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Research article

On reducing and finding solutions of nonlinear evolutionary equations via generalized symmetry of ordinary differential equations

Ivan Tsyfra and Wojciech Rzeszut*

AGH University of Science and Technology, Faculty of Applied Mathematics, aleja Adama Mickiewicza 30, 30-059 Kraków, Poland

* **Correspondence:** Email: wojciech.rzeszut@gmail.com.

Abstract: We study symmetry reductions of nonlinear partial differential equations that can be used for describing diffusion processes in heterogeneous medium. We find ansatzes reducing these equations to systems of ordinary differential equations. The ansatzes are constructed using generalized symmetries of second-order ordinary differential equations. The method applied gives the possibility to find exact solutions which cannot be obtained by virtue of the classical Lie method. Such solutions are constructed for nonlinear diffusion equations that are invariant with respect to one-parameter and two-parameter Lie groups of point transformations. We prove a theorem relating the property of invariance of a found solution to the dimension of the Lie algebra admitted by the corresponding equation. We also show that the method is applicable to non-evolutionary partial differential equations and ordinary differential equations.

Keywords: generalized symmetry; symmetry reduction; ansatz; ordinary differential equation; nonlinear differential equation

1. Introduction

It is of common knowledge, that the most effective method for constructing solutions of nonlinear partial differential equations (PDEs) of mathematical physics is the symmetry reduction method. The method can be both classical [1] and non-classical [2–7]. In these cases the construction of a proper ansatz (by which we mean a general form of an invariant solution) boils down to solving a quasilinear first-order differential equation, therefore an ansatz includes one arbitrary function and the initial equation reduces to a single differential equation with fewer independent variables, especially an ordinary differential equation (ODE). Reductions of differential equations to algebraic equations were considered in detail in [8] and [9].

Papers [10] and [11] presented a concept of conditional symmetry of evolution equations, which is a natural generalization of nonclassical symmetry. By using this method one can reduce nonlinear evolution equations with two independent variables to a system of ODEs. In [11] Zhdanov proved the theorem on the connection between the generalized conditional symmetry and reduction of evolutionary equations to a system of ODEs. It is worth pointing out that the number of differential equations in this system is equal to the number of unknown functions. The approach is used to construct exact solutions of nonlinear diffusion equations in [12]. The relationship between the generalized conditional symmetry of evolution equations and compatibility for overdetermined system of differential equations is studied in [13].

An approach on symmetry reduction of evolutionary equations is well developed. However, the problem of reducing of non-evolutionary equations is significantly less studied. The relation between the compatibility and reduction of partial differential equations in two independent and one dependent variables has been studied in [14]. Svirshchevskii [15] put forward the inverse symmetry reduction method for evolution equations of the form

$$u_t = K[u],$$

where $u = u(t, x)$ and $K[u] := K\left(t, x, u, \frac{\partial u}{\partial x}, \ldots, \frac{\partial^s u}{\partial x^s}\right)$. One can use this approach if $K[u]\partial_u$ is the generalized infinitesimal symmetry of a linear homogeneous ODE. In [16] the generalization of Svirshchevskii's method was proposed. It provides an explicit characterization of all nonlinear differential operators that leave a given subspace of functions invariant. It turns out that the inverse symmetry reduction method is also applicable to non-evolutionary differential equation [17–19]. We are looking for the ansatz reducing PDEs by solving ODEs that do not necessarily have to be linear.

More specifically, let $u_{(k)}$ denote the set of all kth order partial derivatives of $u = u(t, x)$ with respect to (t, x). Suppose that a generalized vector field

$$X = \eta(t, x, u, u_{(1)}, \ldots, u_{(k)})\partial_u \tag{1.1}$$

is a generalized symmetry of an ordinary differential equation

$$H\left(t, x, u, \frac{\partial u}{\partial x}, \ldots, \frac{\partial^p u}{\partial x^p}\right) = 0, \tag{1.2}$$

where H is a smooth function of its arguments, and t plays the role of a parameter [17]. Then an ansatz reducing the equation

$$\eta(t, x, u, u_{(1)}, \ldots, u_{(k)}) = 0 \tag{1.3}$$

to a system of ODEs can be constructed via finding the general solution of the ODE (1.2) [17]. Integrating the reduced system, one obtain exact solutions of PDEs (1.3). Exact solutions of differential equations are useful for understanding physical phenomena described by these equations and testing approximate and numerical methods for solving them. The method can be applied to non-evolutionary equations and even to ODEs. Moreover, in the framework of our approach one can construct an ansatz that reduces non-evolutionary PDEs to a system of ODEs and also the number of equations is smaller than the number of unknown functions. This enabled us to find solutions depending on arbitrary functions, which will be shown in Section 3.1. It is worth pointing out that the suggested method is applicable in multi-dimensional case as well [17,20,21]. We see that the ODE (1.2) includes the parametric variable t, apart from dependent and independent variables u and x. This allowed us to construct integrable PDEs. For example the equation

$$u_{xx} - a(t, x)u = 0 \tag{1.4}$$

admits the symmetry operator $X = \eta\partial_u = (u_t - u_{xxx} + 3\frac{u_{xx}u}{u_x})\partial_u$ provided that $a(t, x)$ satisfies the KdV equation

$$a_t = a_{xxx} - 6aa_x \tag{1.5}$$

as was shown in [17]. Then it is clear that the method is related to the inverse scattering transformation method. We emphasize that η depends on u_t and the condition of invariance of Eq (1.4) with respect to the group with generator X in classical sense leads to a determining equation which is the well-known integrable KdV equation. This approach belongs to a class of nonclassical symmetry methods in a sense that it produces results unobtainable within the classical methods. The idea is to use generalized symmetries of ODEs for constructing solutions of evolutionary equations. In this paper we consider a nonlinear evolutionary equation that describes transport phenomena in heterogeneous medium and apply a reduction method based on the symmetries of second-order nonlinear ODEs.

Extended symmetry analysis of porous medium equations with absorption and convection terms, including nonclassical symmetries, was carried out in papers [22–24]. Those equations included a factor dependent on a spatial variable x. We also studied diffusion-type equations for which the right-hand side involves a spatial variable and found solutions which are not invariant in classical sense by using the suggested method. We present the results obtained for the model medium with exponential and polynomial heterogeneity. Within the method applied, nonlinear transport equation is reduced to a system of two ODEs. After integrating (solving) the system of ODEs, we obtain exact solution of the initial equation. Right now main two differences from classical symmetry reduction should become apparent. With classical symmetry reduction there is only one reduced equation, not a system, and the solution from classical reduction will always be classically invariant. Since the method applied differs from the classical Lie method, it is not suitable for constructing algorithms for the generation of new solutions, or production of conservation laws. Its only advantage is the preservation of the reduction property. In addition, it doesn't ensure that none of the obtained solutions could also be found within the classical method. Therefore there is a very important question of distinguishing truly new solutions obtained within the method proposed. Many new local and nonlocal symmetries have been found for nonlocally related PDE systems [25]. Nonlocal symmetries alongside Bäcklund transformations helped finding solutions for the (2+1)-dimensional KdV–mKdV equation [26].

Based on the fact that a set of point and generalized symmetry operators (of the ODE) form a Lie algebra, we distinguish a class of diffusion equations whose solutions, obtained with the help of the aforementioned approach, cannot be obtained through the classical Lie method. Furthermore, it can be used to construct a large class of nonlinear evolution equation all of which are reduced to systems of ODEs by the same ansatz and possess solutions which are not invariant in the classical Lie sense. We emphasize that evolutionary equations are widely used and referenced in mathematical biology (see for instance [27] and references given there).

The organization of the article is as follows: We give the definition of generalized symmetry of an ODE and outline and explanation of the method in Section 2. In Section 3.1 we discuss applying of the method to PDEs. We find the classes of nonlinear evolutionary equations for which the method can be applied in the Section 3.2. In Section 3.3 we show the application of the method for finding solutions and obtain the solutions which cannot be constructed by the classical Lie method for the modified diffusion equations. The Theorem on a sufficient condition for the solution to be invariant one in classical sense is given too. In Section 4 we discuss the obtained results and provide some conclusions.

2. Materials and methods

Consider the differential equation (1.2). Let us denote by L the set of all differential consequences of (1.2) with respect to t and x. Let

$$\tilde{\eta} = \tilde{\eta}(t, x, u, \tfrac{\partial u}{\partial t}, \tfrac{\partial u}{\partial x}, \tfrac{\partial^2 u}{\partial t \partial x}, \ldots, \tfrac{\partial^k u}{\partial x^k}).$$

We treat (1.2) as it was a PDE which does not include the partial derivatives with respect to t.

Definition 1. *We say that a generalized vector field*

$$X = \tilde{\eta}(t, x, u, \tfrac{\partial u}{\partial t}, \tfrac{\partial u}{\partial x}, \tfrac{\partial^2 u}{\partial t \partial x}, \ldots, \tfrac{\partial^k u}{\partial x^k})\partial_u$$

is a generalized infinitesimal symmetry of equation (1.2) *if the condition*

$$X^{[p]} H\left(t, x, u, \tfrac{\partial u}{\partial x}, \ldots \tfrac{\partial^p u}{\partial x^p}\right)\Big|_L = 0 \tag{2.1}$$

holds, where

$$X^{[p]} = \sum_{i=0}^{p} D_x^i \tilde{\eta}\, \partial_{u_i}$$

is the standard pth prolongation (pth extension) of X and $D_x = \frac{\partial}{\partial x} + u_1 \frac{\partial}{\partial u} + u_{1t} \frac{\partial}{\partial u_t} + u_2 \frac{\partial}{\partial u_1} + \ldots$ *is the operator of total derivative with respect to x,* $u_0 = u$, $u_i = \frac{\partial^i u}{\partial x^i}$, $D_x^0 = 1$, $D_x^{j+1} = D_x(D_x^j)$, $i, j \in \mathbb{N}$.

For a more elaborate take on the basics of symmetry methods please refer to [28]. The invariance property (2.1) ensures a reduction of the equation

$$\tilde{\eta}(t, x, u, \tfrac{\partial u}{\partial t}, \tfrac{\partial u}{\partial x}, \tfrac{\partial^2 u}{\partial t \partial x}, \ldots, \tfrac{\partial^k u}{\partial x^k}) = 0 \tag{2.2}$$

to a system of ordinary differential equations. Let

$$u = F(x, t, \varphi_1(t), \varphi_2(t), \ldots, \varphi_p(t)), \tag{2.3}$$

where F is a smooth function of $(x, t, \varphi_1, \ldots, \varphi_p)$, be the general solution of Eq (1.2). Then the ansatz (2.3), where $\varphi_1(t), \ldots, \varphi_p(t)$ are unknown functions of the variable t, reduces partial differential equation (2.2) to the system of k_1 ordinary differential equations, where $k_1 \leq p$ [17]. For evolutionary equations $k_1 = p$, and for non-evolutionary equations, this number may be less than p. This property is illustrated by Eqs (3.9) and (3.13). The necessary computation was made in Maple™ with the addition of a software *Jets* for differential calculus on jet spaces and diffieties, which was created by H. Baran and M. Marvan and is distributed under the GNU General Public License. The tool and the user guide can be found at https://jets.math.slu.cz/.

3. Results

3.1. On application of the generalized symmetry reduction method to differential equations with two independent variables

In this subsection we show how to apply the method to PDE's. At first we consider differential equations obtained with the help of symmetry operators of Eq (1.4). One can find the solution $a = \frac{2}{x^2}$

of Eq (1.5) when $\frac{\partial a}{\partial t} = 0$ (the stationary Korteweg–de Vries equation). Then the equation

$$u_{xx} - \frac{2}{x^2}u = 0 \tag{3.1}$$

admits the three-dimensional Lie algebra with basic operators $Q_1 = u_t\partial_u$, $Q_2 = (u_{xxx} - 3\frac{u_{xx}u}{u_x})\partial_u$, $Q_3 = u\partial_u$ in this case. Equation (3.1) is integrable by quadratures and thus we obtain the ansatz

$$u = \varphi_1(t)x^2 + \frac{\varphi_2(t)}{x}, \tag{3.2}$$

where $\varphi_1(t)$, $\varphi_2(t)$ are unknown functions, which reduces nonlinear evolution equation

$$u_t - u_{xxx} + 3\frac{u_{xx}u_x}{u} - \lambda u = 0, \tag{3.3}$$

with λ = const, to the system of two ODEs

$$\varphi_1'(t) = \lambda\varphi_1, \quad \varphi_2'(t) = \lambda\varphi_2 - 12\varphi_1.$$

The solution of this system has the form

$$\varphi_1 = C_1 e^{\lambda t}, \quad \varphi_2 = (C_2 - 12C_1 t)e^{\lambda t}, C_1, C_2 = \text{const}. \tag{3.4}$$

Substituting (3.4) into (3.2) one can obtain the solution of (3.3). Note that for $a = w_x$, where w is a solution of the stationary Calogero–Bogoyavlenskii–Shiff equation

$$w_{zxxx} - 2w_z w_{xx} - 4w_{zx}w_x = 0,$$

Eq (1.4) is also integrable by quadratures [29]. The reduction method can also be applied for reducing the Cauchy problem for Eq (3.3) with an initial condition $u(0, x) = f(x)$ to the Cauchy problem for a system of ODEs.

Our next goal is to show that the method is applicable to non-evolutionary type PDEs. For this purpose we first consider the following differential equation:

$$u_{xx} - \left(\frac{x}{6(t+1)} + \frac{5}{16x^2}\right)u = 0. \tag{3.5}$$

It is invariant with respect to the two-parameter Lie group of point transformations with generators

$$Y_0 = u\partial_u, \quad Y_1 = \left(\frac{2}{\sqrt{x}}u_x + \frac{1}{2}x^{-3/2}u\right)\partial_u = \eta_1\partial_u.$$

From this it immediately follows that the Eq (3.5) is integrable by quadratures. To obtain non-evolutionary equations we look for symmetry operators of the form $Y = \eta(t, x, u, u_x, u_t, u_{xt})\partial_u$. We prove that the Eq (3.5) admits the generalized infinitesimal symmetries given by

$$Y_2 = \left(\frac{2}{\sqrt{x}}u_{xt} + \frac{1}{2}x^{-3/2}u_t + \frac{x^{3/2}}{9(t+1)^2}u\right)\partial_u = \eta_2\partial_u, \quad Y_3 = \sqrt{x}\left(\frac{3(t+1)}{2}\eta_1^2 - u^2\right)u\partial_u = \eta_3\partial_u$$

as well. We emphasize that the coefficient η_2 depends explicitly on the (parametric) variable t and the derivative u_{tx}. This enabled us to find non-evolutionary nonlinear differential equations that can be reduced to a system of ODEs by appropriate ansatz. One can use linear combinations or commutators of the operators Y_1, Y_2 and Y_3. We consider the symmetry operator $Y_2 - Y_3$. Then the corresponding non-evolutionary differential equation takes the form

$$\eta_2 = \eta_3. \tag{3.6}$$

By using symmetry properties we integrate Eq (3.5) and thus obtain the ansatz

$$u = x^{-1/4}\left(\varphi_1(t)\mathrm{e}^{k(t)\omega} + \varphi_2(t)\mathrm{e}^{-k(t)\omega}\right), \tag{3.7}$$

where $\omega = x^{3/2}$, $k(t) = \frac{\sqrt{2}}{3\sqrt{3(t+1)}}$, and $\varphi_1(t)$ and $\varphi_2(t)$ are unknown functions, which reduces (3.6) to the system of ODEs

$$\varphi_1' - \frac{\varphi_1}{2(t+1)} = -2\sqrt{6(t+1)}\varphi_1^2\varphi_2, \quad \varphi_2' - \frac{\varphi_2}{2(t+1)} = 2\sqrt{6(t+1)}\varphi_1\varphi_2^2. \tag{3.8}$$

The application of the same method to non-evolutionary nonlinear hyperbolic equation

$$u_{x_1x_2} = u_{x_1}F(u_{x_1} - u), \tag{3.9}$$

with F being an arbitrary function of $u_{x_1} - u$, yields the solution

$$u = -\varphi_1(x_2) + C\mathrm{e}^{x_1 + \int F(\varphi_1(x_2))\mathrm{d}x_2}, \quad C = \mathrm{const}, \tag{3.10}$$

parameterized by an arbitrary function $\varphi_1(x_2)$. This solution is obtained by using the generalized infinitesimal symmetries $Q_1 = u_{x_1x_2}\partial_u$ and $Q_2 = u_{x_1}F(u_{x_1} - u)\partial_u$ of the ODE $u_{x_1x_1} - u_{x_1} = 0$.

It is clear that an ODE generating an ansatz may be nonlinear. For example, the equation

$$u_{x_1x_1} + u_{x_1x_1}^2 = 0 \tag{3.11}$$

admits the generalized infinitesimal symmetries $Q_1 = \frac{u_{x_1x_2}}{u_{x_1}^2}\partial_u$ and $Q_2 = F(u + \ln u_{x_1})\partial_u$, where F is an arbitrary function of $u + \ln u_{x_1}$. Thus, it admits the generalized infinitesimal symmetry $Q = Q_1 - Q_2$ as well. The general solution of (3.11) gives the ansatz

$$u = \ln(x_1 + \Phi_1(x_2)) + \Phi_2(x_2), \tag{3.12}$$

where $\Phi_1(x_2)$ and $\Phi_2(x_2)$ are arbitrary smooth functions, which reduces the nonlinear PDE

$$u_{x_1x_2} = u_{x_1}^2 F(u + \ln u_{x_1}) \tag{3.13}$$

to the single ODE

$$\Phi_1' = -F(\Phi_2).$$

From this we obtain the family of solutions of (3.13)

$$u = \ln\left(x_1 - \int F(\Phi_2(x_2))\mathrm{d}x_2\right) + \Phi_2(x_2), \tag{3.14}$$

which is parameterized by the arbitrary smooth function $\Phi_2(x_2)$. Therefore, the method allows us to construct solutions, which contain arbitrary functions. We recall that second order in time nonlinear partial differential equations, which have only one higher symmetry and have a solution depending on one arbitrary function were called partially integrable equations in [30].

Note that setting $F(u + \ln u_{x_1}) = e^{-2(u+\ln u_{x_1})}$ in (3.13), we obtain the equation $u_{x_1x_2} = e^{-2u}$ which is transformed to the classical Liouville equation $v_{xt} = e^v$ by means of the change of variables $v = -2u$, $x = -2x_1, t = x_2$.

It is obvious that Eq (3.11) is invariant with respect to the Lie group of translations in t, which is generated by the vector field $Q_3 = u_t\partial_u$. Therefore, the generalized vector field $Q_3 + Q_1 - Q_2$ is a generalized infinitesimal symmetry of Eq (3.11). Then the ansatz

$$u = \ln(x_1 + \varphi_1(x_2, t)) + \varphi_2(x_2, t) \tag{3.15}$$

reduces the partial differential equation with three independent variables of evolution type

$$u_t + \frac{u_{x_1x_2}}{u_{x_1}^2} = F(u + \ln u_{x_1}) \tag{3.16}$$

to the system of two partial differential equations with two independent variables

$$\varphi'_{1_t} = 0, \quad \varphi'_{2t} - \varphi'_{1x_2} = F(\varphi_2). \tag{3.17}$$

At the same time we do not obtain such reduction for the non-evolutionary equation

$$u_{tt} + \frac{u_{x_1x_2}}{u_{x_1}^2} = F(u + \ln u_{x_1}). \tag{3.18}$$

Note that the generalized vector field $Q_3 = u_{tt}\partial_u$ is not an infinitesimal symmetry of Eq (3.11). In [16], it was shown that the reduction method can be applied to partial differential equations of the form $u_{tt} = Q[u]$ or the more general form $T[u] = Q[u]$, where T is a linear ordinary differential operator in t. Each of ansatzes used in [16] in this case to reduce partial differential equations is a solution of a linear ordinary differential equation whose coefficients do not depend on t and, therefore, the generalized vector fields $u_{tt}\partial_u$ and $T[u]\partial_u$ are the generalized symmetries of this equation. These facts substantiate the reduction of considered partial differential equations $u_{tt} = Q[u]$ and $T[u] = Q[u]$.

The above examples demonstrate rather strikingly that the method can be applied to non-evolutionary type PDE. The main idea of applying the method to evolutionary equations can be illustrated by the example of the Korteweg–de Vries equation. One can verify that the equation

$$u_{xx} + \frac{u^2}{2} = 0 \tag{3.19}$$

is invariant with respect to the generalized vector field $X = (u_{xxx} + uu_x)\partial_u$. Note that we can obtain any desired coefficients for the terms of Eq (1.5) by rescaling dependent variable and independent variables t and x. We have proved that the Eq (3.19) is invariant with respect to a two-parameter group of contact transformations. A basis of the corresponding Lie algebra consists of the generalized vector fields

$$Q_1 = u_x h_1(3u_x^2 + u^3)\partial_u, \tag{3.20}$$

$$Q_2 = u_x h_2(3u_x^2 + u^3) \int_0^u \frac{ds}{(u_x^2 + \frac{u^3}{3} - \frac{s^3}{3})^{\frac{3}{2}}}\partial_u, \tag{3.21}$$

where h_1 and h_2 are arbitrary smooth functions. The advantage of using the approach lies in the fact that a linear combination $\alpha_1 X + \alpha_2 Q_1 + \alpha_3 Q_2$ with arbitrary real constants α_1, α_2 and α_3 is also a symmetry operator of (3.19) and, therefore, the method can be applied to the nonlinear differential equation

$$u_t = u_{xxx} + uu_x + u_x h_1(3u_x^2 + u^3) + u_x h_2(3u_x^2 + u^3) \int_0^u \frac{\mathrm{d}s}{(u_x^2 + \frac{u^3}{3} - \frac{s^3}{3})^{\frac{3}{2}}}. \tag{3.22}$$

Then the ansatz

$$\int_0^u \frac{\mathrm{d}s}{\sqrt{\varphi_1(t) - \frac{s^3}{3}}} = x + \varphi_2(t) \tag{3.23}$$

generated by (3.19) reduces the PDE (3.22) to the system of two ODEs

$$\varphi_1'(t) = 2h_2(3\varphi_1(t)), \quad \varphi_2'(t) = h_1(3\varphi_1(t)). \tag{3.24}$$

Equation (3.22) in the general form is invariant with respect to the two-dimensional Lie algebra $\langle \partial_t, \partial_x \rangle$. Since the derivatives $\frac{\partial u}{\partial \varphi_1}$ and $\frac{\partial u}{\partial \varphi_2}$ are linearly independent, one can easily see that among solutions of (3.22) constructed with the ansatz (3.23), there are those that are invariant with respect to no one-parameter Lie group with generator in $\langle \partial_t, \partial_x \rangle$. Hence we conclude that the method enables us to construct solutions to equations from class (3.22) that are not invariant in the classical Lie sense.

3.2. The classes of evolutionary equations for which the method can be applied

Next we consider a nonlinear evolutionary equation which can be used for describing diffusion processes in heterogeneous medium and for which the method can be applied. We are looking for the second-order ODEs of the form

$$u_{xx} = U(x, u, u_x) \tag{3.25}$$

(which belong to the class (1.2)) admitting a generalized symmetry with a generator of the form $X = (\frac{H(x)}{u})_{xx}\partial_u$ corresponding to the right hand side of our diffusion equation. Note that the Eq (3.25) is invariant with respect to the translation group $t' = t + a$, where a is a group parameter and therefore it admits the symmetry operator $X_1 = u_t\partial_u$. From this it follows that the method is applicable to any equation from the class $k_1 u_t + k_2(\frac{H(x)}{u})_{xx} = 0$, where k_1, k_2 are arbitrary constants. We assume H and U are some sufficiently smooth functions and H is nonzero. Function U should satisfy the determining equation

$$X^{[2]}(u_{xx} - U(x, u, u_x))\Big|_L = 0$$

as stated in Definition 1, for a given function H, with L being the set of all differential consequences of Eq (3.25) with respect to the variable x. For the sake of being able to split the equation above into an overdetermined system of differential equations, we restrict our search to a function $U = \sum_{i,j\in\mathbb{Z}} A_{ij}(x)u^i u_x^j$, that is a power series in both u and u_x. We focus not a complete classification but rather particular H and U for which this method may produce nonclassical solutions. Due to a great number of determining equations and cases to consider we will move past the explanation of calculations and present some of the results in the following proposition.

Proposition 1. *An equation $u_{xx} = U(x, u, u_x)$ admits the LBS operator $X = (\frac{H(x)}{u})_{xx}\partial_u$ if $H(x)$ and U have the form* *a)* $H(x) = e^{\beta x}$, $U = 3\frac{u_x^2}{u} - 3\beta u_x + \beta^2 u$,

b) $H(x) = (x+\gamma)^\alpha$, $U = 3\frac{u_x^2}{u} + \frac{3-3\alpha}{x+\gamma} u_x + \frac{(\alpha-2)(\alpha-1)}{(x+\gamma)^2} u$,

c) $H(x) = (x+\gamma)^\alpha$, $U = 3\frac{u_x^2}{u} + \frac{1-3\alpha}{x+\gamma} u_x + \frac{(\alpha-2)\alpha}{(x+\gamma)^2} u$,

d) $H(x) = (x+\gamma)^{-2}$, $U = 3\frac{u_x^2}{u} + \frac{7}{x+\gamma} u_x + \frac{8}{(x+\gamma)^2} u$,

e) $H(x) = (x+\gamma)^{-2}$, $U = 3\frac{u_x^2}{u} + \frac{10}{x+\gamma} u_x + \frac{15}{(x+\gamma)^2} u$.

Terms α, β, γ are all constants.

It must be noted that because of the said restrictions this is not a complete classification. Solutions of ODEs can be used as ansatzes which produce a reduction of the equation

$$u_t = \left(\frac{H(x)}{u}\right)_{xx} \tag{3.26}$$

to a system of two ODEs. Equation (3.26) can be written in the form

$$u_t = (D(x,u)u_x)_x + P(x,u)u_x + Q(x,u), \tag{3.27}$$

where $D(x,u) = -\frac{H(x)}{u^2}$, $P(x,u) = -\frac{H'(x)}{u^2}$, $Q(x,u) = \frac{H''(x)}{u}$. Assumption of constancy D is not always justified – often the diffusion coefficient depends on the concentration of the diffusant, concentration gradient, spatial coordinate and time of the diffusion experiment (and sometimes - from all these parameters together). In [31] the reaction-diffusion equation of the typical form

$$u_t = D\Delta u + f(u) \tag{3.28}$$

is considered. $u = u(x,t)$ is a state variable and describes density/concentration of a substance, a population at point x: $x \in \Omega \subset \mathbb{R}^n$ (Ω is an open set). Δ denotes the Laplace operator. The second term, $f(u)$ is a smooth function, and describes processes with really "change" the present u, i.e., something happens to it (birth, death, chemical reaction), not just diffuse in the space. It is also possible, that the reaction term depends not only on u, but also on the first derivative of u, i.e., ∇u and/or explicitly on x. Instead of a scalar equation, one can also introduce systems of reaction diffusion equations, which are of the form

$$u_t = D\Delta u + f(x, u, \nabla u), \tag{3.29}$$

where $u(x,t) \in \mathbb{R}^m$ [31]. The system of equations

$$\begin{aligned} u_t &= ((d_1 + d_{11}u + d_{12}v)u)_{xx} + (K_x u)_x + u(k_1 + k_2 u + k_3 v), \\ v_t &= ((d_2 + d_{21}u + d_{22}v)v)_{xx} + (K_x v)_x + u(k_4 + k_5 u + k_6 v) \end{aligned} \tag{3.30}$$

is used in [32] to describe the densities of two biological species, considering nonlinear movements of the individuals of populations. Here $K(x)$ is environmental potential describing the heterogeneity of environment. Equations (3.26) and (3.27) belong to the class of Eq (3.29) and the method is applicable to any equation from this class or (3.30). Note that Eq (3.26) can be used for describing fast diffusion in heterogeneous medium if $H(x) \neq const$ and homogeneous medium if $H(x) = const$. We use the fact that the equation

$$u_t = (\tfrac{H}{u})_{xx} + \eta(x, u, u_x)$$

also allows reduction with the same ansatz, for η such that $\eta\partial_u$ is the symmetry of the ODE. Note that if we take $H = -1$ and $\eta = 0$ then we get the well-known equation describing nonlinear diffusion processes, which possesses an infinite generalized symmetry. We consider separate cases from Proposition 1 depending on the type of heterogeneity of the medium $H(x)$ and the ODE (which is not unique for the choice of H), calculate the full contact symmetry of the ODE and obtain the reduced equations. From now on A_i are arbitrary smooth functions on their two arguments. We put $\gamma = 0$ in these cases.

(i) When $H(x) = \mathrm{e}^{\beta x}$ and

$$u_{xx} = 3\tfrac{u_x^2}{u} - 3\beta u_x + \beta^2 u \tag{3.31}$$

the most general formula for $\eta(x, u, u_x)$ is $\eta = \sum_{i=1}^{2} \mathrm{e}^{-i\beta x} u^3 A_i \left(2\frac{\mathrm{e}^{\beta x}}{u^3}\left(u - \frac{u_x}{\beta}\right), \frac{\mathrm{e}^{2\beta x}}{u^3}\left(2\frac{u_x}{\beta} - u\right)\right)$. From what already has been shown it follows that the ansatz

$$u(x, t) = \frac{\pm \mathrm{e}^{\beta x}}{\sqrt{\varphi_1(t)\mathrm{e}^{\beta x} + \varphi_2(t)}}, \tag{3.32}$$

which is general solution of (3.31), reduces the equation

$$u_t = \left(\frac{\mathrm{e}^{\beta x}}{u}\right)_{xx} + \sum_{i=1}^{2} \mathrm{e}^{-i\beta x} u^3 A_i \left(2\frac{\mathrm{e}^{\beta x}}{u^3}\left(u - \frac{u_x}{\beta}\right), \frac{\mathrm{e}^{2\beta x}}{u^3}\left(2\frac{u_x}{\beta} - u\right)\right) \tag{3.33}$$

to the system of ODEs

$$\varphi_1' + \tfrac{\beta^2}{2}\varphi_1^2 + 2A_1(\varphi_1, \varphi_2) = 0, \tag{3.34}$$

$$\varphi_2' + \beta^2\varphi_1\varphi_2 + 2A_2(\varphi_1, \varphi_2) = 0. \tag{3.35}$$

(ii) When $H(x) = x^\alpha$, $\alpha \neq 0$, and $u_{xx} = 3\frac{u_x^2}{u} + \frac{3-3\alpha}{x}u_x + \frac{(\alpha-2)(\alpha-1)}{x^2}u$ we obtain, in a similar way, the equation

$$u_t = \left(\frac{x^\alpha}{u}\right)_{xx} + \sum_{i=1}^{2} x^{2-i\alpha} u^3 A_i \left(\frac{2x^{\alpha-2}}{\alpha u^3}((\alpha-1)u - xu_x), \frac{x^{2\alpha-2}}{\alpha u^3}((2-\alpha)u + 2xu_x)\right) \tag{3.36}$$

and the ansatz

$$u(x, t) = \frac{\pm x^{\alpha-1}}{\sqrt{\varphi_1(t)x^\alpha + \varphi_2(t)}}. \tag{3.37}$$

Substituting (3.36) into (3.37) yields the reduced system

$$\varphi_1' + \tfrac{\alpha(\alpha+1)}{2}\varphi_1^2 + 2A_1(\varphi_1, \varphi_2) = 0,$$

$$\varphi_2' + \alpha(\alpha+1)\varphi_1\varphi_2 + 2A_2(\varphi_1, \varphi_2) = 0.$$

Analysis similar to that in the first two cases gives the following results:

(iii) When $H(x) = x^{\alpha},\ \alpha \neq -2$ and $u_{xx} = 3\frac{u_x^2}{u} + \frac{1-3\alpha}{x}u_x + \frac{(\alpha-2)\alpha}{x^2}u$, the equation

$$u_t = \left(\frac{x^{\alpha}}{u}\right)_{xx} + \sum_{i=1}^{2} x^{4-i(\alpha+2)}u^3A_i\left(\frac{2x^{\alpha-2}(\alpha u - xu_x)}{(\alpha+2)u^3}, \frac{x^{2\alpha}((2-\alpha)u + 2xu_x)}{(\alpha+2)u^3}\right)$$

is reduced by the ansatz

$$u(x,t) = \frac{\pm x^{\alpha}}{\sqrt{\varphi_1(t)x^{\alpha+2} + \varphi_2(t)}},$$

to the system

$$\varphi_1' + \frac{\alpha(\alpha+1)}{2}\varphi_1^2 + 2A_1(\varphi_1,\varphi_2) = 0,$$

$$\varphi_2' + (\alpha+1)(\alpha+2)\varphi_1\varphi_2 + 2A_2(\varphi_1,\varphi_2) = 0.$$

(iv) When $H(x) = x^{-2}$ and $u_{xx} = 3\frac{u_x^2}{u} + \frac{7}{x}u_x + \frac{8}{x^2}u$, the equation

$$u_t = \left(\frac{1}{x^2u}\right)_{xx} + \sum_{i=1}^{2} x^4\ln(x)^{2-i}u^3A_i\left(-2\frac{xu_x + 2u}{x^4u^3}, \frac{2x\ln xu_x + (4\ln x + 1)u}{x^4u^3}\right)$$

is reduced by the ansatz

$$u(x,t) = \frac{\pm 1}{x^2\sqrt{\varphi_1(t)\ln(x) + \varphi_2(t)}},$$

to the system

$$\varphi_1' - \varphi_1^2 + 2A_1(\varphi_1,\varphi_2) = 0,$$

$$\varphi_2' - \frac{1}{2}\varphi_1^2 - \varphi_1\varphi_2 + 2A_2(\varphi_1,\varphi_2) = 0.$$

(v) When $H(x) = x^{-2}$ and $u_{xx} = 3\frac{u_x^2}{u} + \frac{10}{x}u_x + \frac{15}{x^2}u$, the equation

$$u_t = \left(\frac{1}{x^2u}\right)_{xx} + \sum_{i=1}^{2} x^{7-i}u^3A_i\left(-\frac{5u + 2xu_x}{x^6u^3}, 2\frac{3u + xu_x}{x^5u^3}\right)$$

is reduced by the ansatz

$$u(x,t) = \frac{\pm 1}{x^2\sqrt{\varphi_1(t)x^2 + \varphi_2(t)x}},$$

to the system

$$\varphi_1' - \frac{1}{2}\varphi_1^2 + 2A_1(\varphi_1,\varphi_2) = 0,$$

$$\varphi_2' + 2A_2(\varphi_1,\varphi_2) = 0.$$

We calculated all $\eta(x,u,u_x)$ for which the method can be applied. In general, $\eta(x,u,u_x)$ is a nonlinear function on its arguments. Note that the nonlinear terms are essential in realistic models [33]. In special cases we can obtain from $\eta(x,u,u_x)$ the terms describing external forces, absorbent rate, sources or sinks and convection (advection) processes. We show that the well known nonlinearity $\lambda\, u\ln u$ [34] can be obtained in the framework of the approach by using the stationary solution of equation

$$u_t = (\tfrac{u_x}{u})_x. \tag{3.38}$$

Indeed, it is obvious that the vector field $Q = (\frac{u_x}{u})_x \partial_u$ is the generalized symmetry of ordinary differential equation

$$(\tfrac{u_x}{u})_x = 0. \tag{3.39}$$

The solution of (3.39) satisfies condition $u_t = 0$ and is the stationary solution of (3.38) The contact symmetry of (3.39) is given by vector field $Q_1 = \eta_1 \partial_u = (x\,u\,F_1(\omega_1, \omega_2) + u\,F_2(\omega_1, \omega_2))\partial_u$, where F_1, F_2 are arbitrary smooth functions on two variables $\omega_1 = \frac{u_x}{u}$, $\omega_2 = \ln u - \frac{x u_x}{u}$. Imposing conditions $\frac{\partial \eta_1}{\partial x} = 0$ and $\frac{\partial \eta_1}{\partial u_x} = 0$ yields $F_1 = \lambda\omega_1$, $F_2 = \lambda\omega_2$, where λ is an arbitrary real constant. Then we have the diffusion equation

$$u_t = (\tfrac{u_x}{u})_x + \lambda u \ln u. \tag{3.40}$$

This equation possessing point nonclassical symmetry has been obtained in [34]. If $F_1 = \lambda\omega_1$, $F_2 = \lambda\omega_2 + h(\omega_1)$, where h is a smooth function then we obtain the generalization of (3.40)

$$u_t = (\tfrac{u_x}{u})_x + \lambda u \ln u + u h(\tfrac{u_x}{u}). \tag{3.41}$$

The ansatz

$$u = e^{\varphi_1(t)x}\varphi_2(t)$$

generated by the stationary solution of (3.38) reduces Eq (3.41) to system of two ODEs

$$\varphi_1'(t) = \lambda\varphi_1(t), \quad \varphi_2'(t) = \lambda\varphi_2(t)\ln\varphi_2(t) + \varphi_2(t)h(\varphi_1(t))$$

Applying the method to the two-component diffusive system we have

$$u_t = (\tfrac{u_x}{u})_x + u(\lambda_1 \ln u + \lambda_2 \ln v) \quad v_t = (\tfrac{v_x}{v})_x + v(\lambda_3 \ln v + \lambda_4 \ln u) \tag{3.42}$$

Next consider the equation

$$u_t = (\tfrac{1}{u})_{xx} \tag{3.43}$$

when $H(x) = 1$. It is obvious that the vector field $Q = (\frac{1}{u})_{xx}\partial_u$ is the generalized symmetry of ordinary differential equation

$$(\tfrac{1}{u})_{xx} = 0.$$

This equation is also invariant with respect to the Lie group of point symmetry given by vector field $Q_2 = (ku^2 + k_1 u + u_x(k_2 + \frac{k_3}{u}))\partial_u$, where k, k_1, k_2, k_3 are real constants. From the stationary solution of (3.43) we obtain the ansatz

$$u = \frac{1}{\varphi_1(t)x - \varphi_2(t)}, \tag{3.44}$$

which reduces nonlinear diffusion equation

$$u_t = (\tfrac{1}{u})_{xx} + ku^2 + k_1 u + u_x(k_2 + \tfrac{k_3}{u}) \tag{3.45}$$

to the system of ordinary differential equations

$$-\varphi_1'(t) = k_1\varphi_1 - k_3\varphi_1^2, \quad -\varphi_2'(t) = k + k_1\varphi_2 - k_2\varphi_1 - k_3\varphi_1\varphi_2.$$

In the framework of this approach one can obtain the two-component diffusive system

$$u_t = (\tfrac{1}{u})_{xx} + ku^2 + k_1 u + u_x(k_2 + \tfrac{k_3}{u}) + k_4\tfrac{u^2}{v}, \quad v_t = (\tfrac{1}{v})_{xx} + k_5 v^2 + k_6 v + v_x(k_7 + \tfrac{k_8}{v}) + k_9\tfrac{v^2}{u},$$

where $k_4, k_5, k_6, k_7, k_8, k_9$ are real constants. Note that the method based on stationary solutions, in general, is not applicable to the non-evolutionary equations. Although the approach described here is purely mathematical, however, it is sufficiently general to be applied to differential equations used in various fields of science including biological. For example, the diffusion coefficients which are used in [35] and are proportional to $C^\beta z^m$, where the values of β, m may be positive, negative or zero, z is the scaled height and C is the scaled concentration of the diffusing material, closely correlate with results described in (ii), (iii), (iv) and (v). There is a plausible physical justification for an inverse relation between the diffusion coefficient and the concentration C [35]. The relevant nonlinear differential equation is demonstrated as a model of turbulent dispersion in the atmosphere in [35], thermal wave propagation in plasma physics and fluid flow in porous media [36]. The approach can be also applied to the systems of Lotka–Volterra type which are used in mathematical biology [27] and equations describing fast diffusion processes. Here we use the method which rather belong to nonclassical symmetry approach since it provides solutions that cannot be obtained by using the classical Lie group method. In general A_i may depend on parametric variable t and the corresponding diffusion equation may be used for simulation of nonstationary media. It is obvious that the method is applicable to the ODE $(\frac{H}{u})_{xx} + \eta(x, u, u_x) = 0$. The corresponding ansatz reduces an ODE to a system of algebraic (not differential) equations in this case. For example ansatz (3.32) reduces the ODE

$$\left(\frac{e^{\beta x}}{u}\right)_{xx} + \sum_{i=1}^{2} e^{-i\beta x} u^3 A_i \left(2\frac{e^{\beta x}}{u^3}\left(u - \frac{u_x}{\beta}\right), \frac{e^{2\beta x}}{u^3}\left(2\frac{u_x}{\beta} - u\right)\right) = 0 \tag{3.46}$$

to the system of two algebraic equations

$$\frac{\beta^2}{2}\varphi_1^2 + 2A_1(\varphi_1, \varphi_2) = 0, \quad \beta^2\varphi_1\varphi_2 + 2A_2(\varphi_1, \varphi_2) = 0. \tag{3.47}$$

One can easily choose such A_1, A_2 that the system (3.47) will not have solutions. In general, the method ensures the reduction of ODEs to a system of algebraic equations but does not guarantee the existence of even one solution of reduced system and consequently solution of overdetermined system given by the ODE under study and the ODE possessing the corresponding generalized symmetry. It means that the ordinary differential equation is reduced to the system of algebraic equations but the appropriate overdetermined system of ordinary differential equations is not compatible.

3.3. How to construct solutions using a 2^{nd} order ODE

In this subsection we show the application of the method for finding solutions and obtain the solutions which cannot be constructed by the classical Lie method for the modified diffusion equations. We modify our original diffusion equation by some selected characteristics of point symmetries representing some physical properties of nonlinearity on heterogeneity. Let's start with the equation

$$u_{xx} = 3\frac{u_x^2}{u} - 3\beta u_x + \beta^2 u, \quad \beta \neq 0. \tag{3.48}$$

The ansatz is

$$u(x,t) = \frac{e^{\beta x}}{\sqrt{\varphi_1(t)e^{\beta x} + \varphi_2(t)}}. \tag{3.49}$$

We will consider equation

$$u_t = \left(\frac{e^{\beta x}}{u}\right)_{xx} + a_1 u + a_2 u_x + a_3 u^3 e^{-\beta x} + a_4 u^3 e^{-2\beta x} + a_5(\beta u e^{-\beta x} - e^{-\beta x} u_x), \quad a_1, a_2, \ldots, a_5 = \text{const} \tag{3.50}$$

which is obtained from (3.33) by letting

$$A_1 = -a_1 I_1 - a_2 \frac{\beta}{2} I_1 + a_3,$$

$$A_2 = -a_1 I_2 + a_2 \beta I_2 + a_4 + a_5 \frac{\beta}{2} I_1,$$

where $I_1 = 2\frac{e^{\beta x}}{u^3}\left(u - \frac{u_x}{\beta}\right)$, $I_2 = \frac{e^{2\beta x}}{u^3}\left(2\frac{u_x}{\beta} - u\right)$. By choosing only those five of all eight characteristics of the symmetry operators it is easy to solve the reduced equations. Note, that the Eq (3.48) can be linearized using a point change of variables, but we don't do such thing, because it would change also the term $\left(\frac{e^{\beta x}}{u}\right)_{xx}$. The property of reduction for Eq (3.50) is not limited to linear ODEs, it is also valid for nonlinear ODEs. For example, Korteweg–de Vries equation admits reduction with an ansatz obtained from a solution to an ODE which is an arbitrary linear combination of higher-order symmetries. One can also use any linear combination of higher-order symmetries of KdV to produce the ansatz. After substituting the ansatz (3.49) into (3.50) we obtain reduced equations:

$$2\varphi_1' + \beta^2\varphi_1^2 + 2(2a_1 + \beta a_2)\varphi_1 + 4a_3 = 0, \tag{3.51}$$

$$2\varphi_2' + 2\beta^2\varphi_1\varphi_2 + 4a_1\varphi_2 + 4\beta a_2\varphi_2 + 4a_4 + 2\beta a_5\varphi_1 = 0. \tag{3.52}$$

Now, the evolutionary equation (3.50) admits the following symmetry operators:

$$Y_1 = \partial_t,$$

$$Y_2 = \tfrac{1}{\beta}\partial_x + \tfrac{u}{2}\partial_u, \text{ if } a_4 = a_5 = 0,$$

$$Y_2 = t\partial_t + \tfrac{2}{\beta}\partial_x + \tfrac{3}{2}u\partial_u, \text{ if } a_1 = a_2 = a_3 = a_5 = 0,\ a_4 \neq 0,$$

$$Y_2 = t\partial_t + \tfrac{1}{\beta}\partial_x + u\partial_u, \text{ if } a_1 = a_2 = a_3 = a_4 = 0,\ a_5 \neq 0,$$

$$Y_3 = e^{(a_2\beta+2a_1)t}\partial_t - a_2 e^{(a_2\beta+2a_1)t}\partial_x + a_1 u e^{(a_2\beta+2a_1)t}\partial_u, \text{ if } a_3 = a_4 = a_5 = 0,\ a_2\beta + 2a_1 \neq 0,$$

$$Y_3 = t\partial_t - a_2 t\partial_x + (a_1 t + \tfrac{1}{2})u\partial_u \text{ if } a_3 = a_4 = a_5 = 0,\ a_2\beta + 2a_1 = 0.$$

Depending on the choice of a_i the reduced system is integrable and we can construct the solutions to the modified evolutionary equation.

For $a_2 \neq 0$, $\gamma = \pm\sqrt{(a_2\beta + 2a_1)^2 - 4\beta^2 a_3}$, $\gamma \neq 0$, $\delta = \frac{a_1a_5 - a_4\beta + a_2a_5\beta}{a_1^2 + a_1a_2\beta - a_3\beta^2}$, $a_1^2 + a_1a_2\beta - a_3\beta^2 \neq 0$ the reduced system (3.51) and (3.52) has the following solution:

$$\varphi_1(t) = \frac{\gamma\tanh\left(\frac{\gamma}{2}(t+s_1)\right) - \beta a_2 - 2a_1}{\beta^2},$$

$$\varphi_2(t) = \frac{s_2 e^{-\beta a_2 t} - (2a_5 + \beta a_2\delta)\cosh(\gamma(t+s_1)) + \delta\gamma\sinh(\gamma(t+s_1)) + \frac{4}{\beta a_2}(a_1a_5 - \beta a_4) + 2a_5}{4\beta\cosh^2\left(\frac{\gamma}{2}(t+s_1)\right)}.$$

Substituting φ_1, φ_2 into (3.49) gives the solution of (3.50) with the restrictions above.

The construction of solutions in the rest of the cases runs as before.

The imposition of condition $a_2 = 0$ on (3.50) slightly simplifies the reduced system as well as its solution, which then is:

$$\varphi_1(t) = \tfrac{\gamma}{\beta^2}\tanh\left(\tfrac{\gamma}{2}(t + s_1)\right) - \tfrac{2a_1}{\beta^2},$$

$$\varphi_2(t) = \frac{s_2+(a_1a_5-\beta a_4)\big(\gamma(t+s_1)+\sinh(\gamma(t+s_1))\big)}{\beta\gamma\cosh^2\left(\frac{\gamma}{2}(t+s_1)\right)} - \frac{a_5}{\beta},$$

γ now representing $\pm 2\sqrt{a_1^2-\beta^2 a_3} \neq 0$. Equating in (3.50) a nonzero term a_3 to $(\frac{a_2}{2}+\frac{a_1}{\beta})^2$ leads to quite a different solution:

$$\varphi_1(t) = \frac{2}{\beta^2(t+s_1)} - \frac{a_2\beta+2a_1}{\beta^2},$$

$$\varphi_2(t) = \frac{s_2 e^{-\beta a_2 t}+4(a_5(a_2\beta+a_1)-\beta a_4)(1-\beta a_2(t+s_1))}{\beta^4 a_2^3(t+s_1)^2} + \frac{a_5(a_2\beta+2a_1)-2\beta a_4}{\beta^2 a_2}.$$

Similarly, putting in (3.50) $a_2=0$ and $a_3=\frac{a_1^2}{\beta^2}$ results in solutions:

$$\varphi_1(t) = \frac{2}{\beta^2(t+s_1)} - \frac{2a_1}{\beta^2},$$

$$\varphi_2(t) = \frac{s_2}{(t+s_1)^2} + \frac{2(a_1a_5-a_4)}{3\beta}(t+s_1) - \frac{a_5}{\beta}.$$

Lastly, for $a_1=0$ and $a_2+\beta a_3=0$ in (3.50):

$$\varphi_1(t) = \frac{2a_2}{1+e^{-a_2\beta(t+s_1)}},$$

$$\varphi_2(t) = -\frac{2(2a_4+a_2a_5)e^{a_2\beta(t+s_1)}+(a_4+a_2a_5)e^{2a_2\beta(t+s_1)}+2a_2a_4t+s_2}{a_2(1+e^{a_2\beta(t+s_1)})^2}.$$

When $\exists_{i\in\{4,5\}}\ :\ a_i\neq 0$ and $\exists_{j\in\{1,2,3,4,5\},j\neq i}\ :\ a_j\neq 0$ the maximal Lie invariance algebra of the Eq (3.50) is one-dimensional and is spanned by ∂_t. The presented solutions are clearly not invariant under translations of the variable t. When exactly one of the constants a_4, a_5 is nonzero and $a_1=a_2=a_3=0$ or a_3 is nonzero and both a_4 and a_5 are zeros, the maximal Lie invariance algebra of the Eq (3.50) is two-dimensional, and invariance under one-parameter symmetry group with the generator $\alpha_1Y_1+\alpha_2Y_2$ must be verified from the definition, that is solution $u=u(x,t)$ is invariant when there exist real numbers $\alpha_1,\ \alpha_2$, at least one nonzero, that $(\alpha_1Y_1+\alpha_2Y_2)(u-u(x,t))\big|_{u=u(x,t)}=0$. Otherwise, the solution is not invariant. Instead of checking the invariance by the definition, we will compare them with the invariant solutions in the class (3.49). Functions φ_i for invariant solutions are as follows.

For $a_3\neq 0$, $a_4=a_5=0$

$$\varphi_1 = c_1,\quad \varphi_2 = c_2\exp(\tfrac{\alpha_2}{\alpha_1}t),$$

for $a_4\neq 0$, $a_1=a_2=a_3=a_5=0$

$$\varphi_1 = \frac{c_1}{\alpha_1+\alpha_2 t},\quad \varphi_2 = \frac{c_2}{\alpha_1+\alpha_2 t},$$

and for $a_5\neq 0$, $a_1=a_2=a_3=a_4=0$

$$\varphi_1 = \frac{c_1}{\alpha_1+\alpha_2 t},\quad \varphi_2 = c_2$$

where c_1, c_2, α_1, α_2 are all constants. By plain comparison, in those 3 cases, none of the five solutions in general form obtained from the reduced equations is invariant under $\alpha_1Y_1+\alpha_2Y_2$ (Y_2 depending on the choice of nonzero a_i).

One does not always have to solve the reduced equations to determine if the solution is or isn't invariant. Let's for example take equation

$$u_t = \Big(\frac{e^{\beta x}}{u}\Big)_{xx} + e^{\beta x}(\tfrac{\beta}{2}u-u_x)(a_6+e^{\beta x}u^{-2}a_7)\ +a_8\,(\tfrac{e^{\beta x}}{u})_x,\ a_i=\text{const},\ i=6,7,8 \tag{3.53}$$

which is obtained from (3.33) by letting

$$A_1 = -a_6\frac{\beta}{2}I_2 - a_7\frac{\beta}{2}I_1I_2 + a_8\frac{\beta}{2}I_1^2,$$

$$A_2 = -a_7\frac{\beta}{2}I_2^2 + a_8\frac{\beta}{2}I_1I_2.$$

It admits the same ansatz as in the previous example,

$$u(x,t) = \frac{e^{\beta x}}{\sqrt{\varphi_1(t)e^{\beta x}+\varphi_2(t)}}, \tag{3.54}$$

and is reduced to a system

$$\varphi_1' - \beta a_6\varphi_2 - \beta a_7\varphi_1\varphi_2 + \beta(a_8 + \tfrac{1}{2}\beta)\varphi_1^2 = 0,$$

$$\varphi_2' - \beta a_7\varphi_2^2 + \beta(a_8 + \beta)\varphi_1\varphi_2 = 0.$$

The Eq (3.53) possesses three symmetry operators when $a_6 = a_7 = 0$ and only one symmetry operator (∂_t) when both $a_6 \neq 0$ and $a_7 \neq 0$. Let's consider the case $a_6 = 0$, $a_7 \neq 0$ with two-dimensional Lie algebra. Here $Q_1 = \partial_t$, $Q_2 = t\partial_t + \frac{1}{2}u\partial_u$ is the basis of the algebra for Eq (3.53) (with $a_6 = 0$). Invariance criterion in terms of functions φ_i after splitting with respect to the powers of $e^{\beta x}$ is

$$(\alpha_1 + \alpha_2 t)\varphi_1' + \alpha_2\varphi_1 = 0,$$

$$(\alpha_1 + \alpha_2 t)\varphi_2' + \alpha_2\varphi_2 = 0.$$

Reduced equations do have an explicit solution but it is invariant, because $\varphi_1 = 0$. They also have an implicit solution. At this point all we need is to solve the reduced equations for the derivatives of φ_i and substitute those into the system above. The result is another system,

$$\varphi_1(\beta(\alpha_1 + \alpha_2 t)(a_7\varphi_2 - a_8\varphi_1 - \tfrac{1}{2}\beta\varphi_1) + \alpha_2) = 0,$$

$$\varphi_2(\beta(\alpha_1 + \alpha_2 t)(a_7\varphi_2 - a_8\varphi_1 - \beta\varphi_1) + \alpha_2) = 0.$$

The two equations are very similar. After dividing the ith equation by $\beta\varphi_1\varphi_i$ and subtracting one from another we have

$$\tfrac{1}{2}(\alpha_1 + \alpha_2 t) = 0,$$

meaning

$$\alpha_1 = \alpha_2 = 0,$$

so the Eq (3.54), where both φ_i are nonzero solutions of the reduced equations, wouldn't be an invariant solution.

Now we will find the solution to the system of the reduced equation. But firstly, we will show how to find a symmetry of the reduced equations having the symmetry of the PDE. We consider a case, when the symmetry operator Q of the PDE is admitted also by the ODE (3.48) (we treat the ODE as a PDE that does not include the derivative u_t) and condition

$$Q^{[1]}I_1 = F_1(x,u,u_x) = f_1(I_1,I_2), \qquad Q^{[1]}I_2 = F_2(x,u,u_x) = f_2(I_1,I_2) \tag{3.55}$$

holds, where $Q^{[1]}$ is the first prolongation of operator Q, $f_1(I_1, I_2)$, $f_2(I_1, I_2)$ are arbitrary smooth functions,

$$I_1 = \varphi_1 = e^{\beta x}\{\tfrac{2}{u^2} - \tfrac{2}{\beta}\tfrac{u_x}{u^3}\}, \qquad I_2 = \varphi_2 = e^{2\beta x}\{\tfrac{2}{\beta}\tfrac{u_x}{u^3} - \tfrac{1}{u^2}\},$$

are the first integrals of Eq (3.48). For an independent variable we impose the condition

$$Qt = m(t), \tag{3.56}$$

where $m(t)$ is arbitrary smooth function. One can construct the symmetry operator for reduced system in the form $\tilde{Q} = m(t)\partial_t + f_1(\varphi_1, \varphi_2)\partial_{\varphi_1} + f_2(\varphi_1, \varphi_2)\partial_{\varphi_2}$ if the conditions (3.55) and (3.56) are fulfilled. Obviously, we obtain a nontrivial symmetry for a system of reduced equations, if f_1, f_2 and m are not identically zeros. It's pretty obvious that Q_1 i Q_2 satisfy all the conditions, namely for Q_1 we have $f_1 = 0$, $f_2 = 0$ and $m = 1$, and for Q_2 we have $f_1 = -I_1$, $f_2 = -I_2$ and $m = t$. Thus we conclude that the ODE system

$$\varphi_1' - \beta a_7\varphi_1\varphi_2 + \beta(a_8 + \tfrac{1}{2}\beta)\varphi_1^2 = 0,$$

$$\varphi_2' - \beta a_7\varphi_2^2 + \beta(a_8 + \beta)\varphi_1\varphi_2 = 0.$$

is invariant with respect to 2-parameter Lie group of point transformations whose Lie algebra is given by basic elements

$$X_1 = \partial_t,$$

$$X_2 = t\partial_t - \varphi_1\partial_{\varphi_1} - \varphi_2\partial_{\varphi_2},$$

which are obtained from Q_1, Q_2. It means that the system is solvable in quadratures. A point transformations

$$T = \tfrac{\varphi_1}{\varphi_2},$$

$$W(T) = t,$$

$$Z(T) = \ln\varphi_2$$

maps the symmetry operators X_1, X_2 into

$$Y_1 = \partial_W,$$

$$Y_2 = W\partial_W - \partial_Z.$$

The transformed system can be simplified into

$$W_T = \tfrac{2}{e^Z\beta^2T^2},$$

$$Z_T = \tfrac{2}{\beta T^2}\{-(a_8 + \beta)T + a_7\}.$$

We can easily solve for Z,

$$Z(T) = \ln\{c\,T^{\frac{-2(a_8+\beta)}{\beta}}\exp\left(\tfrac{-2a_7}{\beta T}\right)\},$$

which in initial coordinates is an algebraic equation

$$\varphi_2 = c\left(\tfrac{\varphi_1}{\varphi_2}\right)^{\frac{-2(a_8+\beta)}{\beta}}\exp\left(\tfrac{-2a_7}{\beta}\tfrac{\varphi_2}{\varphi_1}\right), \quad c = \text{const}. \tag{3.57}$$

Solution

$$W(T) = 2\frac{c}{\beta^2}\int T^{\frac{2a_8}{\beta}}\exp(\tfrac{2a_7}{\beta T})\mathrm{d}T + c_0, \quad c_0, c = \text{const}, \tag{3.58}$$

is trickier to utilize once going back to the original coordinates. If we were to use $W_T = \frac{2}{e^Z\beta^2T^2}$, this equation would imply a differential equation

$$(\tfrac{\varphi_1}{\varphi_2})' = \frac{\beta^2}{2}\frac{\varphi_1^2}{\varphi_2} \tag{3.59}$$

that can be alternatively written as

$$(\tfrac{\varphi_2}{\varphi_1})' = -\frac{\beta^2}{2}\varphi_2. \tag{3.60}$$

The only invariant solution of the form $u(x,t) = \pm\frac{e^{\beta x}}{\sqrt{\varphi_1 e^{\beta x}+\varphi_2}}$ is the one where $\varphi_i = \frac{c_i}{\alpha_1+\alpha_2 t}$, $c_i, \alpha_j =$ const, $i, j = 1, 2$. In that case $\frac{\varphi_1}{\varphi_2} = \frac{c_1}{c_2} =$ const, and it is visible that such φ_i do not satisfy Eqs (3.57) to (3.60).

If we take $a_8 = -\beta$, then

$$Z = \ln c - \frac{2a_7}{\beta T}$$

but most importantly the integral in (3.58) can be easily calculated,

$$W = -\frac{1}{\beta c a_7}e^{\frac{2a_7}{\beta T}} + c_0.$$

From this we obtain

$$-\frac{2a_7}{\beta T} = \ln\big(\frac{1}{\beta a_7 c\,(c_0-W)}\big).$$

Taking the equations above into account, we see that

$$e^Z = \frac{1}{\beta a_7(c_0-W)},$$

$$T = \frac{2a_7}{\beta\ln(\beta a_7 c\,(c_0-W))}.$$

Because $W = t$, $\varphi_2 = e^Z$, $\varphi_1 = T\varphi_2$, the solution of the reduced system in original coordinates is

$$\varphi_2 = \frac{1}{\beta a_7(c_0-t)},$$

$$\varphi_1 = \frac{2}{\beta^2(c_0-t)\ln(\beta a_7 c\,(c_0-t))}.$$

The solutions we constructed have the property that corresponding solutions φ_1, φ_2 are not (identically) zero. At least we can say that there exist values of a_8 such that the proposed method allows us to construct non-invariant solutions, i.e., solutions that cannot be obtained by the classical Lie method. In conclusion, we have found φ_i's for which the solution $u(x,t) = u(x, \varphi_1(t), \varphi_2(t))$ is not invariant under the point symmetries of the evolution equation. Consider the equation

$$u_t = (\frac{e^{\beta x}}{u})_{xx}. \tag{3.61}$$

A basis of its Lie algebra A_3 of point symmetries consists of the vector fields

$$X_1 = \partial_t, \quad X_2 = 2t\partial_t + u\partial_u, \quad X_3 = 2\partial_x + \beta u\partial_u. \tag{3.62}$$

The solution to the reduced equations

$$\varphi_1' + \tfrac{1}{2}\beta^2\varphi_1^2 = 0, \quad \varphi_2' + \beta^2\varphi_1\varphi_2 = 0. \tag{3.63}$$

is

$$\varphi_1 = \tfrac{2}{\beta^2 t + c_1}, \quad \varphi_2 = \tfrac{c_2}{(\beta^2 t + c_1)^2}, \quad c_1,\ c_2 = \text{const}. \tag{3.64}$$

We call ansatz (3.54) together with solutions (3.64) a particular solution of (3.61).

Theorem 1. *Any particular solution of* (3.61) *given by* (3.54) *and* (3.64) *is invariant with respect to a one-parameter Lie invariance group of* (3.61) .

Proof of Theorem 1. Note that in the general case, the derivatives $\frac{\partial u}{\partial \varphi_1}$ and $\frac{\partial u}{\partial \varphi_2}$ are linearly independent. Otherwise, for some β_1, β_2, equation $\beta_1 \frac{\partial u}{\partial \varphi_1} + \beta_2 \frac{\partial u}{\partial \varphi_2} = 0$ would be true, and therefore u would be dependent on only one constant,

$$u = f(x, t, \beta_2\varphi_1 - \beta_1\varphi_2),$$

which is impossible, because it is the general solution of second-order ODE (3.48) Moreover the derivatives $\frac{\partial u}{\partial c_1}$ and $\frac{\partial u}{\partial c_2}$ are linearly independent since the pair (φ_1, φ_2) is the general solution of (3.63) and has the form (3.64).

The action of the operator $X = \xi_j(x,u)\frac{\partial}{\partial x_j} + \eta(x,u)\partial_u$, is the following

$$X(h(x) - u))\Big|_{u=h(x)} = (\xi_j(x,u)\frac{\partial h(x)}{\partial x_j} - \eta(x,u))\Big|_{u=h(x)}$$

where $x = (x_1, \ldots, x_n)$ for some integer n, $h(x)$ is a differentiable function. Then we show that

$$Q(f(x,t) - u))\Big|_{u=f(x,t)} \in W_2 \tag{3.65}$$

where $W_2 = \text{span}\{\frac{\partial u}{\partial c_1}, \frac{\partial u}{\partial c_2}\}$ and $Q \in A_3$. To prove this, it is enough to show this property for each of the basis elements X_1, X_2, X_3.

By direct computation we show that for $X_1 = \partial_t$

$$X_1(f(x,t) - u))\Big|_{u=f(x,t)} = \beta^2 \frac{\partial u}{\partial c_1},$$

for $X_2 = 2t\partial_t + u\partial_u$

$$X_2(f(x,t) - u))\Big|_{u=f(x,t)} = -2c_1\frac{\partial u}{\partial c_1} - 2c_2\frac{\partial u}{\partial c_2}$$

and for $X_3 = 2\partial_x + \beta u\partial_u$

$$X_3(f(x,t) - u))\Big|_{u=f(x,t)} = -2\beta c_2\frac{\partial u}{\partial c_2},$$

where $f(x,t)$ is the solution of (3.61) given by (3.54) and (3.64).

From the fact that any three vectors in two-dimensional vector space are linearly dependent, it follows that for any special solution (3.54) and (3.64) there can be selected α_1, α_2, α_3 such that $X(f(x,t) - u))\Big|_{u=f(x,t)} = 0$, where $X = \alpha_1X_1 + \alpha_2X_2 + \alpha_3X_3$ and not all α_i are equal to zero.

We conclude that every solution given by (3.54) and (3.64) can be found using a classical method of invariant solutions with respect to a one-parameter Lie group $\alpha_1X_1 + \alpha_2X_2 + \alpha_3X_3$.

This theorem is only sufficient but not necessary condition for a solution to be invariant in a classical sense. In fact, let us consider the two-dimensional abelian subalgebra of Lie algebra A_3, with basis elements Q_1, Q_2, where

$$Q_1 = 2c_1X_1 + \beta^2X_2 = 2(\beta^2t + c_1)\partial_t + \beta^2u\partial_u,$$
$$Q_2 = X_3 = 2\partial_x + \beta u\partial_u$$

(these operators clearly commute, $[Q_1, Q_2] = 0$). Because

$$Q_1(f(x,t) - u))\big|_{u=f(x,t)} = -2c_2\beta^2\frac{\partial u}{\partial c_2}$$

and

$$Q_2(f(x,t) - u))\big|_{u=f(x,t)} = -2\beta c_2\frac{\partial u}{\partial c_2},$$

the solution $u = f(x,t)$ is invariant with respect to a linear combination $Q_1 - \beta Q_2$. On the other hand, it is obvious that this solution cannot be obtained with a classical method using just any two-dimensional subalgebra, because not every two-dimensional algebra has the aforementioned properties, for example $\{X_1, X_3\}$. No nonzero linear combination of X_1 and X_3 leaves the solution invariant.

4. Discussion

We have constructed solutions of nonlinear evolution equations which can be used for describing the diffusion processes in heterogeneous medium by using the method based on the generalized symmetry of ODEs [17]. We show that the method gives us the possibility to obtain solutions which are not invariant ones in the classical Lie sense. We use the generalized symmetry of the second-order ODEs. The corresponding ansatzes reduce nonlinear diffusion equations to systems of two ODEs. One can obtain the solutions which can not be constructed by the classical Lie method in the cases when the dimension of the invariance Lie algebra is equal to 1 or 2. When the Lie algebra of the Lie invariance group of studied diffusion equation is three-dimensional then the solutions obtained by using our method could also be obtained via the classical Lie symmetry method as follows from Theorem 1. We have found wide classes of diffusion-type equations (and exact solutions) for which this method can be used. It is also shown that the method is applicable to non-evolutionary PDEs and enables us to construct solution depending on an arbitrary function for nonlinear hyperbolic (wave-type) equation. In fact, we show that the approach extends the applicability of the symmetry method for constructing exact solutions to PDEs. Exact solutions of differential equations are useful for understanding physical phenomena described by these equations and testing approximate and numerical methods for solving them.

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Conflict of interest

The authors declare there is no conflict of interest.

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