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Research article

Extremal problems on exponential vertex-degree-based topological indices

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Abstract: In this work we obtain new lower and upper optimal bounds for general (exponential) indices of a graph. In the same direction, we show new inequalities involving some well-known topological indices like the generalized atom-bound connectivity index ABC_α and the generalized second Zagreb index M_2^α. Moreover, we solve some extremal problems for their corresponding exponential indices (e^{ABC_α} and $e^{M_2^\alpha}$).

Keywords: general sum connectivity index; generalized second Zagreb index; topological indices; exponential indices

1. Introduction

A topological descriptor is a single number that represents a chemical structure in graph-theoretical terms via the molecular graph, they play a significant role in mathematical chemistry, especially in the QSPR/QSAR investigations. A topological descriptor is called a topological index if it correlates with a molecular property. Topological indices are used to understand physico-chemical properties of chemical compounds, since they capture the essence of some properties of a molecule in a single number.

Several of the well-known topological indices based its computing as a sum of terms which correspond to edges where each term is obtained by a symmetric expression of the degree of both end points, e.g., harmonic, Randić connectivity, first and second Zagreb, atom-bound connectivity and Platt indices. Lately, researchers in chemistry and pharmacology have focused in topological indices based on degrees of vertices obtaining good results and showing that a number of these indices has turned out to be a useful tool, see [1] and references therein. Probably, Randić connectivity index (R) [2] is the most known. In fact, there exist hundreds of works about this molecular descriptor (see, e.g., [3–8] and the references therein). Trying to improve the predictive power of this index during many years, scientists proposed a great number of new topological descriptors based on degrees of vertices, similar to the Randić index. There are a lot of works showing the interest on these indices, see e.g., [9–17].

The study of the exponential vertex-degree-based topological indices was initiated in [18] and has been successfully studied in [19–25]. The study of topological indices associated to exponential representations has been successfully studied in [12, 14, 20, 22, 26, 27]. Cruz et al. mentioned in 2020 some open problems on the exponential vertex-degree-based topological indices of trees [28] and Das et al. show in 2021 the solution of two of those problems in [23]. In this sense, we work with generic exponential topological indices described below that allows us to improve some of the bounds given in [23] as well as obtain some new results.

Along this work, given a graph $G = (V(G), E(G))$ and a symmetric function $f : [1, \infty)\times[1, \infty) \to \mathbb{R}$, we consider the general topological indices

$$F(G) = \sum_{uv\in E(G)} f(d_u, d_v), \qquad e^F(G) = \sum_{uv\in E(G)} e^{f(d_u,d_v)}.$$

As usual, uv denotes an edge of the graph G connecting the vertices u and v, and d_x denotes the degree of the vertex x. The family of the topological indices like the one to the right above are called *exponential* for obvious reasons.

Two examples of those topological indices that have been extended to its exponential indices are the generalized atom-bound connectivity index, ABC_α, and the generalized second Zagreb index, M_2^α, defined respectively as

$$ABC_\alpha(G) = \sum_{uv\in E(G)} \Big(\frac{d_u + d_v - 2}{d_u\, d_v}\Big)^\alpha,$$

$$M_2^\alpha(G) = \sum_{uv\in E(G)} (d_u\, d_v)^\alpha.$$

Then, their corresponding exponential indices are

$$e^{ABC_\alpha}(G) = \sum_{uv\in E(G)} e^{\left(\frac{d_u+d_v-2}{d_u d_v}\right)^\alpha},$$

$$e^{M_2^\alpha}(G) = \sum_{uv\in E(G)} e^{(d_u\, d_v)^\alpha}.$$

The study of topological indices is often formalized mathematically as optimization problems on graphs. In general, those problems have been proven to be useful and quite difficult. Indeed, to obtain quasi-minimizing or quasi-maximizing graphs is a good strategy that is commonly used. Therefore, frequently, it is needed to find bounds on them which involve several parameters.

Topological indices have been successfully applied to different branches of knowledge, such as chemistry, physics, biology, social sciences, etc. see [29–33]. In this direction, two of the most important theoretical and practical problems facing the study of topological indices are the following: the study of the extremal problems associated with topological indices and obtaining new inequalities relating different indices. This research has as fundamental contribution to face the above mentioned problems associated with the study of exponential vertex-degree-based topological indices. In particular, we solve some extremal problems related with the exponential vertex-degree-based topological indices $e^{M_2^\alpha}$ and e^{ABC_α}. Also, we obtain new lower and upper optimal bounds for these general exponential indices.

2. Optimization problems on the exponential index $e^{M_2^\alpha}$

Throughout this work, $G = (V(G), E(G))$ denotes a (non-oriented) finite simple graph without isolated vertices. Given a graph G and $v \in V(G)$, we denote by $N(v)$ the set of *neighbors* of v, i.e., $N(v) = \{u \in V(G) | \, uv \in E(G)\}$. We denote by Δ, δ, n, m the maximum degree, the minimum degree and the cardinality of the set of vertices and edges of G, respectively; thus, $1 \leq \delta \leq \Delta < n$.

We study in this section some optimization problems on the exponential index $e^{M_2^\alpha}$.

Recall that we denote by $N(w)$ the set of neighbors of w.

Proposition 2.1. *Let G be a graph with nonadjacent vertices u and v. If f is a symmetric function which is increasing in each variable, then $e^F(G + uv) > e^F(G)$.*

Proof. Since f is increasing in each variable, we have $e^{f(d_u+1,d_w)} \geq e^{f(d_u,d_w)}$, $e^{f(d_v+1,d_w)} \geq e^{f(d_v,d_w)}$ and

$$\begin{aligned} e^F(G + uv) - e^F(G) = \\ \sum_{w\in N(u)} \left(e^{f(d_u+1,d_w)} - e^{f(d_u,d_w)}\right) + \sum_{w\in N(v)} \left(e^{f(d_v+1,d_w)} - e^{f(d_v,d_w)}\right) + e^{f(d_u+1,d_v+1)} \\ \geq e^{f(d_u+1,d_v+1)} > 0. \end{aligned}$$

Hence,

$$e^F(G + uv) > e^F(G).$$

For example, if we take $\alpha > 0$, $f(x, y) = (xy)^\alpha$, the cycle C_n with $n > 2$ and remove one edge we obtain the path P_n, and we have $e^{M_2^\alpha}(C_n) = ne^{4^\alpha}$, $e^{M_2^\alpha}(P_n) = (n-3)e^{4^\alpha} + 2e^{2^\alpha}$ and $e^{M_2^\alpha}(C_n) - e^{M_2^\alpha}(P_n) = 3e^{4^\alpha} - 2e^{2^\alpha} > 0$. Now if we take the complete graph K_n with $n > 2$ and H is the graph obtained by removing an edge from K_n, we have $e^{M_2^\alpha}(K_n) = \frac{n(n-1)}{2} e^{(n-1)^{2\alpha}}$, $e^{M_2^\alpha}(H) = \left(\frac{(n-2)(n-3)}{2}\right) e^{(n-1)^{2\alpha}} + 2(n-2)e^{[(n-1)(n-2)]^\alpha}$ and $e^{M_2^\alpha}(K_n) - e^{M_2^\alpha}(H) = (2n-3)e^{(n-1)^{2\alpha}} - 2(n-2)e^{[(n-1)(n-2)]^\alpha} > 0$. Thus, we have $e^{(n-1)^\alpha - (n-2)^\alpha} > 1 > \frac{2n-4}{2n-3}$.

Given an integer $n \geq 2$, let $\mathcal{G}(n)$ be the set of graphs with n vertices.

Given integers $1 \leq \delta < n$, let $\mathcal{H}(n, \delta)$ be the set of graphs with n vertices and minimum degree δ

We consider the optimization problem for the exponential index $e^{M_2^\alpha}$ on $\mathcal{G}(n)$.

Theorem 2.2. *Consider $\alpha > 0$ and an integer $n \geq 2$.*

(1) *The unique graph that maximizes the $e^{M_2^\alpha}$ index on $\mathcal{G}(n)$ is the complete graph K_n.*

(2) *If n is even, then the unique graph that minimizes the $e^{M_2^\alpha}$ index on $\mathcal{G}(n)$ is the disjoint union of $n/2$ paths P_2. If n is odd, then the unique graph that minimizes the $e^{M_2^\alpha}$ index on $\mathcal{G}(n)$ is the disjoint union of $(n-3)/2$ paths P_2 and a path P_3.*

Proof. Since $\alpha > 0$ we have that $f(x, y) = (xy)^\alpha$ is an increasing function in each variable and so, we can apply Proposition 2.1. The first item is a direct consequence of Proposition 2.1.

Assume first that n is even. Handshaking lemma gives $2m \geq n\delta \geq n$. We have for any graph $G \in \mathcal{G}(n)$

$$e^{M_2^\alpha}(G) = \sum_{uv\in E(G)} e^{(d_u d_v)^\alpha} \geq \sum_{uv\in E(G)} e = m\, e \geq \frac{1}{2}\, n\, e,$$

and the equality in the bound is attained if and only if $d_u = 1$ for every $u \in V(G)$, i.e., G is the disjoint union of $n/2$ path graphs P_2.

Assume now that n is odd, and consider a graph $G \in \mathcal{G}(n)$. If $d_u = 1$ for every $u \in V(G)$, then handshaking lemma gives $2m = n$, a contradiction since n is odd. Thus, there exists a vertex w with $d_w \geq 2$. Handshaking lemma gives $2m \geq (n-1)\delta + 2 \geq n+1$. We have

$$\begin{aligned}
e^{M_2^\alpha}(G) &= \sum_{u \in N(w)} e^{(d_u d_w)^\alpha} + \sum_{uv \in E(G), u,v \neq w} e^{(d_u d_v)^\alpha} \\
&\geq \sum_{u \in N(w)} e^{2^\alpha} + \sum_{uv \in E(G), u,v \neq w} e \\
&\geq 2e^{2^\alpha} + (m-2)e \\
&\geq 2e^{2^\alpha} + \Big(\frac{n+1}{2} - 2\Big)e \\
&= 2e^{2^\alpha} + \frac{n-3}{2}\, e,
\end{aligned}$$

and the equality in the bound is attained if and only if $d_u = 1$ for every $u \in V(G) \setminus \{w\}$, and $d_w = 2$, i.e., G is the disjoint union of $(n-3)/2$ path graphs P_2 and a path graph P_3.

Note that for $\alpha = 1$, the result in Theorem 2.2 was obtained in [20, Theorem 2.2].

If $1 \leq \delta < \Delta$ are integers, we say that a graph G is (Δ, δ)-*pseudo-regular* if there exists $v \in V(G)$ with $d_v = \Delta$ and $d_u = \delta$ for every $u \in V(G) \setminus \{v\}$.

In [34] appears the following result.

Lemma 2.3. *Consider integers $2 \leq k < n$.*
(1) *If kn is even, then there is a connected k-regular graph with n vertices.*
(2) *If kn is odd, then there is a connected $(k+1, k)$-pseudo-regular graph with n vertices.*

Given integers $1 \leq \delta < n$, denote by K_n^δ the n-vertex graph with maximum degree $n-1$ and minimum degree δ, obtained from the complete graph K_{n-1} and an additional vertex v in the following way: Fix δ vertices $u_1, \dots, u_\delta \in V(K_{n-1})$ and let $V(K_n^\delta) = V(K_{n-1}) \cup \{v\}$ and $E(K_n^\delta) = E(K_{n-1}) \cup \{u_1 v, \dots, u_\delta v\}$.
Figure 1 illustrates this construction by showing the graphs K_6^2 and K_6^3.

We consider now the optimization problem for the exponential index $e^{M_2^\alpha}$ on $\mathcal{H}(n, \delta)$.

Theorem 2.4. *Consider $\alpha > 0$ and integers $1 \leq \delta < n$.*
(1) *Then the unique graph in $\mathcal{H}(n, \delta)$ that maximizes the $e^{M_2^\alpha}$ index is K_n^δ.*
(2) *If $\delta \geq 2$ and δn is even, then the unique graphs in $\mathcal{H}(n, \delta)$ that minimize the $e^{M_2^\alpha}$ index are the δ-regular graphs.*
(3) *If $\delta \geq 2$ and δn is odd, then the unique graphs in $\mathcal{H}(n, \delta)$ that minimize the $e^{M_2^\alpha}$ index are the $(\delta+1, \delta)$-pseudo-regular graphs.*

Proof. Given a graph $G \in \mathcal{H}(n, \delta) \setminus \{K_n^\delta\}$, fix any vertex $u \in V(G)$ with $d_u = \delta$. Since

$$G \neq G \cup \{vw : v, w \in V(G) \setminus \{u\} \text{ and } vw \notin E(G)\} = K_n^\delta,$$

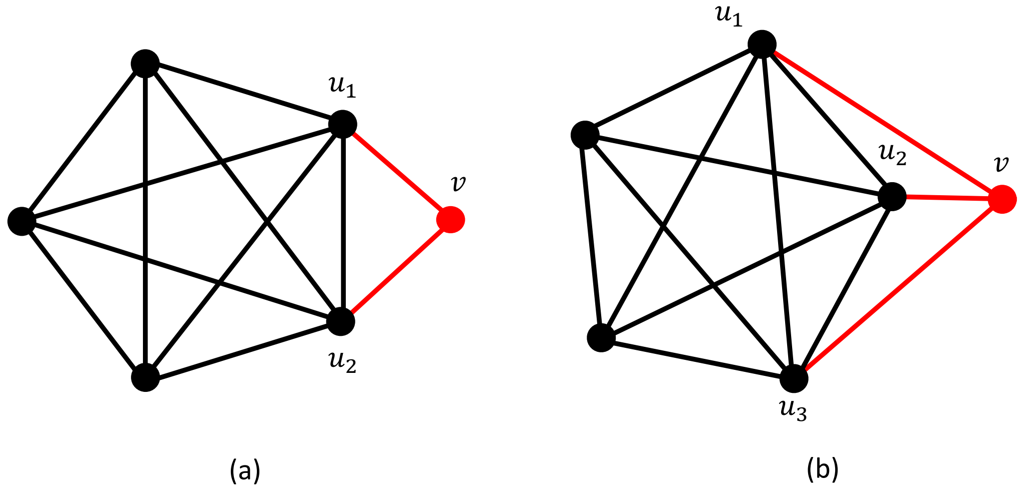

Figure 1. The graphs (a) K_6^2 , (b) K_6^3.

Proposition 2.1 gives $e^{M_2^\alpha}(K_n^\delta) > e^{M_2^\alpha}(G)$. This proves item (1).

Handshaking lemma gives $2m \geq n\delta$.

Since $d_u \geq \delta$ for every $u \in V(G)$, we obtain

$$e^{M_2^\alpha}(G) = \sum_{uv\in E(G)} e^{(d_u d_v)^\alpha} \geq \sum_{uv\in E(G)} e^{\delta^{2\alpha}} = m\, e^{\delta^{2\alpha}} \geq \frac{1}{2} n\delta e^{\delta^{2\alpha}},$$

and the equality in the bound is attained if and only if $d_u = \delta$ for every $u \in V(G)$.

If δn is even, then Lemma 2.3 gives that there is a δ-regular graph with n vertices. Hence, the unique graphs in $\mathcal{H}(n, \delta)$ that minimize the $e^{M_2^\alpha}$ index are the δ-regular graphs.

If δn is odd, then handshaking lemma gives that there is no regular graph. Hence, there exists a vertex w with $d_w \geq \delta + 1$. Since $d_u \geq \delta$ for every $u \in V(G)$, handshaking lemma gives $2m \geq (n-1)\delta + \delta + 1 = n\delta + 1$. We have

$$e^{M_2^\alpha}(G) = \sum_{u\in N(w)} e^{(d_u d_w)^\alpha} + \sum_{uv\in E(G),u,v\neq w} e^{(d_u d_v)^\alpha} \geq \sum_{u\in N(w)} e^{\delta^\alpha(\delta+1)^\alpha} + \sum_{uv\in E(G),u,v\neq w} e^{\delta^{2\alpha}}.$$

From the above and since w has at least $\delta + 1$ neighbors we have

$$e^{M_2^\alpha}(G) \geq (\delta + 1)e^{\delta^\alpha(\delta+1)^\alpha} + (m - \delta - 1)e^{\delta^{2\alpha}},$$

now using $2m \geq n\delta + 1$, we have

$$e^{M_2^\alpha}(G) \geq (\delta + 1)e^{\delta^\alpha(\delta+1)^\alpha} + \Big(\frac{n\delta + 1}{2} - \delta - 1\Big)e^{\delta^{2\alpha}} = (\delta + 1)e^{\delta^\alpha(\delta+1)^\alpha} + \frac{1}{2}(\delta(n-2) - 1)e^{\delta^{2\alpha}},$$

and equality in the bound is attained if and only if $d_u = \delta$ for every $u \in V(G) \setminus \{w\}$, and $d_w = \delta + 1$. Lemma 2.3 gives that there is a $(\delta + 1, \delta)$-pseudo-regular graph with n vertices. Therefore, the unique graphs in $\mathcal{H}(n, \delta)$ that minimize the $e^{M_2^\alpha}$ index are the $(\delta + 1, \delta)$-pseudo-regular graphs.

3. Optimal inequalities for exponential generalized atom-bound connectivity indices

In [23, Theorem 2.1] appears the inequality

$$e^{ABC}(G) \geq \Delta\Big(e^{\frac{\sqrt{2(\Delta-1)}}{\Delta}} - e^{\sqrt{1-\frac{1}{\Delta}}}\Big) + m\, e^{\frac{ABC(G)}{m}}.$$

The equality holds in this bound if and only if G is a disjoint union of isolated edges or each connected component of G is a path graph P_k ($k \geq 3$) or a cycle graph C_k ($k \geq 3$).

Proposition 3.1. *If G is a graph with size m, then*

$$e^{ABC}(G) \geq m\, e^{\frac{ABC(G)}{m}}.$$

The equality holds if every edge of G is incident to a vertex of degree 2, or G is regular or biregular.

Proof. $ABC(G) = \sum_{uv\in E(G)} \Big(\dfrac{d_u + d_v - 2}{d_u\, d_v}\Big)$. Note that the exponential function $\exp(x) = e^x$ is a strictly convex function, and Jensen's inequality gives

$$\exp\Big(\frac{1}{m}\sum_{uv\in E(G)} \frac{d_u + d_v - 2}{d_u\, d_v}\Big) \leq \frac{1}{m}\sum_{uv\in E(G)} e^{\frac{d_u+d_v-2}{d_u\, d_v}},$$

and the equality in this bound is attained if and only if $\frac{d_u+d_v-2}{d_u\, d_v} = \frac{d_w+d_z-2}{d_w\, d_z}$ for every $uv, wz \in E(G)$.

If every edge of G is incident to a vertex of degree 2, then $\frac{d_u+d_v-2}{d_u\, d_v} = \frac{1}{2}$ for every $uv \in E(G)$. If G is a regular or biregular graph, then $\frac{d_u+d_v-2}{d_u\, d_v} = \frac{\Delta+\delta-2}{\Delta\delta}$ for every $uv \in E(G)$.

Since $\frac{\sqrt{2(\Delta-1)}}{\Delta} < \sqrt{1-\frac{1}{\Delta}}$ if $\Delta > 2$, Propostion 3.1 improves [23, Theorem 2.1].

The following result relates the exponential generalized atom-bound connectivity indices with positive and negative parameters.

Theorem 3.2. *Let G be a graph with m edges, minimum degree δ, maximum degree $\Delta > 2$, and without isolated edges, and let $\alpha, \beta \in \mathbb{R}$ with $\alpha > 0 > \beta$. Then*

$$e^{\left(\left(\frac{2(\Delta-1)}{\Delta^2}\right)^{\alpha} - \left(\frac{2(\Delta-1)}{\Delta^2}\right)^{\beta}\right)} e^{ABC_\beta}(G) \leq e^{ABC_\alpha}(G),$$

and the equality holds if and only if G is regular.
If $\delta = 1$, then

$$e^{ABC_\alpha}(G) \leq e^{\left(\left(1-\frac{1}{\Delta}\right)^{\alpha} - \left(1-\frac{1}{\Delta}\right)^{\beta}\right)} e^{ABC_\beta}(G),$$

and the equality holds if and only if G is the disjoint union of star graphs $K_{1,\Delta}$.
If $\delta \geq 2$, then

$$e^{ABC_\alpha}(G) \leq e^{\left(\left(\frac{2(\delta-1)}{\delta^2}\right)^{\alpha} - \left(\frac{2(\delta-1)}{\delta^2}\right)^{\beta}\right)} e^{ABC_\beta}(G).$$

If $\delta > 2$, then the equality holds if and only if G is regular. If $\delta = 2$, then the equality holds if and only if each edge in G is incident to a vertex with degree two.

Proof. For each fixed α and β with $\alpha > 0 > \beta$, we are going to compute the extremal values of the function $g : [\delta, \Delta] \times [\max\{2, \delta\}, \Delta]$ (with $\Delta \geq 3$) given by

$$g(x,y) = e^{\left(\frac{x+y-2}{xy}\right)^{\alpha}} e^{-\left(\frac{x+y-2}{xy}\right)^{\beta}} = e^{\left(\left(\frac{x+y-2}{xy}\right)^{\alpha} - \left(\frac{x+y-2}{xy}\right)^{\beta}\right)}.$$

We have

$$\frac{\partial g}{\partial x} = e^{\left(\left(\frac{x+y-2}{xy}\right)^{\alpha} - \left(\frac{x+y-2}{xy}\right)^{\beta}\right)} \left(\alpha \left(\frac{x+y-2}{xy}\right)^{\alpha-1} - \beta \left(\frac{x+y-2}{xy}\right)^{\beta-1}\right) \frac{1}{x^2} \left(\frac{2}{y} - 1\right) \leq 0.$$

Then the function $g(x,y)$ is decreasing on $x \in [\delta, \Delta]$ for each fixed $y \in [\max\{2, \delta\}, \Delta]$ and consequently

$$g(\Delta, y) \leq g(x,y) \leq g(\delta, y).$$

Let us define

$$g_1(y) = g(\Delta, y) = e^{\left(\left(\frac{y+\Delta-2}{\Delta y}\right)^{\alpha} - \left(\frac{y+\Delta-2}{\Delta y}\right)^{\beta}\right)}.$$

We have

$$\begin{aligned} g_1'(y) =& e^{\left(\left(\frac{y+\Delta-2}{\Delta y}\right)^{\alpha} - \left(\frac{y+\Delta-2}{\Delta y}\right)^{\beta}\right)} \times \\ & \left(\alpha \left(\frac{y+\Delta-2}{\Delta y}\right)^{\alpha-1} - \beta \left(\frac{y+\Delta-2}{\Delta y}\right)^{\beta-1}\right) \frac{1}{y^2} \left(\frac{2}{\Delta} - 1\right) < 0. \end{aligned}$$

Then $g_1(y)$ is strictly decreasing, and consequently $g_1(\Delta) \leq g_1(y)$. Therefore, for each $uv \in E(G)$ we have

$$\begin{aligned} e^{\left(\left(\frac{2(\Delta-1)}{\Delta^2}\right)^{\alpha} - \left(\frac{2(\Delta-1)}{\Delta^2}\right)^{\beta}\right)} &\leq e^{\left(\frac{d_u+d_v-2}{d_u d_v}\right)^{\alpha}} e^{-\left(\frac{d_u+d_v-2}{d_u d_v}\right)^{\beta}}, \\ e^{\left(\left(\frac{2(\Delta-1)}{\Delta^2}\right)^{\alpha} - \left(\frac{2(\Delta-1)}{\Delta^2}\right)^{\beta}\right)} e^{\left(\frac{d_u+d_v-2}{d_u d_v}\right)^{\beta}} &\leq e^{\left(\frac{d_u+d_v-2}{d_u d_v}\right)^{\alpha}}, \\ e^{\left(\left(\frac{2(\Delta-1)}{\Delta^2}\right)^{\alpha} - \left(\frac{2(\Delta-1)}{\Delta^2}\right)^{\beta}\right)} e^{ABC_\beta}(G) &\leq e^{ABC_\alpha}(G), \end{aligned}$$

and the equality in the last inequality holds if and only if $d_u = d_v = \Delta$ for each $uv \in E(G)$, i.e., G is a regular graph.

Let us define

$$g_2(y) = g(\delta, y) = e^{\left(\left(\frac{y+\delta-2}{\delta y}\right)^{\alpha} - \left(\frac{y+\delta-2}{\delta y}\right)^{\beta}\right)}.$$

We have

$$\begin{aligned} g_2'(y) =& e^{\left(\left(\frac{y+\delta-2}{\delta y}\right)^{\alpha} - \left(\frac{y+\delta-2}{\delta y}\right)^{\beta}\right)} \times \\ & \left(\alpha \left(\frac{y+\delta-2}{\delta y}\right)^{\alpha-1} - \beta \left(\frac{y+\delta-2}{\delta y}\right)^{\beta-1}\right) \frac{1}{y^2} \left(\frac{2}{\delta} - 1\right). \end{aligned}$$

If $\delta = 1$, then $g_2'(y) > 0$ and so, $g_2(y)$ is strictly increasing. Consequently, $g_2(y) \leq g_2(\Delta)$, and we have for each $uv \in E(G)$

$$\begin{aligned} e^{\left(\frac{d_u+d_v-2}{d_u d_v}\right)^{\alpha}} e^{-\left(\frac{d_u+d_v-2}{d_u d_v}\right)^{\beta}} &\leq e^{\left(\left(1-\frac{1}{\Delta}\right)^{\alpha} - \left(1-\frac{1}{\Delta}\right)^{\beta}\right)}, \\ e^{\left(\frac{d_u+d_v-2}{d_u d_v}\right)^{\alpha}} &\leq e^{\left(\left(1-\frac{1}{\Delta}\right)^{\alpha} - \left(1-\frac{1}{\Delta}\right)^{\beta}\right)} e^{\left(\frac{d_u+d_v-2}{d_u d_v}\right)^{\beta}}, \\ e^{ABC_\alpha}(G) &\leq e^{\left(\left(1-\frac{1}{\Delta}\right)^{\alpha} - \left(1-\frac{1}{\Delta}\right)^{\beta}\right)} e^{ABC_\beta}(G), \end{aligned}$$

and the equality in the last inequality holds if and only if $d_u = 1$ and $d_v = \Delta$ o viceversa for each $uv \in E(G)$, i.e., G is the disjoint union of star graphs $K_{1,\Delta}$.

If $\delta = 2$, then $g_2'(y) = 0$ and so, $g_2(y)$ is a constant function. We have

$$
\begin{aligned}
g(x,y) \le g(2,y) = g_2(y) = e^{\left(\left(\frac{1}{2}\right)^{\alpha}-\left(\frac{1}{2}\right)^{\beta}\right)} &= e^{\left(\left(\frac{2(\delta-1)}{\delta^2}\right)^{\alpha}-\left(\frac{2(\delta-1)}{\delta^2}\right)^{\beta}\right)},\\
e^{\left(\frac{d_u+d_v-2}{d_ud_v}\right)^{\alpha}} e^{-\left(\frac{d_u+d_v-2}{d_ud_v}\right)^{\beta}} &\le e^{\left(\left(\frac{2(\delta-1)}{\delta^2}\right)^{\alpha}-\left(\frac{2(\delta-1)}{\delta^2}\right)^{\beta}\right)},\\
e^{\left(\frac{d_u+d_v-2}{d_ud_v}\right)^{\alpha}} &\le e^{\left(\left(\frac{2(\delta-1)}{\delta^2}\right)^{\alpha}-\left(\frac{2(\delta-1)}{\delta^2}\right)^{\beta}\right)} e^{\left(\frac{d_u+d_v-2}{d_ud_v}\right)^{\beta}},\\
e^{ABC_{\alpha}}(G) &\le e^{\left(\left(\frac{2(\delta-1)}{\delta^2}\right)^{\alpha}-\left(\frac{2(\delta-1)}{\delta^2}\right)^{\beta}\right)} e^{ABC_{\beta}}(G),
\end{aligned}
$$

and the equality in the last inequality holds if and only if each edge in G is incident to a vertex with degree two.

If $\delta > 2$, then $g_2'(y) < 0$ and so, $g_2(y)$ is strictly decreasing. Consequently, $g_2(y) \le g_2(\delta)$, and we have for each $uv \in E(G)$

$$
\begin{aligned}
e^{\left(\frac{d_u+d_v-2}{d_ud_v}\right)^{\alpha}} e^{-\left(\frac{d_u+d_v-2}{d_ud_v}\right)^{\beta}} &\le e^{\left(\left(\frac{2(\delta-1)}{\delta^2}\right)^{\alpha}-\left(\frac{2(\delta-1)}{\delta^2}\right)^{\beta}\right)},\\
e^{\left(\frac{d_u+d_v-2}{d_ud_v}\right)^{\alpha}} &\le e^{\left(\left(\frac{2(\delta-1)}{\delta^2}\right)^{\alpha}-\left(\frac{2(\delta-1)}{\delta^2}\right)^{\beta}\right)} e^{\left(\frac{d_u+d_v-2}{d_ud_v}\right)^{\beta}},\\
e^{ABC_{\alpha}}(G) &\le e^{\left(\left(\frac{2(\delta-1)}{\delta^2}\right)^{\alpha}-\left(\frac{2(\delta-1)}{\delta^2}\right)^{\beta}\right)} e^{ABC_{\beta}}(G),
\end{aligned}
$$

and the equality in the last inequality holds if and only if $d_u = d_v = \delta$ for each $uv \in E(G)$, i.e., G is a regular graph.

The hypothesis $\Delta > 2$ in Theorem 3.2 is not an important restriction: If $\Delta = 2$, then G is is the disjoint union of path and cycle graphs with three or more vertices, and we have $e^{ABC_{\alpha}(G)} = me^{\left(\frac{1}{2}\right)^{\alpha}}$. Hence,

$$
e^{ABC_{\alpha}}(G) = e^{\left(\left(\frac{1}{2}\right)^{\alpha}-\left(\frac{1}{2}\right)^{\beta}\right)} e^{ABC_{\beta}}(G).
$$

4. Conclusions

We have studied some properties of the generalized exponential indices. For the exponential index $e^{M_2^{\alpha}}$ with $\alpha > 0$, we characterize the graphs with extreme values in the class of graphs with a fixed number of vertices and in the class of graphs with a fixed minimum degree and a fixed number of vertices.

In addition, we found some optimal inequalities involving the exponential atom-bound connectivity index. In particular, we found a bound that improves a result in [23]. Also, we obtained an inequality concerning the indices $e^{ABC_{\alpha}}$ for different values of the parameter.

As an open problem it remains to find the extremal graphs and to obtain optimal bounds for other generalized exponential vertex-degree-based topological indices. In particular, for the index $e^{M_2^{\alpha}}$ to find the extremal graphs in other classes, for example the class of graphs with n vertices and fixed maximum degree.

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Conflict of interest

The author declare there is no conflict of interest.

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