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Research article

Asymptotic behavior of the solutions for a stochastic SIRS model with information intervention

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Abstract: In this paper, a stochastic SIRS epidemic model with information intervention is considered. By constructing an appropriate Lyapunov function, the asymptotic behavior of the solutions for the proposed model around the equilibria of the deterministic model is investigated. We show the average in time of the second moment of the solutions of the stochastic system is bounded for a relatively small noise. Furthermore, we find that information interaction response rate plays an active role in disease control, and as the intensity of the response increases, the number of infected population decreases, which is beneficial for disease control.

Keywords: information intervention; SIRS model; stochastic perturbation; asymptotic behavior

1. Introduction

Infectious diseases seriously affect human's daily life and cause great losses to society and economy. It is one of the biggest public enemies threatening human life and health. According to the World Health Organization (WHO) Report, 2.6 million people died from lower respiratory tract infections worldwide in 2019 [1], and 10 million people were diagnosed with tuberculosis and 1.4 million people died of tuberculosis [2]. Severe Acute Respiratory Syndrome (SARS) caused approximately US$60 billion in losses to Asian countries in 2003 [3].There are more than 354 million people suffering from chronic hepatitis in the world, with an average of more than 8000 new hepatitis B and C patients every day, and more than 1 million deaths due to advanced liver disease and liver cancer each year [4]. And the sudden outbreak of COVID-19 in early 2020 brought a catastrophe to humanity. As of October 3, 2021, the cumulative number of confirmed cases reported globally exceeded 234 million, and the cumulative death toll was just under 4.8 million [5].

In recent decades, infectious disease dynamics models have become an effective tool for people to understand the spread and evolution of various infectious diseases [6–12]. These models have played an important role in the prevention and control of infectious diseases [10]. People usually adopt drug intervention (vaccination and drug treatment) to prevent and control the spread of infectious diseases, which has a good effect on the long-term control of infectious diseases. However, in recent years, researchers have found that in the absence of drug control, control through information plays a crucial role in the initial stages of the spread of infectious diseases. During the spread of an infectious disease, individual behaviors change in response to various infectious disease information. Such behavioral changes play an important role in the spread of emergent infectious diseases. Because after obtaining information on infectious diseases, people will try to reduce the possibility of infection by reducing gatherings, wearing masks, maintaining indoor ventilation, and strengthening disinfection of daily necessities to protect themselves. Therefore, although changes in personal behavior cannot eradicate infectious diseases, they can effectively delay the peak period of infectious disease outbreaks and reduce their severity [13–15]. It seems that the combined use of drug intervention and information intervention is better than a single intervention in controlling the spread of infectious diseases [14], and many researchers have noticed the role of information intervention in controlling the spread of infectious diseases [16–18].

Information control as a means of intervention can be spread through educational campaigns, the media, or the population itself through social activities [14]. Researchers considered different methods to incorporate the impact of infectious disease information into the mathematical model of infectious diseases. The impact of media coverage on disease transmission can be accounted by considering a correction in the incidence rate with a saturated function of infective individuals in [19–26]. Considering the impact of disease epidemic information on individuals' behavioral responses, and the dissemination of such disease information through various media, including television, newspapers, social activities, and positive education programs by the government and local organizations, Anuj Kumar et al. [14] established a model as follows,

$$\begin{cases} \frac{dS(t)}{dt} = \Lambda - \mu S - \beta S I - \mu_1 mZS + \delta_0 R, \\ \frac{dI(t)}{dt} = \beta S I - (\mu + \delta + \gamma) I, \\ \frac{dR(t)}{dt} = \gamma I + \mu_1 mZS - (\mu + \delta_0) R, \\ \frac{dZ(t)}{dt} = \frac{aI}{1 + bI} - a_0 Z, \end{cases} \tag{1.1}$$

where $S(t)$, $I(t)$ and $R(t)$ is the numbers of susceptible, infected, and recovered, respectively. $Z(t)$ represents the density of information. For the biological significance of all parameters, please refer to Anuj Kumar et al. [14]. For model (1.1), Anuj Kumar et al. [14] difined the basic reproductive number $R_0 = \frac{\beta\Lambda}{\mu(\mu+\delta+\gamma)}$, and discussed the globally asymptotically stability of the equilibriua. The authors proved that if $R_0 < 1$, model (1.1) has a unique globally asymptotically stable disease-free equilibrium $E_0 = (S_0, 0, 0, 0) = (\frac{\Lambda}{\mu}, 0, 0, 0)$; if $R_0 > 1$, model (1.1) has a unique globally asymptotically stable endemic equilibrium $E^* = (S^*, I^*, R^*, Z^*)$.

However, the above models are deterministic models, but the real life world is full of randomness (such as fluctuations, noise, earthquakes, tsunamis, etc.), so it is particularly important to consider the random factors in the process of disease transmission. The stochastic infectious disease models are more realistic and have more dynamic properties. In recent years, many scholars have paid great attention to the study of stochastic infectious disease models and have produced many good results [27–33]. For example, Zhao et al. [34] and Liu et al. [35] have investigated a stochastic SIVS epidemic model and a SIS epidemic model with nonlinear saturation rates respectively. In the model, they assumed that the contact rate β was perturbed by white noise and studied the extinction and persistence of the disease by constructing suitable Lyapunov functions. Zhou et al. [36] have considered a stochastic SIR epidemic model with a general nonlinear incidence rate, the stationary distribution of the solutions of the system under certain parameter constraints is obtained. Zhou et al. [37] have established the threshold dynamics of a stochastic SIS model with Lévy jumps.

Recently, Zhao et al. [38] hypothesized that the exposure rate of infectious diseases is affected by white noise, and proposed a stochastic SIS epidemic model combining media coverage and environmental fluctuations. They found that environmental fluctuations may significantly affect the threshold dynamic behavior of infectious diseases and the fluctuation of populations of different sizes. Media coverage in low-intensity noise environment has an important influence on the smooth distribution of infectious diseases. Jin et al. [39] introduce stochastic perturbations into model (1.1) and get the following stochastic SIRS model with information intervention:

$$\begin{cases} dS(t) = [\Lambda - \mu S - \beta SI - \mu_1 mZS + \delta_0 R]dt + \sigma_1 S\,dB_1(t), \\ dI(t) = [\beta SI - (\mu + \delta + \gamma)I]dt + \sigma_2 I dB_2(t), \\ dR(t) = [\gamma I + \mu_1 mZS - (\mu + \delta_0)R]dt + \sigma_3 R dB_3(t), \\ dZ(t) = [\frac{aI}{1+bI} - a_0 Z]dt + \sigma_4 Z dB_4(t), \end{cases} \tag{1.2}$$

where $B_i(t)$ are independent standard Brownian motions and $B_i(0) = 0$ $(i = 1, 2, 3, 4)$, the corresponding stochastic integrals w.r.t. the Brownian motions under consideration are all of the Itô's formula, and $\sigma_i > 0$ $(i = 1, 2, 3, 4)$ represent the intensities of the noise to the susceptible, infected, recovered and information, respectively.

The authors investigated the dynamics of model (1.2) including the extinction and persistence of the disease, and the existence of stationary distribution. For stochastic systems, it is of great significance to study the asymptotic behavior of the solution near the equilibria, because there are no equilibria for the stochastic system, and the solutions of the stochastic system do not reach a fixed value over time, but oscillate continuously around some value. For example, Yu et al. [40] considered a type of two-group stochastic SIR model with white noise, the authors showed that the oscillation of the solution of the system decreases with the decrease of the noise level. Then motivated by Yu et al. [40], we mainly discuss the asymptotic behavior of the solution for stochastic model (1.2) near the disease-free equilibrium and disease equilibrium of deterministic model (1.1), which is different from Jin et al. [39].

The main parts of the paper are arranged as follows. In Section 2, we give some preliminaries including some notations and lemmas. In Section 3, we discuss the asymptotic behavior of the solutions for the proposed model around the equilibria of the deterministic model by using some suitable Lyapunov functions and inequality techniques. In Section 4, we carry out some numerical simulations to verify the main results. In Section 5, we give the strengths and weaknesses of the study and the next research topics.

2. Preliminaries

In the section, we give some notations and lemmas.

Let $(\Omega, \mathcal{F}, \mathcal{F}_{t\geq 0}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}\}_{t\geq 0}$ satisfying the usual conditions(i.e., it is increasing and right continuous while $\mathcal{F}_0$ contains all $\mathbb{P}-$ *null* sets). Further suppose $B(t)$ is defined on Ω.

For the given n-dimensional stochastic differential equation [41]

$$dx(t) = f_1(x(t), t)dt + f_2(x(t), t)dB(t) \text{ on } t \geq t_0 \tag{2.1}$$

with the initial value $x(0) = x_0 \in \mathbb{R}^n$. Let $C^{2,1}(\mathbb{R}^n \times [t_0, \infty); \mathbb{R}_+)$ represent the family of all non-negative functions $U(x, t)$ defined on $\mathbb{R}^n \times [t_0, \infty)$ such that they are continuously twice differentiable in x and once in t. Define the differential operator $\mathcal{L}$ associated with (2.1) by

$$\mathcal{L} = \frac{\partial}{\partial t} + \sum_{i=1}^{n} f_{1i}(x, t)\frac{\partial}{\partial x_i} + \frac{1}{2}\sum_{i,j=1}^{n}\left[f_2^T(x, t)f_2(x, t)\right]_{i,j}\frac{\partial^2}{\partial x_i \partial x_j}.$$

For a function $U \in C^{2,1}(\mathbb{R}^n \times [t_0, \infty); \mathbb{R}_+)$, we get

$$\mathcal{L}U(x, t) = U_t(x, t) + U_x(x, t)f_1(x, t) + \frac{1}{2}\operatorname{trace}\left[f_2^T(x, t)U_{xx}(x, t)f_2(x, t)\right],$$

here

$$U_t = \tfrac{\partial U}{\partial t}, \quad U_x = \left(\tfrac{\partial U}{\partial x_1}, \ldots, \tfrac{\partial U}{\partial x_n}\right),$$
$$U_{xx} = \left(\tfrac{\partial^2 U}{\partial x_i \partial x_j}\right)_{n\times n} = \begin{pmatrix} \frac{\partial^2 U}{\partial x_1 \partial x_1} & \cdots & \frac{\partial^2 U}{\partial x_1 \partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 U}{\partial x_n \partial x_1} & \cdots & \frac{\partial^2 U}{\partial x_n \partial x_n} \end{pmatrix}.$$

By using the Itô's formula, we have

$$dU(x(t), t) = \mathcal{L}U(x(t), t)dt + U_x(x(t), t)f_2(x(t), t)dB(t),$$

where

$$x(t) \in \mathbb{R}^n, \mathbb{R}^n_+ = \{(x_1, x_2, ..., x_n) \in \mathbb{R}^n : x_i > 0, i = 1, 2, ..., n\}.$$

For an integrable function $x(t) \in [0, +\infty)$, $\langle x(t)\rangle = \frac{1}{t}\int_0^t x(r)dr.$

Lemma 2.1. *[39] For any initial value $(S(0), I(0), R(0), Z(0)) \in \mathbb{R}^4_+$, the model (1.2) has a unique positive solution $(S(t), I(t), R(t), Z(t))$ on $t \geq 0$, and the solution will remain in $\mathbb{R}^4_+$ with probability one.*

Lemma 2.2. *[39] Let $(S(t), I(t), R(t), Z(t))$ be the solution of model (1.2) with the initial value $(S(0), I(0), R(0), Z(0)) \in \mathbb{R}^4_+$. Then*

$$\lim_{t\to\infty}\frac{S(t)}{t} = 0, \lim_{t\to\infty}\frac{I(t)}{t} = 0, \lim_{t\to\infty}\frac{R(t)}{t} = 0, \lim_{t\to\infty}\frac{Z(t)}{t} = 0,$$

and

$$\lim_{t\to\infty}\frac{\int_0^t S(r)dB_1(r)}{t} = 0, \lim_{t\to\infty}\frac{\int_0^t I(r)dB_2(r)}{t} = 0, \lim_{t\to\infty}\frac{\int_0^t R(r)dB_3(r)}{t} = 0, \lim_{t\to\infty}\frac{\int_0^t Z(r)dB_4(r)}{t} = 0 \quad a.s.$$

Lemma 2.3. *Let $(S(t), I(t), R(t), Z(t))$ be the solution of model (1.2) with initial value $(S(0), I(0), R(0), Z(0)) \in \mathbb{R}^4_+$. Then*

$$\limsup_{t\to\infty}\langle S(t)\rangle \leq \frac{\Lambda}{\mu}, \limsup_{t\to\infty}\langle I(t)\rangle \leq \frac{\Lambda}{\mu}, \limsup_{t\to\infty}\langle R(t)\rangle \leq \frac{\Lambda}{\mu}, \lim_{t\to\infty}\langle Z(t)\rangle \leq \frac{a\Lambda}{a_0(\mu + b\Lambda)} \quad a.s.$$

Proof. Summing the first three variables in model (1.2) yields

$$d(S+I+R)=[\Lambda-\mu S-(\mu+\delta)I-\mu R]dt+\sigma_1 S\,dB_1+\sigma_2 IdB_2+\sigma_3 RdB_3,$$

taking integral form 0 to t, we have

$$\begin{aligned}\frac{S(t)+I(t)+R(t)}{t}=&\Lambda-\mu\langle S(t)\rangle-(\mu+\delta)\langle I(t)\rangle-\mu\langle R(t)\rangle+\frac{\int_0^t\sigma_1 S(s)dB_1(s)}{t}+\frac{\int_0^t\sigma_2 I(s)dB_2(s)}{t}\\&+\frac{\int_0^t\sigma_3 R(s)dB_3(s)}{t}+\frac{S(0)+I(0)+R(0)}{t},\end{aligned}$$

thus we get

$$\Lambda-\mu\langle S(t)\rangle-(\mu+\delta)\langle I(t)\rangle-\mu\langle R(t)\rangle=\varphi_1(t),$$

where

$$\begin{aligned}\varphi_1(t)=&\frac{\int_0^t\sigma_1 S(s)dB_1(s)}{t}+\frac{\int_0^t\sigma_2 I(s)dB_2(s)}{t}+\frac{\int_0^t\sigma_3 R(s)dB_3(s)}{t}\\&-\frac{S(t)+I(t)+R(t)}{t}+\frac{S(0)+I(0)+R(0)}{t}.\end{aligned}$$

According to the Lemma 2.2, one has

$$\lim_{t\to\infty}\varphi_1(t)=0,$$

then

$$\limsup_{t\to\infty}\langle S(t)\rangle+\limsup_{t\to\infty}\langle I(t)\rangle+\limsup_{t\to\infty}\langle R(t)\rangle\le\frac{\Lambda}{\mu},$$

then

$$\limsup_{t\to\infty}\langle S(t)\rangle\le\frac{\Lambda}{\mu},\quad\limsup_{t\to\infty}\langle I(t)\rangle\le\frac{\Lambda}{\mu},\quad\limsup_{t\to\infty}\langle R(t)\rangle\le\frac{\Lambda}{\mu}.$$

According to the fourth equation in model (1.2), one yields

$$\frac{Z(t)}{t}\le\frac{a\langle I(t)\rangle}{1+b\langle I(t)\rangle}-a_0\langle Z(t)\rangle+\frac{\int_0^t\sigma_4 Z(s)dB_4(s)}{t}+\frac{Z(0)}{t},$$

then, we have

$$\frac{a\langle I(t)\rangle}{1+b\langle I(t)\rangle}-a_0\langle Z(t)\rangle\ge\varphi_2(t),$$

where

$$\varphi_2(t)=-\frac{\int_0^t\sigma_4 Z(s)dB_4(s)}{t}+\frac{Z(t)}{t}-\frac{Z(0)}{t}.$$

According to the Lemma 2.2, we get

$$\lim_{t\to\infty}\varphi_2(t)=0,$$

then

$$\limsup_{t\to\infty}\langle Z(t)\rangle\le\frac{a\Lambda}{a_0(\mu+b\Lambda)}\quad\text{a.s.}$$

The proof of Lemma 2.3 is completed.

3. Dynamics behavior of the stochastic model

For the deterministic model (1.1), if $R_0 < 1$, then disease-free equilibrium E_0 is globally asymptotically stable, and if $R_0 > 1$, the unique disease equilibrium E^* is globally asymptotically stable. A very interesting question is, what kind of effect does stochastic disturbance have on the model (1.1), what happens to the solution of the stochastic model (1.2). The following theorems answers these questions. According to Theorems 3.1 and 3.2, we will find that for relatively small stochastic disturbances, the stochastic system solution will oscillate around the equilibria of the deterministic system.

Theorem 3.1. *Let $(S(t), I(t), R(t), Z(t))$ be the solution of model (1.2) with any initial value $(S(0), I(0), R(0), Z(0)) \in \mathbb{R}^4_+$. If $R_0 \le 1 - \frac{ax_2}{y_1(\mu+\delta+\gamma)}$, and the following conditions are satisfied*

$$\sigma_1^2 < \mu,$$
$$\sigma_2^2 + x_1\sigma_2^2 + a < 2(\mu+\delta) + 2(\mu+\delta+\gamma)x_1,$$
$$\sigma_3^2 < 2\mu,$$
$$\sigma_4^2 + a < 2a_0,$$

then

$$\limsup_{t\to\infty} \frac{1}{t} E\int_0^t \left[\left(S(\tau) - \frac{\Lambda}{\mu}\right)^2 + I^2(\tau) + R^2(\tau) + Z^2(\tau)\right] d\tau \le \frac{1}{K_3}\frac{(1+x_1)\Lambda^2\sigma_1^2}{\mu^2} \quad a.s.,$$

where

$$x_1 = \frac{2\mu}{\delta_0},$$
$$x_2 = \frac{\mu_1 m x_1 \Lambda^2}{a_0\mu^2},$$
$$y_1 = \frac{2\mu+\delta+(2\mu+\delta+\gamma)x_1}{\beta},$$

$$K_3 = \min\left\{(1+x_1)(\mu-\sigma_1^2), \mu+\delta+(\mu+\delta+\gamma)x_1 - \frac{a+\sigma_2^2+x_1\sigma_2^2}{2}, \mu - \frac{\sigma_3^2}{2}, a_0 - \frac{a+\sigma_4^2}{2}\right\}.$$

Proof. Consider the following Lyapunov function

$$V(S, I, R, Z) = V_1 + V_2 + V_3 + V_4,$$

where

$$V_1 = \frac{1}{2}\left[\left(S - \frac{\Lambda}{\mu}\right) + I + R\right]^2, V_2 = \frac{1}{2}x_1\left(S - \frac{\Lambda}{\mu} + I\right)^2, V_3 = y_1 I + x_2 Z, V_4 = \frac{1}{2}Z^2,$$

and

$$x_1 = \frac{2\mu}{\delta_0},$$
$$x_2 = \frac{\mu_1 m x_1 \Lambda^2}{a_0\mu^2},$$
$$y_1 = \frac{2\mu+\delta+(2\mu+\delta+\gamma)x_1}{\beta}.$$

Applying Itô formula to V_1, V_2, V_3 and V_4, respectively, one has

$$
\begin{aligned}
dV_1 =& LV_1dt + \left(S - \frac{\Lambda}{\mu} + I + R\right)(\sigma_1 S\,dB_1(t) + \sigma_2 IdB_2(t) + \sigma_3 RdB_3(t)),\\
dV_2 =& LV_2dt + x_1\left(S - \frac{\Lambda}{\mu} + I\right)(\sigma_1 S\,dB_1(t) + \sigma_2 IdB_2(t)),\\
dV_3 =& LV_3dt + y_1\sigma_2 IdB_2(t) + x_2\sigma_4 ZdB_4(t),\\
dV_4 =& LV_4dt + \sigma_4 Z^2 dB_4(t).
\end{aligned} \tag{3.1}
$$

Firstly, one can derive that

$$
\begin{aligned}
LV_1 =& \left(S - \frac{\Lambda}{\mu} + I + R\right)[\Lambda - \mu S - (\mu+\delta)I - \mu R] + \frac{1}{2}\sigma_1^2 S^2 + \frac{1}{2}\sigma_2^2 I^2 + \frac{1}{2}\sigma_3^2 R^2\\
=& \left(S - \frac{\Lambda}{\mu} + I + R\right)\left[-\mu\left(S - \frac{\Lambda}{\mu}\right) - (\mu+\delta)I - \mu R\right] + \frac{1}{2}\sigma_1^2 S^2 + \frac{1}{2}\sigma_2^2 I^2 + \frac{1}{2}\sigma_3^2 R^2\\
=& -\mu\left(S - \frac{\Lambda}{\mu}\right)^2 - (\mu+\delta)\left(S - \frac{\Lambda}{\mu}\right)I - \mu\left(S - \frac{\Lambda}{\mu}\right)R - \mu\left(S - \frac{\Lambda}{\mu}\right)I - (\mu+\delta)I^2\\
& - \mu IR - \mu\left(S - \frac{\Lambda}{\mu}\right)R - (\mu+\delta)IR - \mu R^2 + \frac{1}{2}\sigma_1^2 S^2 + \frac{1}{2}\sigma_2^2 I^2 + \frac{1}{2}\sigma_3^2 R^2\\
=& -\mu\left(S - \frac{\Lambda}{\mu}\right)^2 - (\mu + \delta - \frac{1}{2}\sigma_2^2)I^2 - (\mu - \frac{1}{2}\sigma_3^2)R^2 - (2\mu+\delta)\left(S - \frac{\Lambda}{\mu}\right)I\\
& - 2\mu\left(S - \frac{\Lambda}{\mu}\right)R - (2\mu+\delta)IR + \frac{1}{2}\sigma_1^2\left[(S - \frac{\Lambda}{\mu}) + \frac{\Lambda}{\mu}\right]^2.
\end{aligned}
$$

Using the inequality $(a+b)^2 \le 2a^2 + 2b^2$ for $a, b \in \mathbb{R}$, we get

$$
\frac{1}{2}\sigma_1^2\left[(S - \frac{\Lambda}{\mu}) + \frac{\Lambda}{\mu}\right]^2 \le \sigma_1^2\left(S - \frac{\Lambda}{\mu}\right)^2 + \frac{\Lambda^2\sigma_1^2}{\mu^2},
$$

then

$$
\begin{aligned}
LV_1 \le& -\mu\left(S - \frac{\Lambda}{\mu}\right)^2 - (\mu + \delta - \frac{1}{2}\sigma_2^2)I^2 - (\mu - \frac{1}{2}\sigma_3^2)R^2 - (2\mu+\delta)\left(S - \frac{\Lambda}{\mu}\right)I\\
& - 2\mu\left(S - \frac{\Lambda}{\mu}\right)R - (2\mu+\delta)IR + \sigma_1^2\left(S - \frac{\Lambda}{\mu}\right)^2 + \frac{\Lambda^2\sigma_1^2}{\mu^2}\\
=& -(\mu - \sigma_1^2)\left(S - \frac{\Lambda}{\mu}\right)^2 - (\mu + \delta - \frac{1}{2}\sigma_2^2)I^2 - (\mu - \frac{1}{2}\sigma_3^2)R^2 - (2\mu+\delta)\left(S - \frac{\Lambda}{\mu}\right)I\\
& - 2\mu\left(S - \frac{\Lambda}{\mu}\right)R - (2\mu+\delta)IR + \frac{\Lambda^2\sigma_1^2}{\mu^2}.
\end{aligned}
$$

Secondly, one has

$$
\begin{aligned}
LV_2 =& x_1\left(S-\frac{\Lambda}{\mu}+I\right)[\Lambda-\mu S-\mu_1 mZS+\delta_0 R-(\mu+\delta+\gamma)I]+\frac{1}{2}x_1\sigma_1^2 S^2+\frac{1}{2}x_1\sigma_2^2 I^2\\
=& x_1\left(S-\frac{\Lambda}{\mu}+I\right)\left[-\mu\left(S-\frac{\Lambda}{\mu}\right)-\mu_1 m(S-\frac{\Lambda}{\mu})Z-\mu_1 m\frac{\Lambda}{\mu}Z+\delta_0 R-(\mu+\delta+\gamma)I\right]\\
&+\frac{1}{2}x_1\sigma_1^2 S^2+\frac{1}{2}x_1\sigma_2^2 I^2\\
=&-\mu x_1\left(S-\frac{\Lambda}{\mu}\right)^2-\mu_1 m x_1\left(S-\frac{\Lambda}{\mu}\right)^2 Z-\mu_1 m\frac{\Lambda}{\mu}x_1\left(S-\frac{\Lambda}{\mu}\right)Z+\delta_0 x_1\left(S-\frac{\Lambda}{\mu}\right)R\\
&-(\mu+\delta+\gamma)x_1\left(S-\frac{\Lambda}{\mu}\right)I-\mu x_1\left(S-\frac{\Lambda}{\mu}\right)I-\mu_1 m x_1\left(S-\frac{\Lambda}{\mu}\right)ZI-\mu_1 m\frac{\Lambda}{\mu}x_1 ZI\\
&+\delta_0 x_1 IR-(\mu+\delta+\gamma)x_1 I^2+\frac{1}{2}x_1\sigma_1^2 S^2+\frac{1}{2}x_1\sigma_2^2 I^2\\
=&-\mu x_1\left(S-\frac{\Lambda}{\mu}\right)^2-\mu_1 m x_1\left(S-\frac{\Lambda}{\mu}\right)^2 Z-\mu_1 m\frac{\Lambda}{\mu}x_1\left(S-\frac{\Lambda}{\mu}\right)Z+\delta_0 x_1\left(S-\frac{\Lambda}{\mu}\right)R\\
&-(2\mu+\delta+\gamma)x_1\left(S-\frac{\Lambda}{\mu}\right)I-\mu_1 m x_1\left(S-\frac{\Lambda}{\mu}\right)ZI-\mu_1 m\frac{\Lambda}{\mu}x_1 ZI+\delta_0 x_1 IR\\
&-\left[(\mu+\delta+\gamma)x_1-\frac{1}{2}x_1\sigma_2^2\right]I^2+\frac{1}{2}x_1\sigma_1^2\left[(S-\frac{\Lambda}{\mu})+\frac{\Lambda}{\mu}\right]^2\\
\le&-\mu x_1\left(S-\frac{\Lambda}{\mu}\right)^2-\left[(\mu+\delta+\gamma)x_1-\frac{1}{2}x_1\sigma_2^2\right]I^2-\mu_1 m\frac{\Lambda}{\mu}x_1\left(S-\frac{\Lambda}{\mu}\right)Z\\
&+\delta_0 x_1\left(S-\frac{\Lambda}{\mu}\right)R-(2\mu+\delta+\gamma)x_1\left(S-\frac{\Lambda}{\mu}\right)I+\delta_0 x_1 IR+\frac{1}{2}x_1\sigma_1^2\left[(S-\frac{\Lambda}{\mu})+\frac{\Lambda}{\mu}\right]^2.
\end{aligned}
$$

Since $\frac{1}{2}x_1\sigma_1^2\left[(S-\frac{\Lambda}{\mu})+\frac{\Lambda}{\mu}\right]^2\le x_1\sigma_1^2\left(S-\frac{\Lambda}{\mu}\right)^2+\frac{x_1\Lambda^2\sigma_1^2}{\mu^2}$, then

$$
\begin{aligned}
LV_2 \le&-\mu x_1\left(S-\frac{\Lambda}{\mu}\right)^2-\left[(\mu+\delta+\gamma)x_1-\frac{1}{2}x_1\sigma_2^2\right]I^2-\mu_1 m\frac{\Lambda}{\mu}x_1\left(S-\frac{\Lambda}{\mu}\right)Z\\
&+\delta_0 x_1\left(S-\frac{\Lambda}{\mu}\right)R-(2\mu+\delta+\gamma)x_1\left(S-\frac{\Lambda}{\mu}\right)I+\delta_0 x_1 IR+x_1\sigma_1^2\left(S-\frac{\Lambda}{\mu}\right)^2+\frac{x_1\Lambda^2\sigma_1^2}{\mu^2}\\
=&-(\mu x_1-x_1\sigma_1^2)\left(S-\frac{\Lambda}{\mu}\right)^2-\left[(\mu+\delta+\gamma)x_1-\frac{1}{2}x_1\sigma_2^2\right]I^2-\mu_1 m\frac{\Lambda}{\mu}x_1\left(S-\frac{\Lambda}{\mu}\right)Z\\
&+\delta_0 x_1\left(S-\frac{\Lambda}{\mu}\right)R-(2\mu+\delta+\gamma)x_1\left(S-\frac{\Lambda}{\mu}\right)I+\delta_0 x_1 IR+\frac{x_1\Lambda^2\sigma_1^2}{\mu^2}\\
=&-(\mu x_1-x_1\sigma_1^2)\left(S-\frac{\Lambda}{\mu}\right)^2-\left[(\mu+\delta+\gamma)x_1-\frac{1}{2}x_1\sigma_2^2\right]I^2+\mu_1 m\frac{x_1\Lambda^2}{\mu^2}Z\\
&+\delta_0 x_1\left(S-\frac{\Lambda}{\mu}\right)R-(2\mu+\delta+\gamma)x_1\left(S-\frac{\Lambda}{\mu}\right)I+\delta_0 x_1 IR+\frac{x_1\Lambda^2\sigma_1^2}{\mu^2}.
\end{aligned}
$$

Next, let us consider LV_3, one has

$$\begin{aligned}LV_3 =& y_1 \left[\beta S I - (\mu + \delta + \gamma)I\right] + \frac{x_2 a I}{1 + bI} - x_2 a_0 Z\\ \leq& y_1 \left[\beta I\left(S - \frac{\Lambda}{\mu}\right) + \left(\beta\frac{\Lambda}{\mu} - (\mu + \delta + \gamma)\right) I\right] + x_2 a I - x_2 a_0 Z\\ =& y_1\beta I\left(S - \frac{\Lambda}{\mu}\right) - \left[y_1(\mu + \delta + \gamma)(1 - R_0) - x_2 a\right] I - x_2 a_0 Z,\end{aligned}$$

then,

$$\begin{aligned}L(V_1 + V_2 + V_3) =& -(1 + x_1)(\mu - \sigma_1^2)\left(S - \frac{\Lambda}{\mu}\right)^2 - \left[\mu + \delta + x_1(\mu + \delta + \gamma) - \frac{1}{2}\sigma_2^2 - x_1\frac{1}{2}\sigma_2^2\right] I^2\\ &- (\mu - \frac{1}{2}\sigma_3^2)R^2 - [2\mu + \delta + (2\mu + \delta + \gamma)x_1 - y_1\beta]\left(S - \frac{\Lambda}{\mu}\right) I\\ &- (2\mu - \delta_0 x_1)\left(S - \frac{\Lambda}{\mu}\right) R - [y_1(\mu + \delta + \gamma)(1 - R_0) - x_2 a] I\\ &- (2\mu + \delta - \delta_0 x_1) IR - \left(x_2 a_0 - \mu_1 m\frac{x_1\Lambda^2}{a_0\mu^2}\right) Z + \frac{(1 + x_1)\Lambda^2\sigma_1^2}{\mu^2}.\end{aligned}$$

Since $R_0 \leq 1 - \frac{a x_2}{y_1(\mu+\delta+\gamma)}$, choose

$$\begin{aligned}x_1 =& \frac{2\mu}{\delta_0},\\ x_2 =& \frac{\mu_1 m x_1 \Lambda^2}{a_0\mu^2},\\ y_1 =& \frac{2\mu + \delta + (2\mu + \delta + \gamma)x_1}{\beta},\end{aligned}$$

such that

$$\begin{aligned}2\mu + \delta + (2\mu + \delta + \gamma)x_1 - y_1\beta =& 0,\\ 2\mu - \delta_0 x_1 =& 0,\\ 2\mu + \delta - \delta_0 x_1 \geq& 0,\\ y_1(\mu + \delta + \gamma)(1 - R_0) - x_2 a \geq& 0,\\ x_2 a_0 - \mu_1 m\frac{x_1\Lambda^2}{a_0\mu^2} =& 0,\end{aligned}$$

then

$$\begin{aligned}L(V_1 + V_2 + V_3) \leq& -(1 + x_1)(\mu - \sigma_1^2)\left(S - \frac{\Lambda}{\mu}\right)^2 - \left[\mu + \delta + x_1(\mu + \delta + \gamma) - \frac{1}{2}\sigma_2^2 - x_1\frac{1}{2}\sigma_2^2\right] I^2\\ &- (\mu - \frac{1}{2}\sigma_3^2)R^2 + \frac{(1 + x_1)\Lambda^2\sigma_1^2}{\mu^2}.\end{aligned}$$

Similarly, we have

$$\begin{aligned}LV_4 =& Z\left[\frac{aI}{1+bI} - a_0 Z\right] + \frac{1}{2}\sigma_4^2 Z^2\\ \le& aIZ - a_0 Z^2 + \frac{1}{2}\sigma_4^2 Z^2\\ \le& -\left(a_0 - \frac{1}{2}a - \frac{1}{2}\sigma_4^2\right) Z^2 + \frac{a}{2} I^2,\end{aligned}$$

then

$$\begin{aligned}LV \le& -(1+x_1)(\mu - \sigma_1^2)\left(S - \frac{\Lambda}{\mu}\right)^2 - \left[\mu + \delta + (\mu+\delta+\gamma)x_1 - \frac{1}{2}a - \frac{1}{2}\sigma_2^2 - \frac{1}{2}x_1\sigma_2^2\right] I^2\\ &-(\mu - \frac{1}{2}\sigma_3^2)R^2 - \left(a_0 - \frac{1}{2}a - \frac{1}{2}\sigma_4^2\right) Z^2 + \frac{(1+x_1)\Lambda^2\sigma_1^2}{\mu^2}.\end{aligned}$$

Therefore,

$$\limsup_{t\to\infty} \frac{1}{t} E\int_0^t \left[\left(S(\tau) - \frac{\Lambda}{\mu}\right)^2 + I^2(\tau) + R^2(\tau) + Z^2(\tau)\right] d\tau \le \frac{1}{K_3}\frac{(1+x_1)\Lambda^2\sigma_1^2}{\mu^2} \quad \text{a.s.},$$

where

$$K_3 = \min\left\{(1+x_1)(\mu-\sigma_1^2), \mu+\delta+(\mu+\delta+\gamma)x_1 - \frac{1}{2}a - \frac{1}{2}\sigma_2^2 - \frac{1}{2}x_1\sigma_2^2, \mu - \frac{1}{2}\sigma_3^2, a_0 - \frac{1}{2}a - \frac{1}{2}\sigma_4^2\right\}.$$

Theorem 3.2. *Let $(S(t), I(t), R(t), Z(t))$ be the solution of model (1.2) with any initial value $(S(0), I(0), R(0), Z(0)) \in \mathbb{R}_+^4$, If $R_0 = \frac{\beta\Lambda}{\mu(\mu+\delta+\gamma)} > 1$ and the following conditions are satisfied*

$$(1+\frac{2\mu}{\delta_0})\sigma_1^2 < \frac{2\mu^2}{\delta_0} + \mu - \frac{\mu\mu_1 m S^*}{\delta_0},$$

$$\sigma_2^2 < \mu + \delta - a,$$

$$\sigma_3^2 < \mu,$$

$$\sigma_4^2 < a_0 - a - \frac{\mu\mu_1 m S^*}{\delta_0},$$

then

$$\limsup_{t\to\infty} \frac{1}{t} E\int_0^t \left[(S-S^*)^2 + (I-I^*)^2 + (R-R^*)^2 + (Z-Z^*)^2\right] d\tau \le \frac{A_1}{K_4} \quad a.s.,$$

where

$$A_1 = \left[\frac{2\mu+\delta_0}{\delta_0}(S^*)^2\sigma_1^2 + \frac{\mu I^* S^* + \delta_0 (I^*)^2}{\delta_0}\sigma_2^2 + \frac{2\mu+\delta}{2\beta}\sigma_2^2 I^* + (R^*)^2\sigma_3^2 + (Z^*)^2\sigma_4^2 + \frac{(2\mu+\delta)\Lambda}{\mu}(I^*+R^*)\right],$$

$$K_4 = \min\left\{\frac{2\mu(\mu-\sigma_1^2)}{\delta_0} + \mu - \sigma_1^2 - \frac{\mu\mu_1 m S^*}{\delta_0}, \mu+\delta-\sigma_2^2 - a, \mu - \sigma_3^2, a_0 - a - \sigma_4^2 - \frac{\mu\mu_1 m S^*}{\delta_0}\right\}.$$

Proof. Noting that (S^*, I^*, R^*, Z^*) is the positive equilibrium of the model (1.1), then

$$\begin{cases} \Lambda - \mu S^* - \beta S^* I^* - \mu_1 m Z^* S^* + \delta_0 R^* = 0, \\ \beta S^* I^* - (\mu + \delta + \gamma) I^* = 0, \\ \gamma I^* + \mu_1 m Z^* S^* - (\mu + \delta_0) R^* = 0, \\ \dfrac{a I^*}{1 + b I^*} - a_0 Z^* = 0. \end{cases}$$

Define

$$V = \frac{\mu}{\delta_0}(S - S^*)^2 + \frac{2\mu S^*}{\delta_0}\left(I - I^* - I^* \ln \frac{I}{I^*}\right) + \frac{1}{2}\left[(S - S^*) + (I - I^*) + (R - R^*)\right]^2 + \frac{1}{2}(Z - Z^*)^2,$$

then

$$\begin{aligned} dV =& LVdt + \frac{2\mu}{\delta_0}(S - S^*)\,\sigma_1 S\,dB_1(t) + \frac{2\mu S^*}{\delta_0}(I - I^*)\,\sigma_2 dB_2(t) \\ &+ \left[(S - S^*) + (I - I^*) + (R - R^*)\right]\left[\sigma_1 S\,dB_1(t) + \sigma_2 I dB_2(t) + \sigma_3 R dB_3(t)\right] \\ &+ (Z - Z^*)\,\sigma_4 Z dB_4(t). \end{aligned}$$

Define

$$V_5 = \frac{1}{2}(S - S^*)^2,$$

then

$$\begin{aligned} LV_5 =& (S - S^*)\left[\Lambda - \beta S I - \mu S - \mu_1 m Z S + \delta_0 R\right] + \frac{1}{2}\sigma_1^2 S^2 \\ =& (S - S^*)\left[-\beta(SI - S^* I^*) - \mu(S - S^*) - \mu_1 m (ZS - Z^* S^*) + \delta_0 (R - R^*)\right] + \frac{1}{2}\sigma_1^2 S^2 \\ =& (S - S^*)\left[-\beta I(S - S^*) - \beta S^*(I - I^*) - \mu(S - S^*) - \mu_1 m Z(S - S^*)\right. \\ &\left. - \mu_1 m S^*(Z - Z^*) + \delta_0 (R - R^*)\right] + \frac{1}{2}\sigma_1^2 S^2 \\ =& -\beta I(S - S^*)^2 - \beta S^*(S - S^*)(I - I^*) - \mu(S - S^*)^2 - \mu_1 m Z(S - S^*)^2 \\ &- \mu_1 m S^*(S - S^*)(Z - Z^*) + \delta_0 (S - S^*)(R - R^*) + \frac{1}{2}\sigma_1^2 S^2 \\ \le& -\mu(S - S^*)^2 - \beta S^*(S - S^*)(I - I^*) - \mu_1 m S^*(S - S^*)(Z - Z^*) \\ &+ \delta_0 (S - S^*)(R - R^*) + (S - S^*)^2 \sigma_1^2 + (S^*)^2 \sigma_1^2 \\ =& -\left(\mu - \sigma_1^2\right)(S - S^*)^2 - \beta S^*(S - S^*)(I - I^*) - \mu_1 m S^*(S - S^*)(Z - Z^*) \\ &+ \delta_0 (S - S^*)(R - R^*) + (S^*)^2 \sigma_1^2. \end{aligned}$$

Define

$$V_6 = I - I^* - I^* \ln \frac{I}{I^*},$$

then

$$\begin{aligned} LV_6 =& (I - I^*)[\beta S - (\mu + \delta + \gamma)] + \frac{1}{2}\sigma_2^2 I^* \\ =& \beta(I - I^*)(S - S^*) + \frac{1}{2}\sigma_2^2 I^*. \end{aligned}$$

Define

$$V_7 = V_5 + S^* V_6,$$

then

$$\begin{aligned} LV_7 =& -\left(\mu - \sigma_1^2\right)(S - S^*)^2 - \beta S^* (S - S^*)(I - I^*) - \mu_1 m S^* (S - S^*)(Z - Z^*) \\ &+ \delta_0 (S - S^*)(R - R^*) + \sigma_1^2 (S^*)^2 + \beta S^* (S - S^*)(I - I^*) + \frac{1}{2} I^* S^* \sigma_2^2 \\ =& -\left(\mu - \sigma_1^2\right)(S - S^*)^2 - \mu_1 m S^* (S - S^*)(Z - Z^*) + \delta_0 (S - S^*)(R - R^*) \\ &+ \sigma_1^2 (S^*)^2 + \frac{1}{2} I^* S^* \sigma_2^2. \end{aligned}$$

Define

$$V_8 = \frac{1}{2}\left[(S - S^*) + (I - I^*) + (R - R^*)\right]^2,$$

then

$$\begin{aligned} LV_8 =& [(S - S^*) + (I - I^*) + (R - R^*)]\,[\Lambda - \beta S I - \mu S - \mu_1 m Z S + \delta_0 R + \beta S I - (\mu + \delta + \gamma) I \\ &+ \gamma I + \mu_1 m Z S - (\mu + \delta_0) R] + \frac{1}{2}\sigma_1^2 S^2 + \frac{1}{2}\sigma_2^2 I^2 + \frac{1}{2}\sigma_3^2 R^2 \\ =& [(S - S^*) + (I - I^*) + (R - R^*)][\Lambda - \mu S - (\mu + \delta) I - \mu R] + \frac{1}{2}\sigma_1^2 S^2 + \frac{1}{2}\sigma_2^2 I^2 + \frac{1}{2}\sigma_3^2 R^2 \\ =& [(S - S^*) + (I - I^*) + (R - R^*)]\,[-\mu (S - S^*) - (\mu + \delta)(I - I^*) - \mu (R - R^*)] \\ &+ \frac{1}{2}\sigma_1^2 S^2 + \frac{1}{2}\sigma_2^2 I^2 + \frac{1}{2}\sigma_3^2 R^2 \\ \leq& -\mu (S - S^*)^2 - (\mu + \delta)(S - S^*)(I - I^*) - \mu (S - S^*)(R - R^*) - \mu (I - I^*)(S - S^*) \\ &- (\mu + \delta)(I - I^*)^2 - \mu (I - I^*)(R - R^*) - \mu (S - S^*)(R - R^*) - (\mu + \delta)(R - R^*)(I - I^*) \\ &- \mu (R - R^*)^2 + (S - S^*)^2 \sigma_1^2 + (I - I^*)^2 \sigma_2^2 + (R - R^*)^2 \sigma_3^2 + \sigma_1^2 (S^*)^2 + \sigma_2^2 (I^*)^2 + \sigma_3^2 (R^*)^2 \\ =& -\left(\mu - \sigma_1^2\right)(S - S^*)^2 - \left(\mu + \delta - \sigma_2^2\right)(I - I^*)^2 - \left(\mu - \sigma_3^2\right)(R - R^*)^2 - (2\mu + \delta)(I - I^*)(S - S^*) \\ &- 2\mu (S - S^*)(R - R^*) - (2\mu + \delta)(I - I^*)(R - R^*) + \sigma_1^2 (S^*)^2 + \sigma_2^2 (I^*) + \sigma_3^2 (R^*). \end{aligned}$$

Define

$$V_9 = \frac{2\mu + \delta}{\beta} V_6 + \frac{2\mu}{\delta_0} V_7 + V_8,$$

then

$$
\begin{aligned}
LV_9 =&(2\mu+\delta)(I-I^*)(S-S^*)+\frac{2\mu+\delta}{2\beta}\sigma_2^2 I^* - \frac{2\mu(\mu-\sigma_1^2)}{\delta_0}(S-S^*)^2 \\
&-\frac{2\mu\mu_1 m S^*}{\delta_0}(S-S^*)(Z-Z^*)+2\mu(R-R^*)(S-S^*)+\frac{2\mu}{\delta_0}(S^*)^2\sigma_1^2+\frac{\mu}{\delta_0}I^*S^*\sigma_2^2 \\
&-(\mu-\sigma_1^2)(S-S^*)^2-(\mu+\delta-\sigma_2^2)(I-I^*)^2-(\mu-\sigma_3^2)(R-R^*)^2-(2\mu+\delta)(S-S^*)(I-I^*) \\
&-2\mu(S-S^*)(R-R^*)-(2\mu+\delta)(R-R^*)(I-I^*)+(S^*)^2\sigma_1^2+(I^*)^2\sigma_2^2+(R^*)^2\sigma_3^2 \\
=&-\left[\frac{2\mu(\mu-\sigma_1^2)}{\delta_0}+\mu-\sigma_1^2\right](S-S^*)^2-(\mu+\delta-\sigma_2^2)(I-I^*)^2-(\mu-\sigma_3^2)(R-R^*)^2 \\
&-\frac{2\mu\mu_1 m S^*}{\delta_0}(S-S^*)(Z-Z^*)-(2\mu+\delta)(R-R^*)(I-I^*) \\
&+\frac{2\mu+\delta_0}{\delta_0}(S^*)^2\sigma_1^2+\frac{\mu I^*S^*+\delta_0(I^*)^2}{\delta_0}\sigma_2^2+(R^*)^2\sigma_3^2+\frac{2\mu+\delta}{2\beta}\sigma_2^2 I^*.
\end{aligned}
$$

By using the Young inequality, one has

$$
-\frac{2\mu\mu_1 m S^*}{\delta_0}(S-S^*)(Z-Z^*)\le\frac{\mu\mu_1 m S^*}{\delta_0}(S-S^*)^2+\frac{\mu\mu_1 m S^*}{\delta_0}(Z-Z^*)^2,
$$

then

$$
\begin{aligned}
LV_9 \le&-\left[\frac{2\mu(\mu-\sigma_1^2)}{\delta_0}+\mu-\sigma_1^2-\frac{\mu\mu_1 m S^*}{\delta_0}\right](S-S^*)^2-\left(\mu+\delta-\sigma_2^2\right)(I-I^*)^2 \\
&-\left(\mu-\sigma_3^2\right)(R-R^*)^2+\frac{\mu\mu_1 m S^*}{\delta_0}(Z-Z^*)^2-(2\mu+\delta)(R-R^*)(I-I^*) \\
&+\frac{2\mu+\delta_0}{\delta_0}(S^*)^2\sigma_1^2+\frac{\mu I^*S^*+\delta_0(I^*)^2}{\delta_0}\sigma_2^2+\frac{2\mu+\delta}{2\beta}\sigma_2^2 I^*+(R^*)^2\sigma_3^2 \\
\le&-\left[\frac{2\mu(\mu-\sigma_1^2)}{\delta_0}+\mu-\sigma_1^2-\frac{\mu\mu_1 m S^*}{\delta_0}\right](S-S^*)^2-\left(\mu+\delta-\sigma_2^2\right)(I-I^*)^2 \\
&-\left(\mu-\sigma_3^2\right)(R-R^*)^2+\frac{\mu\mu_1 m S^*}{\delta_0}(Z-Z^*)^2+(2\mu+\delta)RI^*+(2\mu+\delta)IR^* \\
&+\frac{2\mu+\delta_0}{\delta_0}(S^*)^2\sigma_1^2+\frac{\mu I^*S^*+\delta_0(I^*)^2}{\delta_0}\sigma_2^2+\frac{2\mu+\delta}{2\beta}\sigma_2^2 I^*+(R^*)^2\sigma_3^2.
\end{aligned}
$$

Define

$$
LV_{10}=\frac{1}{2}(Z-Z^*)^2,
$$

then

$$
\begin{aligned}
LV_{10} =&(Z-Z^*)\left[\frac{aI}{1+bI}-a_0Z\right]+\frac{1}{2}\sigma_4^2Z^2 \\
=&(Z-Z^*)\left[\frac{aI}{1+bI}-\frac{aI^*}{1+bI^*}-a_0(Z-Z^*)\right]+\frac{1}{2}\sigma_4^2Z^2 \\
=&(Z-Z^*)\left(\frac{aI}{1+bI}-\frac{aI^*}{1+bI^*}\right)-a_0(Z-Z^*)^2+\frac{1}{2}\sigma_4^2Z^2.
\end{aligned}
$$

By using the Young inequality, one has

$$\begin{aligned}(Z-Z^*)\left(\frac{aI}{1+bI}-\frac{aI^*}{1+bI^*}\right)=&\frac{a(I-I^*)(Z-Z^*)}{(1+bI)(1+bI^*)}\\ =&\frac{\sqrt{a}(I-I^*)}{1+bI}\frac{\sqrt{a}(Z-Z^*)}{1+bI^*}\\ \leq&\frac{a(I-I^*)^2}{2(1+bI)^2}+\frac{a(Z-Z^*)^2}{2(1+bI^*)^2}\\ \leq&a(I-I^*)^2+a(Z-Z^*)^2,\end{aligned}$$

then

$$\begin{aligned}LV_{10}\leq&-(a_0-a)(Z-Z^*)^2+a(I-I^*)^2+\frac{1}{2}\sigma_4^2Z^2\\ \leq&-(a_0-a)(Z-Z^*)^2+a(I-I^*)^2+(Z-Z^*)^2\sigma_4^2+(Z^*)^2\sigma_4^2\\ =&-(a_0-a-\sigma_4^2)(Z-Z^*)^2+a(I-I^*)^2+(Z^*)^2\sigma_4^2.\end{aligned}$$

Define

$$V_{11}=V_9+V_{10},$$

then

$$\begin{aligned}LV_{11}=&-\left[\frac{2\mu(\mu-\sigma_1^2)}{\delta_0}+\mu-\sigma_1^2-\frac{\mu\mu_1 mS^*}{\delta_0}\right](S-S^*)^2-\left(\mu+\delta-\sigma_2^2-a\right)(I-I^*)^2\\ &-\left(\mu-\sigma_3^2\right)(R-R^*)^2-\left(a_0-a-\sigma_4^2-\frac{\mu\mu_1 mS^*}{\delta_0}\right)(Z-Z^*)^2+(2\mu+\delta)RI^*\\ &+(2\mu+\delta)IR^*+\frac{2\mu+\delta_0}{\delta_0}(S^*)^2\sigma_1^2+\frac{\mu I^*S^*+\delta_0(I^*)^2}{\delta_0}\sigma_2^2+\frac{2\mu+\delta}{2\beta}\sigma_2^2I^*+(R^*)^2\sigma_3^2+(Z^*)^2\sigma_4^2.\end{aligned}$$

According to Lemma 2.3, one can obtain

$$\limsup_{t\to\infty}\frac{1}{t}E\int_0^t\left[(S-S^*)^2+(I-I^*)^2+(R-R^*)^2+(Z-Z^*)^2\right]d\tau\leq\frac{A_1}{K_4}\quad \text{a.s.},$$

where

$$A_1=\left[\frac{2\mu+\delta_0}{\delta_0}(S^*)^2\sigma_1^2+\frac{\mu I^*S^*+\delta_0(I^*)^2}{\delta_0}\sigma_2^2+\frac{2\mu+\delta}{2\beta}\sigma_2^2I^*+(R^*)^2\sigma_3^2+(Z^*)^2\sigma_4^2+\frac{(2\mu+\delta)\Lambda}{\mu}(I^*+R^*)\right],$$

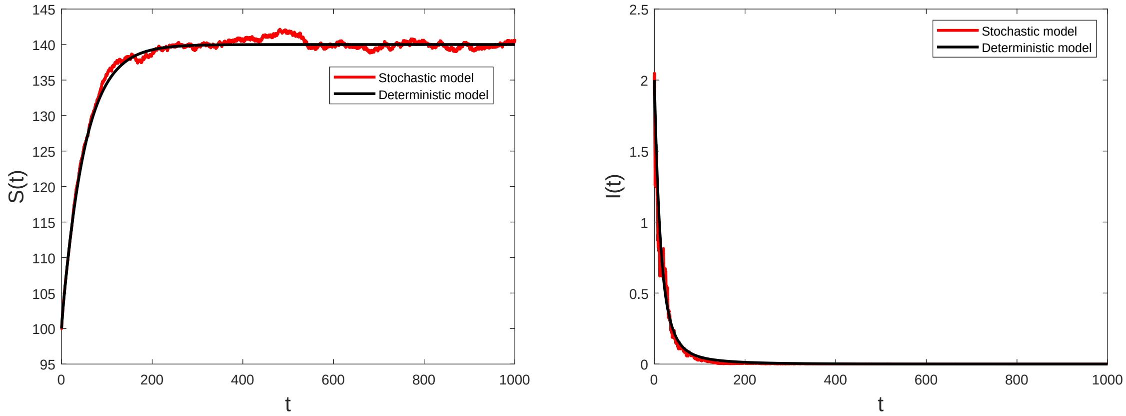
$$K_4=\min\left\{\frac{2\mu(\mu-\sigma_1^2)}{\delta_0}+\mu-\sigma_1^2-\frac{\mu\mu_1 mS^*}{\delta_0},\mu+\delta-\sigma_2^2-a,\mu-\sigma_3^2,a_0-a-\sigma_4^2-\frac{\mu\mu_1 mS^*}{\delta_0}\right\}.$$

4. Numerical simulations

In this section, we verify the theoretical results by some numerical examples. Firstly, for the deterministic system, the parameters are chosen as

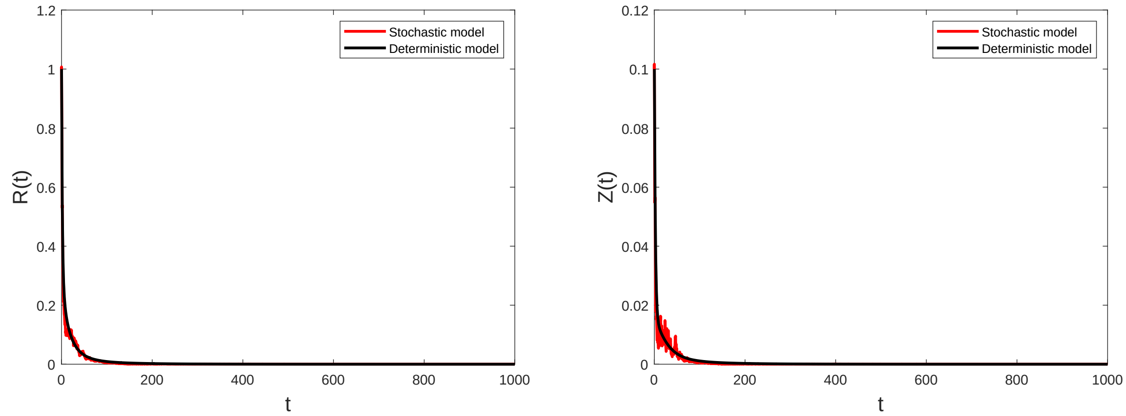
$$\mu=0.02,\Lambda=2.8,\beta=0.0015,\delta=0.15,\gamma=0.1,\mu_1=0.9,$$

$$m=0.017,\delta_0=0.8,a=0.01,b=1,a_0=0.45.$$



(a) Time series of S for the deterministic model and the stochastic model.

(b) Time series of I for the deterministic model and the stochastic model.



(c) Time series of R for the deterministic model and the stochastic model.

(d) Time series of Z for the deterministic model and the stochastic model.

Figure 1. Comparison of time series of the deterministic model and the stochastic model.

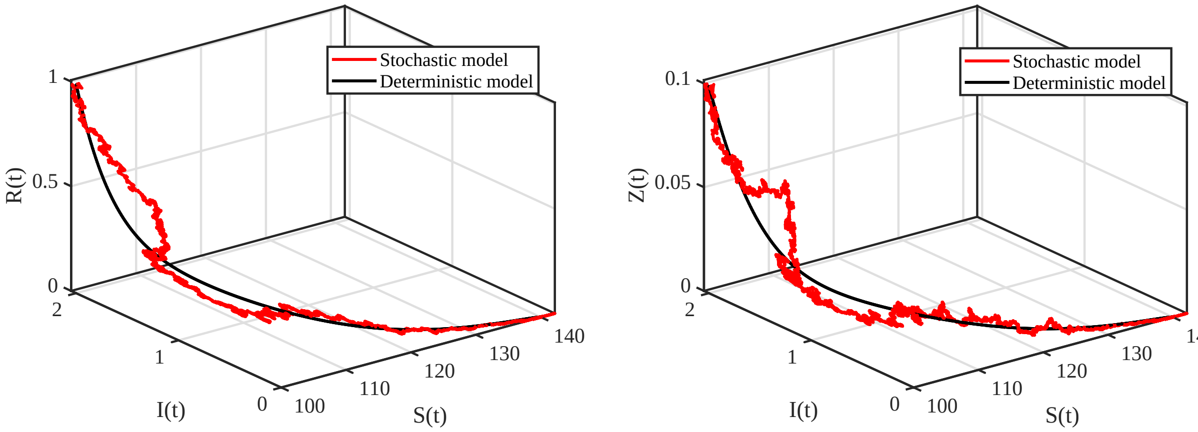
By direct calculations, we have $R_0 = 0.7778 < 1 - \frac{ax_2}{y_1(\mu+\delta+\gamma)} = 0.9909$, then the deterministic system has a globally asymptotically stable disease-free equilibrium $E_0 = (140, 0, 0, 0)$, please see the black curves in Figures 1 and 2. For the stochastic system, let $\sigma_1 = 0.001, \sigma_2 = 0.1, \sigma_3 = 0.1, \sigma_4 = 0.25$. Simple calculations show

$$
\begin{aligned}
\sigma_1^2 =& 0.000001 < \mu = 0.02,\\
\frac{1}{2}\sigma_2^2 + \frac{1}{2}x_1\sigma_2^2 + \frac{1}{2}a =& 0.0103 < \mu + \delta + (\mu + \delta + \gamma)x_1 = 0.1835,\\
\frac{1}{2}\sigma_3^2 =& 0.005 < \mu = 0.02,\\
\frac{1}{2}a + \frac{1}{2}\sigma_4^2 =& 0.0362 < a_0 = 0.45,\\
K_3 =& 0.015 > 0,
\end{aligned}
$$

then according to Theorem 3.1, we get

$$\limsup_{t\to\infty}\frac{1}{t}E\int_0^t\left[\left(S(\tau)-\frac{\Lambda}{\mu}\right)^2+I^2(\tau)+R^2(\tau)+Z^2(\tau)\right]d\tau\le 1.3720,$$

the solutions of the stochastic system oscillate around the equilibrium of the deterministic system for a relatively small noise, please see the red curves in Figures 1 and 2.



(a) Three-dimensional diagram of S, I and R for the deterministic model and the stochastic model.

(b) Three-dimensional diagram of S, I and Z for the deterministic model and the stochastic model.

Figure 2. Comparison of three-dimensional diagram of the deterministic model and the stochastic model.

Next, we give a numerical example to illustrate Theorem 3.2. For the deterministic system, we choose the parameters as the following

$$\Lambda = 12.8, \mu = 0.4, \beta = 0.019, \delta = 0.1, \gamma = 0.1, \mu_1 = 0.009,$$

$$m = 0.017, \delta_0 = 0.5, a = 0.01, b = 1, a_0 = 0.45.$$

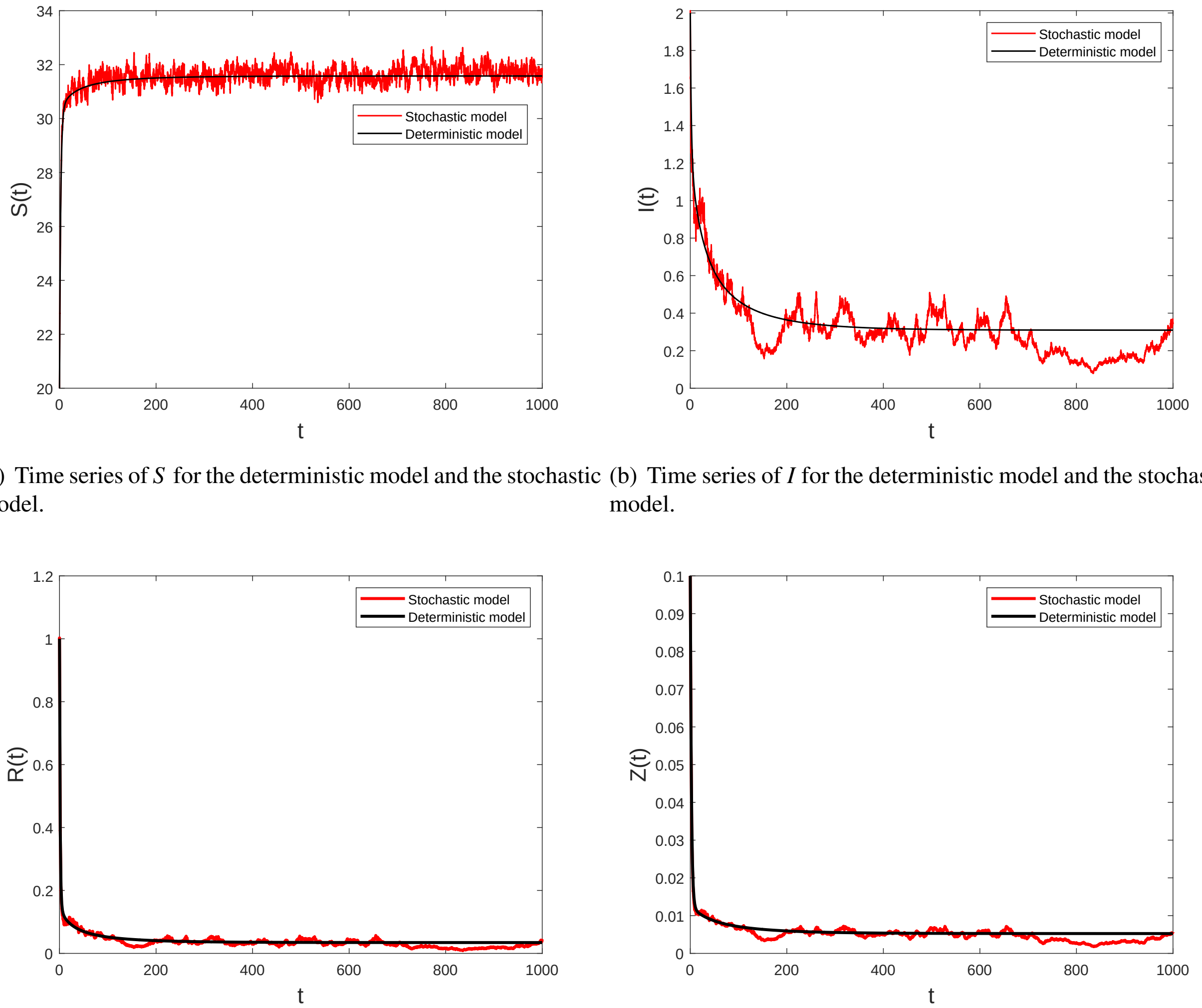
By direct calculations, we have $R_0 = 1.0133 > 1$, then the deterministic system has a globally asymptotically stable disease equilibrium $E^* = (31.5789, 0.3093, 0.0344, 0.0052)$, please see the black curves in Figures 3 and 4. For the stochastic system, let $\sigma_1 = 0.008, \sigma_2 = 0.06, \sigma_3 = 0.058, \sigma_4 = 0.025$. Simple calculations show

$$\begin{aligned}
\sigma_1^2+\frac{2\mu\sigma_1^2}{\delta_0} &= 0.0001664 < \frac{2\mu^2}{\delta_0}+\mu-\frac{\mu\mu_1 mS^*}{\delta_0} = 1.0361,\\
\sigma_2^2 &= 0.0036 < \mu+\delta-a = 0.49,\\
\sigma_3^2 &= 0.0034 < \mu = 0.4,\\
\sigma_4^2 &= 0.000625 < a_0-a-\frac{\mu\mu_1 mS^*}{\delta_0} = 0.4361,\\
K_4 &= 0.3966 > 0,
\end{aligned}$$

then according to Theorem 3.2, we obtain

$$\limsup_{t\to\infty}\frac{1}{t}E\int_0^t\left[(S-S^*)^2+(I-I^*)^2+(R-R^*)^2+(Z-Z^*)^2\right]d\tau\le 25.5145,$$

the solutions of the stochastic system oscillate around the equilibrium of the deterministic system for a relatively small noise, please see the red curves in Figures 3 and 4.



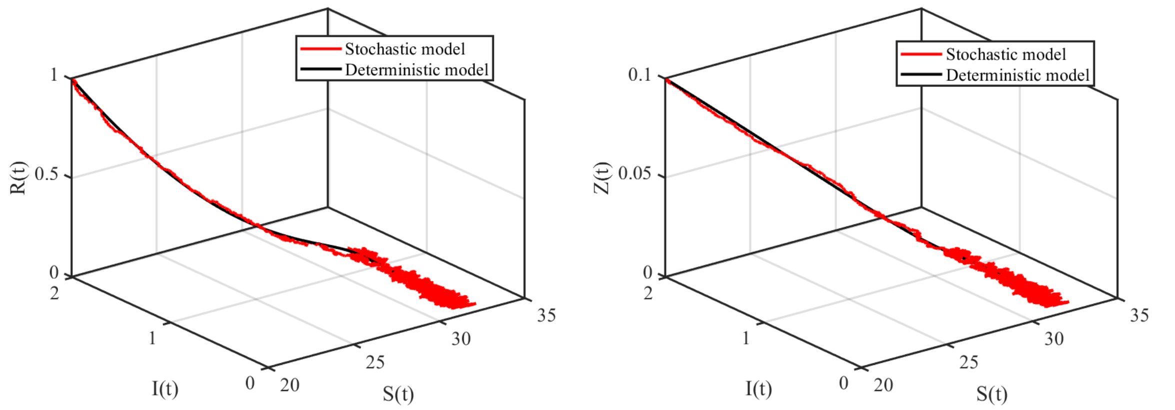
(a) Time series of S for the deterministic model and the stochastic model.

(b) Time series of I for the deterministic model and the stochastic model.

(c) Time series of R for the deterministic model and the stochastic model.

(d) Time series of Z for the deterministic model and the stochastic model.

Figure 3. Comparison of time series of the deterministic model and the stochastic model.



(a) Three-dimensional diagram of S, I and R for the deterministic model and the stochastic model.

(b) Three-dimensional diagram of S, I and Z for the deterministic model and the stochastic model.

Figure 4. Comparison of three-dimensional diagram of the deterministic model and the stochastic model.

We also noticed that the increase in information interventions (such as media coverage) will reduce the number of infected individuals. This shows that information intervention can reduce the spread of diseases among the population and has a positive significance for disease control. (see Figure 5)

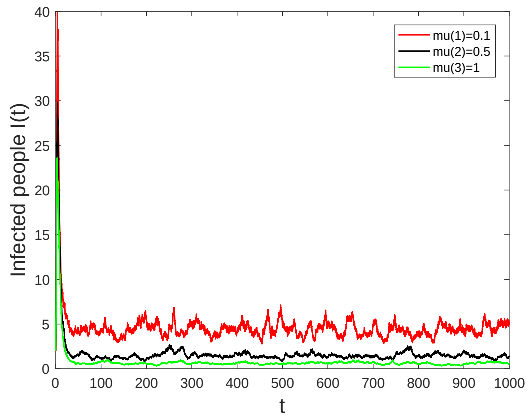


Figure 5. Time series of I for the stochastic model with different parameter values of μ_1.

5. Conclusions

In this paper, a stochastic SIRS epidemic model with information intervention is investigated. The asymptotic behavior of the solutions near the disease-free equilibrium and endemic equilibrium of the deterministic model is explored mathematically. We find that the solutions of the stochastic system will oscillate around the equilibria of the deterministic system for a relatively small noise, and we get an estimate of the upper bound of the amplitude. In addition, through numerical simulation, we also find that as the intensity of information intervention increases, the number of infected patients decreases.

This means that information intervention plays important roles in the outbreak of sudden infectious diseases. For example, media reports can be used to provide the public with information about the current situation of the epidemic and the effective prevention and control measures proposed by experts. Outbreaks of infectious diseases have led to a dramatic increase in information interventions, which in turn can help raise awareness and change their behaviors for better implementation of mitigation measures. People will adopt relatively conservative behaviors to reduce the possibility of infection, and individual behavior can effectively delay the peak period of infectious disease outbreaks and reduce the severity of infectious disease outbreaks. However, this study only focuses on the qualitative analysis of stochastic models, and the estimation of several parameters, such as people's acceptance of information intervention μ_1, contact rate β, etc., is insufficient. Some parameters, such as individual behavioral change constants [42], are very important parameters for disease control that are not considered in the current model and we leave it as our future work.

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Conflict of interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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