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Research article

On an initial boundary value problem for fractional pseudo-parabolic equation with conformable derivative

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Abstract: In this paper, we study the initial boundary value problem of the pseudo-parabolic equation with a conformable derivative. We focus on investigating the existence of the global solution and examining the derivative's regularity. In addition, we contributed two interesting results. Firstly, we proved the convergence of the mild solution of the pseudo-parabolic equation to the solution of the parabolic equation. Secondly, we examine the convergence of solution when the order of the derivative of the fractional operator approaches 1^-. Our main techniques used in this paper are Banach fixed point theorem and Sobolev embedding. We also apply different techniques to evaluate the convergence of generalized integrals encountered.

Keywords: conformable derivative; pseudo-parabolic equation; well-posedness; regularity estimates

1. Introduction

Fractional calculus is one of today's most popular mathematical tools to model real-world problems. More specifically, it has been applied to model evolutionary systems involving memory effects on dynamical systems. Partial differential equations (PDEs) with fractional operators are important in describing phenomena in many fields such as physics, biology and chemistry [1, 2]. Based on the generalizations of fractional derivatives by famous mathematicians such as Euler, Lagrange, Laplace, Fourier, Abel and Liouville, today's mathematicians have explored and introduced many more types of fractional derivatives such as Riemann-Liouville, Caputo, Liouville, Weyl, Riesz and Hifler [3–6].

PDEs with conformable derivatives attract interested mathematicians using different approaches because of their wide range of applications, such as electrical circuits [7] and chaotic systems in dynamics [8]. We recognize that the conformable and classical derivatives have a close relationship. There is an interesting observation that: If f is a real function and $s > 0$, then f has a conformable fractional derivative of order β at s if and only if it is (classically) differentiable at s, and

$$\frac{{}^{C}\partial^{\beta} f(s)}{\partial s^{\beta}} = s^{1-\beta}\frac{\partial f(s)}{\partial s}, \tag{1.1}$$

where $0 < \beta \leq 1$. Another surprising observation is that Eq (1.1) will not hold if f is defined in a general Banach space. We can better understand why the ODEs with the conformable derivative on $\mathbb{R}$ have been studied so much. In addition, the relevant research in infinite-dimensional spaces, such as Banach or Hilbert space, is still limited, which motivates us to investigate some types of PDEs with conformable derivatives in Hilbert or Sobolev spaces.

Besides, in some phenomena, the conformable derivative is better simulated than the classical derivative. In [9], the authors considered the conformable diffusion equation

$$\frac{{}^{C}\partial^{\alpha}}{\partial t^{\alpha}} u(x,t) = D_{\alpha}\frac{\partial^2}{\partial x^2} u(x,t), \tag{1.2}$$

where $0 < \alpha \leq 1$, $x > 0$, $t > 0$, $u(x,t)$ is the concentration, and D_α represents the generalized diffusion coefficient, which was applied in the description of a subdiffusion process. In particular, the conformable diffusion equation (1.2) reduces to the normal diffusion equation if $\alpha = 1$. A natural and fundamental question is, "Does the conformable diffusion model predict better than the normal diffusion model?". The results in [9] show that the conformable derivative model agrees better with the experimental data than the normal diffusion equation.

Let $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ be a bounded domain with smooth boundary $\partial\Omega$, and $T > 0$ is a given positive number. In this paper, we investigate the Sobolev equation with a conformable derivative as follows

$$\begin{cases} \dfrac{{}^{C}\partial^{\alpha}}{\partial t^{\alpha}} u + (-\Delta)^{\beta} u(x,t) - m\dfrac{{}^{C}\partial^{\alpha}}{\partial t^{\alpha}}\Delta u = F(u(x,t)), & x \in \Omega, \ \ t \in (0,T), \\ u(x,t) = 0, & x \in \partial\Omega, t \in (0,T), \\ u(x,0) = u_0(x) & x \in \Omega \end{cases} \tag{1.3}$$

where $\alpha, \beta \in (0,1]$, $m > 0$, the time fractional derivative $\frac{{}^{C}\partial^{\alpha}}{\partial t^{\alpha}}$ is the conformable derivative of order α, defined in Definition 2.1. The function F represents the external forces or the advection term of a diffusion phenomenon, etc., and the function u_0 is the initial condition specified later. The operator $(-\Delta)^\beta$ is the fractional Laplacian operator, which is well-defined in [10] (see page 3). In the sense of distribution, the study of weak solutions to Problem (1.3) is still limited compared to the classical problem. Thus, the fundamental knowledge in the distributive sense for Problem (1.3) is still open and challenging, which is the main reason and motivation for us to study the problem from the perspective of the semigroup.

Next, we mention some results related to Problem (1.3). There are two interesting observations regarding Problem (1.3).

• If we take $m = 0$ in Problem (1.3), then we obtain an initial boundary value problem of the parabolic equation with a conformable operator as follows

$$\begin{cases} \dfrac{{}^C\partial^\alpha}{\partial t^\alpha} u + (-\Delta)^\beta u(x,t) = F(u(x,t)), & x \in \Omega,\ \ t \in (0,T), \\ u(x,t) = 0, & x \in \partial\Omega, t \in (0,T), \\ u(x,0) = u_0(x) & x \in \Omega. \end{cases} \tag{1.4}$$

The latest results on the well-posedness of solutions to Problem (1.4) are shown in more detail in [11], and the authors used the Hilbert scales space technique to prove the local existence of the mild solution to Problem (1.4).

• If $\alpha = 1$, the main equation of Problem (1.3) becomes the classical equation

$$u_t + (-\Delta)^\beta u(x,t) - m\Delta u_t = F(u(x,t)). \tag{1.5}$$

Equation (1.5) is familiar to mathematicians about PDEs, called the pseudo-parabolic equation, also known as the Sobolev equation. The pseudo-parabolic equation describes a series of important physical processes, such as the permeation of a homogeneous liquid through fractured rock, population aggregation and one-way propagation of the nonlinear dispersion length wave [12, 13]. Equation (1.5) has been studied extensively; for details, see [12–14] and references given there. Concerning the study of the existence and blowup of solutions to pseudo-parabolic equations, we refer the reader to [15–17].

For the convenience of readers, we next list some interesting results related to pseudo-parabolic with fractional derivative. Luc et al. in [18] considered fractional pseudo-parabolic equation with Caputo derivative

$$\begin{cases} D_t^\alpha(u + k\mathcal{A}u) + \mathcal{A}^\beta u = F(t,x,u), & \text{in} \quad (0,T] \times \Omega, \\ u(t,x) = 0, & \text{on} \quad (0,T] \times \partial\Omega, \\ u(0,x) = u_0(x), & \text{in} \quad \Omega, \end{cases} \tag{1.6}$$

where $0 < \alpha < 1$, $\mathcal{A} = -\Delta$, D^α is Caputo fractional derivative operator of order α. They studied the local and global existence of solutions to Problem (1.6) when the nonlinear term F is the global Lipschitz. Based on the work in [18] , there are many related results in the spirit of a semigroup representation of the form of the Fourier series. In [12], the authors studied nonlinear time-fractional pseudo-parabolic equations with Caputo derivative on both bounded and unbounded domains by different methods and techniques from [18]. In [19], the authors studied the nonlocal in time problem for a pseudo-parabolic equation with fractional time and space in the linear case. Tuan et al. [20]

derived the nonlinear pseudo-parabolic equation with a nonlocal type of integral condition. In [21], the authors considered the time-space pseudo-parabolic equation with the Riemann-Liouville time-fractional derivative, and they applied the Galerkin method to show the global and local existence of solutions.

As far as we know, there has not been any work that considers the initial boundary value problem (1.3) with a conformable derivative. The main results and methods of the present paper are described in detail as follows

- Firstly, we prove the existence of the global solution to Problem (1.3). The main idea is to use Banach fixed point theorem with the new weighted norm used in [22]. In order to prove the regularity and the derivative of the mild solution, we need to apply some complicated techniques on Hilbert scales for nonlinearity terms.Compared with [11], our method has very different characteristics. It is important to emphasize that proving the existence of the global solution is difficult, which is demonstrated in our current paper, but not in the paper [11].
- Secondly, we investigate the convergence of solution to Problem (1.3) when $m \to 0^+$, which does not appear in the works related to fractional pseudo-parabolic equation. This result allows us to get the relationship between the solution of Sobolev equation and parabolic diffusion equation. To overcome the difficulty, we need to control the improper integrals and control the parameters. This pioneering work can open up some new research directions for finding the relationship between the solutions of the pseudo-parabolic equation and the parabolic equation.
- Finally, we prove the convergence of the solution when the order of derivative $\beta \to 1^-$. This direction of research was motivated by the recent paper [23]. Since the current model has a nonlinear source function, the processing technique for the proof in this paper seems to be more complicated than that of [23].

The greatest difficulty in solving this problem is the study of many integrals containing singular terms, such as $s^{\alpha-1}$ or $(t^\alpha - s^\alpha)^{-m}$. To overcome these difficulties, we need to use ingenious calculations and techniques to control the convergence of several generalized integrals.

This paper is organized as follows. Section 2 provides some definitions. In Section 3, we give the definition of the mild solution and some important lemmas for the proof of the main results. Section 4 shows the global existence of the solution to Problem (1.3). In addition, we present the regularity result for the derivative of the mild solution. Section 5 shows the convergence of the solution to Problem (1.3) when $m \to 0^+$. In Section 6, we investigate the convergence of mild solutions when $\beta \to 1^-$.

2. Preliminary results

Definition 2.1. *Conformable derivative model: Let B be a Banach space, and the function $f : [0, \infty) \to B$. Let $\frac{{}^C\partial^\beta}{\partial t^\beta}$ be the conformable derivative operator of order $\beta \in (0, 1]$, locally defined by*

$$\frac{{}^C\partial^\beta f(t)}{\partial t^\beta} := \lim_{h\to 0} \frac{f(t + ht^{1-\beta}) - f(t)}{h} \quad \text{in } B$$

for each $t > 0$. (For more details on the above definition, we refer the reader to [24–27].)

In this section, we introduce the notation and the functional setting used in our paper. Recall the spectral problem

$$\begin{cases} -\Delta e_n(x) = \lambda_n e_n(x), & x \in \Omega, \\ e_n(x) = 0, & x \in \partial\Omega, \end{cases}$$

admits the eigenvalues $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq \ldots$ with $\lambda_n \to \infty$ as $n \to \infty$. The corresponding eigenfunctions are $e_n \in H_0^1(\Omega)$.

Definition 2.2. *(Hilbert scale space). We recall the Hilbert scale space, which is given as follows*

$$\mathbb{H}^s(\Omega) = \left\{ f \in L^2(\Omega) \middle| \sum_{n=1}^{\infty} \lambda_n^{2s} \Big(\int_\Omega f(x) e_n(x) \mathrm{d}x \Big)^2 < \infty \right\}$$

for any $s \geq 0$. It is well-known that $\mathbb{H}^s(\Omega)$ is a Hilbert space corresponding to the norm

$$\|f\|_{\mathbb{H}^s(\Omega)} = \left(\sum_{n=1}^{\infty} \lambda_n^{2s} \Big(\int_\Omega f(x) e_n(x) \mathrm{d}x \Big)^2 \right)^{1/2}, \quad f \in \mathbb{H}^s(\Omega).$$

Definition 2.3. *Let $\mathbf{X}_{a,q,\alpha}((0,T];B)$ denote the weighted space of all the functions $\psi \in C((0,T];X)$ such that*

$$\|\psi\|_{\mathbf{X}_{a,q,\alpha}((0,T];B)} := \sup_{t \in (0,T]} t^a e^{-qt^\alpha} \|\psi(t,\cdot)\|_B < \infty,$$

where $a, q > 0$ and $0 < \alpha \leq 1$ (see [22]). If $q = 0$, we denote $\mathbf{X}_{a,q}((0,T];B)$ by $\mathbb{X}_a((0,T];B)$.

3. The mild solution and some lemmas

In order to find a precise formulation for solutions, we consider the mild solution in terms of the Fourier series

$$u(x,t) = \sum_{n=1}^{\infty} \langle u(.,t), e_n \rangle e_n(x).$$

Taking the inner product of Problem (1.3) with e_n gives

$$\begin{cases} \dfrac{{}^C\partial^\alpha}{\partial t^\alpha} \langle u(.,t), e_n \rangle + \lambda_n^\beta \langle u(.,t), e_n \rangle + m\lambda_n \dfrac{{}^C\partial^\alpha}{\partial t^\alpha} \langle u(.,t), e_n \rangle & = \langle F(.,t), e_n \rangle, \qquad t \in (0,T), \\ \langle u(.,0), e_n \rangle & = \langle u_0, e_n \rangle, \end{cases} \tag{3.1}$$

where we repeat that

$$\langle \Delta u(.,t), e_n \rangle = -\lambda_n \langle u(.,t), e_n \rangle.$$

The first equation of (3.1) is a differential equation with a conformable derivative as follows

$$\frac{{}^C\partial^\alpha}{\partial t^\alpha} \langle u(.,t), e_n \rangle + \frac{\lambda_n^\beta}{1 + m\lambda_n} \langle u(.,t), e_n \rangle = \frac{1}{1 + m\lambda_n} \langle F(.,t), e_n \rangle.$$

In view of the result in Theorem 5, [26] and Theorem 3.3, [28] the solution to Problem (1.3) is

$$\Big\langle u(.,t), e_n \Big\rangle = \exp\left(-\frac{\lambda_n^\beta}{1+m\lambda_n}\frac{t^\alpha}{\alpha}\right)\Big\langle u_0, e_n\Big\rangle + \frac{1}{1+m\lambda_n}\int_0^t \nu^{\alpha-1}\exp\left(\frac{\lambda_n^\beta}{1+m\lambda_n}\frac{\nu^\alpha - t^\alpha}{\alpha}\right)\Big\langle F(.,\nu), e_n\Big\rangle d\nu.$$

To simplify the solution formula, we will express the solution in operator equations. Let us set the following operators

$$\mathbf{S}_{m,\alpha,\beta}(t)f = \sum_{n\in\mathbb{N}}\exp\left(-\frac{\lambda_n^\beta}{1+m\lambda_n}\frac{t^\alpha}{\alpha}\right)\langle f, e_n\rangle e_n,$$

and

$$\mathbf{P}_m f = \sum_{n\in\mathbb{N}}(1+m\lambda_n)^{-1}\langle f, e_n\rangle e_n$$

for any $f \in L^2(\Omega)$ in the form $f = \sum_{n\in\mathbb{N}}\langle f, e_n\rangle e_n$. Then the inverse operator of $\mathbf{S}_{m,\alpha,\beta}(t)$ is defined by

$$\Big(\mathbf{S}_{m,\alpha,\beta}(t)\Big)^{-1} f = \sum_{n\in\mathbb{N}}\exp\left(\frac{\lambda_n^\beta}{1+m\lambda_n}\frac{t^\alpha}{\alpha}\right)\langle f, e_n\rangle e_n$$

The mild solution is given by

$$u(t) = \mathbf{S}_{m,\alpha,\beta}(t)u_0 + \int_0^t \nu^{\alpha-1}\mathbf{P}_m\mathbf{S}_{m,\alpha,\beta}(t)\Big(\mathbf{S}_{m,\alpha,\beta}(\nu)\Big)^{-1}F(\nu)d\nu. \tag{3.2}$$

For a qualitative analysis of the solution to (3.2), we need the bounded result for the operators in Hilbert scales space.

Lemma 3.1. *(a) Let $v \in \mathbb{H}^{s+k-\beta k}(\Omega)$ for any $k > 0$. Then we get*

$$\Big\|\mathbf{S}_{m,\alpha,\beta}(t)v\Big\|_{\mathbb{H}^s(\Omega)} \le C_k\alpha^k m^{-k}t^{-\alpha k}\Big\|v\Big\|_{\mathbb{H}^{s+k-\beta k}(\Omega)}, \tag{3.3}$$

and for $0 \le \nu \le t \le T$,

$$\Big\|\mathbf{S}_{m,\alpha,\beta}(t)\Big(\mathbf{S}_{m,\alpha,\beta}(\nu)\Big)^{-1}v\Big\|_{\mathbb{H}^s(\Omega)} \le C_k\alpha^k m^{-k}\left(t^\alpha - \nu^\alpha\right)^{-k}\Big\|v\Big\|_{\mathbb{H}^{s+k-\beta k}(\Omega)}. \tag{3.4}$$

(b) If $v \in \mathbb{H}^s(\Omega)$, then

$$\Big\|\mathbf{S}_{m,\alpha,\beta}(t)v\Big\|_{\mathbb{H}^s(\Omega)} \le \Big\|v\Big\|_{\mathbb{H}^s(\Omega)} \tag{3.5}$$

and

$$\Big\|\mathbf{S}_{m,\alpha,\beta}(t)\Big(\mathbf{S}_{m,\alpha,\beta}(\nu)\Big)^{-1}v\Big\|_{\mathbb{H}^s(\Omega)} \le \Big\|v\Big\|_{\mathbb{H}^s(\Omega)} \tag{3.6}$$

for $0 \le \nu \le t \le T$.
(c) If $v \in \mathbb{H}^s(\Omega)$, then we have

$$\Big\|\mathbf{P}_m v\Big\|_{\mathbb{H}^s(\Omega)} \le \Big\|v\Big\|_{\mathbb{H}^s(\Omega)}$$

for any s.

Proof. For (a), in view of Parseval's equality, we get that

$$\left\|\mathbf{S}_{m,\alpha,\beta}(t)v\right\|^2_{\mathbb{H}^s(\Omega)} = \sum_{n\in\mathbb{N}} \lambda_n^{2s} \exp\left(-2\frac{\lambda_n^\beta}{1+m\lambda_n}\frac{t^\alpha}{\alpha}\right)\langle v, e_n\rangle^2. \tag{3.7}$$

Using the inequality $e^{-y} \leq C_k y^{-k}$, we find that

$$\exp\left(-\frac{\lambda_n^\beta}{1+m\lambda_n}\frac{t^\alpha}{\alpha}\right) \leq C_k\left(\frac{\lambda_n^\beta}{1+m\lambda_n}\right)^{-k}\frac{t^{-\alpha k}}{\alpha^{-k}} = C_k\alpha^k\left(m^k+\lambda_1^{-1}\right)\lambda_n^{(1-\beta)k}t^{-\alpha k}, \tag{3.8}$$

where we use

$$(1+m\lambda_n)^k \leq C_k\Big(1+m^k\lambda_n^k\Big) \leq C_k\left(m^k+\lambda_1^{-1}\right)\lambda_n^k.$$

It follows from (3.7) that

$$\begin{aligned}\left\|\mathbf{S}_{m,\alpha,\beta}(t)v\right\|^2_{\mathbb{H}^s(\Omega)} &\leq \left(C_k\alpha^k\right)^2\left(m^k+\lambda_1^{-1}\right)^2 t^{-2\alpha k}\sum_{n\in\mathbb{N}}\lambda_n^{2s+2k-2\beta k}\langle f, e_n\rangle^2\\ &= \left(C_k\alpha^k\right)^2\left(m^k+\lambda_1^{-1}\right)^2 t^{-2\alpha k}\|v\|^2_{\mathbb{H}^{s+k-\beta k}(\Omega)}.\end{aligned}$$

By a similar explanation, we also get that for $0 \leq \nu \leq t \leq T$,

$$\begin{aligned}\left\|\mathbf{S}_{m,\alpha,\beta}(t)\Big(\mathbf{S}_{m,\alpha,\beta}(\nu)\Big)^{-1}v\right\|^2_{\mathbb{H}^s(\Omega)} &= \sum_{n\in\mathbb{N}}\lambda_n^{2s}\exp\left(\frac{\lambda_n^\beta}{1+m\lambda_n}\frac{2\nu^\alpha-2t^\alpha}{\alpha}\right)\langle v, e_n\rangle^2\\ &\leq \left(C_k\alpha^k\right)^2\left(m^k+\lambda_1^{-1}\right)^2(t^\alpha-\nu^\alpha)^{-2k}\sum_{n\in\mathbb{N}}\lambda_n^{2s+2k-2\beta k}\langle f, e_n\rangle^2\\ &= \left(C_k\alpha^k\right)^2\left(m^k+\lambda_1^{-1}\right)^2(t^\alpha-\nu^\alpha)^{-2k}\|v\|^2_{\mathbb{H}^{s+k-\beta k}(\Omega)},\end{aligned}$$

where we use the fact that

$$\begin{aligned}\exp\left(\frac{\lambda_n^\beta}{1+m\lambda_n}\frac{\nu^\alpha-t^\alpha}{\alpha}\right) &\leq C_k\left(\frac{\lambda_n^\beta}{1+m\lambda_n}\right)^{-k}\frac{(t^\alpha-\nu^\alpha)^{-k}}{\alpha^{-k}}\\ &= C_k\alpha^k\left(m^k+\lambda_1^{-1}\right)\lambda_n^{(1-\beta)k}(t^\alpha-\nu^\alpha)^{-k}.\end{aligned}$$

Hence, (a) is proved.

For (b), in view of Parseval's equality, we get that

$$\left\|\mathbf{S}_{\alpha,\beta}(t)v\right\|^2_{\mathbb{H}^s(\Omega)} = \sum_{n\in\mathbb{N}}\lambda_n^{2s}\exp\left(-2\frac{\lambda_n^\beta}{1+m\lambda_n}\frac{t^\alpha}{\alpha}\right)\langle v, e_n\rangle^2 \leq \sum_{n\in\mathbb{N}}\lambda_n^{2s}\langle v, e_n\rangle^2 = \|v\|^2_{\mathbb{H}^s(\Omega)},$$

which allows us to conclude the proof of (3.5). The proof of (3.6) is similar to (3.5), and we omit it here. For (c), noting that $(1+m\lambda_n)^{-1} < 1$, we can claim it as follows

$$\left\|\mathbf{P}_m v\right\|^2_{\mathbb{H}^s(\Omega)} = \sum_{n\in\mathbb{N}}\lambda_n^{2s}(1+m\lambda_n)^{-2}\langle v, e_n\rangle^2 \leq \sum_{n\in\mathbb{N}}\lambda_n^{2s}\langle v, e_n\rangle^2 = \|v\|^2_{\mathbb{H}^s(\Omega)}.$$

The proof of Lemma 3.1 is completed. □

4. Well-posedness of Problem (1.3)

Let $G : \mathbb{H}^r(\Omega) \to \mathbb{H}^s(\Omega)$ such that $G(\mathbf{0}) = \mathbf{0}$ and

$$\|G(w_1) - G(w_2)\|_{\mathbb{H}^s(\Omega)} \leq L_g \|w_1 - w_2\|_{\mathbb{H}^r(\Omega)}, \tag{4.1}$$

for any $w_1, w_2 \in \mathbb{H}^r(\Omega)$ and L_g is a postive constant.

Theorem 4.1. *(i) Let $G : \mathbb{H}^r(\Omega) \to \mathbb{H}^s(\Omega)$ such that (4.1) holds. Here r, s satisfy that $s \geq r + k - \beta k$ for $0 < k < \frac{1}{2}$. Let the initial datum $u_0 \in \mathbb{H}^{r+k-\beta k}(\Omega)$. Then Problem (1.3) has a unique solution $u_{m,\alpha,\beta} \in L^p(0, T; \mathbb{H}^r(\Omega))$, where*

$$1 < p < \frac{1}{b}, \quad \alpha k \leq b < \alpha - \alpha k.$$

In addition, we get

$$\left\| u_{m,\alpha,\beta}(t) \right\|_{\mathbb{H}^r(\Omega)} \leq 2 C_k \alpha^k \left(m^k + \lambda_1^{-1} \right) e^{\mu_0 T^\alpha} T^{b-\alpha k} t^{-b} \left\| u_0 \right\|_{\mathbb{H}^{r+k-\beta k}(\Omega)}, \tag{4.2}$$

where C_k depends on k.
(ii) Let us assume that $b < \frac{\alpha}{2}$ and $u_0 \in \mathbb{H}^{s+k-\beta k+\beta-1}(\Omega) \cap \mathbb{H}^{r+k-\beta k}(\Omega)$. Then we have

$$\left\| \frac{\partial}{\partial t} u_{m,\alpha,\beta}(.,t) \right\|_{\mathbb{H}^s(\Omega)} \lesssim t^{\alpha-\alpha k-1} \left\| u_0 \right\|_{\mathbb{H}^{s+k-\beta k+\beta-1}(\Omega)} + \left(t^{\alpha-1-b} + t^{2\alpha-b-k\alpha-1} \right) \left\| u_0 \right\|_{\mathbb{H}^{r+k-\beta k}(\Omega)}. \tag{4.3}$$

Here the hidden constant depends on $k, b, \alpha, m, p, \beta, L_g$ (L_g is defined in (4.1)).

Proof. Let us define $B : \mathbf{X}_{b,\mu,\alpha}((0,T]; \mathbb{H}^r(\Omega)) \to \mathbf{X}_{b,\mu,\alpha}((0,T]; \mathbb{H}^r(\Omega))$, $\mu > 0$ by

$$Bw(t) := \mathbf{S}_{m,\alpha,\beta}(t) u_0 + \int_0^t \nu^{\alpha-1} \mathbf{P}_m \mathbf{S}_{m,\alpha,\beta}(t) \Big(\mathbf{S}_{m,\alpha,\beta}(\nu) \Big)^{-1} G(w(\nu)) \mathrm{d}\nu. \tag{4.4}$$

Let the zero function $w_0(t) = 0$. From the fact $G(0) = 0$, we know that

$$Bw_0(t) = \mathbf{S}_{m,\alpha,\beta}(t) u_0.$$

In view of (3.4) as in Lemma 3.3, we obtain the following estimate

$$\left\| Bw_0(t) \right\|_{\mathbb{H}^r(\Omega))} = \left\| \mathbf{S}_{m,\alpha,\beta}(t) u_0 \right\|_{\mathbb{H}^r(\Omega))} \leq C_k \alpha^k \left(m^k + \lambda_1^{-1} \right) t^{-\alpha k} \left\| u_0 \right\|_{\mathbb{H}^{r+k-\beta k}(\Omega)}.$$

Hence, multiplying both sides of the above expression by $t^b e^{-\mu t^\alpha}$, we have that

$$\begin{aligned} t^b e^{-\mu t^\alpha} \left\| Bw_0(t) \right\|_{\mathbb{H}^r(\Omega))} &\leq C_k \alpha^k \left(m^k + \lambda_1^{-1} \right) t^{b-\alpha k} \left\| u_0 \right\|_{\mathbb{H}^{r+k-\beta k}(\Omega)} \\ &\leq C_k \alpha^k \left(m^k + \lambda_1^{-1} \right) T^{b-\alpha k} \left\| u_0 \right\|_{\mathbb{H}^{r+k-\beta k}(\Omega)}, \end{aligned} \tag{4.5}$$

where we use $b \geq \alpha k$. This implies that $Bw_0 \in \mathbf{X}_{b,\mu,\alpha}((0,T]; \mathbb{H}^r(\Omega))$. Let any two functions $w_1, w_2 \in \mathbf{X}_{b,\mu,\alpha}((0,T]; \mathbb{H}^r(\Omega))$. From (4.4) and (3.3), we obtain that

$$
\begin{aligned}
&\left\|Bw_1(t)-Bw_2(t)\right\|_{\mathbb{H}^r(\Omega))}\\
=&\left\|\int_0^t \nu^{\alpha-1}\mathbf{P}_m\mathbf{S}_{m,\alpha,\beta}(t)\Big(\mathbf{S}_{m,\alpha,\beta}(\nu)\Big)^{-1}\Big(G(w_1(\nu))-G(w_2(\nu))\Big)\mathrm{d}\nu\right\|_{\mathbb{H}^r(\Omega))}\\
\le& C_k\alpha^k m^{-k}\int_0^t \nu^{\alpha-1}\left(t^\alpha-\nu^\alpha\right)^{-k}\left\|G(w_1(\nu))-G(w_2(\nu))\right\|_{\mathbb{H}^{r+k-\beta k}(\Omega))}\mathrm{d}\nu. \qquad (4.6)
\end{aligned}
$$

Since the constraint $s\ge r+k-\beta k$, we know that Sobolev embedding

$$
\mathbb{H}^s(\Omega)\hookrightarrow \mathbb{H}^{r+k-\beta k}(\Omega).
$$

From some above observations and noting (4.1), we get that

$$
\begin{aligned}
&t^b e^{-\mu t^\alpha}\left\|Bw_1(t)-Bw_2(t)\right\|_{\mathbb{H}^r(\Omega))}\\
\le& C_k\alpha^k\left(m^k+\lambda_1^{-1}\right)t^b e^{-\mu t^\alpha}\int_0^t \nu^{\alpha-1}\left(t^\alpha-\nu^\alpha\right)^{-k}\left\|G(w_1(\nu))-G(w_2(\nu))\right\|_{\mathbb{H}^s(\Omega))}\mathrm{d}\nu\\
\le& C_kL_g\alpha^k\left(m^k+\lambda_1^{-1}\right)t^b e^{-\mu t^\alpha}\int_0^t \nu^{\alpha-1}\left(t^\alpha-\nu^\alpha\right)^{-k}\left\|w_1(\nu)-w_2(\nu)\right\|_{\mathbb{H}^r(\Omega))}\mathrm{d}\nu\\
=& C_kL_g\alpha^k\left(m^k+\lambda_1^{-1}\right)t^b\int_0^t \nu^{\alpha-1-b}\left(t^\alpha-\nu^\alpha\right)^{-k}e^{-\mu(t^\alpha-\nu^\alpha)}\nu^b e^{-\mu\nu^\alpha}\left\|w_1(\nu)-w_2(\nu)\right\|_{\mathbb{H}^r(\Omega))}\mathrm{d}\nu. \qquad (4.7)
\end{aligned}
$$

From the fact that

$$
\left\|w_1-w_2\right\|_{\mathbf{X}_{b,\mu,\alpha}((0,T];\mathbb{H}^r(\Omega))}=\sup_{0\le\nu\le T}\nu^b e^{-\mu\nu^\alpha}\left\|w_1(\nu)-w_2(\nu)\right\|_{\mathbb{H}^r(\Omega))},
$$

we follows from (4.7) that

$$
\begin{aligned}
&\sup_{0\le t\le T} t^b e^{-\mu t^\alpha}\left\|Bw_1(t)-Bw_2(t)\right\|_{\mathbb{H}^r(\Omega))}\\
\le&\overline{C}\left\|w_1-w_2\right\|_{\mathbf{X}_{b,\mu,\alpha}((0,T];\mathbb{H}^r(\Omega))}\sup_{0\le t\le T}\Big[t^b\int_0^t \nu^{\alpha-1-b}\left(t^\alpha-\nu^\alpha\right)^{-k}e^{-\mu(t^\alpha-\nu^\alpha)}\mathrm{d}\nu\Big], \qquad (4.8)
\end{aligned}
$$

where $\overline{C}=C_kL_g\alpha^k m^{-k}$. Let us continue to treat the integral term as follows

$$
J_{1,\mu}(t)=t^b\int_0^t \nu^{\alpha-1-b}\left(t^\alpha-\nu^\alpha\right)^{-k}e^{-\mu(t^\alpha-\nu^\alpha)}\mathrm{d}\nu.
$$

In order to control the above integral, we need to change the variable $\nu=t\xi^{\frac{1}{\alpha}}$. Then we get the following statement

$$
J_{1,\mu}(t)=\frac{1}{\alpha}t^{\alpha-\alpha k}\int_0^1 \xi^{\frac{-b}{\alpha}}(1-\xi)^{-k}e^{-\mu t^\alpha(1-\xi)}\mathrm{d}\xi.
$$

Next, we provide the following lemma which can be found in [22], Lemma 8, page 9.

Lemma 4.1. *Let $a_1 > -1$, $a_2 > -1$ such that $a_1 + a_2 \geq -1$, $\rho > 0$ and $t \in [0, T]$. For $h > 0$, the following limit holds*

$$\lim_{\rho\to\infty}\left(\sup_{t\in[0,T]} t^h \int_0^1 \nu^{a_1}(1-\nu)^{a_2} e^{-\rho t(1-\nu)}\mathrm{d}\nu\right) = 0.$$

Since $0 < b < \alpha$ and $0 < k < \min(\frac{b}{\alpha},\ 1 - \frac{b}{\alpha})$, we easily to verify that the following conditions hold

$$\begin{cases} \alpha - \alpha k > 0, \\ \frac{-b}{\alpha} > -1,\ -k > -1, \\ \frac{-b}{\alpha} - k \geq -1. \end{cases} \tag{4.9}$$

By Lemma 4.1 and (4.9), we have

$$\lim_{\mu\to+\infty} \sup_{0\leq t\leq T} J_{1,\mu}(t) = 0.$$

This statement shows that there exists a μ_0 such that

$$\overline{C} \sup_{0\leq t\leq T} J_{1,\mu_0}(t) \leq \frac{1}{2}. \tag{4.10}$$

Combining (4.8) and (4.10), we obtain

$$\left\|Bw_1 - Bw_2\right\|_{\mathbf{X}_{b,\mu_0}((0,T];\mathbb{H}^r(\Omega))} \leq \frac{1}{2}\left\|w_1 - w_2\right\|_{\mathbf{X}_{b,\mu_0}((0,T];\mathbb{H}^r(\Omega))} \tag{4.11}$$

for any $w_1, w_2 \in \mathbf{X}_{b,\mu,\alpha}((0,T];\mathbb{H}^r(\Omega))$. This statement tells us that B is the mapping from $\mathbf{X}_{b,\mu,\alpha}((0,T];\mathbb{H}^r(\Omega))$ to itself. By applying Banach fixed point theorem, we deduce that B has a fixed point $u_{m,\alpha,\beta} \in \mathbf{X}_{b,\mu_0,\alpha}((0,T];\mathbb{H}^r(\Omega))$. Hence, we can see that

$$u_{m,\alpha,\beta}(t) = \mathbf{S}_{m,\alpha,\beta}(t)u_0 + \int_0^t \nu^{\alpha-1}\mathbf{P}_m\mathbf{S}_{m,\alpha,\beta}(t)\Big(\mathbf{S}_{m,\alpha,\beta}(\nu)\Big)^{-1} G(u_{m,\alpha,\beta}(\nu))\mathrm{d}\nu. \tag{4.12}$$

Let us show the regularity property of the mild solution $u_{m,\alpha,\beta}$. Indeed, using the triangle inequality and (4.11) and noting that $B(\nu = 0) = \mathbf{S}_{m,\alpha,\beta}(t)u_0$, we obtain

$$\begin{aligned}\left\|u_{m,\alpha,\beta}\right\|_{\mathbf{X}_{b,\mu_0,\alpha}((0,T];\mathbb{H}^r(\Omega))} &= \left\|Bu_{m,\alpha,\beta}\right\|_{\mathbf{X}_{b,\mu_0,\alpha}((0,T];\mathbb{H}^r(\Omega))} \\ &\leq \frac{1}{2}\left\|u_{m,\alpha,\beta}\right\|_{\mathbf{X}_{b,\mu_0}((0,T];\mathbb{H}^r(\Omega))} + \left\|B(\nu = 0)\right\|_{\mathbf{X}_{b,\mu_0,\alpha}((0,T];\mathbb{H}^r(\Omega))} \\ &= \frac{1}{2}\left\|u_{m,\alpha,\beta}\right\|_{\mathbf{X}_{b,\mu_0,\alpha}((0,T];\mathbb{H}^r(\Omega))} + \left\|\mathbf{S}_{m,\alpha,\beta}(t)u_0\right\|_{\mathbf{X}_{b,\mu_0,\alpha}((0,T];\mathbb{H}^r(\Omega))},\end{aligned}$$

which combined with (4.5), we get

$$\left\|u_{m,\alpha,\beta}\right\|_{\mathbf{X}_{b,\mu_0,\alpha}((0,T];\mathbb{H}^r(\Omega))} \leq 2C_k\alpha^k\left(m^k + \lambda_1^{-1}\right)T^{b-\alpha k}\left\|u_0\right\|_{\mathbb{H}^{r+k-\beta k}(\Omega)},$$

which allows us to get that

$$\left\|u_{m,\alpha,\beta}(t)\right\|_{\mathbb{H}^r(\Omega)} \leq \widetilde{C}_1 t^{-b}\left\|u_0\right\|_{\mathbb{H}^{r+k-\beta k}(\Omega)}, \tag{4.13}$$

where $\widetilde{C}_1$ depends on k, μ_0, b, α, m and

$$\widetilde{C}_1 = 2C_k\alpha^k\left(m^k + \lambda_1^{-1}\right)e^{\mu_0 T^\alpha}T^{b-\alpha k},$$

and we remind that C_k depends on k. Note that the improper integral $\int_0^T t^{-pb}\mathrm{d}t$ is convergent for $1 < p < \frac{1}{b}$, we deduce that

$$u_{m,\alpha,\beta} \in L^p(0,T;\mathbb{H}^r(\Omega)),$$

and the following regularity holds

$$\left\|u_{m,\alpha,\beta}\right\|_{L^p(0,T;\mathbb{H}^r(\Omega))} \le \widetilde{C}\left\|u_0\right\|_{\mathbb{H}^{r+k-\beta k}(\Omega)},$$

where $\widetilde{C}$ depends on $k, \mu_0, b, \alpha, m, p$. Our next aim is to claim the derivative of the mild solution $u_{m,\alpha,\beta}$. Applying the following formula

$$\mathrm{d}\left(\int_0^t \mathcal{K}(t,s)\mathrm{d}s\right) = \int_0^t \partial_t\mathcal{K}(t,s)\mathrm{d}s + \mathcal{K}(t,t)\mathrm{d}t,$$

we obtain the following equality

$$\begin{aligned}\frac{\partial}{\partial t}u_{m,\alpha,\beta}(.,t) =& t^{\alpha-1}\mathbf{Q}_{m,\alpha,\beta}(t)u_0 + t^{\alpha-1}G(u_{m,\alpha,\beta}(x,t))\\ &+ t^{\alpha-1}\int_0^t \nu^{\alpha-1}\mathbf{P}_m\mathbf{Q}_{m,\alpha,\beta}(t)\left(\mathbf{S}_{m,\alpha,\beta}\right)^{-1}(\nu)G(u_{m,\alpha,\beta}(\nu))\mathrm{d}\nu,\end{aligned} \tag{4.14}$$

where the operator $\mathbf{Q}_{m,\alpha,\beta}(t)$ is defined by

$$\mathbf{Q}_{m,\alpha,\beta}(t)v = -\sum_{n\in\mathbb{N}}\frac{\lambda_n^\beta}{1+m\lambda_n}\exp\left(-\frac{\lambda_n^\beta}{1+m\lambda_n}\frac{t^\alpha}{\alpha}\right)\langle v, e_n\rangle e_n$$

for any $v \in L^2(\Omega)$. In view of Parseval's equality, we get that

$$\left\|\mathbf{Q}_{m,\alpha,\beta}(t)v\right\|^2_{\mathbb{H}^s(\Omega)} = \sum_{n\in\mathbb{N}}\lambda_n^{2s}\left(\frac{\lambda_n^\beta}{1+m\lambda_n}\right)^2\exp\left(-2\frac{\lambda_n^\beta}{1+m\lambda_n}\frac{t^\alpha}{\alpha}\right)\langle v, e_n\rangle^2.$$

Using (3.8) and noting that $\frac{\lambda_n^\beta}{1+m\lambda_n} \le m^{-1}\lambda_n^{\beta-1}$, we get that

$$\left\|\mathbf{Q}_{m,\alpha,\beta}(t)v\right\|^2_{\mathbb{H}^s(\Omega)} \le \left(C_k\alpha^k m^{-k-1}\right)^2 t^{-2\alpha k}\sum_{n\in\mathbb{N}}\lambda_n^{2s+2k-2\beta k+2\beta-2}\langle v, e_n\rangle^2,$$

which implies that

$$\left\|\mathbf{Q}_{m,\alpha,\beta}(t)v\right\|_{\mathbb{H}^s(\Omega)} \le C_k\alpha^k m^{-k-1}t^{-\alpha k}\left\|v\right\|_{\mathbb{H}^{s+k-\beta k+\beta-1}(\Omega)} \tag{4.15}$$

for any $v \in \mathbb{H}^{s+k-\beta k+\beta-1}(\Omega)$. In a similar technique as above, we also get that

$$\left\|\mathbf{Q}_{m,\alpha,\beta}(t)\left(\mathbf{S}_{m,\alpha,\beta}\right)^{-1}(\nu)v\right\|_{\mathbb{H}^s(\Omega)} \le C_k\alpha^k m^{-k-1}\left(t^\alpha - \nu^\alpha\right)^{-k}\left\|v\right\|_{\mathbb{H}^{s+k-\beta k+\beta-1}(\Omega)}, \tag{4.16}$$

for any $0 \le \nu \le t$. Let us go back to the right hand side of (4.14). By (4.15), we evaluate the first term on the right hand side of (4.14) as follows

$$\left\| t^{\alpha-1}\mathbf{Q}_{m,\alpha,\beta}(t)u_0 \right\|_{\mathbb{H}^s(\Omega)} \le C_k \alpha^k m^{-k-1} t^{\alpha-\alpha k-1} \left\| u_0 \right\|_{\mathbb{H}^{s+k-\beta k+\beta-1}(\Omega)}. \tag{4.17}$$

Using global Lipschitz of G as in (4.1) and the fact that $G(\mathbf{0}) = \mathbf{0}$, the second term on the right hand side of (4.14) is estimated as follows

$$\left\| t^{\alpha-1} G(u_{m,\alpha,\beta}(x,t)) \right\|_{\mathbb{H}^s(\Omega)} \le \widetilde{C}_1 t^{\alpha-1} L_g \left\| u_{m,\alpha,\beta} \right\|_{\mathbb{H}^r(\Omega)} \le \widetilde{C}_1 L_g t^{\alpha-1-b} \left\| u_0 \right\|_{\mathbb{H}^{r+k-\beta k}(\Omega)}, \tag{4.18}$$

where we use (4.13). Let us now to treat the third integral term in (4.14). By (4.16), we obtain that

$$\begin{aligned} &\left\| \int_0^t \nu^{\alpha-1} \mathbf{P}_m \mathbf{Q}_{m,\alpha,\beta}(t) \left(\mathbf{S}_{m,\alpha,\beta}\right)^{-1}(\nu) G(u_{m,\alpha,\beta}(\nu)) \mathrm{d}\nu \right\|_{\mathbb{H}^s(\Omega)} \\ \le & C_k \alpha^k m^{-k-1} \int_0^t \nu^{\alpha-1} \left(t^\alpha - \nu^\alpha\right)^{-k} \left\| G(u_{m,\alpha,\beta}(\nu)) \right\|_{\mathbb{H}^{s+k-\beta k+\beta-1}(\Omega)} \mathrm{d}\nu. \end{aligned} \tag{4.19}$$

Since $\beta \le 1$ and $0 < k < 1$, we can easily verify that $s + k - \beta k + \beta - 1 \le s$, which implies the Sobolev embedding $\mathbb{H}^s(\Omega) \hookrightarrow \mathbb{H}^{s+k-\beta k+\beta-1}(\Omega)$ is true. From these above observations and using (4.13), we derive that

$$\begin{aligned} &\int_0^t \nu^{\alpha-1} \left(t^\alpha - \nu^\alpha\right)^{-k} \left\| G(u_{m,\alpha,\beta}(\nu)) \right\|_{\mathbb{H}^{s+k-\beta k+\beta-1}(\Omega)} \mathrm{d}\nu \\ \le & C(s,k,\beta) \int_0^t \nu^{\alpha-1} \left(t^\alpha - \nu^\alpha\right)^{-k} \left\| G(u_{m,\alpha,\beta}(\nu)) \right\|_{\mathbb{H}^s(\Omega)} \mathrm{d}\nu \\ \le & L_g C(s,k,\beta) \int_0^t \nu^{\alpha-1} \left(t^\alpha - \nu^\alpha\right)^{-k} \left\| u_{m,\alpha,\beta}(\nu) \right\|_{\mathbb{H}^r(\Omega)} \mathrm{d}\nu \\ \le & L_g C(s,k,\beta) \widetilde{C}_1 \left\| u_0 \right\|_{\mathbb{H}^{r+k-\beta k}(\Omega)} \left(\int_0^t \nu^{\alpha-1-b} \left(t^\alpha - \nu^\alpha\right)^{-k} \mathrm{d}\nu \right). \end{aligned} \tag{4.20}$$

Let us now treat the integral term on the right hand side of (4.20). Controlling it is really not that simple task. By applying Hölder inequality, we find that

$$\begin{aligned} \left(\int_0^t \nu^{\alpha-1-b} \left(t^\alpha - \nu^\alpha\right)^{-k} \mathrm{d}\nu \right)^2 &= \left(\int_0^t \nu^{\frac{\alpha-1}{2}-b} \nu^{\frac{\alpha-1}{2}} \left(t^\alpha - \nu^\alpha\right)^{-k} \mathrm{d}\nu \right)^2 \\ &\le \left(\int_0^t \nu^{\alpha-1-2b} \mathrm{d}\nu \right) \left(\int_0^t \nu^{\alpha-1} \left(t^\alpha - \nu^\alpha\right)^{-2k} \mathrm{d}\nu \right) \\ &= \frac{t^{\alpha-2b}}{\alpha-2b} \left(\int_0^t \nu^{\alpha-1} \left(t^\alpha - \nu^\alpha\right)^{-2k} \mathrm{d}\nu \right), \end{aligned} \tag{4.21}$$

where $\alpha > 2b$. By changing to a new variable $z = \nu^\alpha$, we derive that $\mathrm{d}z = \alpha \nu^{\alpha-1} \mathrm{d}\nu$. Hence, we infer that

$$\int_0^t \nu^{\alpha-1} \left(t^\alpha - \nu^\alpha\right)^{-2k} \mathrm{d}\nu = \frac{1}{\alpha} \int_0^{t^\alpha} \left(t^\alpha - z\right)^{-2k} \mathrm{d}z = \frac{t^{\alpha(1-2k)}}{\alpha(1-2k)}, \tag{4.22}$$

where we note that $0 < k < \frac{1}{2}$. Combining (4.21) and (4.22), we obtain that the following inequality

$$\int_0^t \nu^{\alpha-1-b} (t^\alpha - \nu^\alpha)^{-k} \mathrm{d}\nu \le \frac{t^{\alpha-b-k\alpha}}{\sqrt{\alpha(1-2k)(\alpha-2b)}}. \tag{4.23}$$

By (4.23) and following from (4.19) and (4.20), we have

$$\begin{aligned}
&\left\| \int_0^t \nu^{\alpha-1} \mathbf{Q}_{m,\alpha,\beta}(t) \left(\mathbf{S}_{m,\alpha,\beta}\right)^{-1} (\nu) G(u_{m,\alpha,\beta}(\nu)) \mathrm{d}\nu \right\|_{\mathbb{H}^s(\Omega)} \\
\le & C_k \alpha^k m^{-k-1} \int_0^t \nu^{\alpha-1} (t^\alpha - \nu^\alpha)^{-k} \left\| G(u_{m,\alpha,\beta}(\nu)) \right\|_{\mathbb{H}^{s+k-\beta k+\beta-1}(\Omega)} \mathrm{d}\nu \\
\le & \frac{C_k \alpha^k m^{-k-1} L_g C(s,k,\beta) \widetilde{C}_1}{\sqrt{\alpha(1-2k)(\alpha-2b)}} t^{\alpha-b-k\alpha} \left\| u_0 \right\|_{\mathbb{H}^{r+k-\beta k}(\Omega)}.
\end{aligned} \tag{4.24}$$

Summarizing the above results (4.14), (4.17), (4.18), (4.24) and using the triangle inequality, we obtain the following assertion

$$\begin{aligned}
\left\| \frac{\partial}{\partial t} u_{m,\alpha,\beta}(.,t) \right\|_{\mathbb{H}^s(\Omega)} \le & \left\| t^{\alpha-1} \mathbf{Q}_{m,\alpha,\beta}(t) u_0 \right\|_{\mathbb{H}^s(\Omega)} + \left\| t^{\alpha-1} G(u_{m,\alpha,\beta}(x,t)) \right\|_{\mathbb{H}^s(\Omega)} \\
& + t^{\alpha-1} \left\| \int_0^t \nu^{\alpha-1} \mathbf{P}_m \mathbf{Q}_{m,\alpha,\beta}(t) \left(\mathbf{S}_{m,\alpha,\beta}\right)^{-1} (\nu) G(u_{m,\alpha,\beta}(\nu)) \mathrm{d}\nu \right\|_{\mathbb{H}^s(\Omega)} \\
\le & C_k \alpha^k m^{-k-1} t^{\alpha-\alpha k-1} \left\| u_0 \right\|_{\mathbb{H}^{s+k-\beta k+\beta-1}(\Omega)} + \widetilde{C}_1 L_g t^{\alpha-1-b} \left\| u_0 \right\|_{\mathbb{H}^{r+k-\beta k}(\Omega)} \\
& + \frac{C_k \alpha^k m^{-k-1} L_g C(s,k,\beta) \widetilde{C}_1}{\sqrt{\alpha(1-2k)(\alpha-2b)}} t^{2\alpha-b-k\alpha-1} \left\| u_0 \right\|_{\mathbb{H}^{r+k-\beta k}(\Omega)}
\end{aligned}$$

which shows (4.3). The proof is completed. □

5. The convergence of the mild solution when $m \to 0^+$

The main purpose of this section is to investigate the convergence of mild solutions to Problem (1.3) when $m \to 0^+$. Our result gives us an interesting connection between the solution of the Sobolev equation and the parabolic equation.

Theorem 5.1. *Let $G : \mathbb{H}^r(\Omega) \to \mathbb{H}^s(\Omega)$ such that $G(\mathbf{0}) = \mathbf{0}$ and* (4.1) *holds. Here r, s satisfy that $s \ge r + k - \beta k$ for $0 < k < \frac{1}{2}$. Let the initial datum $u_0 \in \mathbb{H}^{r+k-\beta k}(\Omega)$. Let $u_{m,\alpha,\beta}$ and $u^*_{\alpha,\beta}$ be the mild solutions to Problem* (1.3) *with $m > 0$ and $m = 0$ respectively. Then we get the following estimate*

$$\left\| u_{m,\alpha,\beta}(t) - u^*_{\alpha,\beta}(t) \right\|_{\mathbb{H}^r(\Omega)} \lesssim \left[m^{\frac{2\gamma-\varepsilon\gamma+\varepsilon l}{2}} + m^k \right] \mathbb{T}^*(\alpha,\gamma,l,b,k) E_{1,\alpha}\left(L_g t\right),$$

where $\alpha k \le b < \frac{\alpha}{2}$, $\gamma = \frac{k(1-\beta)+l\left(\beta-\frac{\varepsilon}{2}\right)}{\beta+1-\frac{\varepsilon}{2}}$, $0 < l < k(1-\beta)$ and $1 < \varepsilon < \min\left(2, \frac{2k(1-\beta)+2l\beta}{l}\right)$.

Remark 5.1. *Note that $m^{\frac{2\gamma-\varepsilon\gamma+\varepsilon l}{2}} + m^k$ tends to zero when $m \to 0^+$. Hence, we can deduce that $\left\| u_{m,\alpha,\beta}(t) - u^*_{\alpha,\beta}(t) \right\|_{\mathbb{H}^r(\Omega)} \to 0$ when $m \to 0^+$.*

Proof. In the case $m = 0$, thanks for the results on [11], the mild solution to Problem (1.3) is given by the following operator equation

$$u^*_{\alpha,\beta}(t) = \mathbf{S}^*{}_{\alpha,\beta}(t)u_0 + \int_0^t \nu^{\alpha-1}\mathbf{S}^*{}_{\alpha,\beta}(t)\Big(\mathbf{S}^*{}_{\alpha,\beta}(\nu)\Big)^{-1} G(u^*_{\alpha,\beta}(\nu))\mathrm{d}\nu, \tag{5.1}$$

where we provide two operators that have the following Fourier series representation as follows

$$\mathbf{S}^*{}_{\alpha,\beta}(t)f = \sum_{n\in\mathbb{N}} \exp\left(\frac{-\lambda_n^\beta t^\alpha}{\alpha}\right)\langle f, e_n\rangle e_n$$

and

$$\Big(\mathbf{S}^*{}_{\alpha,\beta}(\nu)\Big)^{-1} f = \sum_{n\in\mathbb{N}} \exp\left(\frac{\lambda_n^\beta \nu^\alpha}{\alpha}\right)\langle f, e_n\rangle e_n.$$

It's worth emphasizing that the existence of the solution to Equation (5.1) has been demonstrated in [11]. Subtracting (5.1) from (4.12), we get the following equality by some simple calculations

$$\begin{aligned}
&u_{m,\alpha,\beta}(t) - u^*_{\alpha,\beta}(t)\\
=&\Big(\mathbf{S}_{m,\alpha,\beta}(t) - \mathbf{S}^*{}_{\alpha,\beta}(t)\Big)u_0\\
&+\int_0^t \nu^{\alpha-1}\mathbf{S}^*{}_{\alpha,\beta}(t)\Big(\mathbf{S}^*{}_{\alpha,\beta}(\nu)\Big)^{-1}\Big(G(u_{m,\alpha,\beta}(\nu)) - G(u^*_{\alpha,\beta}(\nu))\Big)\mathrm{d}\nu\\
&+\int_0^t \nu^{\alpha-1}\left[\mathbf{P}_m\mathbf{S}_{m,\alpha,\beta}(t)\Big(\mathbf{S}_{m,\alpha,\beta}(\nu)\Big)^{-1} - \mathbf{P}_m\mathbf{S}^*{}_{\alpha,\beta}(t)\Big(\mathbf{S}^*{}_{\alpha,\beta}(\nu)\Big)^{-1}\right]G(u_{m,\alpha,\beta}(\nu))\mathrm{d}\nu\\
&+\int_0^t \nu^{\alpha-1}\,(\mathbf{P}_m - I)\,\mathbf{S}^*{}_{\alpha,\beta}(t)\Big(\mathbf{S}^*{}_{\alpha,\beta}(\nu)\Big)^{-1}G(u_{m,\alpha,\beta}(\nu))\mathrm{d}\nu\\
=&M_1 + M_2 + M_3 + M_4.
\end{aligned} \tag{5.2}$$

Next, we estimate the four terms on the right hand of (5.2) in $\mathbb{H}^r(\Omega)$ space. We divide this process into four steps as below.

Step 1. Estimate of M_1. In view of the inequality $\left|e^{-c} - e^{-d}\right| \le C_\gamma \max(e^{-c}, e^{-d})|c-d|^\gamma$ with $\gamma > 0$, let $c = \frac{\lambda_n^\beta}{1+m\lambda_n}\frac{t^\alpha}{\alpha}$ and $d = \lambda_n^\beta\frac{t^\alpha}{\alpha}$, we get the following inequality

$$\begin{aligned}
&\left|\exp\left(-\frac{\lambda_n^\beta}{1+m\lambda_n}\frac{t^\alpha}{\alpha}\right) - \exp\left(-\lambda_n^\beta\frac{t^\alpha}{\alpha}\right)\right|\\
\le&C_\gamma \exp\left(-\frac{\lambda_n^\beta}{1+m\lambda_n}\frac{t^\alpha}{\alpha}\right)\left(\lambda_n^\beta - \frac{\lambda_n^\beta}{1+m\lambda_n}\right)^\gamma t^{\alpha\gamma}\alpha^{-\gamma}\\
\le&C_\gamma C_l\left(\frac{\lambda_n^\beta}{1+m\lambda_n}\right)^{-l}\left(\frac{t^\alpha}{\alpha}\right)^{-l}\left(\lambda_n^\beta - \frac{\lambda_n^\beta}{1+m\lambda_n}\right)^\gamma t^{\alpha\gamma}\alpha^{-\gamma}\\
\le&C_\gamma C_l\alpha^{l-\gamma}\lambda_n^{-\beta l}\left(m\lambda_n^{1+\beta}\right)^\gamma\left(\frac{1}{1+m\lambda_n}\right)^{\gamma-l} t^{\alpha\gamma-\alpha l}
\end{aligned} \tag{5.3}$$

by $e^{-c} > e^{-d}$ and (3.8).

Here γ and l are two positive constants that are later chosen. For $1 < \varepsilon < 2$ and any $z > 0$, we easily verify that $(1+z)^{\frac{2}{\varepsilon}} \geq 1 + z > z$, which implies

$$(1+z)^{\gamma-l} = (1+z)^{\frac{2}{\varepsilon}\frac{\varepsilon(\gamma-l)}{2}} > z^{\frac{\varepsilon(\gamma-l)}{2}} \tag{5.4}$$

for any $\gamma > l > 0$.

In (5.4), by choosing $z = 1 + m\lambda_n$ and after some simple calculation, we have $(1+m\lambda_n)^{\gamma-l} > (m\lambda_n)^{\frac{\varepsilon(\gamma-l)}{2}}$, which leads to the following inequality

$$\left(\frac{1}{1+m\lambda_n}\right)^{\gamma-l} \leq (m\lambda_n)^{\frac{\varepsilon(l-\gamma)}{2}}. \tag{5.5}$$

Combining (5.3) and (5.5), we find that

$$\begin{aligned} &\left|\exp\left(-\frac{\lambda_n^\beta}{1+m\lambda_n}\frac{t^\alpha}{\alpha}\right) - \exp\left(-\lambda_n^\beta\frac{t^\alpha}{\alpha}\right)\right| \\ \leq & C(\alpha,\gamma,l) m^{\frac{2\gamma-\varepsilon\gamma+\varepsilon l}{2}} t^{\alpha\gamma-\alpha l} \lambda_n^{-\beta l+\beta\gamma+\gamma+\frac{\varepsilon(l-\gamma)}{2}}. \end{aligned} \tag{5.6}$$

Therefore, we have the following estimate

$$\begin{aligned} &\left\|\left(\mathbf{S}_{m,\alpha,\beta}(t) - \mathbf{S}^*{}_{\alpha,\beta}(t)\right)u_0\right\|^2_{\mathbb{H}^r(\Omega))} \\ =& \sum_{n\in\mathbb{N}} \lambda_n^{2r}\left|\exp\left(-\frac{\lambda_n^\beta}{1+m\lambda_n}\frac{t^\alpha}{\alpha}\right) - \exp\left(-\lambda_n^\beta\frac{t^\alpha}{\alpha}\right)\right|^2 \langle u_0, e_n\rangle^2 \\ \leq& \left|C(\alpha,\gamma,l)\right|^2 m^{2\gamma-\varepsilon\gamma+\varepsilon l} t^{2\alpha\gamma-2\alpha l} \sum_{n\in\mathbb{N}} \lambda_n^{2r-2\beta l+2\beta\gamma+2\gamma+\varepsilon(l-\gamma)} \langle u_0, e_n\rangle^2, \end{aligned}$$

which implies that

$$\left\|\left(\mathbf{S}_{m,\alpha,\beta}(t) - \mathbf{S}^*{}_{\alpha,\beta}(t)\right)u_0\right\|_{\mathbb{H}^r(\Omega))} \leq C(\alpha,\gamma,l) m^{\frac{2\gamma-\varepsilon\gamma+\varepsilon l}{2}} t^{\alpha\gamma-\alpha l} \left\|u_0\right\|_{\mathbb{H}^{r-\beta l+\beta\gamma+\gamma+\frac{\varepsilon(l-\gamma)}{2}}(\Omega)}. \tag{5.7}$$

Next, we explain how to choose the parameters l, ε, γ. Since $\beta < 1$, we can choose l such that

$$0 < l < \min\left(k(1-\beta), \frac{1-\beta}{\beta}k\right) = k(1-\beta),$$

which implies that $2k(1-\beta) + 2l\beta > l$, that is $\frac{2k(1-\beta)+2l\beta}{l} > 1$. Then we can choose ε such that

$$1 < \varepsilon < \min\left(2, \frac{2k(1-\beta)+2l\beta}{l}\right).$$

Let us choose γ such that

$$\gamma = \frac{k(1-\beta) + l\left(\beta - \frac{\varepsilon}{2}\right)}{\beta + 1 - \frac{\varepsilon}{2}}.$$

It is easy to verify that $\gamma > \frac{l+l\left(\beta-\frac{\varepsilon}{2}\right)}{\beta+1-\frac{\varepsilon}{2}} = l$ and the following equality

$$r - \beta l + \beta\gamma + \gamma + \frac{\varepsilon(l-\gamma)}{2} = r + k - \beta k.$$

Then the estimate (5.7) becomes

$$\begin{aligned}\left\|M_1\right\|_{\mathbb{H}^r(\Omega)} &= \left\|\left(\mathbf{S}_{m,\alpha,\beta}(t) - \mathbf{S}^*{}_{\alpha,\beta}(t)\right)u_0\right\|_{\mathbb{H}^r(\Omega))} \\ &\le C(\alpha,\gamma,l)m^{\frac{2\gamma-\varepsilon\gamma+\varepsilon l}{2}} t^{\alpha\gamma-\alpha l}\left\|u_0\right\|_{\mathbb{H}^{r+k-\beta k}(\Omega)}. \end{aligned} \tag{5.8}$$

Step 2. Estimate of M_2. Let $\psi \in \mathbb{H}^r(\Omega)$. For $0 \le \nu \le t \le T$, we can also get

$$\left\|\mathbf{S}^*{}_{\alpha,\beta}(t)\left(\mathbf{S}^*{}_{\alpha,\beta}(\nu)\right)^{-1}\psi\right\|^2_{\mathbb{H}^r(\Omega)} = \sum_{n\in\mathbb{N}} \lambda_n^{2r} \exp\left(2\lambda_n^{\beta}\frac{\nu^\alpha - t^\alpha}{\alpha}\right)\langle\psi, e_n\rangle^2 \le \left\|\psi\right\|^2_{\mathbb{H}^r(\Omega)}, \tag{5.9}$$

where we use that

$$\exp\left(\frac{\lambda_n^\beta}{1+m\lambda_n}\frac{\nu^\alpha - t^\alpha}{\alpha}\right) \le 1.$$

By (5.9) and Sobolev embedding $\mathbb{H}^s(\Omega)) \hookrightarrow \mathbb{H}^r(\Omega)$, we find that

$$\begin{aligned}&\left\|\int_0^t \nu^{\alpha-1}\mathbf{S}^*{}_{\alpha,\beta}(t)\left(\mathbf{S}^*{}_{\alpha,\beta}(\nu)\right)^{-1}\left(G(u_{m,\alpha,\beta}(\nu)) - G(u^*_{\alpha,\beta}(\nu))\right)\mathrm{d}\nu\right\|_{\mathbb{H}^r(\Omega)} \\ &\le \int_0^t \nu^{\alpha-1}\left\|G(u_{m,\alpha,\beta}(\nu)) - G(u^*_{\alpha,\beta}(\nu))\right\|_{\mathbb{H}^r(\Omega)}\mathrm{d}\nu \\ &\le C(r,s)\int_0^t \nu^{\alpha-1}\left\|G(u_{m,\alpha,\beta}(\nu)) - G(u^*_{\alpha,\beta}(\nu))\right\|_{\mathbb{H}^s(\Omega)}\mathrm{d}\nu. \end{aligned} \tag{5.10}$$

By the global Lipschitz property of G as in (4.1) and noting (5.10), it follows that

$$\left\|M_2\right\|_{\mathbb{H}^r(\Omega)} \le L_g C(r,s)\int_0^t \nu^{\alpha-1}\left\|u_{m,\alpha,\beta}(\nu) - u^*_{\alpha,\beta}(\nu)\right\|_{\mathbb{H}^r(\Omega)}\mathrm{d}\nu. \tag{5.11}$$

Step 3. Estimate of M_3. Let $f \in \mathbb{H}^{r+k-\beta k}(\Omega)$. Then using Parseval' s equality, we have the following identity

$$\begin{aligned}&\left\|\left[\mathbf{P}_m\mathbf{S}_{m,\alpha,\beta}(t)\left(\mathbf{S}_{m,\alpha,\beta}(\nu)\right)^{-1} - \mathbf{P}_m\mathbf{S}^*{}_{\alpha,\beta}(t)\left(\mathbf{S}^*{}_{\alpha,\beta}(\nu)\right)^{-1}\right]f\right\|^2_{\mathbb{H}^r(\Omega))} \\ &= \sum_{n\in\mathbb{N}} \frac{\lambda_n^{2r}}{(1+m\lambda_n)^2}\left|\exp\left(-\frac{\lambda_n^\beta}{1+m\lambda_n}\frac{t^\alpha-\nu^\alpha}{\alpha}\right) - \exp\left(-\lambda_n^\beta\frac{t^\alpha-\nu^\alpha}{\alpha}\right)\right|^2\langle f, e_n\rangle^2. \end{aligned} \tag{5.12}$$

By a similar explanation as in (5.6), we find that

$$
\begin{aligned}
&\left|\exp\left(-\frac{\lambda_n^\beta}{1+m\lambda_n}\frac{t^\alpha-\nu^\alpha}{\alpha}\right)-\exp\left(-\lambda_n^\beta\frac{t^\alpha-\nu^\alpha}{\alpha}\right)\right| \\
\leq &C(\alpha,\gamma,l)m^{\frac{2\gamma-\varepsilon\gamma+\varepsilon l}{2}}\left(t^\alpha-\nu^\alpha\right)^{\gamma-l}\lambda_n^{-\beta l+\beta\gamma+\gamma+\frac{\varepsilon(l-\gamma)}{2}}.
\end{aligned} \tag{5.13}
$$

By (5.12) and (5.13), we get that

$$
\begin{aligned}
&\left\|\left[\mathbf{P}_m\mathbf{S}_{m,\alpha,\beta}(t)\left(\mathbf{S}_{m,\alpha,\beta}(\nu)\right)^{-1}-\mathbf{P}_m\mathbf{S}^*{}_{\alpha,\beta}(t)\left(\mathbf{S}^*{}_{\alpha,\beta}(\nu)\right)^{-1}\right]f\right\|_{\mathbb{H}^r(\Omega))} \\
\leq &C(\alpha,\gamma,l)m^{\frac{2\gamma-\varepsilon\gamma+\varepsilon l}{2}}\left(t^\alpha-\nu^\alpha\right)^{\gamma-l}\left\|f\right\|_{\mathbb{H}^{r+k-\beta k}(\Omega)}.
\end{aligned} \tag{5.14}
$$

In view of (5.14) and $s\geq r+k-\beta k$, we derive that

$$
\begin{aligned}
&\left\|\left[\mathbf{P}_m\mathbf{S}_{m,\alpha,\beta}(t)\left(\mathbf{S}_{m,\alpha,\beta}(\nu)\right)^{-1}-\mathbf{P}_m\mathbf{S}^*{}_{\alpha,\beta}(t)\left(\mathbf{S}^*{}_{\alpha,\beta}(\nu)\right)^{-1}\right]G(u^*_{\alpha,\beta}(\nu))\right\|_{\mathbb{H}^r(\Omega))} \\
\leq &C(\alpha,\gamma,l)m^{\frac{2\gamma-\varepsilon\gamma+\varepsilon l}{2}}\left(t^\alpha-\nu^\alpha\right)^{\gamma-l}\left\|G(u_{m,\alpha,\beta}(\nu))\right\|_{\mathbb{H}^{r+k-\beta k}(\Omega)} \\
\leq &C(\alpha,\gamma,l,s)m^{\frac{2\gamma-\varepsilon\gamma+\varepsilon l}{2}}\left(t^\alpha-\nu^\alpha\right)^{\gamma-l}\left\|G(u_{m,\alpha,\beta}(\nu))\right\|_{\mathbb{H}^{s}(\Omega)}.
\end{aligned} \tag{5.15}
$$

By using global Lipschitz property of G as in (4.1) and noting (4.2), we infer that

$$
\begin{aligned}
\left\|G(u_{m,\alpha,\beta}(\nu))\right\|_{\mathbb{H}^s(\Omega)} &\leq L_g\left\|u_{m,\alpha,\beta}(\nu)\right\|_{\mathbb{H}^r(\Omega)} \\
&\leq 2C_kL_g\alpha^k\left(m^k+\lambda_1^{-1}\right)e^{\mu_0T^\alpha}T^{b-\alpha k}\nu^{-b}\left\|u_0\right\|_{\mathbb{H}^{r+k-\beta k}(\Omega)} \\
&\lesssim\left(m^k+\lambda_1^{-1}\right)\nu^{-b}\left\|u_0\right\|_{\mathbb{H}^{r+k-\beta k}(\Omega)},
\end{aligned} \tag{5.16}
$$

where the hidden constant depends on k,α,L_g,μ_0,b. Combining (5.15) and (5.16), we get the following estimate

$$
\left\|M_3\right\|_{\mathbb{H}^r(\Omega)}\lesssim m^{\frac{2\gamma-\varepsilon\gamma+\varepsilon l}{2}}\left(m^k+\lambda_1^{-1}\right)\left\|u_0\right\|_{\mathbb{H}^{r+k-\beta k}(\Omega)}\int_0^t\nu^{\alpha-1-b}\left(t^\alpha-\nu^\alpha\right)^{\gamma-l}\mathrm{d}\nu. \tag{5.17}
$$

Since $\gamma>l$ and noting that $b<\alpha$, we infer that

$$
\int_0^t\nu^{\alpha-1-b}\left(t^\alpha-\nu^\alpha\right)^{\gamma-l}\mathrm{d}\nu\leq t^{\alpha(\gamma-l)}\int_0^t\nu^{\alpha-1-b}\mathrm{d}\nu=\frac{t^{\alpha(\gamma-l)+\alpha-b}}{\alpha-b}. \tag{5.18}
$$

By (5.17) and (5.18), we obtain

$$
\left\|M_3\right\|_{\mathbb{H}^r(\Omega)}\lesssim m^{\frac{2\gamma-\varepsilon\gamma+\varepsilon l}{2}}\left(m^k+\lambda_1^{-1}\right)\frac{t^{\alpha(\gamma-l)+\alpha-b}}{\alpha-b}\left\|u_0\right\|_{\mathbb{H}^{r+k-\beta k}(\Omega)}. \tag{5.19}
$$

Step 4. Estimate of M_4. By Parseval's equality, we know that

$$
\left\|\left(\mathbf{P}_m-I\right)\mathbf{S}^*{}_{\alpha,\beta}(t)\left(\mathbf{S}^*{}_{\alpha,\beta}(\nu)\right)^{-1}f\right\|^2_{\mathbb{H}^r(\Omega))}=\sum_{n\in\mathbb{N}}\lambda_n^{2r}\Big(\frac{m\lambda_n}{1+m\lambda_n}\Big)^2\exp\left(-2\lambda_n^\beta\frac{t^\alpha-\nu^\alpha}{\alpha}\right)\langle f,e_n\rangle^2. \tag{5.20}
$$

Using the inequality $e^{-z} \le C(\varepsilon_0) z^{-\varepsilon_0}$, we get the following inequality

$$\exp\left(-2\lambda_n^\beta \frac{t^\alpha - \nu^\alpha}{\alpha}\right) \le C(\varepsilon_0, \alpha) \lambda_n^{-2\beta\varepsilon_0} \left(t^\alpha - \nu^\alpha\right)^{-2\varepsilon_0}. \tag{5.21}$$

By the inequality $(1+z)^2 > z^{\varepsilon_1}$ for $1 < \varepsilon_1 < 2$, we find that the following inequality

$$\Big(\frac{m\lambda_n}{1+m\lambda_n}\Big)^2 \le m^{2-\varepsilon_1} \lambda_n^{2-\varepsilon_1}. \tag{5.22}$$

By (5.20)–(5.22), we derive that

$$\begin{aligned}
&\left\| (\mathbf{P}_m - I)\, \mathbf{S}^*{}_{\alpha,\beta}(t) \Big(\mathbf{S}^*{}_{\alpha,\beta}(\nu)\Big)^{-1} f \right\|^2_{\mathbb{H}^r(\Omega))} \\
\le & C(\varepsilon_0, \alpha) m^{2-\varepsilon_1} \left(t^\alpha - \nu^\alpha\right)^{-2\varepsilon_0} \sum_{n\in\mathbb{N}} \lambda_n^{2r+2-\varepsilon_1-2\beta\varepsilon_0} \langle f, e_n\rangle^2.
\end{aligned}$$

Hence, by using Parseval's equality, we obtain the following estimate

$$\left\| (\mathbf{P}_m - I)\, \mathbf{S}^*{}_{\alpha,\beta}(t) \Big(\mathbf{S}^*{}_{\alpha,\beta}(\nu)\Big)^{-1} f \right\|_{\mathbb{H}^r(\Omega))} \le C(\varepsilon_0, \alpha) m^{1-\frac{\varepsilon_1}{2}} \left(t^\alpha - \nu^\alpha\right)^{-\varepsilon_0} \left\| f \right\|_{\mathbb{H}^{r-\beta\varepsilon_0+1-\frac{\varepsilon_1}{2}}(\Omega))}. \tag{5.23}$$

Next, we need to choose the appropriate parameters $\varepsilon_0, \varepsilon_1$. Let $\varepsilon_0 = k \in (0, \frac{1}{2})$ and $\varepsilon_1 = 2 - 2k$, we can verify that $1 < \varepsilon_1 < 2$. Hence, it follows from (5.23) that

$$\left\| (\mathbf{P}_m - I)\, \mathbf{S}^*{}_{\alpha,\beta}(t) \Big(\mathbf{S}^*{}_{\alpha,\beta}(\nu)\Big)^{-1} f \right\|_{\mathbb{H}^r(\Omega))} \le C(k, \alpha) m^k \left(t^\alpha - \nu^\alpha\right)^{-k} \left\| f \right\|_{\mathbb{H}^{r-\beta k+k}(\Omega))}.$$

By (5.16) and Sobolev embedding $\mathbb{H}^s(\Omega)) \hookrightarrow \mathbb{H}^{r-\beta k+k}(\Omega)$, we derive that

$$\begin{aligned}
&\left\| (\mathbf{P}_m - I)\, \mathbf{S}^*{}_{\alpha,\beta}(t) \Big(\mathbf{S}^*{}_{\alpha,\beta}(\nu)\Big)^{-1} G(u_{m,\alpha,\beta}(\nu)) \right\|_{\mathbb{H}^r(\Omega))} \\
\le & C(k, \alpha) m^k \left(t^\alpha - \nu^\alpha\right)^{-k} \left\| G(u_{m,\alpha,\beta}(\nu)) \right\|_{\mathbb{H}^{r-\beta k+k}(\Omega))} \\
\le & C(k, \alpha, s) m^k \left(t^\alpha - \nu^\alpha\right)^{-k} \left\| G(u_{m,\alpha,\beta}(\nu)) \right\|_{\mathbb{H}^s(\Omega))} \\
\lesssim & m^k \left(t^\alpha - \nu^\alpha\right)^{-k} \left(m^k + \lambda_1^{-1}\right) \nu^{-b} \left\| u_0 \right\|_{\mathbb{H}^{r+k-\beta k}(\Omega)},
\end{aligned}$$

which implies that

$$\begin{aligned}
\left\| M_4 \right\|_{\mathbb{H}^r(\Omega)} &\lesssim m^k \left(m^k + \lambda_1^{-1}\right) \left\| u_0 \right\|_{\mathbb{H}^{r+k-\beta k}(\Omega)} \int_0^t \nu^{\alpha-1-b} \left(t^\alpha - \nu^\alpha\right)^{-k} \mathrm{d}\nu \\
&\lesssim m^k \left(m^k + \lambda_1^{-1}\right) \frac{t^{\alpha-b-k\alpha}}{\sqrt{\alpha(1-2k)(\alpha-2b)}} \left\| u_0 \right\|_{\mathbb{H}^{r+k-\beta k}(\Omega)},
\end{aligned} \tag{5.24}$$

where we use (4.23). Combining (5.8), (5.11), (5.19) and (5.24), we obtain that

$$\begin{aligned}
\left\| u_{m,\alpha,\beta}(t) - u^*_{\alpha,\beta}(t) \right\|_{\mathbb{H}^r(\Omega)} \lesssim & \left\| M_1 \right\|_{\mathbb{H}^r(\Omega)} + \left\| M_2 \right\|_{\mathbb{H}^r(\Omega)} + \left\| M_3 \right\|_{\mathbb{H}^r(\Omega)} + \left\| M_4 \right\|_{\mathbb{H}^r(\Omega)} \\
\le & C(\alpha, \gamma, l) m^{\frac{2\gamma-\varepsilon\gamma+\varepsilon l}{2}} t^{\alpha\gamma-\alpha l} \left\| u_0 \right\|_{\mathbb{H}^{r+k-\beta k}(\Omega)}
\end{aligned}$$

$$+ m^{\frac{2\gamma-\varepsilon\gamma+\varepsilon l}{2}}\left(m^k+\lambda_1^{-1}\right)\frac{t^{\alpha(\gamma-l)+\alpha-b}}{\alpha-b}\left\|u_0\right\|_{\mathbb{H}^{r+k-\beta k}(\Omega)}$$
$$+ m^k\left(m^k+\lambda_1^{-1}\right)\frac{t^{\alpha-b-k\alpha}}{\sqrt{\alpha(1-2k)(\alpha-2b)}}\left\|u_0\right\|_{\mathbb{H}^{r+k-\beta k}(\Omega)}$$
$$+ L_g C(r,s)\int_0^t \nu^{\alpha-1}\left\|u_{m,\alpha,\beta}(\nu)-u^*_{\alpha,\beta}(\nu)\right\|_{\mathbb{H}^r(\Omega)}d\nu. \tag{5.25}$$

Here we note that $1-2k>0$ and $\alpha-2b>0$. Since the fact that $\gamma>l$ and $b<\min(\alpha,(1-k)\alpha)$, it is obvious to see that

$$t^{\alpha\gamma-\alpha l}\le T^{\alpha\gamma-\alpha l},$$
$$\frac{t^{\alpha(\gamma-l)+\alpha-b}}{\alpha-b}\le\frac{T^{\alpha(\gamma-l)+\alpha-b}}{\alpha-b},$$
$$\frac{t^{\alpha-b-k\alpha}}{\sqrt{\alpha(1-2k)(\alpha-2b)}}\le\frac{T^{\alpha-b-k\alpha}}{\sqrt{\alpha(1-2k)(\alpha-2b)}},$$

which motivate us to put

$$\mathbb{T}^*(\alpha,\gamma,l,b,k)=\max\left(T^{\alpha\gamma-\alpha l},\ \frac{T^{\alpha(\gamma-l)+\alpha-b}}{\alpha-b},\frac{T^{\alpha-b-k\alpha}}{\sqrt{\alpha(1-2k)(\alpha-2b)}}\right).$$

It follows from (5.25) that

$$\left\|u_{m,\alpha,\beta}(t)-u^*_{\alpha,\beta}(t)\right\|_{\mathbb{H}^r(\Omega)}\lesssim\left[m^{\frac{2\gamma-\varepsilon\gamma+\varepsilon l}{2}}+m^k\right]\mathbb{T}^*(\alpha,\gamma,l,b,k)$$
$$+ L_g\int_0^t \nu^{\alpha-1}\left\|u_{m,\alpha,\beta}(\nu)-u^*_{\alpha,\beta}(\nu)\right\|_{\mathbb{H}^r(\Omega)}\mathrm{d}\nu. \tag{5.26}$$

To continue to go further in the proof, we now need to recall the following Lemma introduced in [29].

Lemma 5.1. *Let $v\in L^1[0,T]$. Consider some postive constant A,B,β',γ' such that $\beta'+\gamma'>1$ and*

$$v(t)\le A+B\int_0^t (t-r)^{\beta'-1}r^{\gamma'-1}v(r)\mathrm{d}r.$$

Then for $0<t\le T$, we get

$$v(t)\le AE_{\beta',\gamma'}\left(B(\Gamma(\beta'))^{\frac{1}{\beta'+\gamma'-1}}t\right)$$

Looking Lemma 5.1 and (5.26), we set

$$v(t)=\left\|u_{m,\alpha,\beta}(t)-u^*_{\alpha,\beta}(t)\right\|_{\mathbb{H}^r(\Omega)},\ \ A=\left[m^{\frac{2\gamma-\varepsilon\gamma+\varepsilon l}{2}}+m^{\frac{2\gamma-\varepsilon\gamma+\varepsilon l}{2}}+m^k\right]\mathbb{T}^*(\alpha,\gamma,l,b,k),$$

$B=L_g,\ \ \beta'=1$ and $\gamma'=\alpha$. Then we deduce that

$$\left\|u_{m,\alpha,\beta}(t)-u^*_{\alpha,\beta}(t)\right\|_{\mathbb{H}^r(\Omega)}\lesssim\left[m^{\frac{2\gamma-\varepsilon\gamma+\varepsilon l}{2}}+m^k\right]\mathbb{T}^*(\alpha,\gamma,l,b,k)E_{1,\alpha}\left(L_g t\right).$$

The proof of Theorem 5.1 is completed. □

6. The convergence of the mild solution when $\beta \to 1^-$

Theorem 6.1. *Let* $G : \mathbb{H}^r(\Omega) \to \mathbb{H}^s(\Omega)$ *such that* (4.1) *holds. Here* r, s *satisfy that* $s > r + k - \beta k$ *for* $0 < k < \frac{1}{2}$. *Let the initial datum* $u_0 \in \mathbb{H}^{r+k-\beta k+\varepsilon}(\Omega)$ *for any* $\varepsilon > 0$. *Then we get the following estimate*

$$\left\| u_{m,\alpha,\beta} - u^{**}_{m,\alpha} \right\|_{\mathbb{X}_{\alpha k}(0,T;\mathbb{H}^r(\Omega))} \lesssim \mathbf{D}_\beta(\varepsilon, k) \left\| u_0 \right\|_{\mathbb{H}^{r+k-\beta k+\varepsilon}(\Omega)} + \left| \mathbf{E}_\beta(r, s, k) \right| \left\| u_0 \right\|_{\mathbb{H}^{r+k-\beta k}(\Omega)},$$

where the hidden constants depends on α, k, T, b. *Here*

$$\mathbf{D}_\beta(\varepsilon, k) = \left| 1 - \lambda_1^{1-\beta} \right|^{\frac{2k-\beta k+\varepsilon}{\beta}} + (1-\beta)^\varepsilon, \qquad \mathbf{E}_\beta(r, s, k) = \left| 1 - \lambda_1^{1-\beta} \right|^{\frac{k+s-r}{\beta}} + (1-\beta)^{s-r-k+\beta k}$$

for any $\varepsilon > 0$.

Proof. For $m > 0$, let $u_{m,\alpha,\beta}$ and $u^{**}_{m,\alpha}$ be the mild solutions to Problem (1.3) with $0 < \beta < 1$ and $\beta = 1$ respectively. Let us recall the formula of these two solutions

$$u_{m,\alpha,\beta}(t) := \mathbf{S}_{m,\alpha,\beta}(t) u_0 + \int_0^t \nu^{\alpha-1} \mathbf{P}_m \mathbf{S}_{m,\alpha,\beta}(t) \Big(\mathbf{S}_{m,\alpha,\beta}(\nu) \Big)^{-1} G(u_{m,\alpha,\beta}(\nu)) \mathrm{d}\nu. \tag{6.1}$$

and

$$u^{**}_{m,\alpha}(t) := \mathbf{S}_{m,\alpha,1}(t) u_0 + \int_0^t \nu^{\alpha-1} \mathbf{P}_m \mathbf{S}_{m,\alpha,1}(t) \Big(\mathbf{S}_{m,\alpha,1}(\nu) \Big)^{-1} G(u^{**}_{m,\alpha}(\nu)) \mathrm{d}\nu. \tag{6.2}$$

Subtracting (6.1) from (6.2) on each side, we derive that

$$\begin{aligned}
& u_{m,\alpha,\beta}(t) - u^{**}_{m,\alpha}(t) \\
=& \Big(\mathbf{S}_{m,\alpha,\beta}(t) u_0 - \mathbf{S}_{m,\alpha,1}(t) u_0 \Big) \\
& + \int_0^t \nu^{\alpha-1} \Big[\mathbf{P}_m \mathbf{S}_{m,\alpha,\beta}(t) \Big(\mathbf{S}_{m,\alpha,\beta}(\nu) \Big)^{-1} - \mathbf{P}_m \mathbf{S}_{m,\alpha,1}(t) \Big(\mathbf{S}_{m,\alpha,1}(\nu) \Big)^{-1} \Big] G(u^{**}_{m,\alpha}(\nu)) \mathrm{d}\nu \\
& + \int_0^t \nu^{\alpha-1} \mathbf{P}_m \mathbf{S}_{m,\alpha,\beta}(t) \Big(\mathbf{S}_{m,\alpha,\beta}(\nu) \Big)^{-1} \Big[G(u_{m,\alpha,\beta}(\nu)) - G(u^{**}_{m,\alpha}(\nu)) \Big] \mathrm{d}\nu \\
=& N_1 + N_2 + N_3.
\end{aligned} \tag{6.3}$$

Step 1. Estimate of N_1. By Parseval' s equality, we have that the following equality

$$\begin{aligned}
\left\| N_1 \right\|^2_{\mathbb{H}^r(\Omega)} =& \left\| \Big(\mathbf{S}_{m,\alpha,\beta}(t) u_0 - \mathbf{S}_{m,\alpha,1}(t) u_0 \Big) \right\|^2_{\mathbb{H}^r(\Omega)} \\
=& \sum_{n \in \mathbb{N}} \lambda_n^{2r} \left| \exp\left(-\frac{\lambda_n^\beta}{1 + m\lambda_n} \frac{t^\alpha}{\alpha} \right) - \exp\left(-\frac{\lambda_n}{1 + m\lambda_n} \frac{t^\alpha}{\alpha} \right) \right|^2 \langle u_0, e_n \rangle^2 \\
=& \sum_{\lambda_n > 1} \lambda_n^{2r} \left| \exp\left(-\frac{\lambda_n^\beta}{1 + m\lambda_n} \frac{t^\alpha}{\alpha} \right) - \exp\left(-\frac{\lambda_n}{1 + m\lambda_n} \frac{t^\alpha}{\alpha} \right) \right|^2 \langle u_0, e_n \rangle^2 \\
& + \sum_{\lambda_n \le 1} \lambda_n^{2r} \left| \exp\left(-\frac{\lambda_n^\beta}{1 + m\lambda_n} \frac{t^\alpha}{\alpha} \right) - \exp\left(-\frac{\lambda_n}{1 + m\lambda_n} \frac{t^\alpha}{\alpha} \right) \right|^2 \langle u_0, e_n \rangle^2
\end{aligned}$$

$$=N_{1,1}+N_{1,2}. \tag{6.4}$$

For the term $N_{1,1}$, since $\lambda_n > 1$ and $0 < \beta < 1$, we note that $\frac{\lambda_n^\beta}{1+m\lambda_n} < \frac{\lambda_n}{1+m\lambda_n}$. Hence, we have the following inequality

$$\exp\left(-\frac{\lambda_n^\beta}{1+m\lambda_n}\frac{t^\alpha}{\alpha}\right) > \exp\left(-\frac{\lambda_n}{1+m\lambda_n}\frac{t^\alpha}{\alpha}\right).$$

In view of the above inequality

$$\left|e^{-c}-e^{-d}\right| \le C_{\gamma'}\max(e^{-c},e^{-d})|c-d|^{\gamma'}, \quad \gamma'>0$$

with $c=\exp\left(-\frac{\lambda_n^\beta}{1+m\lambda_n}\frac{t^\alpha}{\alpha}\right)$ and $d=\exp\left(-\frac{\lambda_n}{1+m\lambda_n}\frac{t^\alpha}{\alpha}\right)$, we derive that

$$\begin{aligned}
&\left|\exp\left(-\frac{\lambda_n^\beta}{1+m\lambda_n}\frac{t^\alpha}{\alpha}\right)-\exp\left(-\frac{\lambda_n}{1+m\lambda_n}\frac{t^\alpha}{\alpha}\right)\right| \\
\le& C_{\gamma'}\exp\left(-\frac{\lambda_n^\beta}{1+m\lambda_n}\frac{t^\alpha}{\alpha}\right)\left|\frac{\lambda_n-\lambda_n^\beta}{1+m\lambda_n}\right|^{\gamma'} \\
\le& C(\gamma',\varepsilon')\left(\frac{\lambda_n^\beta}{1+m\lambda_n}\frac{t^\alpha}{\alpha}\right)^{-\varepsilon'}\left|\frac{\lambda_n-\lambda_n^\beta}{1+m\lambda_n}\right|^{\gamma'} \\
=& C(\gamma',\varepsilon')\alpha^{-\varepsilon'}t^{-\alpha\varepsilon'}(1+m\lambda_n)^{\varepsilon'-\gamma'}\lambda_n^{-\beta\varepsilon'}\left|\lambda_n-\lambda_n^\beta\right|^{\gamma'}.
\end{aligned} \tag{6.5}$$

Since the fact that $\lambda_n > 1$, we know that $\left|\lambda_n-\lambda_n^\beta\right|^{\gamma'}=\lambda_n^{\gamma'}\left(1-\lambda_n^{\beta-1}\right)^{\gamma'}$. Using the inequality $1-e^{-y}\le C(\mu)y^\mu$ for any $\mu>0$, we find that

$$1-\lambda_n^{\beta-1}=1-\exp\left[-(1-\beta)\log(\lambda_n)\right]\le C(\mu)(1-\beta)^\mu\log^\mu(\lambda_n)\le C(\mu)(1-\beta)^\mu\lambda_n^\mu,$$

where we note that $0<\log(y)\le y$ for any $y>1$, which implies that

$$\left|\lambda_n-\lambda_n^\beta\right|^{\gamma'}=\lambda_n^{\gamma'}\left(1-\lambda_n^{\beta-1}\right)^{\gamma'}\le C(\mu,\gamma')(1-\beta)^{\mu\gamma'}\lambda_n^{\mu\gamma'+\gamma'}. \tag{6.6}$$

Combining (6.5) and (6.6), we derive that

$$\begin{aligned}
&\left|\exp\left(-\frac{\lambda_n^\beta}{1+m\lambda_n}\frac{t^\alpha}{\alpha}\right)-\exp\left(-\frac{\lambda_n}{1+m\lambda_n}\frac{t^\alpha}{\alpha}\right)\right| \\
\le& C(\mu,\gamma',\varepsilon')\alpha^{-\varepsilon'}t^{-\alpha\varepsilon'}(1+m\lambda_n)^{\varepsilon'-\gamma'}(1-\beta)^{\mu\gamma'}\lambda_n^{-\beta\varepsilon'}\lambda_n^{\mu\gamma'+\gamma'}.
\end{aligned}$$

If we make the assumption $\varepsilon'\le\gamma'$, then $(1+m\lambda_n)^{\varepsilon'-\gamma'}\le 1$, which allows us to obtain that

$$\begin{aligned}
N_{1,1}&\le |C(\mu,\gamma',\varepsilon',\alpha)|^2 t^{-2\alpha\varepsilon'}(1-\beta)^{2\mu\gamma'}\sum_{\lambda_n>1}\lambda_n^{2r+2\mu\gamma'+2\gamma'-2\beta\varepsilon'}\langle u_0,e_n\rangle^2 \\
&\le |C(\mu,\gamma',\varepsilon',\alpha)|^2 t^{-2\alpha\varepsilon'}(1-\beta)^{2\mu\gamma'}\left\|u_0\right\|^2_{\mathbb{H}^{r+\mu\gamma'+\gamma'-\beta\varepsilon'}(\Omega)}.
\end{aligned}$$

Let us choose $\mu = \frac{\varepsilon}{k}$ and $\gamma' = \varepsilon' = k$ for any $\varepsilon > 0$. Then we get the following estimate

$$N_{1,1} \le |C(\varepsilon, k, \alpha)|^2 t^{-2\alpha k}(1-\beta)^{2\varepsilon} \left\| u_0 \right\|^2_{\mathbb{H}^{r+k-\beta k+\varepsilon}(\Omega)}. \tag{6.7}$$

Before mention to $N_{1,2}$, we provide a set $\overline{N} = \left\{ n \in \mathbb{N} : \lambda_n \le 1 \right\}$. Let us give the observation that if $\overline{N}$ is an empty set, then $N_{1,2} = 0$. If $\overline{N}$ is a non-empty set, then $\lambda_1 \le 1$. For the term $N_{1,2}$, we note that $\frac{\lambda_n^\beta}{1+m\lambda_n} > \frac{\lambda_n}{1+m\lambda_n}$ since $\lambda_n \le 1$ and $0 < \beta < 1$. Hence, we have that

$$\exp\left(-\frac{\lambda_n^\beta}{1+m\lambda_n}\frac{t^\alpha}{\alpha}\right) < \exp\left(-\frac{\lambda_n}{1+m\lambda_n}\frac{t^\alpha}{\alpha}\right).$$

By using the fact that

$$\left| e^{-c} - e^{-d} \right| \le C_{\gamma_1} \max(e^{-c}, e^{-d}) |c-d|^{\gamma_1}, \quad \gamma_1 > 0$$

with $c = \exp\left(-\frac{\lambda_n^\beta}{1+m\lambda_n}\frac{t^\alpha}{\alpha}\right)$ and $d = \exp\left(-\frac{\lambda_n}{1+m\lambda_n}\frac{t^\alpha}{\alpha}\right)$, we derive that

$$\begin{aligned}
&\left| \exp\left(-\frac{\lambda_n^\beta}{1+m\lambda_n}\frac{t^\alpha}{\alpha}\right) - \exp\left(-\frac{\lambda_n}{1+m\lambda_n}\frac{t^\alpha}{\alpha}\right) \right| \\
\le& C(\gamma_1) \exp\left(-\frac{\lambda_n}{1+m\lambda_n}\frac{t^\alpha}{\alpha}\right) \left| \frac{\lambda_n - \lambda_n^\beta}{1+m\lambda_n} \right|^{\gamma_1} \\
\le& C(\gamma_1, \varepsilon_1) \left(\frac{\lambda_n}{1+m\lambda_n}\frac{t^\alpha}{\alpha}\right)^{-\varepsilon_1} \left| \frac{\lambda_n - \lambda_n^\beta}{1+m\lambda_n} \right|^{\gamma_1} \\
=& C(\gamma_1, \varepsilon_1) \alpha^{-\varepsilon_1} t^{-\alpha\varepsilon_1} (1+m\lambda_n)^{\varepsilon_1 - \gamma_1} \lambda_n^{-\varepsilon_1} \left| \lambda_n - \lambda_n^\beta \right|^{\gamma_1}.
\end{aligned} \tag{6.8}$$

Since $\lambda_1 \le \lambda_n \le 1$, it is obvious to see that

$$\left| \lambda_n - \lambda_n^\beta \right|^{\gamma_1} = \left(\lambda_n^\beta - \lambda_n \right)^{\gamma_1} = \lambda_n^{\beta\gamma_1} \left| 1 - \lambda_n^{1-\beta} \right|^{\gamma_1} \le \lambda_n^{\beta\gamma_1} \left| 1 - \lambda_1^{1-\beta} \right|^{\gamma_1}. \tag{6.9}$$

Combining (6.8) and (6.9), we find that

$$\begin{aligned}
&\left| \exp\left(-\frac{\lambda_n^\beta}{1+m\lambda_n}\frac{t^\alpha}{\alpha}\right) - \exp\left(-\frac{\lambda_n}{1+m\lambda_n}\frac{t^\alpha}{\alpha}\right) \right| \\
\le& C(\gamma_1, \varepsilon_1, \alpha) \left| 1 - \lambda_1^{1-\beta} \right|^{\gamma_1} t^{-\alpha\varepsilon_1} (1+m\lambda_n)^{\varepsilon_1 - \gamma_1} \lambda_n^{-\varepsilon_1 + \beta\gamma_1}.
\end{aligned} \tag{6.10}$$

Hence, it follows from (6.10) that

$$\begin{aligned}
N_{1,2} =& \sum_{\lambda_n \le 1} \lambda_n^{2r} \left| \exp\left(-\frac{\lambda_n^\beta}{1+m\lambda_n}\frac{t^\alpha}{\alpha}\right) - \exp\left(-\frac{\lambda_n}{1+m\lambda_n}\frac{t^\alpha}{\alpha}\right) \right|^2 \langle u_0, e_n \rangle^2 \\
\le& \left| C(\gamma_1, \varepsilon_1, \alpha) \right|^2 \left| 1 - \lambda_1^{1-\beta} \right|^{2\gamma_1} t^{-2\alpha\varepsilon_1} \sum_{n=1}^\infty (1+m\lambda_n)^{2\varepsilon_1 - 2\gamma_1} \lambda_n^{2r - 2\varepsilon_1 + 2\beta\gamma_1} \langle u_0, e_n \rangle^2.
\end{aligned}$$

Let $\varepsilon_1 = k$ and $\gamma_1 = \frac{2k+\varepsilon}{\beta} - k$. Since $0 < \beta < 1$, we know that

$$\varepsilon_1 < \gamma_1, \quad 2r - 2\varepsilon_1 + 2\beta\gamma_1 = 2r + 2k - 2\beta k + 2\varepsilon.$$

Hence, in view of Parseval's equality, we get the following estimate

$$N_{1,2} \le C(k,r,\varepsilon,\alpha)\left|1-\lambda_1^{1-\beta}\right|^{\frac{4k-2\beta k+2\varepsilon}{\beta}} t^{-2\alpha k}\left\|u_0\right\|^2_{\mathbb{H}^{r+k-\beta k+\varepsilon}(\Omega)}. \tag{6.11}$$

Combining (6.4), (6.7) and (6.11), we obtain the following estimate

$$\begin{aligned}\left\|N_1\right\|^2_{\mathbb{H}^r(\Omega)} &= N_{1,1}+N_{1,2}\\ &\le C(k,r,\varepsilon,\alpha)\Big[\left|1-\lambda_1^{1-\beta}\right|^{\frac{4k-2\beta k+2\varepsilon}{\beta}}+(1-\beta)^{2\varepsilon}\Big]t^{-2\alpha k}\left\|u_0\right\|^2_{\mathbb{H}^{r+k-\beta k+\varepsilon}(\Omega)}.\end{aligned}$$

By taking the square root of both sides of the above expression and using the inequality $\sqrt{a+b}\le\sqrt{a}+\sqrt{b}$ for any $a,b\ge 0$, we get

$$\left\|N_1\right\|_{\mathbb{H}^r(\Omega)} \le C(k,r,\varepsilon,\alpha)\mathbf{D}_\beta(\varepsilon,k)t^{-\alpha k}\left\|u_0\right\|_{\mathbb{H}^{r+k-\beta k+\varepsilon}(\Omega)}. \tag{6.12}$$

Here we denote

$$\mathbf{D}_\beta(\varepsilon,k)=\left|1-\lambda_1^{1-\beta}\right|^{\frac{2k-\beta k+\varepsilon}{\beta}}+(1-\beta)^{\varepsilon},\quad \varepsilon>0,$$

where we observe that $\mathbf{D}_\beta(\varepsilon,k)\to 0,\ \beta\to 1^-$.

Step 2. Estimate of N_2. We confirm the following result for any $\varepsilon_0>0$ using the method similar to Step 1,

$$\begin{aligned}&\left\|\Big[\mathbf{P}_m\mathbf{S}_{m,\alpha,\beta}(t)\Big(\mathbf{S}_{m,\alpha,\beta}(\nu)\Big)^{-1}-\mathbf{P}_m\mathbf{S}_{m,\alpha,1}(t)\Big(\mathbf{S}_{m,\alpha,1}(\nu)\Big)^{-1}\Big]\psi\right\|_{\mathbb{H}^r(\Omega)}\\ \le&C(k,r,\varepsilon_0,\alpha)\Big[\left|1-\lambda_1^{1-\beta}\right|^{\frac{2k-\beta k+\varepsilon_0}{\beta}}+(1-\beta)^{\varepsilon_0}\Big](t^\alpha-\nu^\alpha)^{-k}\left\|\psi\right\|_{\mathbb{H}^{r+k-\beta k+\varepsilon_0}(\Omega)}.\end{aligned}$$

Since $s>r+k-\beta k$, we know that $\varepsilon_0=s-r-k+\beta k>0$, and using global Lipschitz property of G, we derive that

$$\begin{aligned}&\left\|\Big[\mathbf{P}_m\mathbf{S}_{m,\alpha,\beta}(t)\Big(\mathbf{S}_{m,\alpha,\beta}(\nu)\Big)^{-1}-\mathbf{P}_m\mathbf{S}_{m,\alpha,1}(t)\Big(\mathbf{S}_{m,\alpha,1}(\nu)\Big)^{-1}\Big]G(u^{**}_{m,\alpha}(\nu))\right\|_{\mathbb{H}^r(\Omega)}\\ \le&\overline{C}_1\mathbf{E}_\beta(r,s,k)\,(t^\alpha-\nu^\alpha)^{-k}\left\|G(u^{**}_{m,\alpha}(\nu))\right\|_{\mathbb{H}^s(\Omega)}\\ \le&K_g\overline{C}_1\mathbf{E}_\beta(r,s,k)\,(t^\alpha-\nu^\alpha)^{-k}\left\|u^{**}_{m,\alpha}(\nu)\right\|_{\mathbb{H}^r(\Omega)},\end{aligned}$$

where $\overline{C}_1=C(k,r,s,\beta,\alpha)$, and we have the following observation

$$\mathbf{E}_\beta(r,s,k)=\left|1-\lambda_1^{1-\beta}\right|^{\frac{k+s-r}{\beta}}+(1-\beta)^{s-r-k+\beta k}\to 0,\quad \beta\to 1^-.$$

In view of (4.2), we obtain that the following upper bound

$$\left\|u^{**}_{m,\alpha}(\nu)\right\|_{\mathbb{H}^r(\Omega)}\le\overline{C}_2\nu^{-b}\left\|u_0\right\|_{\mathbb{H}^{r+k-\beta k}(\Omega)},$$

where $\overline{C}_2=C(k,\alpha,m,\mu_0,T,b)$. From two latter estimations as above, we infer that

$$
\begin{aligned}
&\left\|N_2\right\|_{\mathbb{H}^r(\Omega)} \\
\leq &\int_0^t \nu^{\alpha-1}\left\|\left[\mathbf{P}_m\mathbf{S}_{m,\alpha,\beta}(t)\Big(\mathbf{S}_{m,\alpha,\beta}(\nu)\Big)^{-1} - \mathbf{P}_m\mathbf{S}_{m,\alpha,1}(t)\Big(\mathbf{S}_{m,\alpha,1}(\nu)\Big)^{-1}\right]G(u^{**}_{m,\alpha}(\nu))\right\|_{\mathbb{H}^r(\Omega)}\mathrm{d}\nu \\
\leq &\overline{C}_3\mathbf{E}_\beta(r,s,k)\left\|u_0\right\|_{\mathbb{H}^{r+k-\beta k}(\Omega)}\int_0^t \nu^{\alpha-1-b}\left(t^\alpha-\nu^\alpha\right)^{-k}\mathrm{d}\nu, \qquad (6.13)
\end{aligned}
$$

where $\overline{C}_3 = K_g\overline{C}_1\overline{C}_2$. Using (4.23), we obtain that the following inequality

$$
\int_0^t \nu^{\alpha-1-b}\left(t^\alpha-\nu^\alpha\right)^{-k}\mathrm{d}\nu \leq \frac{t^{\alpha-b-k\alpha}}{\sqrt{\alpha(1-2k)(\alpha-2b)}}. \qquad (6.14)
$$

Combining (6.13) and (6.14), we obtain that

$$
\left\|N_2\right\|_{\mathbb{H}^r(\Omega)} \leq \frac{\overline{C}_3 t^{\alpha-b-k\alpha}}{\sqrt{\alpha(1-2k)(\alpha-2b)}}\mathbf{E}_\beta(r,s,k)\left\|u_0\right\|_{\mathbb{H}^{r+k-\beta k}(\Omega)}. \qquad (6.15)
$$

Step 3. Estimate of N_3. By a similar argument as in (4.6), we find that

$$
\begin{aligned}
&\left\|\int_0^t \nu^{\alpha-1}\mathbf{P}_m\mathbf{S}_{m,\alpha,\beta}(t)\Big(\mathbf{S}_{m,\alpha,\beta}(\nu)\Big)^{-1}\Big[G(u_{m,\alpha,\beta}(\nu)) - G(u^{**}_{m,\alpha}(\nu))\Big]\mathrm{d}\nu\right\|_{\mathbb{H}^r(\Omega))} \\
\leq &C_k\alpha^k m^{-k}\int_0^t \nu^{\alpha-1}\left(t^\alpha-\nu^\alpha\right)^{-k}\left\|G(u_{m,\alpha,\beta}(\nu)) - G(u^{**}_{m,\alpha}(\nu))\right\|_{\mathbb{H}^{r+k-\beta k}(\Omega))}\mathrm{d}\nu.
\end{aligned}
$$

Since $s > r + k - \beta k$ and using global Lipschitz property of G, we obtain that

$$
\begin{aligned}
\left\|G(u_{m,\alpha,\beta}(\nu)) - G(u^{**}_{m,\alpha}(\nu))\right\|_{\mathbb{H}^{r+k-\beta k}(\Omega))} &\leq \left\|G(u_{m,\alpha,\beta}(\nu)) - G(u^{**}_{m,\alpha}(\nu))\right\|_{\mathbb{H}^{s}(\Omega))} \\
&\leq K_g\left\|u_{m,\alpha,\beta}(\nu) - u^{**}_{m,\alpha}(\nu)\right\|_{\mathbb{H}^{r}(\Omega))}.
\end{aligned}
$$

From the two above observations, we confirm the following statement

$$
\left\|N_3\right\|_{\mathbb{H}^r(\Omega)} \leq \overline{C}_3\int_0^t \nu^{\alpha-1}\left(t^\alpha-\nu^\alpha\right)^{-k}\left\|u_{m,\alpha,\beta}(\nu) - u^{**}_{m,\alpha}(\nu)\right\|_{\mathbb{H}^{r}(\Omega))}\mathrm{d}\nu, \qquad (6.16)
$$

where $\overline{C}_3 = C_k\alpha^k m^{-k}K_g$. Combining (6.3), (6.12), (6.15) and (6.16), we deduce the following estimate

$$
\begin{aligned}
&\left\|u_{m,\alpha,\beta}(t) - u^{**}_{m,\alpha}(t)\right\|_{\mathbb{H}^r(\Omega)} \\
\leq &\left\|N_1\right\|_{\mathbb{H}^r(\Omega)} + \left\|N_2\right\|_{\mathbb{H}^r(\Omega)} + \left\|N_3\right\|_{\mathbb{H}^r(\Omega)} \\
\leq &\widetilde{C}t^{-\alpha k}\mathbf{D}_\beta(\varepsilon,k)\left\|u_0\right\|_{\mathbb{H}^{r+k-\beta k+\varepsilon}(\Omega)} + \frac{\overline{C}_3 t^{\alpha-b-k\alpha}}{\sqrt{\alpha(1-2k)(\alpha-2b)}}\mathbf{E}_\beta(r,s,k)\left\|u_0\right\|_{\mathbb{H}^{r+k-\beta k}(\Omega)}
\end{aligned}
$$

$$+\overline{C}_3\int_0^t \nu^{\alpha-1}\left(t^\alpha-\nu^\alpha\right)^{-k}\left\|u_{m,\alpha,\beta}(\nu)-u^{**}_{m,\alpha}(\nu)\right\|_{\mathbb{H}^r(\Omega))}\mathrm{d}\nu, \tag{6.17}$$

where $\widetilde{C}=C(k,r,\varepsilon,\alpha)$. By the Hölder inequality and in combination with (4.22), we get the estimate of the third term on the right hand side of (6.17) as follows

$$\begin{aligned}
&\left(\int_0^t \nu^{\alpha-1}\left(t^\alpha-\nu^\alpha\right)^{-k}\left\|u_{m,\alpha,\beta}(\nu)-u^{**}_{m,\alpha}(\nu)\right\|_{\mathbb{H}^r(\Omega))}\mathrm{d}\nu\right)^2\\
\le&\Big(\int_0^t \nu^{\alpha-1}\left(t^\alpha-\nu^\alpha\right)^{-2k}d\nu\Big)\Big(\int_0^t \nu^{\alpha-1}\left\|u_{m,\alpha,\beta}(\nu)-u^{**}_{m,\alpha}(\nu)\right\|^2_{\mathbb{H}^r(\Omega))}\mathrm{d}\nu\Big)\\
\le&\frac{t^{\alpha(1-2k)}}{\alpha(1-2k)}\Big(\int_0^t \nu^{\alpha-1}\left\|u_{m,\alpha,\beta}(\nu)-u^{**}_{m,\alpha}(\nu)\right\|^2_{\mathbb{H}^r(\Omega))}\mathrm{d}\nu\Big).
\end{aligned} \tag{6.18}$$

Combining (6.17), (6.18) and the inequality $(a+b+c)^2\le 3a^2+3b^2+3c^2$, we derive that

$$\begin{aligned}
\left\|u_{m,\alpha,\beta}(t)-u^{**}_{m,\alpha}(t)\right\|^2_{\mathbb{H}^r(\Omega)}\le&3|\widetilde{C}|^2t^{-2\alpha k}\Big|\mathbf{D}_\beta(\varepsilon,k)\Big|^2\left\|u_0\right\|^2_{\mathbb{H}^{r+k-\beta k+\varepsilon}(\Omega)}\\
&+3|\overline{C}_3|^2\frac{t^{2\alpha-2b-2\alpha k}}{\alpha(1-2k)(\alpha-2b)}\Big|\mathbf{E}_\beta(r,s,k)\Big|^2\left\|u_0\right\|^2_{\mathbb{H}^{r+k-\beta k}(\Omega)}\\
&+|\overline{C}_3|^2\frac{t^{\alpha(1-2k)}}{\alpha(1-2k)}\int_0^t \nu^{\alpha-1}\left\|u_{m,\alpha,\beta}(\nu)-u^{**}_{m,\alpha}(\nu)\right\|^2_{\mathbb{H}^r(\Omega))}\mathrm{d}\nu.
\end{aligned} \tag{6.19}$$

Multiplying both sides of (6.19) by $t^{2\alpha k}$, we get the following estimate

$$\begin{aligned}
t^{2\alpha k}\left\|u_{m,\alpha,\beta}(t)-u^{**}_{m,\alpha}(t)\right\|^2_{\mathbb{H}^r(\Omega)}\le&3|\widetilde{C}|^2\Big|\mathbf{D}_\beta(\varepsilon,k)\Big|^2\left\|u_0\right\|^2_{\mathbb{H}^{r+k-\beta k+\varepsilon}(\Omega)}\\
&+3|\overline{C}_3|^2\frac{t^{2\alpha-2b}}{\alpha(1-2k)(\alpha-2b)}\Big|\mathbf{E}_\beta(r,s,k)\Big|^2\left\|u_0\right\|^2_{\mathbb{H}^{r+k-\beta k}(\Omega)}\\
&+|\overline{C}_3|^2\frac{t^{\alpha}}{\alpha(1-2k)}\int_0^t \nu^{\alpha-1}\left\|u_{m,\alpha,\beta}(\nu)-u^{**}_{m,\alpha}(\nu)\right\|^2_{\mathbb{H}^r(\Omega))}\mathrm{d}\nu,
\end{aligned}$$

which implies that

$$\begin{aligned}
t^{2\alpha k}\left\|u_{m,\alpha,\beta}(t)-u^{**}_{m,\alpha}(t)\right\|^2_{\mathbb{H}^r(\Omega)}\le&3|\widetilde{C}|^2\Big|\mathbf{D}_\beta(\varepsilon,k)\Big|^2\left\|u_0\right\|^2_{\mathbb{H}^{r+k-\beta k+\varepsilon}(\Omega)}\\
&+\frac{3|\overline{C}_3|^2T^{2\alpha-2b}}{\alpha(1-2k)(\alpha-2b)}\Big|\mathbf{E}_\beta(r,s,k)\Big|^2\left\|u_0\right\|^2_{\mathbb{H}^{r+k-\beta k}(\Omega)}\\
&+\frac{|\overline{C}_3|^2T^{\alpha}}{\alpha(1-2k)}\int_0^t \nu^{\alpha-1-2\alpha k}\nu^{2\alpha k}\left\|u_{m,\alpha,\beta}(\nu)-u^{**}_{m,\alpha}(\nu)\right\|^2_{\mathbb{H}^r(\Omega))}\mathrm{d}\nu.
\end{aligned} \tag{6.20}$$

Looking Lemma 5.1 and (6.20), we set $v(t)=t^{2\alpha k}\left\|u_{m,\alpha,\beta}(t)-u^{**}_{m,\alpha}(t)\right\|^2_{\mathbb{H}^r(\Omega)}$,

$$\overline{A}=3|\widetilde{C}|^2\Big|\mathbf{D}_\beta(\varepsilon,k)\Big|^2\left\|u_0\right\|^2_{\mathbb{H}^{r+k-\beta k+\varepsilon}(\Omega)}+\frac{3|\overline{C}_3|^2T^{2\alpha-2b}}{\alpha(1-2k)(\alpha-2b)}\Big|\mathbf{E}_\beta(r,s,k)\Big|^2\left\|u_0\right\|^2_{\mathbb{H}^{r+k-\beta k}(\Omega)}$$

and

$$\overline{B} = \frac{|\overline{C}_3|^2 T^\alpha}{\alpha(1-2k)}, \;\; \beta' = 1, \;\; \gamma' = \alpha - 2\alpha k.$$

By applying Lemma 5.1, we obtain that

$$t^{2\alpha k}\left\|u_{m,\alpha,\beta}(t) - u^{**}_{m,\alpha}(t)\right\|^2_{\mathbb{H}^r(\Omega)} \le \overline{A}E_{1,\alpha-2\alpha k}\left(\overline{B}(\Gamma(\beta'))^{\frac{1}{\beta'+\gamma'-1}}t\right) = \overline{A}E_{1,\alpha-2\alpha k}\left(\overline{B}t\right). \tag{6.21}$$

In view of Lemma 3.1 as in [30], we obtain the following upper bound

$$E_{1,\alpha-2\alpha k}\left(\overline{B}t\right) \le \mathbb{C}_\alpha, \tag{6.22}$$

where $\mathbb{C}_\alpha$ is a positive constant that depends on α. By (6.21) and (6.22), we get

$$t^{\alpha k}\left\|u_{m,\alpha,\beta}(t) - u^{**}_{m,\alpha}(t)\right\|_{\mathbb{H}^r(\Omega)} \le \overline{A}\mathbb{C}_\alpha.$$

From Deinition 2.3, we find

$$\left\|u_{m,\alpha,\beta} - u^{**}_{m,\alpha}\right\|_{\mathbb{X}_{\alpha k}(0,T;\mathbb{H}^r(\Omega))} \le \overline{A}\mathbb{C}_\alpha.$$

The proof is completed. □

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Conflict of interest

The authors declare there is no conflict of interest.

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