac{1}{2}}(z^2 + w^2)^{\frac{1}{2}},$$

by setting $x = \|u_\tau\|_2$, $y = \|v_\tau\|_2$, $z = \|u\|_2$, $w = \|v\|_2$ in (3.41), we get

$$(M'(t))^2 \leq 4\left[\int_0^t (\|u_\tau\|_2^2 + \|v_\tau\|_2^2)^{\frac{1}{2}}(\|u\|_2^2 + \|v\|_2^2)^{\frac{1}{2}} d\tau + 2\beta(t + \gamma)\right]^2.$$

By the Hölder inequality, we get

$$\begin{aligned} (M'(t))^2 &\leq 4\left[\int_0^t (\|u_\tau\|_2^2 + \|v_\tau\|_2^2)^{\frac{1}{2}}(\|u\|_2^2 + \|v\|_2^2)^{\frac{1}{2}} d\tau + 2\beta(t + \gamma)\right]^2 \\ &\leq 4\left[\left(\int_0^t \|u_\tau\|_2^2 + \|v_\tau\|_2^2 d\tau\right)^{\frac{1}{2}}\left(\int_0^t \|u\|_2^2 + \|v\|_2^2 d\tau\right)^{\frac{1}{2}} + 2\beta(t + \gamma)\right]^2 \\ &\leq 4\left[\int_0^t \|u_\tau\|_2^2 + \|v_\tau\|_2^2 d\tau + \beta\right]\left[\int_0^t \|u\|_2^2 + \|v\|_2^2 d\tau d\tau + \beta(t + \gamma)^2\right] \\ &\leq 4M(t)\left[\int_0^t \|u_\tau\|_2^2 + \|v_\tau\|_2^2 d\tau + \beta\right]. \end{aligned} \tag{3.42}$$

From (3.38), (3.40) and (3.42), we have

$$M(t)M''(t) - p(M'(t))^2 \geq [4p(d - J(u_0, v_0)) - 2(2p - 1)\beta]M(t).$$

Restricting β to satisfy

$$0 < \beta \leq \frac{2p(d - J(u_0, v_0))}{2p - 1}, \tag{3.43}$$

we have

$$M(t)M''(t) - p(M'(t))^2 \geq 0, t \in [0, T_1].$$

Define $y(t) := M^{1-p}(t)$ for $t \in [0, T_1]$, then by $M(t) > 0, M'(t) > 0$ we get

$$y'(t) = -(p - 1)M^{-p}(t)M'(t) < 0,$$

$$y''(t) = -(p - 1)M^{-p-1}(t)\left(M''(t)M(t) - p(M'(t))^2\right) < 0$$

for all $t \in [0, T_1]$. It follows from $y''(t) < 0$ that

$$y(T_1) - y(0) = y'(\xi)T_1 < y'(0)T_1,\ \xi \in (0, T_1), \tag{3.44}$$

where

$$y(0) = M^{1-p}(0) > 0,\ y(T_1) = M^{1-p}(T_1) > 0,$$

$$y'(0) = -(p - 1)M^{-p}(0)M'(0) = 2(1 - p)\beta\gamma M^{-p}(0) < 0.$$

Combining (3.44) and the above inequalities, we can deduce

$$T_1 \leq \frac{y(T_1)}{y'(0)} - \frac{y(0)}{y'(0)} < -\frac{y(0)}{y'(0)} = \frac{M(0)}{2(p - 1)\beta\gamma}.$$

Then by the definition of $M(t)$ and above inequality we have

$$T_1 \leq \frac{T(\|u_0\|_2^2 + \|v_0\|_2^2) + \beta\gamma^2}{2(p - 1)\beta\gamma} = \frac{\gamma}{2(p - 1)} + \frac{\|u_0\|_2^2 + \|v_0\|_2^2}{2(p - 1)\beta\gamma}T.$$

Hence, letting $T_1 \to T$, we get

$$T \leq \frac{\gamma}{2(p - 1)} + \frac{\|u_0\|_2^2 + \|v_0\|_2^2}{2(p - 1)\beta\gamma}T. \tag{3.45}$$

For any β satisfying (3.43), let γ be large enough such that

$$\frac{\|u_0\|_2^2 + \|v_0\|_2^2}{2(p - 1)\beta} < \gamma < +\infty, \tag{3.46}$$

then (3.45) lead to

$$T \leq \frac{\gamma}{2(p - 1)}\left(1 - \frac{\|u_0\|_2^2 + \|v_0\|_2^2}{2(p - 1)\beta\gamma}\right)^{-1} = \frac{\beta\gamma^2}{2(p - 1)\beta\gamma - (\|u_0\|_2^2 + \|v_0\|_2^2)}.$$

Let

$$\rho(\beta,\gamma) = \frac{\beta\gamma^2}{2(p-1)\beta\gamma - (\|u_0\|_2^2 + \|v_0\|_2^2)},$$

then

$$T \le \min_{(\beta,\gamma)\in\Phi} \rho(\beta,\gamma),$$

where $\Phi = \{(\beta,\gamma) : \beta, \gamma$ satisfy (3.43) and (3.46) respectively$\}$.
Since

$$\rho'_\beta(\beta,\gamma) = -\frac{\gamma^2(\|u_0\|_2^2 + \|v_0\|_2^2)}{\left(2(p-1)\beta\gamma - (\|u_0\|_2^2 + \|v_0\|_2^2)\right)^2} < 0,$$

i.e., $\rho(\beta,\gamma)$ is decreasing with respect to β. Then we have

$$\min_{(\beta,\gamma)\in\Phi} \rho(\beta,\gamma) = \rho\left(\frac{2p(d - J(u_0,v_0))}{2p-1}, \gamma\right) := \rho_1(\gamma),$$

where

$$\rho_1(\gamma) = \frac{2p(d - J(u_0,v_0))\gamma^2}{4p(p-1)\gamma(d - J(u_0,v_0)) - (2p-1)(\|u_0\|_2^2 + \|v_0\|_2^2)}$$

and

$$\frac{(2p-1)(\|u_0\|_2^2 + \|v_0\|_2^2)}{4p(p-1)(d - J(u_0,v_0))} < \gamma < +\infty.$$

It is easy to get that $\rho_1(\gamma)$ achieves its minimum at

$$\gamma_1 = \frac{(2p-1)(\|u_0\|_2^2 + \|v_0\|_2^2)}{2p(p-1)(d - J(u_0,v_0))},$$

and

$$\rho_1(\gamma_1) = \frac{(2p-1)(\|u_0\|_2^2 + \|v_0\|_2^2)}{2p(p-1)^2(d - J(u_0,v_0))}.$$

Thus, we have

$$T \le \frac{(2p-1)(\|u_0\|_2^2 + \|v_0\|_2^2)}{2p(p-1)^2(d - J(u_0,v_0))}.$$

The proof of Theorem 2 is complete.

4. Critical initial energy $J(u_0, v_0) = d$

In this section, we prove global existence and blow up at finite time of solutions for problem (1.1) with the initial energy $J(u_0, v_0) = d$.

Theorem 3. *Assume $(u_0, v_0) \in H_0^1(\Omega) \times H_0^1(\Omega)$ and (1.2) hold. If $J(u_0, v_0) = d$ and $I(u_0, v_0) \ge 0$, then the problem (1.1) has a global solution $(u(t), v(t)) \in L^\infty(0, +\infty; H_0^1(\Omega) \times H_0^1(\Omega))$ with $(u_t(t), v_t(t)) \in L^2(0, +\infty; L^2(\Omega) \times L^2(\Omega))$ and $(u(t), v(t)) \in \overline{W} = W \cup \partial W$ for $0 \le t < \infty$.*

Proof. Since $J(u_0, v_0) = d$, then $(u_0, v_0) \neq (0, 0)$. Let $\lambda_m = 1 - \frac{1}{m}$, $(u_{0m}, v_{0m}) = \lambda_m(u_0, v_0)$, $m = 1, 2, \cdots$, and consider the following problem:

$$\begin{cases} u_{mt} - \Delta u_m = |v_m|^p |u_m|^{p-2} u_m \log(|u_m v_m|), & x \in \Omega,\ t > 0, \\ v_{mt} - \Delta v_m = |u_m|^p |v_m|^{p-2} v_m \log(|u_m v_m|), & x \in \Omega,\ t > 0, \\ u_m(x, 0) = u_{0m}(x), & x \in \Omega, \\ v_m(x, 0) = v_{0m}(x), & x \in \Omega, \\ u_m(x, t) = v_m(x, t) = 0, & (x, t) \in \partial\Omega \times (0, T]. \end{cases} \tag{4.1}$$

By $I(u_0, v_0) \geq 0$ and Lemma 1, there exists a unique $\lambda^* \geq 1$ such that $I(\lambda^* u_0, \lambda^* v_0) = 0$. Due to the $\lambda_m < 1 < \lambda^*$, we get $I(\lambda_m u_0, \lambda_m v_0) > 0, J(\lambda_m u_0, \lambda_m v_0) < J(u_0, v_0) = d$. From Theorem 1, it follows that for each m problem (4.1) admits a global solution $(u_m(t), v_m(t)) \in L^\infty(0, +\infty; H_0^1(\Omega) \times H_0^1(\Omega))$ with $(u_{mt(t)}, v_{mt}(t)) \in L^2(0, +\infty; L^2(\Omega) \times L^2(\Omega))$ with the initial data

$$u_m(0) = u_{0m} \to u_0 \text{ in } H_0^1(\Omega) \text{ as } m \to +\infty.$$

Furthermore, we have $(u_m(t), v_m(t)) \in W$ for $0 \leq t < +\infty$,

$$(u_{mt}, w_1) + (\nabla u_m, \nabla w_1) = \left(|v_m|^p |u_m|^{p-2} u_m \log(|u_m v_m|), w_1\right),\ \forall w_1 \in H_0^1(\Omega),\ 0 \leq t < +\infty,$$

$$(v_{mt}, w_2) + (\nabla v_m, \nabla w_2) = \left(|u_m|^p |v_m|^{p-2} v_m \log(|u_m v_m|), w_2\right),\ \forall w_2 \in H_0^1(\Omega),\ 0 \leq t < +\infty,$$

and

$$\int_0^t \|u_{m\tau}\|_2^2 + \|v_{m\tau}\|_2^2 \, d\tau + J\left(u_m, v_m\right) \leq J\left(u_{0m}, v_{0m}\right) < d,\ 0 \leqslant t < +\infty. \tag{4.2}$$

From (4.2) and

$$J(u_m(t), v_m(t)) = \frac{p-1}{2p}\left(\|u_m(t)\|_{H_0^1(\Omega)}^2 + \|v_m(t)\|_{H_0^1(\Omega)}^2\right) + \frac{1}{p^2}\|v_m(t)\|_p^p + \frac{1}{2p} I(u_m(t), v_m(t)),$$

we obtain

$$\int_0^t \|u_{m\tau}\|_2^2 + \|v_{m\tau}\|_2^2 d\tau + \frac{p-1}{2p}\left(\|u_m(t)\|_{H_0^1(\Omega)}^2 + \|v_m(t)\|_{H_0^1(\Omega)}^2\right) + \frac{1}{p_2}\|u_m(t) v_m(t)\|_p^p < d, 0 \leqslant t < +\infty.$$

The remainder of the proof is similar to that in the proof of Theorem 1.

The proof of Theorem 3 is complete.

Theorem 4. *Assume* $(u_0, v_0) \in H_0^1(\Omega) \times H_0^1(\Omega)$ *and (1.2) hold. If* $J(u_0, v_0) = d$ *and* $I(u_0, v_0) < 0$*, then the weak solution* $(u(x, t), v(x, t))$ *of the problem (1.1) blows up in finite time, i.e., there exists a* $T > 0$ *such that*

$$\lim_{t \to T} \int_0^t \|u\|_2^2 + \|v\|_2^2 d\tau = +\infty.$$

Proof. By contradiction, we suppose that $(u(t), v(t))$ is a global weak solution of problem (1.1), then $T_{max} = +\infty$. Let

$$G(t) = \int_0^t \|u\|_2^2 + \|v\|_2^2 d\tau.$$

Taking into account to (3.35) still holds, combining the fact $J(u_0, v_0) = d$, we have

$$G(t)G''(t) - p(G'(t))^2 \geq \Big((p-1)cG(t) - 2p(\|u_0\|_2^2 + \|v_0\|_2^2)\Big)G'(t) + \Big((p-1)cG'(t) - 4pd\Big)G(t). \tag{4.3}$$

From continuities of $J(u, v)$ and $I(u, v)$ with respect to t, we know that there exists a sufficient small $t_1 \in (0, +\infty)$ such that $J(u(t_1), v(t_1)) > 0$ and $I(u, v) < 0$ for $0 < t < t_1$. By $(u_t, u) + (v_t, v) = -I(u, v)$, we have $(u_t, u) + (v_t, v) > 0$ and $\|u_t\|_2^2 + \|v_t\|_2^2 > 0$ for $t \in [0, t_1]$. From (2.5), we have $0 < J(u(t_1), v(t_1)) \leq d - \int_0^{t_1}(\|u_t\|_2^2 + \|v_t\|_2^2)dt < d$. Hence we take $t = t_1$ as the initial time, and obtain $(u(t_1), v(t_1)) \in V$. From Lemma 3 we have $I(u(t), v(t)) < 0$ for $t_1 \leq t < +\infty$. Thus from Lemma 2 one has

$$-2I(u(t), v(t)) > 4p(d - J(u(t), v(t))),\ t_1 \leq t < +\infty. \tag{4.4}$$

Combing (4.4) and (2.5) we get

$$G''(t) = -2I(u, v) > 4p(d - J(u, v)) \geq 4p(d - J(u(t_1), v(t_1))) := C_2 > 0,\ t_1 \leq t < +\infty$$

and

$$G'(t) \geq C_2(t - t_1) + G'(t_1) = C_2(t - t_1),\ t_1 \leq t < +\infty,$$
$$G(t) \geq \frac{1}{2}C_2 2t^2 - C_2 t_1 t + G(t_1),\ t_1 \leq t < +\infty.$$

Hence for sufficiently large t, we have

$$(p-1)cG(t) > 2p(\|u_0\|_2^2 + \|v_0\|_2^2) \text{ and } (p-1)cG'(t) > 4pd. \tag{4.5}$$

Combining (4.3) with (4.5), we get

$$G(t)G''(t) - p(G'(t))^2 \geq \Big((P-1)cG(t) - 2p(\|u_0\|_2^2 + \|v_0\|_2^2)\Big)G'(t) + \Big((p-1)cG'(t) - 4pd\Big)G(t) > 0,$$

for sufficiently large t. Then similar to the proof of Theorem 2, i.e., there exists a finite time $T > 0$ such that $\lim_{t\to T}\int_0^t \|u\|_2^2 + \|v\|_2^2 d\tau = +\infty$.

The proof of Theorem 4 is complete.

5. High initial energy $J(u_0, v_0) > d$

In this section, we investigate the conditions that ensure the global existence or blow up of solution for problem (1.1) with the initial energy $J(u_0, v_0) > d$.

Theorem 5. *For any $\alpha \in (d, +\infty)$, the following conclusions hold.*
(i) If $(u_0, v_0) \in \Phi_\alpha$, then the solution of the problem (1.1) exists globally and $(u(t), v(t)) \to (0, 0)$, as $t \to \infty$;
(ii) If $(u_0, v_0) \in \Psi_\alpha$, then the solution of the problem (1.1) blows up in finite or infinite time, where

$$\Phi_\alpha = \mathcal{N}_+ \cap \{(u(t), v(t)) \in H_0^1(\Omega) \times H_0^1(\Omega) | \frac{1}{2}(\|u\|_2^2 + \|v\|_2^2) < \lambda_\alpha, d < J(u, v) \leqslant \alpha\}, \tag{5.1}$$

$$\Psi_\alpha = \mathcal{N}_- \cap \{(u(t), v(t)) \in H_0^1(\Omega) \times H_0^1(\Omega) | \frac{1}{2}(\|u\|_2^2 + \|v\|_2^2) > \Lambda_\alpha, d < J(u, v) \leqslant \alpha\}, \tag{5.2}$$

and

$$\lambda_\alpha = \inf\{\frac{1}{2}(\|u\|_2^2 + \|v\|_2^2) | (u, v) \in \mathcal{N}_\alpha\},\ \Lambda_\alpha = \sup\{\frac{1}{2}(\|u\|_2^2 + \|v\|_2^2) | (u, v) \in \mathcal{N}_\alpha\} \textit{ for all } \alpha > d.$$

Proof. (i) Assume that $(u_0, v_0) \in \Phi_\alpha$, then by the definition of Φ_α and the monotonicity property of λ_α, we have $(u_0, v_0) \in \mathcal{N}_+$, $d < J(u_0, v_0) \leq \alpha$ and

$$\frac{1}{2}(\|u_0\|_2^2 + \|v_0\|_2^2) < \lambda_\alpha \leq \lambda_{J(u_0,v_0)}. \tag{5.3}$$

We claim that $(u(t), v(t)) \in \mathcal{N}_+$. By contradiction, there exists a $t_0 \in (0, T)$ such that $(u(t), v(t)) \in \mathcal{N}_+$ for $t \in [0, t_0)$ and $(u(t_0), v(t_0)) \in \mathcal{N}$. By Lemma 4, we have

$$\frac{1}{2}\frac{d}{dt}(\|u\|_2^2 + \|v\|_2^2) = -I(u, v). \tag{5.4}$$

From the definition of $\mathcal{N}_+$ and (5.4), we know that $\|u\|_2^2 + \|v\|_2^2$ is strictly decreasing on $[0, t_0)$. On the other hand, by (2.5), we know that $J(u, v)$ is nonincreasing with respect to t. Therefore, we have

$$J(u, v) \leq J(u_0, v_0) \text{ for all } t \in [0, T).$$

From (5.3), we get

$$\frac{1}{2}(\|u(t_0)\|_2^2 + \|v(t_0)\|_2^2) < \frac{1}{2}(\|u_0\|_2^2 + \|v_0\|_2^2) < \lambda_{J(u_0,v_0)}. \tag{5.5}$$

By $(u(t_0), v(t_0)) \in \mathcal{N}$ and (5.3), we get $(u(t_0), v(t_0)) \in \mathcal{N}_{J(u_0,v_0)}$. According to the definition of $\lambda_{J(u_0,v_0)}$, we have

$$\lambda_{J(u_0,v_0)} = \inf\{\frac{1}{2}(\|u\|_2^2 + \|v\|_2^2) | (u, v) \in \mathcal{N}_{J(u_0,v_0)}\} \leq \frac{1}{2}(\|u(t_0)\|_2^2 + \|v(t_0)\|_2^2),$$

which contradicts with (5.5) and prove the claim. Hence, we have $(u(t), v(t)) \in \mathcal{N}_+$ for all $t \in [0, T)$ and $(u(t), v(t)) \in J^{J(u_0,v_0)}$, i.e., $(u(t), v(t)) \in J^{J(u_0,v_0)} \cap \mathcal{N}_+$ for all $t \in [0, T)$. From the definition of $\mathcal{N}_\alpha$, we have $(\|u\|_{H_0^1(\Omega)}^2 + \|v\|_{H_0^1(\Omega)}^2) < \frac{2p}{p-2} J(u_0, v_0)$, $\forall t \in [0, T)$, so $T = +\infty$. It indicates that $(u(t), v(t))$ is bounded uniformly in $H_0^1(\Omega) \times H_0^1(\Omega)$. Hence, ω-limit set is not an empty set.

Next, for any $(\omega, \varphi) \in \omega(u_0, v_0)$, by the above discussions, we get

$$J(\omega, \varphi) \leq J(u_0, v_0) \text{ and } \frac{1}{2}(\|\omega\|_2^2 + \|\varphi\|_2^2) < \lambda_{J(u_0,v_0)}.$$

According to first inequality, it implies that $(\omega, \varphi) \in J^{J(u_0,v_0)}$. According to the second inequality and the definition of $\lambda_{J(u_0,v_0)}$, we know that $(\omega, \varphi) \notin \mathcal{N}_{J(u_0,v_0)}$. Since $\mathcal{N}_{J(u_0,v_0)} = \mathcal{N} \cap J^{J(u_0,v_0)}$, we obtain $(\omega, \varphi) \notin \mathcal{N}$. Hence, $\omega(u_0, v_0) \cap \mathcal{N} = \phi$. As $\mathcal{N}$ include the nontrivial solutions of the problem (1.1), we have $\omega(u_0, v_0) = (0, 0)$, i.e., $(u(t), v(t)) \to (0, 0)$, as $t \to \infty$.

(ii) If $(u_0, v_0) \in \Psi_\alpha$, by the definition of Ψ_α, it is clear that $(u_0, v_0) \in \mathcal{N}_-$ and $d < J(u_0, v_0) \leq \alpha$. Combing with the monotonicity of Λ_α, we get

$$\frac{1}{2}(\|u_0\|_2^2 + \|v_0\|_2^2) > \Lambda_\alpha \geq \Lambda_{J(u_0,v_0)}.$$

We claim that $(u(t), v(t)) \in \mathcal{N}_-$ for $t \in [0, T)$. By contradiction, if there exists a $t_1 \in (0, T)$ such that $(u(t), v(t)) \in \mathcal{N}_-$ for $t \in [0, t_1)$ and $(u(t_1), v(t_1)) \in \mathcal{N}$. By Lemma 4, we have

$$\frac{1}{2}\frac{d}{dt}(\|u\|_2^2 + \|v\|_2^2) = -I(u, v).$$

Then by the definition of $\mathcal{N}_-$, we deduce that $\frac{1}{2}(\|u\|_2^2 + \|v\|_2^2)$ is strictly increasing on $[0, t_1)$. It along with (2.5) yields

$$\frac{1}{2}(\|u(t_1)\|_2^2 + \|v(t_1)\|_2^2) > \frac{1}{2}(\|u_0\|_2^2 + \|v_0\|_2^2) > \Lambda_{J(u_0,v_0)},\ J(u(t_1), v(t_1)) \leq J(u_0, v_0). \tag{5.6}$$

By $(u(t_1), v(t_1)) \in \mathcal{N}$ and (5.6), we get $(u(t_1), v(t_1)) \in \mathcal{N}_{J(u_0,v_0)}$. Hence, it follows from the definition of $\Lambda_{J(u_0,v_0)}$ that

$$\Lambda_{J(u_0,v_0)} = \sup\{\frac{1}{2}(\|u\|_2^2 + \|v\|_2^2)|(u, v) \in \mathcal{N}_{J(u_0,v_0)}\} \geq \frac{1}{2}(\|u(t_1)\|_2^2 + \|v(t_1)\|_2^2),$$

which is incompatible with (5.6), so we get $(u(t), v(t)) \in J^{J(u_0,v_0)} \cap \mathcal{N}_-$ for all $t \in [0, T)$.

Next, we assume that $(u(t), v(t))$ exists globally, i.e., $T = +\infty$. For every $(\omega, \varphi) \in \omega(u_0, v_0)$, by the above discussions, we get

$$J(\omega, \varphi) \leq J(u_0, v_0) \text{ and } \frac{1}{2}(\|\omega\|_2^2 + \|\varphi\|_2^2) > \Lambda_{J(u_0,v_0)}.$$

According to first inequality, this shows $(\omega, \varphi) \in J^{J(u_0,v_0)}$. According to the second inequality and the definition of $\Lambda_{J(u_0,v_0)}$, we know that $(\omega, \varphi) \notin \mathcal{N}_{J(u_0,v_0)}$. Since $\mathcal{N}_{J(u_0,v_0)} = \mathcal{N} \cap J^{J(u_0,v_0)}$, we obtain $(\omega, \varphi) \notin \mathcal{N}$. Hence, $\omega(u_0, v_0) \cap \mathcal{N} = \phi$. However, since $dist(0, \mathcal{N}_-) > 0$, we also have $(0, 0) \notin \omega(u_0, v_0)$. Thus, $\omega(u_0, v_0) = \emptyset$, it contraries to the assumption that $(u(t), v(t))$ is a global solution, then $T < \infty$.

The proof of Theorem 5 is complete.

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Conflict of interest

The authors declare there is no conflict of interest.

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