Mathematical Biosciences and Engineering

http://www.aimspress.com/journal/MBE

MBE, 18(6): 8462–8498.
DOI: 10.3934/mbe.2021419
Received: 02 August 2021
Accepted: 08 September 2021
Published: 27 September 2021

Research article

Finite-time stability and optimal control of an impulsive stochastic reaction-diffusion vegetation-water system driven by Lévy process with time-varying delay

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Abstract: In this paper, a reaction-diffusion vegetation-water system with time-varying delay, impulse and Lévy jump is proposed. The existence and uniqueness of the positive solution are proved. Meanwhile, mainly through the principle of comparison, we obtain the sufficient conditions for finite-time stability which reflect the effect of time delay, diffusion, impulse, and noise. Besides, considering the planting, irrigation and other measures, we introduce control variable into the vegetation-water system. In order to save the costs of strategies, the optimal control is analyzed by using the minimum principle. Finally, numerical simulations are shown to illustrate the effectiveness of our theoretical results.

Keywords: vegetation-water model; optimal control; finite-time stability; Lévy process

1. Introduction

Vegetation and water resources are important components of the ecosystem. In arid regions, the lack of water resources and the destruction of vegetation lead to desertification. If desertification is formed, it will seriously affect human survival and economic development. From [1], worldwide, economic losses caused by desertification are more than 40 billion dollars every year. On the other hand, in the rainforest ecosystem, abundant vegetation and water resources provide sufficient oxygen for the survival of life on earth. If the rainforest ecosystem is destroyed, life on earth will inevitably suffer the disaster. Therefore, it is of great significance to model a reasonable dynamic system and analyze its dynamic behavior. Meanwhile, studying the optimal control strategy is helpful for the reasonable and effective protection of vegetation ecology.

In the natural environment, the vegetation-water systems are usually disturbed by human activities and natural disasters, such as planting vegetation, irrigation, pruning vegetation regularly, and so on. These phenomena can be more accurately described by impulsive differential equations. Therefore, in recent years, some results were proposed on modelling impulsive vegetation systems [2–5]. In these references, only some impulse events that reduce the biomass density of vegetation were considered, such as forest fires. In vegetation restoration and protection, we mainly adopt measures, including planting and irrigation, etc. Obviously, these behaviors can increase the density of vegetation and water. However, the corresponding impulsive vegetation-water systems are rarely analyzed.

On the other hand, in ecosystem, delay is also a ubiquitous phenomenon that may cause a dramatic changes on dynamic behavior [6–8]. In recent years, delay has been taken into consideration in research on vegetation systems [9, 10, 18]. In [9], Han et al. took constant delay into the vegetation water system and studied the dynamic behavior. In [10], Wang et al. analyzed the asymptotic stability of the equilibrium and Hopf bifurcation in a constant delay vegetation ecosystem. In [11], the authors considered the delay into vegetation-water system and studied the stability and Hopf bifurcation. However, the papers mentioned all consider constant delay. In fact, in real ecology, the delay can be affected by various factors such as temperature, soil moisture content and so on. Therefore, the delay of penetration is related to time. In this paper, we consider the time-varying delay into the vegetation-water system.

In addition to impulse and delay, there are many achievements evidence that noise also plays a major role in vegetation systems [12, 13]. In the real world, it is known to all that there are various environmental factors (such as organic matter, climate and so on) that can affect the ecosystem, which is manifested by fluctuating ecological material density. Recent research results support the importance of stochastic processes in ecosystems [14–16]. For example, Pan et al. [17] studied the near-optimal control of a stochastic vegetation-water system. Zeng et al. [18] analyzed the catastrophic regime shifts of a stochastic grazing ecosystem to explore the impact of noise on vegetation degradation. However, the stochastic process they mainly consider is Gaussian white noise in the system. The Gaussian noise is suitable to simulate non-abrupt and uniform environmental disturbances such as small-scale rainfall, temperature change, etc. It is worth noting that the phenomenon of large disturbance exists in nature, such as volcanic eruptions and earthquakes [19]. Meanwhile, there is evidence that the transition from forest to drought will not be smooth but will exhibit sudden transitions. For example, in [20, 21], a large-scale, long-term experiment showed that the mortality of vegetation will increase abruptly to 226 and 462 percent in the dry season. These sudden changes may have a profound impact on the natural ecosystems and cannot be ignored [22]. The scholars have done some researches and shown that for abrupt random pulsing phenomenons can be described by the Lévy process [23, 24]. There are several existing works on the impact of the Lévy process on ecosystems. For instance, Zhang et al. [25] considered the Lévy process into the grazing ecosystem and analyzed its impact on system dynamics. Larissa et al. [26] introduced Lévy process to model the Amazon vegetation ecosystem and analyzed metastability of system. However, there been no research that introduced Lévy process into vegetation-water system to analyze dynamic behavior.

In the last several years, the dynamic behaviors of vegetation system were extensively investigated. For example, R. Lefever and O.Lejeune [30] introduced a single-equation (vegetation biomass density) system and studied the bifurcation theory and the stability of the steady-state solution. Klausmeier et al. established a vegetation-water (soil water) system and explored the Turing instability of the system [31]. Rietkerk et al. proposed a vegetation-water (soil water and surface water) system and analyzed the stability of steady-state solution [32]. Obviously, they mainly paid attention to long-term dynamic behaviors. Noteworthy, finite-time stability plays a significant role in modeling real-life problems and arises in a wide range of applications, such as economic-controlled system, neural networks and so on [33–37]. In arid ecosystems, the density of vegetation and water is closely related to eco-quality. Low-level vegetation and water density means desertification. Meanwhile, because the environmental capacity is limited, the high density of vegetation and water will also harm the ecological environment. Therefore, it is of significance to study the finite-time stability of vegetation-water system. On the other hand, as is known to all that controlling drought land and rainforest degradation have posed a huge economic burden. Because of the large affected area, it is costly to use control strategy, such as planting vegetation, rational irrigation, etc. Therefore, from the perspective of ecological economics, how to formulate optimal control strategies to balance the costs and benefits is an important and meaningful question. However, there are few papers introducing control strategies to study optimal control problems in the vegetation system.

In this paper, we propose a new vegetation-water system and analyze finite-time stability by using comparative principles. Then, we introduce the control variables into the system and analyze the optimal control of the controlled vegetation system by using the minimum principle. In summary, our main contributions are as follows:

(i) We propose an impulsive stochastic reaction-diffusion vegetation-water system driven by Lévy process with time-varying delay. Our model is an extension of literature [2, 9, 32].

(ii) The sufficient conditions for finite-time stability are given as theoretical results which reflect the effects of diffusion, impulse, delay, and noise disturbance. Compared with existing work, in the analysis of finite-time stability, our contribution is the study of system with time-varying delay and Lévy noise. In order to deal with time-varying delay, we use the idea of classification.

(iii) The control strategies are considered into the impulsive stochastic vegetation-water system with delay, such as planting vegetation, irrigation, applying chemicals etc. Then, the explicit expression of optimal control is obtained through the minimum principle.

The remaining structure of the paper is organized as follows: in section 2, a stochastic diffusion vegetation-water system, with varying-time delay, impulse, and Lévy jump is established. In section 3, we complete the proof of the existence and uniqueness of the global positive solution. Further, we analyze the finite-time stability of the system and give sufficient conditions for the establishment of stability theorem. In section 4, we analyze the optimal control problem by using the minimum principle under the vegetation-water system with control. In section 5, a numerical simulation is presented to illustrate theoretical results. In section 6, we discuss and summarize the main results of this paper.

2. Model formulation and preparations

2.1. Model formulation

In this section, a vegetation-water system with spatial diffusion, time-varying delay, impulse, noise is proposed. Before driving our system, let us recall a classic vegetation-water system proposed by

Rietkerk in [32]

$$\begin{cases} \dfrac{\partial \bar{u}(x,t)}{\partial t} = d_{\bar{u}}\Delta\bar{u}(x,t) + \dfrac{cg_m\bar{v}(x,t)}{\bar{v}(x,t)+k_1}\bar{u}(x,t) - d\bar{u}(x,t), \\ \dfrac{\partial \bar{v}(x,t)}{\partial t} = d_{\bar{v}}\Delta\bar{v}(x,t) + k_0\dfrac{(\bar{u}(x,t)+k_2 f)}{\bar{u}(x,t)+k_2}\bar{w}(x,t) - \dfrac{g_m\bar{v}(x,t)}{\bar{v}(x,t)+k_1}\bar{u}(x,t) - b\bar{v}(x,t), \\ \dfrac{\partial \bar{w}(x,t)}{\partial t} = d_{\bar{w}}\Delta\bar{w}(x,t) + R_o - k_0\dfrac{(\bar{u}(x,t)+k_2 f)}{\bar{u}(x,t)+k_2}\bar{w}(x,t), \end{cases} \tag{2.1}$$

here $\bar{u}(x,t)$, $\bar{v}(x,t)$, $\bar{w}(x,t)$ represent the vegetation biomass density, soil water density and surface water density, respectively. Δ is the Laplace operator. The $\partial\Gamma$ is the boundary of $\Gamma \in R^2$. All parameters in model (2.1) are assumed non-negative constants and are described in Table 1. In the following, we complete the construction of the new vegetation system.

Table 1. Parameters description.

Symbol	Physical significance	Units
$\bar{u}$	Plant density	g/m^2
$\bar{v}$	Soil water	mm
$\bar{w}$	Surface water	mm
$d_{\bar{u}}$	Plant dispersal	m^2/d
$d_{\bar{v}}$	Diffusion coefficient for soil water	m^2/d
$d_{\bar{w}}$	Diffusion coefficient for surface water	m^2/d
c	Conversion of water uptake by plants to plant growth	$g \cdot mm^{-1} \cdot m^{-2}$
g_m	Maximum specific water uptake	$mm \cdot g^{-1} \cdot m^2 \cdot d^{-1}$
d	Natural loss rate of plant density due to mortality	d^{-1}
k_1	Half saturation constant of plant growth and water uptake	mm
k_2	Rate at which infiltration increases with specific plant density	g/m^2
b	Natural loss rate of soil water due to drainage	d^{-1}
p	Natural loss rate of surface water water due to evaporation	d^{-1}
R_o	Rainfall	mm/d
f	Minimum water infiltration in the absence of plants	$\cdots$
k_0	Proportion of surface water available for infiltration	d^{-1}
y	Perturbation of Poisson process to loss rate	d^{-1}
σ_i $(i = 1, 2, 3)$	Perturbation of random Brownian motion to loss rate	d^{-1}
ρ_i $(i = 1, 2, 3)$	Intensity of the Lévy process	$\cdots$
I_{ku}	Intensity of the impulse applied to the vegetation	$\cdots$
I_{kv}	Intensity of the impulse applied to the soil water	$\cdots$
I_{kw}	Intensity of the impulse applied to the surface water	$\cdots$

(1) Surface water evaporation

In the real world, it is ubiquity for surface water (mainly refers to rivers) to evaporate under the influence of some factors such as temperature, wind, etc. In arid regions, the problems of low rainfall and high evaporation are widespread. The evaporation of surface water can hinder the supply of soil water and further affects the growth of plants. Therefore, they may be the cause of ecological degradation. For example, in Yinchuan, China, the annual evaporation reaches 2000 mm, but the rainfall is only 200-300 mm and the desertification situation here is serious [38]. For this phenomenon, we take the loss rate of surface water into account in vegetation-water system. The system (2.1) can be transformed to

$$\begin{cases} \dfrac{\partial \bar{u}(x,t)}{\partial t} = d_{\bar{u}}\Delta\bar{u}(x,t) + \dfrac{cg_m\bar{v}(x,t)}{\bar{v}(x,t)+k_1}\bar{u}(x,t) - d\bar{u}(x,t), \\ \dfrac{\partial \bar{v}(x,t)}{\partial t} = d_{\bar{v}}\Delta\bar{v}(x,t) + k_0\dfrac{(\bar{u}(x,t)+k_2 f)}{\bar{u}(x,t)+k_2}\bar{w}(x,t) - \dfrac{g_m\bar{v}(x,t)}{\bar{u}(x,t)+k_1}\bar{v}(x,t) - b\bar{v}(x,t), \\ \dfrac{\partial \bar{w}(x,t)}{\partial t} = d_{\bar{w}}\Delta\bar{w}(x,t) + R_o - k_0\dfrac{(\bar{u}(x,t)+k_2 f)}{\bar{u}(x,t)+k_2}\bar{w}(x,t) - p\bar{w}(x,t). \end{cases} \tag{2.2}$$

(2) Time-varying delay

The transfer of surface water to soil water is considered as a time delay process. Meanwhile, because the infiltration rate of surface water is affected by the water content of soil, we take time-varying delay into system (2.2). In Figure 1, we show the time delay from surface water to soil water. Thereby, in infiltration item of system (2.2), we replace $\bar{w}(t)$ with $\bar{w}(t-\tau(t))$ and get the following system

$$\begin{cases} \dfrac{\partial \bar{u}(x,t)}{\partial t} = d_{\bar{u}}\Delta\bar{u}(x,t) + \dfrac{cg_m\bar{v}(x,t)}{\bar{v}(x,t)+k_1}\bar{u}(x,t) - d\bar{u}(x,t), \\ \dfrac{\partial \bar{v}(x,t)}{\partial t} = d_{\bar{v}}\Delta\bar{v}(x,t) + k_0\dfrac{(\bar{u}(x,t)+k_2 f)}{\bar{u}(x,t)+k_2}\bar{w}(x,t-\tau(t)) - \dfrac{g_m\bar{v}(x,t)}{\bar{u}(x,t)+k_1}\bar{v}(x,t) - b\bar{v}(x,t), \\ \dfrac{\partial \bar{w}(x,t)}{\partial t} = d_{\bar{w}}\Delta\bar{w}(x,t) + R_o - k_0\dfrac{(\bar{u}(x,t)+k_2 f)}{\bar{u}(x,t)+k_2}\bar{w}(x,t-\tau(t)) - p\bar{w}(x,t), \end{cases} \tag{2.3}$$

where the $\tau(t)$ is bounded, which implies that there is a constant $\bar{\tau} > 0$, such that $0 < \tau(t) \le \bar{\tau}$. Besides, we assume that $0 \le \dot{\tau}(t) \le \eta < 1$. In fact, the hypothesis about $\dot{\tau(t)} \ge 0$ fits the real situation. Because the time required for surface water to penetrate will increase with time. And when there is enough soil water, the time required surface water infiltration will tend to a fixed value $\bar{\tau}$.

(3) Impulse phenomenon

Impulsive phenomena are very common in vegetation ecosystem. For example, human behavior such as planting and felling vegetation, irrigation and so on can be described by impulse differential equations. In this subsection, we introduce the impulse into the vegetation system. The details are as follows:

(i) We define I_{ku} as the impulse intensity that affects vegetation biomass density. It is worth noting that the planting trees, planting grass and other events correspond to $I_{ku} > 0$ and felling plants correspond to $I_{ku} < 0$. However, based on practical factors, vegetation can not be completely destroyed by impulse events. Meanwhile, the impulse intensity can not be too large. We have reason to assume that $-1 < I_{ku} \le I_{mu}$, where I_{mu} is the maximum allowable impulse on vegetation.

(ii) We define I_{kv}, I_{kw} as the impulse intensities that affects soil water density and surface water density, respectively. Irrigation, rainfall and other events correspond to $I_{kv} > 0$, $I_{kw} > 0$ and industrial water, drainage and other events correspond to $I_{kv} < 0$, $I_{kw} < 0$. However, from reality, soil water and surface water never thoroughly disappear due to impulse events and the impulse intensity can not be

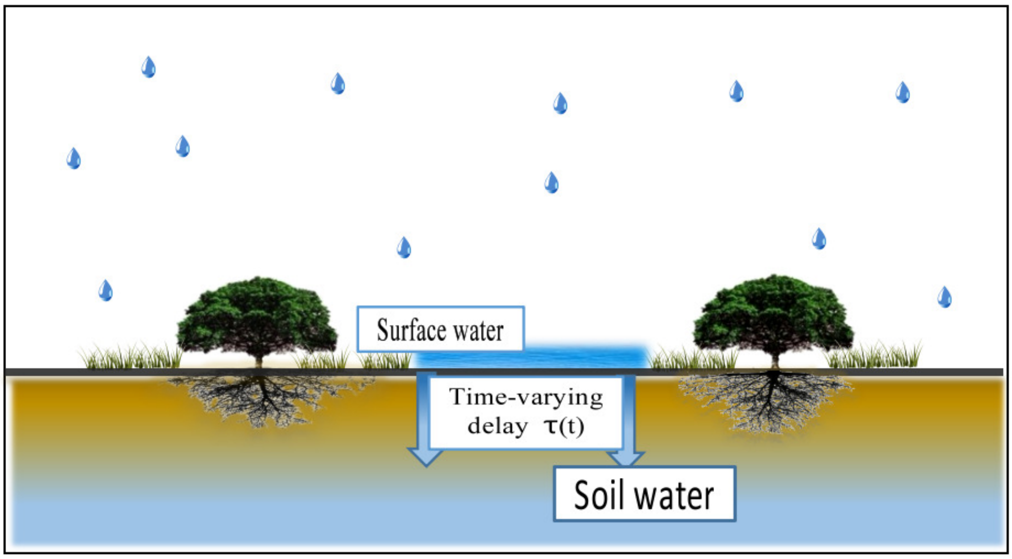


Figure 1. The time delay between surface water and soil water.

too large, which means that $-1 < I_{kv} \leq I_{mv}$, $-1 < I_{kw} \leq I_{mw}$, where I_{mv} and I_{mw} are the maximum allowable impulse on soil water and surface water, respectively.

Therefore, the system (2.3) rewrites as

$$\begin{cases} \left.\begin{aligned} &\frac{\partial \bar{u}(x,t)}{\partial t} = d_{\bar{u}}\Delta\bar{u}(x,t) + \frac{cg_m\bar{v}(x,t)}{\bar{v}(x,t)+k_1}\bar{u}(x,t) - d\bar{u}(x,t),\\ &\frac{\partial \bar{v}(x,t)}{\partial t} = d_{\bar{v}}\Delta\bar{v}(x,t) + k_0\frac{(\bar{u}(x,t)+k_2 f)}{\bar{u}(x,t)+k_2}\bar{w}(x,t-\tau(t)) - \frac{g_m\bar{v}(x,t)}{\bar{v}(x,t)+k_1}\bar{u}(x,t)\\ &\qquad -b\bar{v}(x,t),\\ &\frac{\partial \bar{w}(x,t)}{\partial t} = d_{\bar{w}}\Delta\bar{w}(x,t) + R_o - k_0\frac{(\bar{u}(x,t)+k_2 f)}{\bar{u}(x,t)+k_2}\bar{w}(x,t-\tau(t)) - p\bar{w}(x,t), \end{aligned}\right\} & t\neq t_k,\ (k\in N),\ t>0,\ x\in\Gamma,\\ \left.\begin{aligned} &\bar{u}(x,t_k^+) = (1+I_{ku})\bar{u}(x,t_k),\\ &\bar{v}(x,t_k^+) = (1+I_{kv})\bar{v}(x,t_k),\\ &\bar{w}(x,t_k^+) = (1+I_{kw})\bar{w}(x,t_k), \end{aligned}\right\} t = t_k\ (k\in N). \end{cases} \tag{2.4}$$

where $\{t_k\}$ $(k \in N)$ is impulsive sequence satisfies $0 = t_0 < t_1 < t_2 < \cdots < t_k < \cdots < t_\infty = \infty$, $\vartheta(x,t_k^+) = \lim_{t\to t_k^+}\vartheta(x,t)$ $(\vartheta = u, v, w)$. We define $d_m = \max_{k\in N}\{t_k - t_{k-1}\}$, $d_s = \min_{k\in N}\{t_k - t_{k-1}\}$. $x \in \Gamma \subset R^2$ is a bounded measurable set which means that there are constants $b_i > 0$, such that $|x_i| \leq b_i$, where x_i $(i = 1, 2)$ are components of spatial variables x.

(4) Lévy processes

In the real world, there are physical environmental disturbances such as volcanic eruptions, sudden sandstorms, temperature surges and so on, and biological environmental disturbances such as mass migration of herbivores. It can affect the natural loss rate of species, can be modeled by the Lévy noise. Therefore, we let

$$d \to d + \rho_1 dL_1(t),\ b \to b + \rho_2 dL_2(t),\ p \to p + \rho_3 dL_3(t),$$

where $L_i(t)$ is Lévy process which is composed of a Brownian motion with a linear drift term and a superposition of centered (independent) Poisson processes with different jump sizes $\bar{y} \in \mathbb{Y}$. It follows from the Lévy-Itô decomposition theorem that

$$dL_i(t) = \bar{a}_i dt + \bar{\sigma}_i dB_i(t) + \int_{\mathbb{Y}} \bar{y}\tilde{N}(dt, d\bar{y})\ (i = 1, 2, 3),$$

where $\bar{a}_i\ (day^{-1}) \in R$, $\bar{\sigma}_i\ (day^{-1}) \geq 0$, $B_i(t)$ is standard Brownian motion, $\tilde{N}(dt, d\bar{y}) = N(dt, d\bar{y}) - \lambda(d\bar{y})dt$ is a compensated Poisson process and $N(dt, d\bar{y})$ is a poisson counting measure with characteristic measure λ on a measurable subset $\mathbb{Y} \in (0, \infty)$ with $\lambda(\mathbb{Y}) < \infty$. Thus, the model becomes

$$\begin{cases} \left.\begin{aligned} \frac{\partial \bar{u}(x,t)}{\partial t} &= (d_{\bar{u}}\Delta\bar{u}(x,t) + \frac{cg_m\bar{v}(x,t)}{\bar{v}(x,t)+k_1}\bar{u}(x,t) - \bar{l}_1\bar{u}(x,t))dt - \rho_1\bar{\sigma}_1\bar{u}(x,t)dB_1(t) \\ &\quad - \rho_1\bar{u}(x,t)\int_{\mathbb{Y}}\bar{y}\tilde{N}(dt,d\bar{y}), \\ \frac{\partial \bar{v}(x,t)}{\partial t} &= d_{\bar{v}}\Delta\bar{v}(x,t) + k_0\frac{(\bar{u}(x,t)+k_2 f)}{\bar{u}(x,t)+k_2}\bar{w}(x,t-\tau(t)) - \frac{g_m\bar{v}(x,t)}{\bar{v}(x,t)+k_1}\bar{u}(x,t) \\ &\quad - \bar{l}_2\bar{v}(x,t))dt - \rho_2\bar{\sigma}_2\bar{v}(x,t)dB_2(t) - \rho_2\bar{v}(x,t)\int_{\mathbb{Y}}\bar{y}\tilde{N}(dt,d\bar{y}), \\ \frac{\partial \bar{w}(x,t)}{\partial t} &= d_{\bar{w}}\Delta\bar{w}(x,t) + R_o - k_0\frac{(\bar{u}(x,t)+k_2 f)}{\bar{u}(x,t)+k_2}\bar{w}(x,t-\tau(t)) - \bar{l}_3\bar{w}(x,t))dt \\ &\quad - \rho_3\bar{\sigma}_3\bar{w}(x,t)dB_3(t) - \rho_3\bar{w}(x,t)\int_{\mathbb{Y}}\bar{y}\tilde{N}(dt,d\bar{y}), \end{aligned}\right\} \begin{aligned} &t \neq t_k, \\ &(k \in N), \\ &t > 0, \\ &x \in \Gamma, \end{aligned} \\ \left.\begin{aligned} \bar{u}(x,t_k^+) &= (1+I_{ku})\bar{u}(x,t_k), \\ \bar{v}(x,t_k^+) &= (1+I_{kv})\bar{v}(x,t_k), \\ \bar{w}(x,t_k^+) &= (1+I_{kw})\bar{w}(x,t_k), \end{aligned}\right\} t = t_k\ (k \in N). \end{cases} \tag{2.5}$$

where $\bar{l}_1 = d + \rho_1\bar{a}_1$, $\bar{l}_2 = b + \rho_2\bar{a}_2$, $\bar{l}_3 = p + \rho_3\bar{a}_3$. Besides, we assume that $B_i(t)$ is in dependent of $N(t, d\bar{y})$. The initial value and boundary condition of system (2.5) are given as follows

$$\vartheta(x, s) = \psi_\vartheta(x, s)\ (\vartheta = u, v, w),\ x \in \Gamma,\ s \in (-\bar{\tau}, 0],$$

$$\frac{\partial\vartheta(x,t)}{\partial n} = (\frac{\partial\vartheta(x,t)}{\partial x_1}, \frac{\partial\vartheta(x,t)}{\partial x_2}) = 0\ (\vartheta = u, v, w),\ x \in \partial\Gamma,\ t > 0,$$

where n is the out normal vector of $\partial\Gamma$; $\psi_\vartheta(x, s)$ $(\vartheta = u,\ v,\ w)$ are bounded and continuous functions on $(-\bar{\tau}, 0] \times \Gamma$.

2.2. Preparations

In order to facilitate the subsequent theoretical analysis, we implement the dimensionless processing for the system (2.5) using the method of Zelnik et.al. [12]. Therefore, we obtain the following nondimensional vegetation-water system with time delay and impulse

$$\begin{cases} \left.\begin{aligned} du(x,t) &= (d_u\Delta u(x,t) + \frac{v(x,t)}{v(x,t)+1}u(x,t) - l_1u(x,t))dt - \rho_1\sigma_1 u(x,t)dB_1(t) \\ &\quad - \rho_1 u(x,t)\int_{\mathbb{Y}} y\tilde{N}(dt,dy), \\ dv(x,t) &= (d_v\Delta v(x,t) + \alpha\tfrac{u(x,t)+f}{u(x,t)+1}w(x,t-\tau(t)) - \gamma\tfrac{v(x,t)}{v(x,t)+1}u(x,t) \\ &\quad - l_2v(x,t))dt - \rho_2\sigma_2 v(x,t)dB_2(t) - \rho_2 v(x,t)\int_{\mathbb{Y}} y\tilde{N}(dt,dy), \\ dw(x,t) &= (d_w\Delta w(x,t) + R - \alpha\tfrac{u(x,t)+f}{u(x,t)+1}w(x,t-\tau(t)) - l_3w(x,t))dt \\ &\quad - \rho_3\sigma_3 w(x,t)dB_3(t) - \rho_3 w(x,t)\int_{\mathbb{Y}} y\tilde{N}(dt,dy), \end{aligned}\right\} \begin{array}{l} t\neq t_k, \\ (k\in N), \\ t>0, \\ x\in\Gamma, \end{array} \\ \left.\begin{aligned} u(x,t_k^+) &= (1+I_{ku})u(x,t_k), \\ v(x,t_k^+) &= (1+I_{kv})v(x,t_k), \\ w(x,t_k^+) &= (1+I_{kw})w(x,t_k), \end{aligned}\right\} t = t_k\ (k\in N), \end{cases} \tag{2.6}$$

where $u = \frac{\bar{u}}{k_2}$, $v = \frac{\bar{v}}{k_1}$, $w = \frac{k_0\bar{w}}{cg_mk_1}$, $d_u = \frac{k_0d_{uo}}{d_{wo}cg_m}$, $d_v = \frac{k_0d_{vo}}{d_{wo}cg_m}$, $d_w = 1$, $l_1 = \frac{\bar{l}_1}{cg_m}$, $\gamma = \frac{k_2}{ck_1}$, $l_2 = \frac{\bar{l}_2}{cg_m}$, $R = \frac{R_o}{cg_mk_1}$, $\alpha = \frac{k_0}{cg_m}$, $l_3 = \frac{\bar{l}_3}{cg_m}$, $f = f$, $\sigma_i = \frac{\bar{\sigma}_i}{cg_m}$ $(i = 1,2,3)$, $\bar{y} = \frac{\bar{y}}{cg_m}$, $t = cg_mt_{originnal}$, $x = \sqrt{\frac{d_{wo}}{k_0}}x_{originnal}$. The $t_{originnal}$ and $x_{originnal}$ are the time and space variables before the dimensionless transformation processing.

Let $X = \{(u,v,w) \in W^{2,2}, \frac{\partial(u,v,w)}{\partial n} = 0 \ on\ \partial\Omega\}$. Define C_+^b as a family of bounded and continuous functions. $\mathbb{M}_+ = L^2(\Gamma\times[0,\infty),\mathbb{R}_+^3)$ represents the set of square integrable functions defined on $\Gamma\times[0,\infty)$, which is equipped with the norm $\|\cdot\|$, where $\|y(x,t)\| = (\int_\Gamma y(x,t)y^T(x,t)dx)^{\frac{1}{2}}$. $y(x,t) = (u(x,t), v(x,t), w(x,t))$. Let $(\Omega,\ \mathcal{F},\ (\mathcal{F}_t)_{0\le t\le T},\ \mathcal{P})$ be a complete filtered probability space with a filtration $\{(\mathcal{F}_t)_{0\le t\le T}\}$. E denotes the probability expectation corresponding to $\mathcal{P}$. Additionally, there is a hypothesis that needs to be given.

Assumption 2.1 *There is a positive constant L_i such that $\int_{\mathbb{Y}} \rho_i y(\rho_i y + 2\mathbf{m}(\Gamma) - 2)\lambda(dy)) < L_i < +\infty$ $(i = 1,2,3)$, where $\mathbf{m}(\Gamma)$ is the measure of Γ.*

Remark 2.1 *The assumption 2.1 implies that the intensity of random noise is constrained, which follows the biological background.*

3. Main results

In this section, the positivity, existence and uniqueness of the global solution of system (2.6) is analyzed by a method similar to [39, 40]. Then, we study the finite-time stability of vegetation-water system. In the end, we introduce control variables into the vegetation-water system and study the optimal control of the control system.

3.1. Existence and uniqueness of positive solutions

Theorem 3.1 *For any given initial data $(\psi_u(x,s), \psi_v(x,s), \psi_w(x,s)) \in C_+^b$, there is a unique global positive solution $(u(x,t), v(x,t), w(x,t))$ of system (2.6) on $t \ge 0$ almost surely, which means the solution will remain in $\mathbb{M}_+$ with probability 1.*

The proof of Theorem 3.1 is given in Appendix.

3.2. Finite-time stability

Definition 3.2 *Given positive number T, B_1, B_2 with $B_1 < B_2$, system is said to be finite-time stabile with respect to (T, B_1, B_2), if any $t \in [0, T]$,*

$$\|y(0)\| = \sup_{-\bar{\tau}\leq s\leq 0} \int_{\Gamma} y(x,t)y^T(x,t)dx \leq B_1 \Rightarrow E\|y(t)\| = E\int_{\Gamma} y(x,t)y^T(x,t) \leq B_2,$$

where $y(x,t) = (u(x,t), v(x,t), w(x,t))$.

Remark 3.3 *Definition 3.2 implies that when the initial value of the state variable is within a given limit, it does not exceed the given threshold in a finite time. The image of finite-time stability is displayed in Figure 2.*

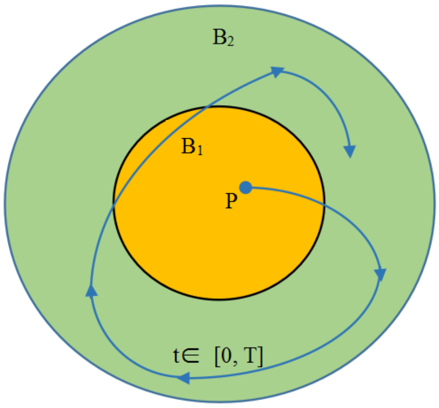


Figure 2. Illustration of finite-time stability. *P* is the initial value of the state variable.

In the following, we give the theorem of finite-time stability of impulsive stochastic reaction-diffusion system with time-varying delay. We present some parallel sufficient conditions of finite-time stability of the system. These conditions reflect the influence of random disturbance and spatial diffusion on finite-time stability. Before proposing the theorem, assign

$$\begin{aligned}
&c_1 = \tfrac{\ln\theta}{d_m} + |K_3|,\ c_2 = \tfrac{\ln\theta}{d_m},\ c_3 = \tfrac{\ln\theta}{d_s} + K_3,\ c_4 = \tfrac{\ln\theta}{d_s},\ \omega = \ln B_2 - \ln(B_1 + |\tfrac{K_2}{K_3}|),\\
&\theta = \max\{(1+I_{ku})^2, (1+I_{kv})^2, (1+I_{kw})^2\},\ \ K_2 = \mathbf{m}(\Gamma),\ K_4 = \alpha(1+f),\\
&K_{3u} = 1 + \rho_1^2\sigma_1^2 + L_1 - l_1 - 2d_u \textstyle\sum_{i=1}^2 b_i^{-2},\ K_{3v} = \alpha(1+f) + \rho_2^2\sigma_2^2 + L_2 - l_2 - 2d_v \textstyle\sum_{i=1}^2 b_i^{-2},\\
&K_{3w} = R^2 + \alpha(1+f) + \rho_3^2\sigma_3^2 + L_3 - l_3 - 2d_w \textstyle\sum_{i=1}^2 b_i^{-2},\ K_3 = \max\{K_{3u}, K_{3v}, K_{3w}\}.
\end{aligned}$$

Theorem 3.4 *The system* (2.6) *is finite-time stable with respect to* (T, B_1, B_2) *if one of the following condition holds:*

C1 : $0 < \theta < 1,\ K_3 \neq 0,\ c_1\theta \leq -K_4 e^{-c_2\bar{\tau}} < 0,\ -\ln\theta \leq \omega,$

C2 : $0 < \theta < 1,\ K_3 \neq 0,\ -K_4 e^{-c_1\bar{\tau}} \leq c_2\theta \leq 0,\ \ (c_1 + \frac{K_4}{\theta(1-\eta)} e^{-c_2\bar{\tau}})T + \frac{K_4}{\theta(1-\eta)} e^{-c_2\bar{\tau}}\bar{\tau} - \ln\theta \leq \omega,$

C3 : $0 < \theta < 1,\ K_3 > 0,\ c_1 > 0,\ (K_4 + c_1)T - \ln\theta \leq \omega,$

C4 : $\theta \geq 1, K_3 > 0,\ c_3 > 0,\ (K_4 + c_3)T \leq \omega,$

C5 : $\theta \geq 1, K_3 < 0,\ c_3 > 0,\ (K_4 + c_4)T \leq \omega,$

C6 : $\theta \geq 1, K_3 < 0,\ c_3 < 0,\ (c_4 + \frac{K_4}{1-\eta} e^{-c_4\bar{\tau}})T + \frac{K_4\bar{\tau}}{1-\eta} e^{-c_4\bar{\tau}} \leq \omega$.

Proof. For a given number B_1, letting $\sup_{-\bar{\tau}\leq s\leq 0} \int_\Gamma u^2(x,s) + v^2(x,s) + w^2(x,s)dx \leq B_1$. Set

$$V(t) = \int_\Gamma u^2(x,t)dx + \int_\Gamma v^2(x,t)dx + \int_\Gamma w^2(x,t)dx.$$

An application of Itô formula yields that

$$\begin{aligned}
&dV(t)\\
&= 2\int_\Gamma d_u u(x,t)\Delta u(x,t) + \frac{v(x,t)}{v(x,t)+1}u(x,t)^2 - l_1 u^2(x,t)dx + 2\int_\Gamma d_v v(x,t)\Delta v(x,t)\\
&\quad + \alpha\frac{u(x,t)+f}{u(x,t)+1}w(x,t-\tau(t))v(x,t) - \gamma\frac{v(x,t)}{v(x,t)+1}u(x,t)v(x,t) - l_2 v^2(x,t)dx\\
&\quad + 2\int_\Gamma d_w w(x,t)\Delta w(x,t) + Rw(x,t) - \alpha\frac{u(x,t)+f}{u(x,t)+1}w(x,t-\tau(t))w(x,t) - l_3 w^2(x,t)dx\\
&\quad + \int_{\mathbb{Y}}\int_\Gamma (1-\rho_1 y)^2 u^2(x,t)dx - \int_\Gamma u^2(x,t)dx + \rho_1 u(x,t)y\int_\Gamma 2u(x,t)dxd\lambda(dy)\\
&\quad + \int_{\mathbb{Y}}\int_\Gamma (1-\rho_2 y)^2 v^2(x,t)dx - \int_\Gamma v^2(x,t)dx + \rho_2 v(x,t)y\int_\Gamma 2v(x,t)dxd\lambda(dy)\\
&\quad + \int_{\mathbb{Y}}\int_\Gamma (1-\rho_3 y)^2 w^2(x,t)dx - \int_\Gamma w^2(x,t)dx + \rho_3 w(x,t)y\int_\Gamma 2w(x,t)dxd\lambda(dy)\\
&\quad - \int_\Gamma \rho_1\sigma_1 u^2(x,t)dB_1(x,t)dx - \int_\Gamma \rho_2\sigma_2 v^2(x,t)dB_2(x,t)dx - \int_\Gamma \rho_3\sigma_3 w^2(x,t)dB_3(x,t)dx\\
&\quad + \int_{\mathbb{Y}}\int_\Gamma (1-\rho_1 y)^2 u^2(x,t)dx - \int_\Gamma u^2(x,t)dx\tilde{N}(dt,dy) + \int_{\mathbb{Y}}\int_\Gamma (1-\rho_2 y)^2 v^2(x,t)dx\\
&\quad - \int_\Gamma v^2(x,t)dx\tilde{N}(dt,dy) + \int_{\mathbb{Y}}\int_\Gamma (1-\rho_3 y)^2 w^2(x,t)dx - \int_\Gamma w^2(x,t)dx\tilde{N}(dt,dy)\\
&\quad + \int_\Gamma \rho_1^2\sigma_1^2 u^2(x,t) + \rho_2^2\sigma_2^2 v^2(x,t) + \rho_3^2\sigma_3^2 w^2(x,t)dx.
\end{aligned}$$

By using some basic inequalities and applying of Green identity (lemma 2, [41]), we have

$$\begin{aligned}
&dV(t)\\
&\leq -2\Big(\sum_{i=1}^m b_i^{-2}\int_\Gamma d_u u^2(x,t)dx + \sum_{i=1}^m b_i^{-2}\int_\Gamma d_v v^2(x,t)dx + \sum_{i=1}^m b_i^{-2}\int_\Gamma d_w w^2(x,t)dx\Big) + \int_\Gamma u^2(x,t)dx\\
&\quad - \int_\Gamma l_1 u^2(x,t)dx + \int_\Gamma \alpha(1+f)(w^2(x,t-\tau(t)) + v^2(x,t))dx - \int_\Gamma l_2 v^2(x,t)dx + \mathbf{m}(\Gamma)\\
&\quad + \int_\Gamma R^2 w^2(x,t)dx + \int_\Gamma \alpha(1+f)(w^2(x,t-\tau(t)) + w^2(x,t))dx - \int_\Gamma l_3 w^2(x,t)dx\\
&\quad + \int_\Gamma\int_{\mathbb{Y}} \rho_1 y(\rho_1 y + 2\mathbf{m}(\Gamma) - 2)d\lambda(dy)u^2(x,t)dx + \int_\Gamma\int_{\mathbb{Y}} \rho_2 y(\rho_2 y + 2\mathbf{m}(\Gamma) - 2)d\lambda(dy)v^2(x,t)dx\\
&\quad + \int_\Gamma\int_{\mathbb{Y}} \rho_3 y(\rho_3 y + 2\mathbf{m}(\Gamma) - 2)d\lambda(dy)w^2(x,t)dx + \int_\Gamma\int_{\mathbb{Y}} \rho_1 y(\rho_1 y - 2)\tilde{N}(dt,dy)u^2(x,t)dx\\
&\quad + \int_\Gamma\int_{\mathbb{Y}} \rho_2 y(\rho_2 y - 2)\tilde{N}(dt,dy)v^2(x,t)dx + \int_\Gamma\int_{\mathbb{Y}} \rho_3 y(\rho_3 y - 2)\tilde{N}(dt,dy)w^2(x,t)dx\\
&\quad - \int_\Gamma \rho_1\sigma_1 u^2(x,t)dB_1(x,t)dx - \int_\Gamma \rho_2\sigma_2 v^2(x,t)dB_2(x,t)dx - \int_\Gamma \rho_3\sigma_3 w^2(x,t)dB_3(x,t)dx\\
&\quad + \int_\Gamma \rho_1^2\sigma_1^2 u^2(x,t) + \rho_2^2\sigma_2^2 v^2(x,t) + \rho_3^2\sigma_3^2 w^2(x,t)dx
\end{aligned} \tag{3.1}$$

$$
\begin{aligned}
&\le K_2 + K_3(\int_\Gamma u^2(x,t)dx + \int_\Gamma v^2(x,t)dx + \int_\Gamma w^2(x,t)dx) + K_4(\int_\Gamma u^2(x,t-\tau(t))dx \\
&\quad + \int_\Gamma v^2(x,t-\tau(t))dx + \int_\Gamma w^2(x,t-\tau(t))dx) + \int_\Gamma\int_{\mathbb{Y}} \rho_1 y(\rho_1 y - 2)\tilde{N}(dt,dy)u^2(x,t)dx \\
&\quad + \int_\Gamma\int_{\mathbb{Y}} \rho_2 y(\rho_2 y - 2)\tilde{N}(dt,dy)v^2(x,t)dx + \int_\Gamma\int_{\mathbb{Y}} \rho_3 y(\rho_3 y - 2)\tilde{N}(dt,dy)w^2(x,t)dx \\
&\quad - \int_\Gamma \rho_1\sigma_1 u^2(x,t)dB_1(x,t)dx - \int_\Gamma \rho_2\sigma_2 v^2(x,t)dB_2(x,t)dx - \int_\Gamma \rho_3\sigma_3 w^2(x,t)dB_3(x,t)dx.
\end{aligned}
$$

where $K_2 = \mathbf{m}(\Gamma)$, $K_4 = \alpha(1+f)$, $K_3 = \max\{1 + \rho_1^2\sigma_1^2 + L_1 - l_1 - 2d_u\sum_{i=1}^2 b_i^{-2}, \alpha(1+f) + \rho_2^2\sigma_2^2 + L_2 - l_2 - 2d_v\sum_{i=1}^2 b_i^{-2}, R^2 + \alpha(1+f) + \rho_3^2\sigma_3^2 + L_3 - l_3 - 2d_w\sum_{i=1}^2 b_i^{-2}\}$.
For $t = t_k$, one can derive that

$$
\begin{aligned}
&V(u(x,t_k^+), v(x,t_k^+), w(x,t_k^+)) \\
&\quad = \int_\Gamma (1+I_{ku})^2 u^2(x,t_k)dx + \int_\Gamma (1+I_{kv})^2 v^2(x,t_k)dx + \int_\Gamma (1+I_{kw})^2 w^2(x,t_k)dx \\
&\quad \le \theta \int_\Gamma V(t_k)dx.
\end{aligned} \tag{3.2}
$$

where $\theta = \max\{(1+I_{ku})^2, (1+I_{kv})^2, (1+I_{kw})^2\}$. Taking expectation on Eq.(3.1), Eq.(3.2), we get

$$dEV(t) \le K_2 + K_3 EV(t) + K_4 EV(t-\tau(t)),\ t \ne t_k\ k \in N^+,$$

$$EV(t_k^+) \le \theta EV(t_k).$$

Next, we choose $b(t)$ satisfies

$$
\begin{cases}
\dot{b}(t) = K_2 + K_3 b(t) + K_4 b(t-\tau(t)) & t \ne t_k, \\
b(t_k^+) = \theta b(t_k) & t = t_k, \\
b(s) = EV(s) & -\bar{\tau} \le s \le 0.
\end{cases} \tag{3.3}
$$

It follows from comparison lemma [42] that

$$EV(t) \le b(t).$$

According to the method of variation of constant on (3.3), we have

$$b(t) = -\theta^{N(t,0)}\frac{K_2}{K_3} + \theta^{N(t,0)}e^{K_3 t}(b(0) + \frac{K_2}{K_3}) + \int_0^t \theta^{N(t,s)}[K_4 b(s-\tau(s))e^{K_3(t-s)}]ds, \tag{3.4}$$

for $t \ge 0$. Noting that

$$\frac{t-s-d_m}{d_m} \le N(t,s) \le \frac{t-s}{d_s}.$$

Therefore, a direct computation gives that

$$
\begin{aligned}
&\exp\{N(t,s)\ln\theta + K_3(t-s)\} \\
&\quad \le \exp\{\frac{t-s-d_m}{d_m}\ln\theta + K_3(t-s)\} \\
&\quad = \exp\{(\frac{\ln\theta}{d_m} + K_3)(t-s) - \ln\theta\}.
\end{aligned}
$$

Hence, based on (3.4), for $t \geq 0$, there is

$$b(t) \leq -\theta^{N(t,0)}\frac{K_2}{K_3} + \frac{e^{(\frac{\ln\theta}{d_m}+K_3)t}}{\theta}(b(0)+\frac{K_2}{K_3}) + \frac{1}{\theta}\int_0^t e^{(\frac{\ln\theta}{d_m}+K_3)(t-s)}[K_4 b(s-\tau(s))]ds. \tag{3.5}$$

In the following, we continue our analysis under two situations.

Situation1 : $0 < \theta < 1$.

Case **C1** $- 1$: $K_3 > 0$.

Take a continuous function $h(\lambda) = K_4 e^{-\lambda\bar{\tau}} - \theta(\lambda - \frac{\ln\theta}{d_m} - K_3)$. We have $h(-\infty) = +\infty$, $h(0) = K_4 + \theta(K_3 + \frac{\ln\theta}{d_m})$. Form **C1**, we can know $\theta(\frac{\ln\theta}{d_m} + K_3) < -K_4 e^{-(\frac{\ln\theta}{d_m}+K_3)\bar{\tau}}$. Therefore, we can easily yield that $h(0) = \theta(\frac{\ln\theta}{d_m} + K_3) + K_4 < 0$. Besides, $\dot{h}(\lambda) = -\bar{\tau}K_4 e^{-\lambda\bar{\tau}} - \theta < 0$. Therefore, there is at least one number $\lambda_1 < 0$ such that $K_4 e^{-\lambda_1\bar{\tau}} = \theta(\lambda_1 - \frac{\ln\theta}{d_m} - K_3)$. It is clearly that

$$b(t) \leq \frac{1}{\theta}(b(0) + \frac{K_2}{K_3})e^{\lambda_1 t}, \quad -\bar{\tau} \leq t < 0.$$

In the following, we prove the inequality

$$b(t) \leq \tfrac{1}{\theta}(b(0) + \tfrac{K_2}{K_3})e^{\lambda_1 t}, \quad t \geq 0. \tag{3.6}$$

If the inequality is not true, there is a t^* such that

$$b(t^*) > \tfrac{1}{\theta}(b(0) + \tfrac{K_2}{K_3})e^{\lambda_1 t^*}, \tag{3.7}$$

and

$$b(t) \leq \frac{1}{\theta}(b(0) + \frac{K_2}{K_3})e^{\lambda_1 t}, \quad t < t^*.$$

However, it follows from (3.4) that

$$\begin{aligned}
b(t^*) &\leq -\theta^{N(t^*,0)}\frac{K_2}{K_3} + \frac{e^{(\frac{\ln\theta}{d_m}+K_3)t^*}}{\theta}(b(0)+\frac{K_2}{K_3}) + \frac{1}{\theta}\int_0^{t^*} e^{(\frac{\ln\theta}{d_m}+K_3)(t^*-s)}[K_4 b(s-\tau(s))]ds \\
&\leq \frac{e^{(\frac{\ln\theta}{d_m}+K_3)t^*}}{\theta}(b(0)+\frac{K_2}{K_3}) + \frac{1}{\theta}\int_0^{t^*} e^{(\frac{\ln\theta}{d_m}+K_3)(t^*-s)}[K_4 b(s-\tau(s))]ds \\
&= \frac{e^{(\frac{\ln\theta}{d_m}+K_3)t^*}}{\theta}\Big(b(0)+\frac{K_2}{K_3} + \int_0^{t^*} e^{-(\frac{\ln\theta}{d_m}+K_3)s}[K_4 b(s-\tau(s))]ds\Big) \\
&\leq \frac{e^{(\frac{\ln\theta}{d_m}+K_3)t^*}}{\theta}\Big(b(0)+\frac{K_2}{K_3} + \int_0^{t^*} e^{-(\frac{\ln\theta}{d_m}+K_3)s}[\frac{K_4}{\theta}(b(o)+\frac{K_2}{K_3})e^{\lambda_1(s-\tau(s))}]ds\Big) \\
&\leq \frac{e^{(\frac{\ln\theta}{d_m}+K_3)t^*}}{\theta}\Big(b(0)+\frac{K_2}{K_3} + \frac{K_4}{\theta}(b(0)+\frac{K_2}{K_3})e^{-\lambda_1\bar{\tau}}\int_0^{t^*} e^{\left(\lambda_1-(\frac{\ln\theta}{d_m}+K_3)\right)s}ds\Big) \\
&\leq \frac{e^{(\frac{\ln\theta}{d_m}+K_3)t^*}}{\theta}\Big(b(0)+\frac{K_2}{K_3} + \frac{K_4(b(o)+\frac{K_2}{K_3})e^{-\lambda_1\bar{\tau}}}{\theta(\lambda_1-(\frac{\ln\theta}{d_m}+K_3))}(e^{\left(\lambda_1-(\frac{\ln\theta}{d_m}+K_3)\right)t^*}-1)\Big) \\
&\leq \frac{1}{\theta}(b(0)+\frac{K_2}{K_3})e^{\lambda_1 t^*}.
\end{aligned} \tag{3.8}$$

It contradicts (3.7), so (3.6) holds. Furthermore, we have

$$EV(t) \leq b(t) \leq \frac{1}{\theta}(b(0)+\frac{K_2}{K_3})e^{\lambda_1 t} \leq \frac{1}{\theta}(B_1 + \frac{K_2}{K_3}).$$

Based on **C1**, we have $-\ln\theta \le \ln B_2 - \ln(B_1 + \frac{K_2}{K_3})$ for $K_3 > 0$. This implies that $EV(t) \le B_2$. It shows the desired result.

Case **C1** $- 2:\ K_3 < 0$.

From (3.4), we have

$$\begin{aligned} b(t) &= -\theta^{N(t,0)}\frac{K_2}{K_3} + \theta^{N(t,0)}e^{K_3 t}(b(0) + \frac{K_2}{K_3}) + \int_0^t \theta^{N(t,s)}[K_4 b(s-\tau(s))e^{K_3(t-s)}]ds \\ &\le \exp\{(\frac{\ln\theta}{d_m})(t-d_m)\}(b(0) - \frac{K_2}{K_3}) + \frac{1}{\theta}\int_0^t e^{(\frac{\ln\theta}{d_m}+K_3)(t-s)}[K_4 b(s-\tau(s))]ds \\ &\le \frac{1}{\theta}e^{\frac{\ln\theta}{d_m}t}(b(0) - \frac{K_2}{K_3}) + \frac{1}{\theta}\int_0^t e^{(\frac{\ln\theta}{d_m})(t-s)}[K_4 b(s-\tau(s))]ds. \end{aligned} \tag{3.9}$$

Based on **C1**, we can obtain $\theta(\frac{\ln\theta}{d_m}) < -K_4 e^{-(\frac{\ln\theta}{d_m})\bar{\tau}} < 0$. By the same discussion as in **C1** $- 1$, we obtain that

$$b(t) \le \frac{1}{\theta}(b(0) - \frac{K_2}{K_3})e^{\lambda_2 t}, \quad t \ge 0, \tag{3.10}$$

where λ_2 is the root of equation $h(\lambda) = K_4 e^{-\lambda\bar{\tau}} - \theta(\lambda - \frac{\ln\theta}{d_m})$ and $\lambda_2 < 0$. **C1** implies that $-\ln\theta \le \ln B_2 - \ln(B_1 - \frac{K_2}{K_3})$ for $K_3 < 0$. This means that

$$EV(t) \le b(t) \le \frac{1}{\theta}(b(0) - \frac{K_2}{K_3}) \le B_2.$$

It is the desired result.

Case **C2** $- 1:\ K_3 > 0$.

Contrary to **C1**-1, we consider $-K_4 e^{-(\frac{\ln\theta}{d_m}+K_3)\bar{\tau}} \le \theta(\frac{\ln\theta}{d_m} + K_3) \le 0$. Assign

$$q_1(t) = b(t)e^{-(\frac{\ln\theta}{d_m}+K_3)t} > 0.$$

Form (3.5), $0 \le \eta < 1$ and Gronwall inequality [43], one has

$$\begin{aligned} q_1(t) &\le \frac{1}{\theta}(b(0) + \frac{K_2}{K_3}) + \frac{1}{\theta}\int_0^t e^{-(\frac{\ln\theta}{d_m}+K_3)(s-\tau(s))}e^{-(\frac{\ln\theta}{d_m}+K_3)\tau(s)}[K_4 b(s-\tau(s))]ds \\ &\le \frac{1}{\theta}(b(0) + \frac{K_2}{K_3}) + \frac{1}{\theta}e^{-(\frac{\ln\theta}{d_m}+K_3)\bar{\tau}}\int_0^t e^{-(\frac{\ln\theta}{d_m}+K_3)(s-\tau(s))}[K_4 b(s-\tau(s))]ds \\ &\le \frac{1}{\theta}(b(0) + \frac{K_2}{K_3}) + \frac{1}{\theta(1-\eta)}e^{-(\frac{\ln\theta}{d_m}+K_3)\bar{\tau}}\int_{-\bar{\tau}}^t [K_4 q_1(s)]ds \\ &\le \tfrac{1}{\theta}(b(0) + \tfrac{K_2}{K_3})\exp\Big\{\tfrac{K_4}{\theta(1-\eta)}e^{-(\frac{\ln\theta}{d_m}+K_3)\bar{\tau}}(t+\bar{\tau})\Big\}. \end{aligned}$$

Then, there is

$$\begin{aligned} EV(t) \le b(t) &= q_1(t)e^{(\frac{\ln\theta}{d_m}+K_3)t} \\ &\le \frac{1}{\theta}(B_1 + \frac{K_2}{K_3})\exp\Big\{(\frac{\ln\theta}{d_m} + K_3 + \frac{K_4}{\theta(1-\eta)}e^{-(\frac{\ln\theta}{d_m}+K_3)\bar{\tau}})t + \frac{K_4\bar{\tau}}{\theta(1-\eta)}e^{-(\frac{\ln\theta}{d_m}+K_3)\bar{\tau}}\Big\}. \end{aligned} \tag{3.11}$$

C2 gives that $((\frac{\ln\theta}{d_m} + K_3) + \frac{K_4}{\theta(1-\eta)}e^{-(\frac{\ln\theta}{d_m}+K_3)\bar{\tau}})T + \frac{K_4\bar{\tau}}{\theta(1-\eta)}e^{-(\frac{\ln\theta}{d_m}+K_3)\bar{\tau}} - \ln\theta \le \ln B_2 - \ln(B_1 + \frac{K_2}{K_3})$. Therefore, we obtain $EV(t) \le B_2$. It means that the system (2.6) is finite-time stability under condition **C1**.

Case **C2** − 2 : $K_3 < 0$.

Contrary to **C1**-2, we consider $-K_4 e^{-(\frac{\ln\theta}{d_m})\bar{\tau}} \le \theta(\frac{\ln\theta}{d_m})$. In this case, choosing

$$q_2(t) = b(t)e^{-\frac{\ln\theta}{d_m}t}.$$

Similar discussion as in **C2** − 1, one obtains

$$\begin{aligned}
q_2(t) &= b(t)e^{-\frac{\ln\theta}{d_m}t}\\
&= \Big(-\theta^{N(t,0)}\frac{K_2}{K_3} + \theta^{N(t,0)}e^{K_3 t}(b(0) + \frac{K_2}{K_3}) + \int_0^t \theta^{N(t,s)}[K_4 b(s-\tau(s))e^{K_3(t-s)}]ds\Big)e^{-\frac{\ln\theta}{d_m}t}\\
&\le \frac{1}{\theta}(b(0) - \frac{K_2}{K_3}) + \frac{1}{\theta}\int_0^t e^{-\frac{\ln\theta}{d_m}s}[K_4 b(s-\tau(s))]ds\\
&\le \frac{1}{\theta}(b(0) - \frac{K_2}{K_3}) + \frac{K_4}{(1-\eta)\theta}e^{-\frac{\ln\theta}{d_m}\bar{\tau}}\int_{\bar{\tau}}^t e^{-\frac{\ln\theta}{d_m}s}b(s)ds\\
&\le \frac{1}{\theta}(b(0) - \frac{K_2}{K_3})\exp\{\frac{K_4}{(1-\eta)\theta}e^{-\frac{\ln\theta}{d_m}\bar{\tau}}(t+\bar{\tau})\}
\end{aligned}$$

Further, we can compute that

$$\begin{aligned}
EV(t) &\le b(t) = q_2(t)e^{\frac{\ln\theta}{d_m}t}\\
&\le \frac{1}{\theta}(B_1 - \frac{K_2}{K_3})\exp\Big\{(\frac{\ln\theta}{d_m} + \frac{K_4}{(1-\eta)\theta}e^{-\frac{\ln\theta}{d_m}\bar{\tau}})T + \frac{K_4\bar{\tau}}{(1-\eta)\theta}e^{-\frac{\ln\theta}{d_m}\bar{\tau}}\Big\}.
\end{aligned} \tag{3.12}$$

In view of **C2**, one can calculate that $(\frac{\ln\theta}{d_m} + \frac{K_4}{(1-\eta)\theta}e^{-(\frac{\ln\theta}{d_m})\bar{\tau}})T + \frac{K_4\bar{\tau}}{(1-\eta)\theta}e^{-(\frac{\ln\theta}{d_m})\bar{\tau}} - \ln\theta \le \ln B_2 - \ln(B_1 - \frac{K_2}{K_3})$. Therefore, $EV(t) \le B_2$.

Case **C3** − 1 : $K_3 > 0$.

Contrary to the above case, we consider $K_3 + \frac{\ln\theta}{d_m} > 0$. Let $f(t)$ satisfy the following equation

$$\begin{cases}
f(t) = -\theta^{N(t,0)}\frac{K_2}{K_3} + \frac{e^{(\frac{\ln\theta}{d_m}+K_3)t}}{\theta}(\tilde{f}(0) + \frac{K_2}{K_3}) + \frac{1}{\theta}\int_0^t e^{(\frac{\ln\theta}{d_m}+K_3)(t-s)}[K_4 f(s-\tau(s))]ds, & t > 0,\\
f(s) = EV(s), & -\bar{\tau} \le t \le 0,
\end{cases} \tag{3.13}$$

where $\tilde{f} = \sup_{-\bar{\tau}<s<0} f(s)$. By virtue of (3.5) and (3.13), we derive $0 \le b(t) \le f(t)$ for $t \ge -\bar{\tau}$. Before the following proof, setting

$$A_1 = \{t\,|t \le \tau(t),\ t \in (0,\bar{\tau}]\},\quad A_2 = \{t\,|t > \tau(t),\ t \in (0,\bar{\tau}]\}.$$

It is obvious that $A_1 \cup A_2 = (0,\bar{\tau}]$. For $t \in A_1$, one can obtain

$$\begin{aligned}
f(t) - f(t-\tau(t)) &\ge f(t) - \frac{1}{\theta}(\tilde{f}(0) + \frac{K_2}{K_3})\\
&= \frac{1}{\theta}(\tilde{f}(0) + \frac{K_2}{K_3})(e^{(\frac{\ln\theta}{d_m}+K_3)t} - 1) + \frac{1}{\theta}\int_0^t e^{(\frac{\ln\theta}{d_m}+K_3)(t-s)}[K_4 f(s-\tau(s))]ds \ge 0.
\end{aligned}$$

For $t \in A_2 \cup (\bar{\tau}, T]$, a direct calculation leads that

$$\begin{aligned}
&f(t) - f(t-\tau(t)) \\
&= (-\theta^{N(t,0)} + \theta^{N(t-\tau(t),0)})\frac{K_2}{K_3} + (\frac{1}{\theta}e^{(\frac{\ln\theta}{d_m}+K_3)t} - \frac{1}{\theta}e^{(\frac{\ln\theta}{d_m}+K_3)(t-\tau(t))})(\tilde{f}(0) + \frac{K_2}{K_3}) \\
&+ \frac{1}{\theta}\int_0^t e^{(\frac{\ln\theta}{d_m}+K_3)(t-s)}[K_4 f(s-\tau(s))]ds - \frac{1}{\theta}\int_0^{t-\tau(t)} e^{(\frac{\ln\theta}{d_m}+K_3)(t-\tau(t)-s)}[K_4 f(s-\tau(s))]ds \\
&= (-\theta^{N(t,0)} + \theta^{N(t-\tau(t),0)})\frac{K_2}{K_3} + \frac{1}{\theta}e^{(\frac{\ln\theta}{d_m}+K_3)t}(1 - \frac{1}{\exp\{(\frac{\ln\theta}{d_m}+K_3)\tau(t)\}})(\tilde{f}(0) + \frac{K_2}{K_3}) \\
&+ \frac{1}{\theta}e^{(\frac{\ln\theta}{d_m}+K_3)t}\int_0^t e^{-(\frac{\ln\theta}{d_m}+K_3)s}[K_4 f(s-\tau(s))]ds - \frac{1}{\theta}e^{(\frac{\ln\theta}{d_m}+K_3)(t-\tau(t))}\int_0^{t-\tau(t)} e^{-(\frac{\ln\theta}{d_m}+K_3)s}\times \\
&[K_4 f(s-\tau(s))]ds \\
&\geq (-\theta^{N(t,0)} + \theta^{N(t-\tau(t),0)})\frac{K_2}{K_3} + \frac{1}{\theta}e^{(\frac{\ln\theta}{d_m}+K_3)t}(1 - \frac{1}{\exp\{(\frac{\ln\theta}{d_m}+K_3)\tau(t)\}})(\tilde{f}(0) + \frac{K_2}{K_3}) \\
&+ \frac{1}{\theta}e^{(\frac{\ln\theta}{d_m}+K_3)(t-\tau(t))}\int_{t-\tau(t)}^t e^{-(\frac{\ln\theta}{d_m}+K_3)s}[K_4 f(s-\tau(s))]ds \geq 0.
\end{aligned}$$

This implies that $f(t) \geq f(t-\tau(t))$ when $t > 0$. Then, in light of (3.13), we can deduce the following inequality

$$f(t) \leq \frac{1}{\theta}(\tilde{f}(0) + \frac{K_2}{K_3})e^{(\frac{\ln\theta}{d_m}+K_3)t} + \frac{1}{\theta}\int_0^t e^{(\frac{\ln\theta}{d_m}+K_3)(t-s)}[K_4 f(s)]ds.$$

The Gronwall inequalities [43] gives that

$$f(t)e^{-(\frac{\ln\theta}{d_m}+K_3)t} \leq \frac{1}{\theta}(\tilde{f}(0) + \frac{K_2}{K_3})e^{K_4 t}.$$

That is to say

$$EV(t) \leq b(t) \leq f(t) \leq \frac{1}{\theta}(\tilde{f}(0) + \frac{K_2}{K_3})e^{(K_4+\frac{\ln\theta}{d_m}+K_3)t} \leq \frac{1}{\theta}(B_1 + \frac{K_2}{K_3})e^{(K_4+\frac{\ln\theta}{d_m}+K_3)T}.$$

By virtue of **C3**, one can see that $(K_4 + \frac{\ln\theta}{d_m} + K_3)T - \ln\theta \leq \ln B_2 - \ln(B_1 + \frac{K_2}{K_3})$. This implies that $EV(t) \leq B_2$. Then we can obtain the required statement.

Situation 2: $\theta \geq 1$.

In this situation, one can calculate that

$$\begin{aligned}
\exp\{N(t,s)\ln\theta + K_3(t-s)\} &\leq \exp\Big\{\frac{t-s}{d_s}\ln\theta + K_3(t-s)\Big\} \\
&= \exp\Big\{(\frac{\ln\theta}{d_s} + K_3)(t-s)\Big\},
\end{aligned} \tag{3.14}$$

and

$$b(t) \leq -\theta^{N(t,0)}\frac{K_2}{K_3} + e^{(\frac{\ln\theta}{d_s}+K_3)t}(b(0) + \frac{K_2}{K_3}) + \int_0^t e^{(\frac{\ln\theta}{d_s}+K_3)(t-s)}[K_4 b(s-\tau(s))]ds. \tag{3.15}$$

It is clear that inequalities (3.15) and (3.5) have the same form and properties. We can use the same method as the discussions in **Situation 1**, and yield the desired result. For the sake of simplicity, we omit the details.

Remark 3.5 *In the theorem 3.4, we have dealt with sufficient conditions that are more stringent than the actual situation. For example, in* **C1***, we show the system 2.6 is finite-time stable when the condition* $c_1\theta \leq -K_4 e^{-c_2\bar{\tau}} < 0$ *holds. In fact, in our proof, we put forward that the system 2.6 is finite time stable under the condition* $c_1\theta \leq -K_4 e^{-c_1\bar{\tau}} < 0$. *It is clearly that* $-K_4 e^{-c_2\bar{\tau}} < -K_4 e^{-c_1\bar{\tau}}$.

Remark 3.6 *For an ecosystem, the initial material (vegetation and water) density* B_1 *can be estimated. Similarly, the desired maximum density* B_2 *of vegetation and water can also be given. In addition, the desired time* T *to keep the density of plants and water between* B_1 *and* B_2 *can also be given. Therefore, we can judge whether the system is finite-time stable through the relationship between the parameters.*

4. Optimal control strategies

Desertification can bring great economic losses. We need to adopt some control strategies to increase the amount of vegetation and water density. There are many strategies for the management of vegetation systems such as replanting, irrigation, and so on. The cost of strategy is inevitable. It is easy to think that the way to save costs is the search for optimal control. In the following, we mainly use the principle of minimum value to find the optimal control in the vegetation system.

Consider $(u(x,t), v(x,t), w(x,t)) \in X$ where X is defined in preparations. We define a control function set as $U = U_1 \bigcup U_2 = \{\pi_i = \pi_i(x,t)\ \ where\ (x,t) \in \Gamma \times \{t|[0,T] - \{t_k,\ (k \in N)\}\}|i = 1,2,3\} \bigcup \{\pi_i = \pi_i(x,t_k)\ where\ x \in \Gamma,\ t_k \in [0,T]\ and\ k \in \{1,\cdots N\}|i = 4,5,6\}$ where the meaning of π_i are listed as follows:

(a) π_1 indicates that the planting strategy is used to increase vegetation density.

(b) π_2 is the strategy of applying aquasorb which can reduce the infiltration and loss of soil water [44].

(c) π_3 is the use of chemical substances such as Hexadecanol, Octadecanol, Cetyl and Stearyl alcohols strategy which can inhibit the evaporation of surface water [45–47].

(d) The control strategy of π_i $(i = 4,5,6)$ can be explained by human control or government intervention.

Due to the limitation of technology or cost, each control strategy π_i has an upper bound π_{max}. A vegetation model with control strategy can be given as

$$\begin{cases} \left.\begin{aligned} du(x,t) &= (d_u\Delta u(x,t) + \pi_1 u(x,t) + \frac{v(x,t)}{v(x,t)+1}u(x,t) - l_1 u(x,t))dt \\ &\quad - \rho_1\sigma_1 u(x,t)dB_1(t) - \rho_1 u(x,t)\int_{\mathbb{Y}} y\tilde{N}(dt,dy), \\ dv(x,t) &= (d_v\Delta v(x,t) + \alpha\tfrac{u(x,t)+f}{u(x,t)+1}w(x,t-\tau(t)) - \gamma\tfrac{v(x,t)}{v(x,t)+1}u(x,t) \\ &\quad - (l_2 - \pi_2)v(x,t))dt - \rho_2\sigma_2 v(x,t)dB_2(t) - \rho_2 v(x,t)\int_{\mathbb{Y}} y\tilde{N}(dt,dy), \\ dw(x,t) &= (d_w\Delta w(x,t) + R - \alpha\tfrac{u(x,t)+f}{u(x,t)+1}w(x,t-\tau(t)) - (l_3 - \pi_3)w(x,t))dt \\ &\quad - \rho_3\sigma_3 w(x,t)dB_3(t) - \rho_3 w(x,t)\int_{\mathbb{Y}} y\tilde{N}(dt,dy), \end{aligned}\right\} & \begin{aligned} &t \in [0,T], \\ &t \neq t_k, \\ &k \in N, \\ &x \in \Gamma, \end{aligned} \\ \left.\begin{aligned} u(x,t_k^+) - u(x,t_k) &= I_{ku}\pi_4 u(x,t_k), \\ v(x,t_k^+) - v(x,t_k) &= I_{kv}\pi_5 v(x,t_k), \\ w(x,t_k^+) - w(x,t_k) &= I_{kw}\pi_6 w(x,t_k), \end{aligned}\right\} t = t_k\ (k \in N), \end{cases} \tag{4.1}$$

The conditions of initial value and boundary are the same as system (4.1). The set **X** is admissible trajectories is given by

$$\mathbf{X} = \{X(\cdot) \in W^{2,2}(\Gamma \times [0,T]; R^3) | \ (4.1) \ is\ satisfied\},$$

and the admissible control set $\mathbf{U}$ is given by

$$\mathbf{U} = \{U(\cdot) \in L^{\infty}(\Gamma \times [0,T]; R^6) | 0 < \pi_i(x,t) \le \pi_{max} < 1, \ \forall (x,t) \in \Gamma \times [0,T]\}.$$

We consider the objective function

$$J(X(\cdot), U_1(\cdot)) = \sum_{k=1}^{N} \int_{t_{k-1}}^{t_k} \int_{\Gamma} (-P_1 u - P_2 v - P_3 w + \frac{1}{2} \sum_{i=1}^{3} Q_i \pi_i(x,t)^2) dx dt,$$

$$J(X(\cdot), U_2(\cdot)) = \sum_{k=1}^{N} \int_{\Gamma} -\bar{P}_1 u - \bar{P}_2 v - \bar{P}_3 w + \frac{1}{2} \sum_{i=4}^{6} Q_i \pi_i(x,t_k)^2 dx.$$

It is worth noting that Q_i $(i = 1, 2 \cdots 6)$ are the weight constants for control strategies, P_i $(\bar{P}_i)$ $(i = 1, 2, 3)$ are positive weight constant of vegetation, soil water, surface water, respectively. $\frac{1}{2} Q_i \pi_i^2$ $(i = 1, 2, \cdots 6)$ is the cost of control strategies. The square of the control variables means that the cost of strategies is gradually increasing [48]. Our goal is to obtain the most plants and the lowest cost of corresponding control strategy. Therefore, optimal control problem is equivalent to finding the optimal control U^* in the allowable control set $\mathbf{U}$ and determining the corresponding vector function $(u^*, v^*, w^*) \in \mathbf{X}$ to satisfy the objective function:

$$J(X(\cdot), U(\cdot)) = \min_{(X(\cdot), U(\cdot)) \in \mathbf{X} \times \mathbf{U}} (J(X(\cdot), U_1(\cdot)) + J(X(\cdot), U_2(\cdot))). \tag{4.2}$$

Further, we introduce adjoint equation and Hamiltonian function [49–52]

$$\begin{cases} H(t, u, v, w, p_1, p_2, p_3) = p_1[d_u \Delta u + \pi_1 u + \frac{v}{v+1} u - l_1 u] + p_2[d_v \Delta v + \alpha \frac{u+f}{u+1} w(t - \tau(t)) \\ \quad - \gamma \frac{v}{v+1} u - (l_2 - \pi_2) v] + p_3[d_w \Delta w + R - \alpha \frac{u+f}{u+1} w(t - \tau(t)) - (l_3 - \pi_3) w] \\ \quad - q_1 \rho_1 \sigma_1 u - q_2 \rho_2 \sigma_2 v - q_3 \rho_3 \sigma_3 w - \int_{\mathbb{Y}} \rho_1 u y r_1(y) \lambda(dy) - \int_{\mathbb{Y}} \rho_2 v y r_2(y) \lambda(dy) \\ \quad - \int_{\mathbb{Y}} \rho_3 w y r_3(y) \lambda(dy) - P_1 u - P_2 v - P_3 w + \frac{1}{2} \sum_{i=1}^{3} Q_i \pi_i^2 \\ IH(t_k, u, v, w, p_1, p_2, p_3) = \frac{1}{2} \sum_{i=4}^{6} Q_i \pi_i(t_k)^2 + p_1(t_k) I_{ku} \pi_4(t_k) u + p_2(t_k) I_{kv} \pi_5(t_k) v \\ \quad + p_3(t_k) I_{kw} \pi_6(t_k) w - \bar{P}_1 u - \bar{P}_2 v - \bar{P}_3 w. \end{cases}$$

Theorem 4.1 *The optimal control problem* (4.2) *with fixed time T admits a unique optimal solution (u^*, v^*, w^*) associated with an optimal control $U(x,t)$ for $(x,t) \in \Gamma \times [0,T]$. Moreover, there are adjoint* *functions* $p_i(\cdot,\cdot)$ $(i=1,2,3)$ *such as*

$$\left\{\begin{array}{l}
\left.\begin{array}{l}
dp_1=-[d_u\Delta p_1+(\pi_1-l_1)p_1+\dfrac{v^*}{v^*+1}(p_1-\gamma p_2)+\alpha\dfrac{1-f}{(u^*+1)^2}w^*(t-\tau(t))(p_2-p_3)\\
\qquad -\rho_1\sigma_1 q_1-\displaystyle\int_{\mathbb{Y}}\rho_1 y r_1(y)\lambda(dy)-P_1]dt+q_1dB_1(t)+\int_{\mathbb{Y}}r_1(y)\tilde{N}(dt,dy)\\
dp_2=-[d_v\Delta p_2+\dfrac{u^*}{(v^*+1)^2}(p_1-\gamma p_2)-(l_2-\pi_2)p_2-\rho_2\sigma_2 q_2-\displaystyle\int_{\mathbb{Y}}\rho_2 y r_2(y)\lambda(dy)\\
\qquad -P_2]dt+q_2dB_2(t)+\displaystyle\int_{\mathbb{Y}}r_2(y)\tilde{N}(dt,dy)\\
dp_3=-[d_w\Delta p_3+\dfrac{\chi_{[0,T-\tau(T)]}(t)}{1-\dot{\tau}(t+\varsigma(t))}(p_2(t+\varsigma(t))-p_3(t+\varsigma(t))\alpha\dfrac{u^*(t+\varsigma(t))+f}{u^*(t+\varsigma(t))+1}-l_3p_3\\
\qquad +\pi_3p_3-\rho_3\sigma_3q_3-\displaystyle\int_{\mathbb{Y}}\rho_3 y r_3(y)\lambda(dy)-P_3]dt+q_3dB_3(t)+\int_{\mathbb{Y}}r_3(y)\tilde{N}(dt,dy)
\end{array}\right\}\begin{array}{l}t\in[0,T],\\ t\neq t_k,\\ (k\in N),\\ x\in\Gamma,\end{array}\\
\left.\begin{array}{l}
p_1(t_k^+)-p_1(t_k)=-I_{ku}\pi_4(t_k)p_1(t_k)-\bar{P}_1,\\
p_2(t_k^+)-p_2(t_k)=-I_{kv}\pi_5(t_k)p_2(t_k)-\bar{P}_2,\\
p_3(t_k^+)-p_3(t_k)=-I_{kw}\pi_6(t_k)p_3(t_k)-\bar{P}_3,
\end{array}\right\}t=t_k\ (k\in N)\ x\in\Gamma,\\
\left.\begin{array}{l}
p_i(T)=0\\
\frac{\partial p_i}{\partial x}=0
\end{array}\right\}(i=1,2,3),
\end{array}\right. \tag{4.3}$$

where $\varsigma(t)$ *is introduced to take into account the function dependence of the time-varying delay* $\tau(t)$ *on time; if* $s=t-\tau(t)$*,* $0\le t\le T$*, is solved for t,* $\varsigma(t)$ *is given by* $t=s+\varsigma(s)$*. Additionally, the* $\chi_{[a,b]}(t)$ *is a characteristic function defined by*

$$\chi_{[a,b]}(t)=\begin{cases}1, & if\ t\in[a,b],\\ 0, & otherwise.\end{cases}$$

Furthermore,

$$\pi_i^*=\max[0,\min(\tilde{\pi}_i,\pi_{max})]\ (i=1,2,3,4,5,6), \tag{4.4}$$

where

$$\begin{aligned}\tilde{\pi}_1&=\frac{-p_1u^*}{Q_1},\ \tilde{\pi}_2=\frac{-p_2v^*}{Q_2},\ \tilde{\pi}_3=\frac{-p_3w^*}{Q_3},\\ \tilde{\pi}_4&=\frac{-p_1I_{ku}u^*}{Q_4},\ \tilde{\pi}_5=\frac{-p_2I_{kv}v^*}{Q_5},\ \tilde{\pi}_6=\frac{-p_3I_{kw}w^*}{Q_6}.\end{aligned} \tag{4.5}$$

The proof is omitted. Interested readers can see the reference [49].

5. Numerical examples

In this section, numerical simulations are given to illustrate our theoretical results. We select the parameters from the Table 2.

5.1. Finite-time stability

In this section, we discuss that the system is finite time stable when the sufficient conditions are satisfied. We take $\Gamma=[-0.25,0.25]$, $d=0.1$, $k_0=0.05$, $R_o=3$, $\rho_i=0.3$, $a_i=0.5$, $\sigma_i=0.9$ $(i=1,2,3)$. Then, one can obtain the $m=1$, $L_1=L_2=L_3=0.09$, $K_2=0.500$, $K_3=0.7809\neq 0$,

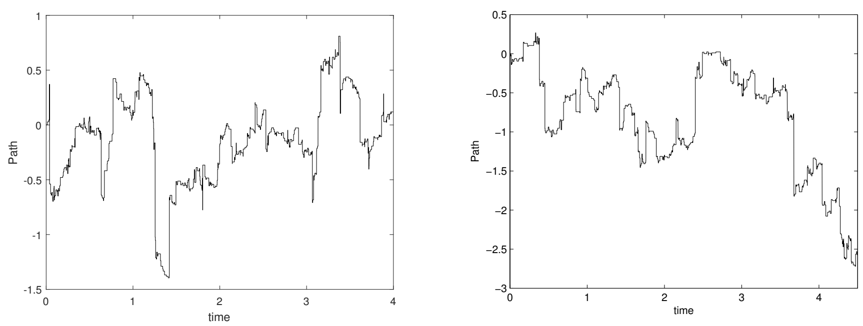
Table 2. Parameters Value.

Symbol	Value	Reference	Symbol	Value	Reference
d_{uo}	$0.1\ m^2/d$	[53]	c	$10\ g\cdot mm^{-1}\cdot m^{-2}$	[53]
d_{vo}	$0.1\ m^2/d$	[53]	g_m	$0.05\ mm\cdot g^{-1}\cdot m^2\cdot d^{-1}$	[53]
d_{wo}	$100\ m^2/d$	[53]	d	$(0,\ 0.5)\ d^{-1}$	[32]
k_1	$2\ mm$	[32]	R_o	$(0,3)\ mm/d$	[53]
k_2	$2\ g/m^2$	[32]	f	0.2	[53]
b	$(0,0.5)\ d^{-1}$	[53]	p	$(0,1)\ d^{-1}$	Estimated
k_0	$(0.05,\ 0.2)\ d^{-1}$	Estimated	$\rho_i\ (i=1,2,3)$	$[0,1]$	Estimated
$a_i\ (i=1,2,3)$	$[0,1]$	Estimated	$\sigma_i\ (i=1,2,3)$	$[0,1]$	Estimated
$I_\vartheta\ (\vartheta=u,v,w)$	$(-1,1)$	Estimated			

$K_4 = 0.12$. Letting $B_1 = 1.44$, $B_2 = 8.41$, $T = 4$, $(u_0(x,t), v_0(x,t), w_0(x,t) = (0.9, 0.9, 1)$ where $t \in (-\bar{\tau}, 0)$ and taking $I_u = I_v = I_w = -0.2$, we can get $c_1 = -0.3348$, $\theta = 0.64 \in (0,1)$ and $y(0) = 1.144$ by simple calculation. We set the impulse sequence $t_k = \{0.4, 0.8, 1.2, 1.6, 2, 2.4, 2.8, 3.2, 3.6, 4, 4.4\}$. Therefore, $d_m = 0.4$, $d_s = 0.4$. Additionally, we choose

$$\tau(t) = \begin{cases} \frac{1}{10f}\sin(\frac{25f^2\pi}{2}t), & t \in [0,1], \\ 1/10f, & t \in [1,T], \end{cases} \tag{5.1}$$

and noise (Figure 3(a)). Through calculation, we have $\bar{\tau} = \frac{1}{10f} = 1/2$, $\eta = \frac{5f\pi}{4} = \pi/4$. For noise, we choose a α stable Lévy process which is randomly generated and shown in Figure 3(a). A directly calculation shows $c_1\theta = -0.2143 < -K_4 e^{-c_2\bar{\tau}} = -0.2096 < -K_4 e^{c_1\bar{\tau}} = -0.1419 < 0$, and $-\ln(\theta) = 0.4463 < \ln(B_2) - \ln(B_1 + \frac{K_2}{K_3}) = 2.8619$. Therefore, the condition **C1** is holds. From Figure 4, we can know $\|y(x,0)\| = 1.144 < \sqrt{B_1} = 1.2 < \max\limits_{\Gamma\times[0,T]} \|y(x,t)\| = 2.8814 < \sqrt{B_2} = 2.9$, which means the system (2.6) is finite-time stable.



(a) (b)

Figure 3. The different state trajectories of α stable lévy process where α=0.9.

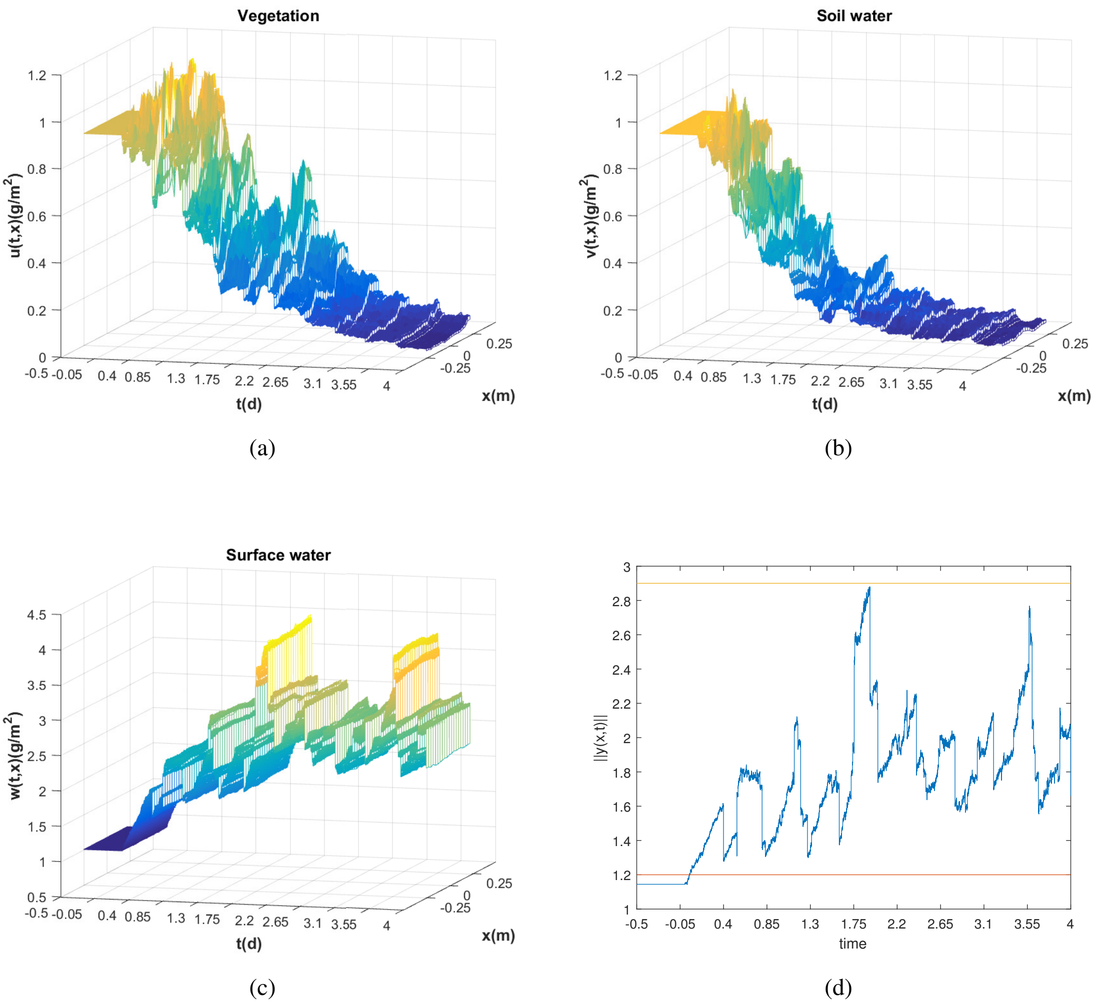


Figure 4. State trajectories of vegetation-water system which is finite-time stable. The unit of time is d (day) and unit of space is m (meter).

(1) The role of impulse

In this section, we consider the impact of impulses on finite-time stability. Obviously, from sufficient conditions, we can find that the finite time stability of system (2.6) can be effected by impulse. In order to intuitively indicate the effect of the impulses through numerical simulation, we keep the system parameters, time delay function $\tau(t)$ and noise (Figure 3 (a)) unchanged and show the variation of the finite-time stability of the system (2.6) under different impulse intensities. Therefore, we choose $I_u = I_v = I_w = 0$.

Through simple calculations, we can get $\theta = 1$, $c_3 = 0.7809 >$, $K_2 = 0.5$, $K_3 = 0.7809 > 0$, $K_4 = 0.12$ and $(K_4 + c_3)T = 4.0541 > \ln(B_2) + \ln(B_1 + K2/K3) = 1.3969$. Therefore, the conditions of theorem 3.4 is not satisfied. The results of the numerical simulation of $I_u = I_v = I_w = 0$ are shown in Figure 5. We can find $y(1.9154) = 4.0297 > \sqrt{B_2} = 2.9$ which means system (2.6) is not finite time stable. Comparing with the results of $I_u = I_v = I_w = -0.2$ which are shown in Figure 4, we can know that the impulse can affect the finite-time stability.

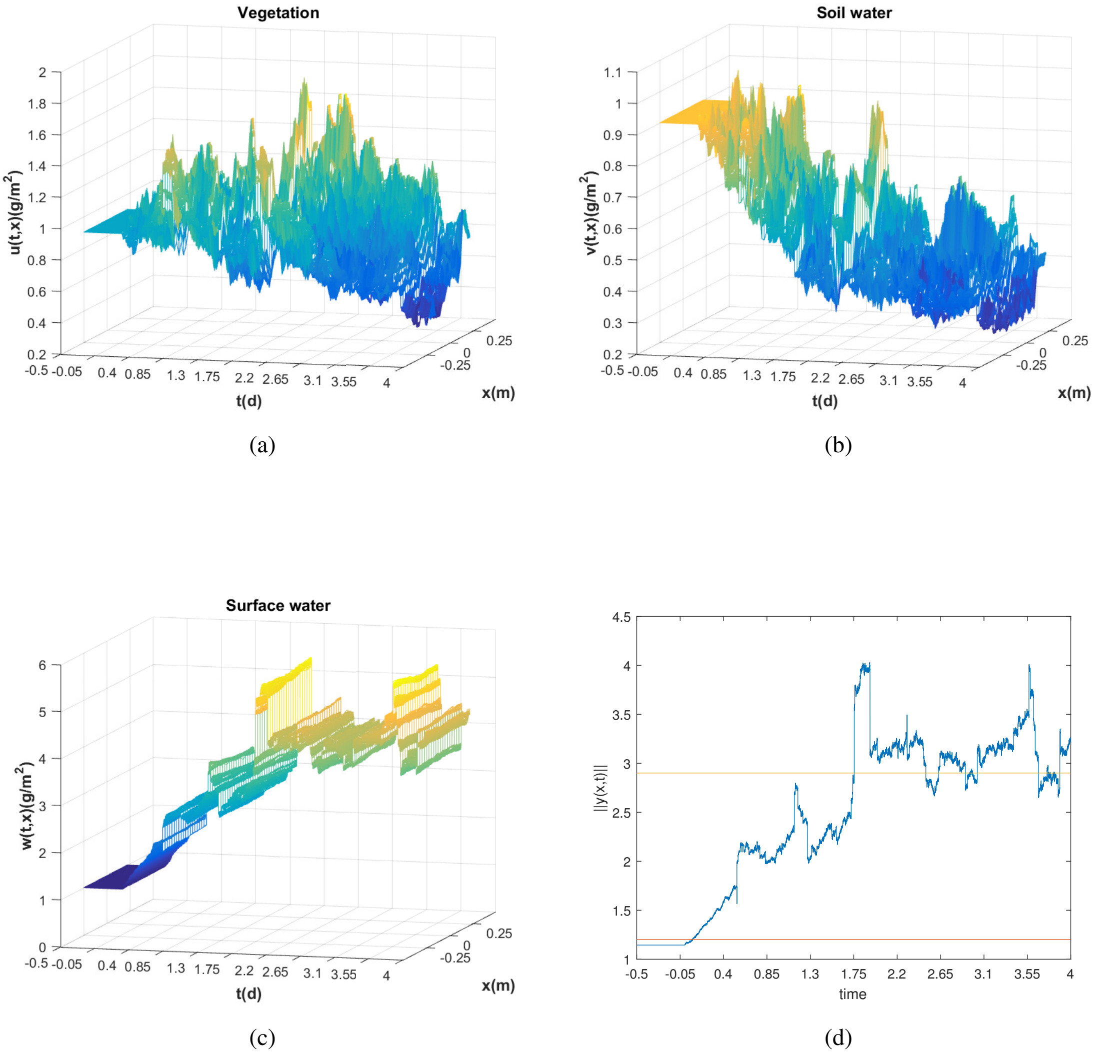


Figure 5. State trajectories of vegetation-water system with $I_u = I_v = I_w = 0$. The unit of time is d (day) and unit of space is m (meter).

(2) The role of time delay

Time delay does affect the finite-time stability of system (2.6). For example, in details, the larger $\bar{\tau}$ plays an opposite role in satisfying the inequality $c_1\theta \leq -K_4 e^{-c_2\bar{\tau}}$ in **C1**. Retaining the system parameters, the impulse intensity and noise (Figure 3 (a)) unchanged, we choose $\tau_2(t) = \bar{\tau}_2 = 4.5$.

Through a direct calculation, it can be known that $\eta = 0$, $\theta = 0.64 < 1$, $K_2 = 0.5$, $K_3 = 0.7809 > 0$, $K_4 = 0.12$, $c_1 = -0.3348$ and $(c_1 + K_4/\theta/(1-\eta)\exp(-c_1\bar{\tau}))T + K_4\bar{\tau}/\theta/(1-\eta)\exp(-c_1\bar{\tau}) - \ln(\theta) = 6.5531 > \ln(B_2) + \ln(B_1 + K_2/K_3) = 1.3969$, which means the conditions of theorem 3.4 is not satisfied. Further, from Figure 6, we find $\|y(3.3980)\| = 4.0454 > 2.9 = \sqrt{B_2}$ which implies the system is not finite-time stable. Compared with $\bar{\tau} = 1/2$ in Figure 4, the change of delay affects the finite-time stability.

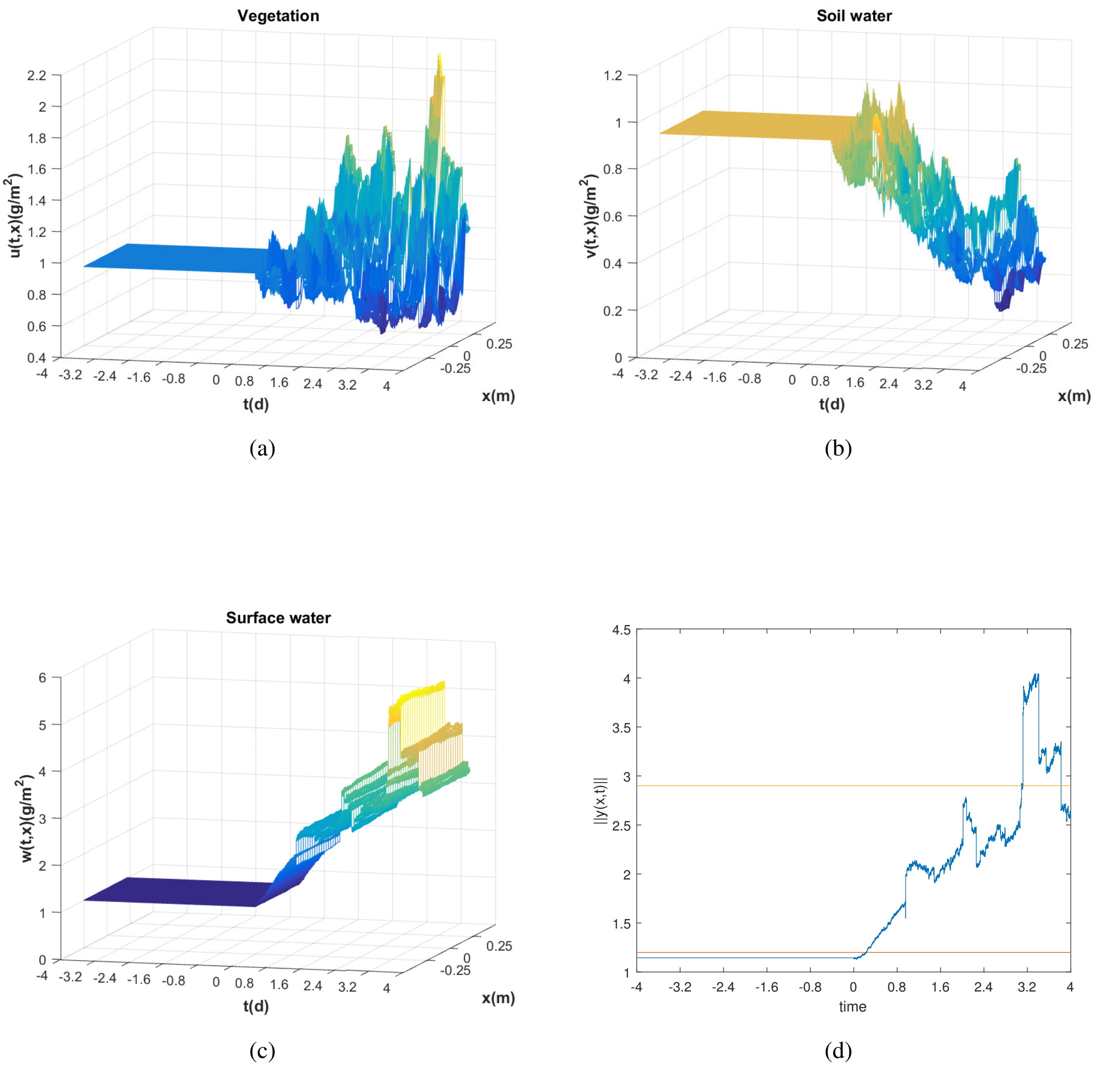
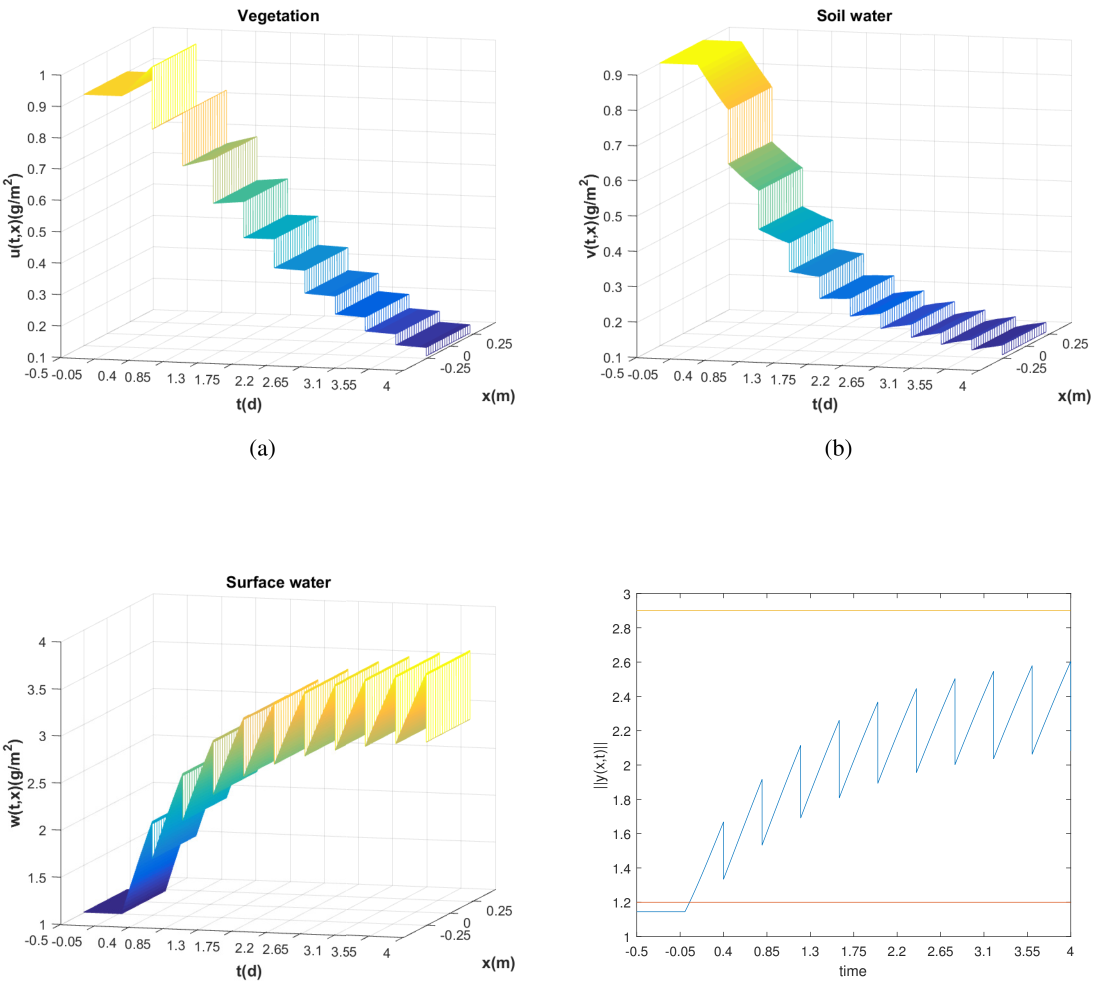


Figure 6. State trajectories of vegetation-water system with $\bar{\tau} = 4.5$. The unit of time is d (day) and unit of space is m (meter).

(3) The role of noise

It is essential to analyze the impact of environmental noise. For comparison, we choose $\rho_i = 0$ and $\rho_i = 0.5$ $(i = 1, 2, 3)$ to carry out numerical simulation. The time delay function $\tau(t)$, noise path as Figure 3(a) and system parameters except ρ_i $(i = 1, 2, 3)$ are also unchanged.

When $\rho_i = 0$ $(i = 1, 2, 3)$, we have $-\ln(\theta) = 0.4463 < \ln(B_2) - \ln(B_1 + \frac{K_2}{K_3}) = 1.3918$ which means the system is finite time stable. Meanwhile, through calculation, when $\rho_i = 0.5$ $(i = 1, 2, 3)$, the sufficient condition for finite-time stability also is not satisfied. The results of the numerical simulation are shown in Figures 7 and 8. Comparing with the $\rho_i = 0.3$ $(i = 1, 2, 3)$ in Figure 4, we can observe that noise intensity does affect the finite-time stability.



(a) (b) (c) (d)

Figure 7. State trajectories of vegetation-water system with noise intensity $\rho_i = 0$ ($i = 1, 2, 3$). The unit of time is d (day) and unit of space is m (meter).

(4) The role of diffusion

In this section we mainly analyze the impact of the diffusion on finite-time stability. In the ecological environment, different types of plants have different diffusion intensities. The vegetation structure of the area can be changed through human planting, etc. However, the diffusion strength of water is fixed and not easily changed. Therefore, we adjust the diffusion coefficient of vegetation to analyze the impact of diffusion. We choose $d_{uo} = 10$ (m^2/d) while keeping all other parameters unchanged.

Through calculation, it can be obtain that $d_u = 0.1$, $\theta = 0.64 < 1$, $K_2 = 0.5$, $K_3 = -0.0991$ and $-\ln(\theta) = 0.4463 > \ln(B_2) - \ln(B_1 + \frac{K_2}{K_3}) = 0.2599$. This is obvious that the conditions of theorem 3.4 is not hold. The results of the numerical simulation are shown in Figure 9 which confirmed the analysis. Comparing with the results of $d_{uo} = 0.1$ (m^2/d) which are shown in Figure 4, we know that diffusion can affect the finite-time stability."

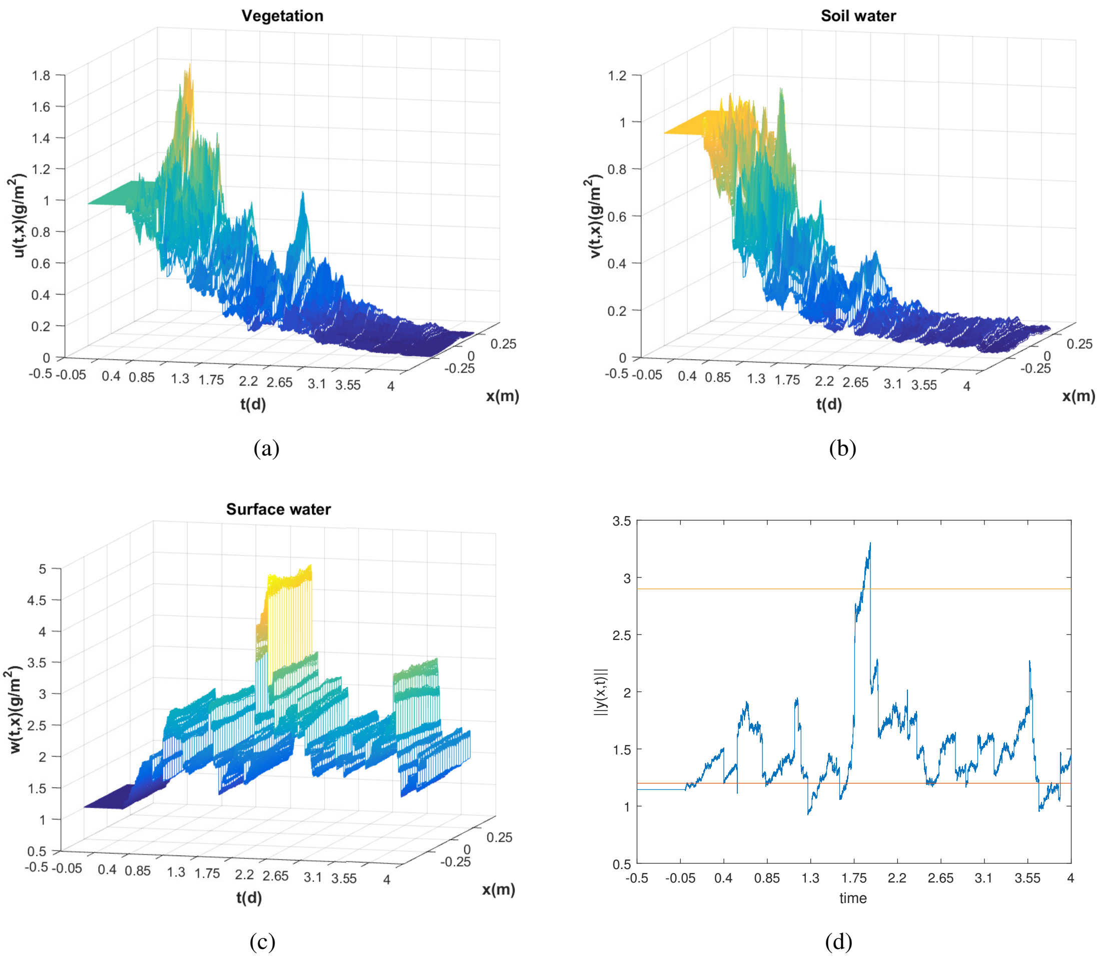
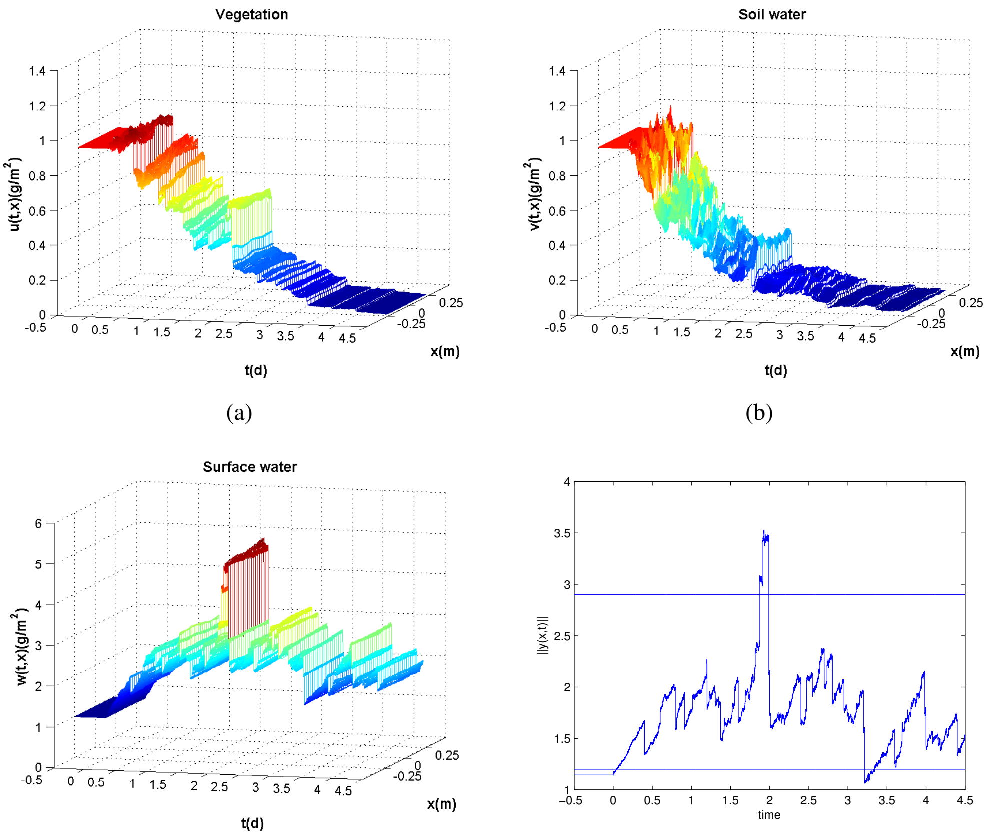


Figure 8. State trajectories of vegetation-water system with noise intensity $\rho_i = 0.5$ ($i = 1, 2, 3$). The unit of time is d (day) and unit of space is m (meter).

5.2. *Optimal control*

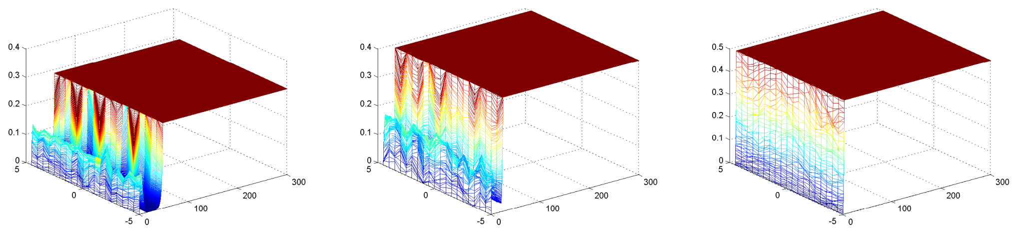
In this section, we mainly show optimal control through numerical simulation. We choose $t \in [0, 300]$, $x \in [-5, 5]$, $d = 0.35$, $b = 0.5$, $z = 0.7$, $a_i = 0.2$, $\sigma_i = 0.2$, $\rho_i = 0.1$, $q_i = 0.2$, $r_i = 0.2$, $P_i = 1$, $\bar{P}_i = 1$, $Q_i = 1$ ($i = 1, 2, 4, 5, 6$), $Q_3 = 5$, $I_{ku} = I_{kv} = I_{kw} = 0.2$ where $i = 1, 2, 3$. We set the impulse sequence $t_k = \{25, 50, 75, 100, 125, 150, 175, 200, 225, 250, 275\}$ and choose noise (Figure 3 (b)). Other parameters can be found in Table 2. Because of technical limitations, we set the maximum value of the control variable $\pi_1 \in (0, 0.3)$, $\pi_2 \in (0, 0.4)$, $\pi_3 \in (0, 0.5)$, $\pi_4 \in (0, 2)$, $\pi_5 \in (0, 2)$, $\pi_6 \in (0, 2)$.

From (4.1), (4.3), (4.5), we can get the numerical solution of optimal control which are shown in Figures 9 and 10. Meanwhile, under optimal control, state trajectories of vegetation-water system is shown in Figure 11 (a). For comparison, we give state trajectories of vegetation-water system without control, which is shown in Figure 11(b). Obviously, the biomass density of vegetation has increased significantly under control. From the view of ecology, this is beneficial to the ecological environment.



(a) (b) (c) (d)

Figure 9. State trajectories of vegetation-water system with $d_{uo} = 10$. The unit of time is d (day) and unit of space is m (meter).

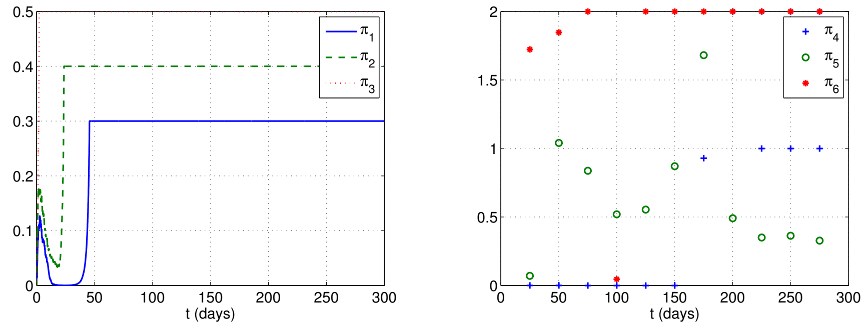


(a) (b) (c)

Figure 10. The three-dimensional diagram of control variable π_1, π_2, π_3.

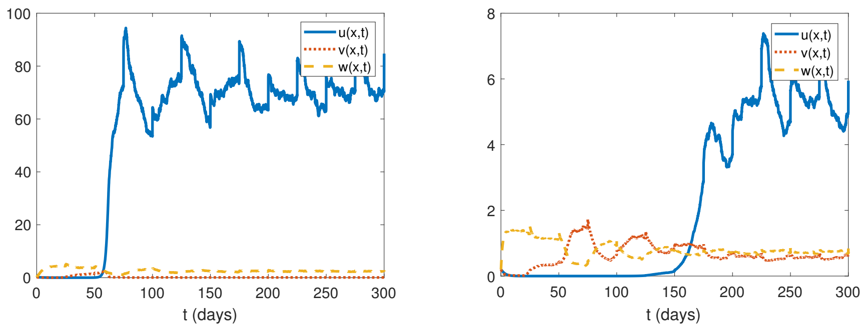
6. Conclusions

The desertification phenomenon caused by the destruction of the ecological environment by human beings is becoming more and more serious. Severe desertification may cause a food crisis and bring the disaster. Therefore, it is necessary for us to study the dynamics of vegetation-water system in arid areas and consider control strategies. In this paper, we propose a vegetation-water system with delay,



(a) (b)

Figure 11. The two-dimensional cross section of control variable π_1, π_2, π_3 and control variable for impulse π_4, π_5, π_6.



(a) (b)

Figure 12. State trajectories of vegetation-water system under optimal control.

impulse and noise. Through the proof, we show that the system has a unique global positive solution. Different from the analysis of the long-term dynamic behavior, we give the sufficient conditions for the finite-time stability of the system. It is worth noting that what we analyze is the finite-time stability of the system with time-varying delay. Some simulations are provided to support the theoretical results. Furthermore, we considered several control strategies and formulated an optimal control strategy to increase the density of vegetation. Through numerical algorithm, the numerical path for optimal control is given.

It is well-known that the initial values and parameters can affect the dynamic behavior of the system [54, 55]. Obviously, this phenomenon can also be observed from the conditions of Theorem 3.4. For example, from **C2**, we can find that delay has a negative impact on the finite-time stability. As the delay increases, the system may lose finite-time stability, which is shown in Figure 6. The effect of diffusion coefficients d_u, d_v, d_w, noise intensities σ_i, L_i $(i = 1, 2, 3)$ and impulse intensities I_u, I_v, I_w on the finite-time stability also can be obtained from Theorem 3.4 via similar discussion. Furthermore, through the analysis, we naturally raise a question. Whether changes in parameters can cause more complex dynamics of the system, such as the change of basins of attraction [54] and the generation of branching phenomena [55]. These will also be our further investigation.

Acknowledgments

The authors thank the editor and referees for their careful reading and valuable comments. The research was supported in part by the National Natural Science Foundation of China (No. 12161068) and Ningxia Natural Science Foundation (No. 2020AAC03065).

Conflict of interest

The authors declare there is no conflict of interest.

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Appendix

The proof of Theorem 3.1 is as follows

Proof Considering the following stochastic partial differential equation without impulse:

$$\begin{cases} dX(x,t) = (d_u \Delta X(x,t) + \frac{A_v(t)Y(x,t)}{A_v(t)Y(x,t)+1} X(x,t) - (l_1 - A_u(t)\ln(1+I_{ku}))X(x,t))dt \\ \qquad - \rho_1\sigma_1 X(x,t)dB_1(t) - \rho_1 X(x,t)\int_{\mathbb{Y}} y\tilde{N}(dt,dy), \\ dY(x,t) = (d_v \Delta Y(x,t) + \alpha A_v(t)^{-1} \dfrac{A_u(t)X(x,t)+f}{A_u(t)X(x,t)+1} A_w(t-\tau(t))Z(x,t-\tau(t)) \\ \qquad - (l_2 - A_v(t)\ln(1+I_{kv}))Y(x,t) - \gamma A_v(t)^{-1}\dfrac{A_v(t)Y(x,t)}{A_v(t)Y(x,t)+1} A_u(t)X(x,t))dt \\ \qquad - \rho_2\sigma_2 Y(x,t)dB_2(t) - \rho_2 Y(x,t)\int_{\mathbb{Y}} y\tilde{N}(dt,dy), \\ dZ(x,t) = (d_w \Delta Z(x,t) + RA_w(t)^{-1} - \alpha \dfrac{A_u(t)X(x,t)+f}{A_u(t)X(x,t)+1}\dfrac{A_w(t-\tau(t))}{A_w(t)} Z(x,t-\tau(t)) \\ \qquad - (l_3 - A_w(t)\ln(1+I_{kw}))Z(x,t))dt - \rho_3\sigma_3 Z(x,t)dB_3(t) - \rho_3 Z(x,t)\int_{\mathbb{Y}} y\tilde{N}(dt,dy), \end{cases} \tag{6.1}$$

with initial value $(X(0), Y(0), Z(0)) = (u(0), v(0), w(0))$, where A_ϑ $(\vartheta = u, v, w)$ can be defined by

$$A_\vartheta(t) = \begin{cases} 1 & t \in [-\bar{\tau}, 0), \\ \left.\begin{matrix} (1+I_{k\vartheta})^{[t]-t} & t \neq t_k \\ (1+I_{k\vartheta})^{-1} & t = t_k \end{matrix}\right\} & t \geq 0 \ (k \in N). \end{cases} \tag{6.2}$$

Clearly, A_ϑ $(\vartheta = u, v, w)$ is left-continuous, bounded and 1-periodic when $t \geq 0$. Next, we explain that system (2.6) and system (6.1) are equivalent. Let $(u(x,t), v(x,t), w(x,t)) = (A_u(t)X(x,t), A_v(t)Y(x,t), A_w(t)Z(x,t))$. It can be easily checked that $(X(x,t), Y(x,t), Z(x,t))$ are continuous on $(k, k+1) \in [0, \infty),\ k \in N$. For $t \neq t_k$, one can compute

$$\begin{aligned} du &= A_u'(t)X(x,t) + A_u(t)dX(x,t) \\ &= A_u(t)\Big((d_u\Delta X(x,t) + \frac{A_v(t)Y(x,t)}{A_v(t)Y(x,t)+1}X(x,t) - (l_1 - A_u(t)\ln(1+I_{ku}))X(x,t))dt \\ &\quad - \rho_1\sigma_1 X(x,t)dB_1(t) - \rho_1 X(x,t)\int_{\mathbb{Y}} y\tilde{N}(dt,dy)\Big) - A_u(t)\ln(1+I_{ku})X(x,t) \\ &= (d_u\Delta u(x,t) + \frac{v(x,t)}{v(x,t)+1}u(x,t) - l_1 u(x,t))dt - \rho_1\sigma_1 u(x,t)dB_1(t) - \rho_1 u(x,t)\int_{\mathbb{Y}} y\tilde{N}(dt,dy)). \end{aligned} \tag{6.3}$$

For k, we have

$$\begin{aligned} u(x,k^-) &= \lim_{t\to k^-} A_u(t)X(x,t) = (1+I_{ku})^{(k-1)-k}X(x,k) = (1+I_{ku})^{-1}X(x,k) = u(x,k), \\ u(x,k^+) &= \lim\nolimits_{t\to k^+} A_u(t)X(x,t) = (1+I_{ku})^{k-k}X(x,k) = X(x,k). \end{aligned}$$

This means that $u(x,k^+) = (1+I_{ku})u(x,k)$ for $t = t_k$. Similarly, we can derive that

$$\begin{aligned} dv(x,t) &= (d_v\Delta v(x,t) + \alpha\frac{u(x,t)+f}{u(x,t)+1}w(x,t)(x,t-\tau(t)) - \gamma\frac{v(x,t)}{v(x,t)+1}u(x,t) - l_2 v(x,t))dt \\ &\quad - \rho_2\sigma_2 v(x,t)dB_2(t) - \rho_2 v(x,t)\int_{\mathbb{Y}} y\tilde{N}(dt,dy), \\ dw(x,t) &= (d_w\Delta w(x,t) + R - \alpha\tfrac{u(x,t)+f}{u(x,t)+1}w(x,t-\tau(t)) - l_3 w(x,t))dt \\ &\quad - \rho_3\sigma_3 w(x,t)dB_3(t) - \rho_2 w(x,t)\int_{\mathbb{Y}} y\tilde{N}(dt,dy). \end{aligned} \tag{6.4}$$

In this way, we have shown that the system (6.1) without impulse is equivalent to system (2.6). Therefore, in the following, we just need to analyze the solution of system (6.1).

Obviously, the coefficients of the system conforming to the local Lipschitz continuous, for any given initial data $(X(x,s),\ Y(x,s),\ Z(x,s)) \in C(\Gamma\times[-\bar{\tau},0];\ R_+^3)$, the system (6.1) has a unique maximal local solution $(X(x,t), Y(x,t), Z(x,t)))$ on $\Gamma\times[-\bar{\tau},\tau_e)$, where τ_e is explosion time. Make $k_0 > 0$ be sufficiently large number for

$$\frac{1}{k_0} < \min_{\Gamma\times[-\bar{\tau},0]}\{X(x,t), Y(x,t), Z(x,t)\} \leq \max_{\Gamma\times[-\bar{\tau},0]}\{X(x,t), Y(x,t), Z(x,t)\} < k_0.$$

Define the stopping time

$$\tau_k = \inf\{t \in [0,\tau_e) : \min_{x\in\Gamma, t\in[0,\tau_e)}\{X(x,t), Y(x,t), Z(x,t)\} \leq \frac{1}{k_0} \text{ or } \max_{x\in\Gamma, t\in[0,\tau_e)}\{X(x,t), Y(x,t), Z(x,t)\} \geq k_0\},$$

for each $k \geq k_0,\ k \in N$. We set $\inf\ \emptyset = \infty$ (usually $\emptyset$ is the empty set). We can easily know that τ_k is increasing as $k \to \infty$. Besides, we set $\lim_{k\to\infty}\tau_k = \tau_\infty$, whence $\tau_\infty < \tau_e$. Hence, if we can show that $\tau_\infty = \infty$, then $\tau_e = \infty$ and the solution of system (6.1) is positive.

Define a $C^2(R_+; R)$ function

$$V(t) = \int_\Gamma X^2(x,t)dx + \int_\Gamma Y^2(x,t)dx + \int_\Gamma Z^2(x,t)dx.$$

For $0 \le t < \tau_k \wedge T$, Applying Itô formula to $V(t)$ leads to

$$
\begin{aligned}
& dV(t) \\
&= 2\int_\Gamma X(x,t)\Big(\big(d_u\Delta X(x,t)+\tfrac{A_v(t)Y(x,t)}{A_v(t)Y(x,t)+1}X(x,t) - (l_1-A_u(t)\ln(1+I_{ku}))X(x,t)\big)dt - \rho_1\sigma_1 X(x,t)\times \\
& dB_1(t)\Big)dx + 2\int_\Gamma Y(x,t)\Big(\big(d_v\Delta Y(x,t)+\alpha A_v(t)^{-1}\frac{A_u(t)X(x,t)+f}{A_u(t)X(x,t)+1}A_w(t-\tau(t))Z(x,t-\tau(t))
\end{aligned}
$$

$$
\begin{aligned}
& -(l_2-A_v(t)\ln(1+I_{kv}))Y(x,t) - \gamma A_v(t)^{-1}\frac{A_v(t)Y(x,t)}{A_v(t)Y(x,t)+1}A_u(t)X(x,t)\big)dt - \rho_2\sigma_2 Y(x,t)dB_2(t)\Big)dx \\
& + 2\int_\Gamma Z(x,t)\Big(\big(d_w\Delta Z(x,t)+RA_w(t)^{-1}-\alpha\frac{A_u(t)X(x,t)+f}{A_u(t)X(x,t)+1}\frac{A_w(t-\tau(t))}{A_w(t)}Z(x,t-\tau(t)) \\
& -(l_3-A_w(t)\ln(1+I_{kw}))Z(x,t)\big)dt - \rho_3\sigma_3 Z(x,t)dB_3(t)\Big)dx + \int_\Gamma(\rho_1^2\sigma_1^2X^2(x,t)+\rho_2^2\sigma_2^2Y^2(x,t) \\
& +\rho_3^2\sigma_3^2Z^2(x,t))dtdx+\int_{\mathbb{Y}}[\int_\Gamma(1-\rho_1 y)^2X^2(x,t)dx-\int_\Gamma X^2(x,t)dx]\tilde{N}(dt,dy)+\int_{\mathbb{Y}}[\int_\Gamma(1-\rho_2 y)^2Y^2(x,t)dx \\
& -\int_\Gamma Y^2(x,t)dx]\tilde{N}(dt,dy)+\int_{\mathbb{Y}}[\int_\Gamma(1-\rho_3 y)^2Z^2(x,t)dx-\int_\Gamma Z^2(x,t)dx]\tilde{N}(dt,dy) \\
& +\int_{\mathbb{Y}}[\int_\Gamma(1-\rho_1 y)^2X(x,t)^2dx-\int_\Gamma X(x,t)^2dx+\int_\Gamma 2X(x,t)dx\rho_1 yX(x,t)]\lambda(dy)dtdx \\
& +\int_\Gamma\int_{\mathbb{Y}}[\int_\Gamma(1-\rho_2 y)^2Y(x,t)^2dx-\int_\Gamma Y(x,t)^2dx+\int_\Gamma 2Y(x,t)dx\rho_2 yY(x,t)]\lambda(dy)dtdx \\
& +\int_\Gamma\int_{\mathbb{Y}}[\int_\Gamma(1-\rho_3 y)^2Z(x,t)^2dx-\int_\Gamma Z(x,t)^2dx+\int_\Gamma 2Z(x,t)dx\rho_3 yZ(x,t)]\lambda(dy)dtdx.
\end{aligned}
$$

Through some simple calculations and Holder inequality, we can get

$$
\begin{aligned}
& dV(t) \\
&= 2\int_\Gamma X(x,t)\Big(\big(d_u\Delta X(x,t)+\frac{A_v(t)Y(x,t)}{A_v(t)Y(x,t)+1}X(x,t) - (l_1-A_u(t)\ln(1+I_{ku}))X(x,t)\big)dt \\
&\quad - \rho_1\sigma_1 X(x,t)dB_1(t)\Big)dx + 2\int_\Gamma Y(x,t)\Big(\big(d_v\Delta Y(x,t)+\alpha A_v(t)^{-1}\frac{A_u(t)X(x,t)+f}{A_u(t)X(x,t)+1}\times \\
&\quad A_w(t-\tau(t))Z(x,t-\tau(t))-(l_2-A_v(t)\ln(1+I_{kv}))Y(x,t) - \gamma A_v(t)^{-1}\frac{A_v(t)Y(x,t)}{A_v(t)Y(x,t)+1}\times \\
&\quad A_u(t)X(x,t)\big)dt - \rho_2\sigma_2 Y(x,t)dB_2(t)\Big)dx + 2\int_\Gamma Z(x,t)\Big(\big(d_w\Delta Z(x,t)+RA_w(t)^{-1} \\
&\quad -\alpha\frac{A_w(t-\tau(t))}{A_w(t)}\frac{A_u(t)X(x,t)+f}{A_u(t)X(x,t)+1}Z(x,t-\tau(t))-(l_3-A_w(t)\ln(1+I_{kw}))Z(x,t)\big)dt \\
&\quad - \rho_3\sigma_3 Z(x,t)dB_3(t)\Big)dx + \int_\Gamma(\rho_1^2\sigma_1^2X^2(x,t)+\rho_2^2\sigma_2^2Y^2(x,t)+\rho_3^2\sigma_3^2Z^2(x,t))dtdx \\
&\quad +\int_\Gamma\int_{\mathbb{Y}}\rho_1 y(\rho_1 y-2)\tilde{N}(dt,dy)X^2(x,t)dx+\int_\Gamma\int_{\mathbb{Y}}\rho_2 y(\rho_2 y-2)\tilde{N}(dt,dy)Y^2(x,t)dx \\
&\quad +\int_\Gamma\int_{\mathbb{Y}}\rho_3 y(\rho_3 y-2)\tilde{N}(dt,dy)Z^2(x,t)dx+\int_\Gamma\int_{\mathbb{Y}}\rho_1 y(\rho_1 y+2\mathbf{m}(\Gamma)-2)\lambda(dy)X(x,t)^2dtdx \\
&\quad +\int_\Gamma\int_{\mathbb{Y}}\rho_2 y(\rho_2 y+2\mathbf{m}(\Gamma)-2)\lambda(dy)Y(x,t)^2dtdx+\int_\Gamma\int_{\mathbb{Y}}\rho_3 y(\rho_3 y+2\mathbf{m}(\Gamma)-2)\lambda(dy)Z(x,t)^2dtdx.
\end{aligned}
$$

Assign

$$\begin{aligned}
\mathcal{L}V(t) = & \int_\Gamma X(x,t)\Big(d_u\Delta X(x,t)+\tfrac{A_v(t)Y(x,t)}{A_v(t)Y(x,t)+1}X(x,t) - (l_1 - A_u(t)\ln(1+I_{ku}))X(x,t)\Big)dx \\
& + \int_\Gamma Y(x,t)\Big(d_v\Delta Y(x,t)+\alpha A_v(t)^{-1}\frac{A_u(t)X(x,t)+f}{A_u(t)X(x,t)+1}A_w(t-\tau(t))Z(x,t-\tau(t)) \\
& -(l_2 - A_v(t)\ln(1+I_{kv}))Y(x,t) - \frac{\gamma A_v(t)^{-1}A_v(t)Y(x,t)}{A_v(t)Y(x,t)+1}A_u(t)X(x,t)\Big)dx \\
& + \int_\Gamma Z(x,t)\Big(d_w\Delta Z(x,t)+RA_w(t)^{-1} - \alpha\frac{A_w(t-\tau(t))}{A_w(t)}\frac{A_u(t)X(x,t)+f}{A_u(t)X(x,t)+1}Z(x,t-\tau(t)) \\
& -(l_3 - A_w(t)\ln(1+I_{kw}))Z(x,t)\Big)dx + \int_\Gamma (\rho_1^2\sigma_1^2X^2(x,t)+\rho_2^2\sigma_2^2Y^2(x,t)+\rho_3^2\sigma_3^2Z^2(x,t))dx
\end{aligned}$$

$$\begin{aligned}
& + \int_\Gamma\int_{\mathbb{Y}} \rho_1 y(\rho_1 y + 2\mathbf{m}(\Gamma) - 2)\lambda(dy)X(x,t)^2dx + \int_\Gamma\int_{\mathbb{Y}} \rho_2 y(\rho_2 y + 2\mathbf{m}(\Gamma) - 2)\lambda(dy)Y(x,t)^2dx \\
& + \int_\Gamma\int_{\mathbb{Y}} \rho_3 y(\rho_3 y + 2\mathbf{m}(\Gamma) - 2)\lambda(dy)Z(x,t)^2dx.
\end{aligned}$$

In view of the partial integral formula, some basic inequalities and hypothesis (H1), we deduce that

$$\begin{aligned}
& \mathcal{L}V(t) \\
\leq & -\int_\Gamma d_u(\nabla X(x,t))^2dx + \int_\Gamma X(x,t)\tfrac{A_v(t)Y(x,t)}{A_v(t)Y(x,t)+1}X(x,t)dx + \int_\Gamma A_u(t)\ln(1+I_{ku})X^2(x,t)dx \\
& - \int_\Gamma Y(x,t)d_v(\nabla Y(x,t))^2dx + \int_\Gamma Y(x,t)\alpha A_v(t)^{-1}\frac{A_u(t)X(x,t)+f}{A_u(t)X(x,t)+1}A_w(t-\tau(t))Z(x,t-\tau(t))dx \\
& + \int_\Gamma \gamma A_v(t)^{-1}\frac{A_v(t)Y(x,t)}{A_v(t)Y(x,t)+1}A_u(t)X(x,t)Y(x,t)dx + \int_\Gamma A_v(t)\ln(1+I_{kv}Y^2(x,t)dx \\
& - \int_\Gamma d_w(\nabla Z(x,t))^2dx + \int_\Gamma RA_w(t)^{-1}Z(x,t)dx + \int_\Gamma \alpha\frac{A_u(t)X(x,t)+f}{A_u(t)X(x,t)+1}\frac{A_w(t-\tau(t))}{A_w(t)}Z(x,t-\tau(t)) \\
& Z(t)dx + \int_\Gamma A_w(t)\ln(1+I_{kw})Z^2(x,t)dx + \int_\Gamma (\rho_1^2\sigma_1^2X^2(x,t)+\rho_2^2\sigma_2^2Y^2(x,t)+\rho_3^2\sigma_3^2Z^2(x,t))dx \\
& + \int_\Gamma\int_{\mathbb{Y}} \rho_1 y(\rho_1 y + 2\mathbf{m}(\Gamma) - 2)\lambda(dy)X(x,t)^2dx + \int_\Gamma\int_{\mathbb{Y}} \rho_2 y(\rho_2 y + 2\mathbf{m}(\Gamma) - 2)\lambda(dy)Y(x,t)^2dx \\
& + \int_\Gamma\int_{\mathbb{Y}} \rho_3 y(\rho_3 y + 2\mathbf{m}(\Gamma) - 2)\lambda(dy)Z(x,t)^2dx \\
\leq & \int_\Gamma (1 + A_u(t)\ln(1+I_{ku}) + \gamma A_v(t)^{-1}A_u(t) + \rho_1^2\sigma_1^2)X^2(x,t)dtdx + RA_w(t)^{-1}\mathbf{m}(\Gamma) \\
& + \int_\Gamma \alpha(A_v(t)^{-1}(1+f)A_w(t-\tau(t)) + A_v(t)\ln(1+I_{kv}) + \gamma A_v(t)^{-1}A_u(t) + \rho_2^2\sigma_2^2)Y^2(x,t)dtdx \\
& + \int_\Gamma (\alpha A_v(t)^{-1}(1+f)A_w(t-\tau(t)) + \alpha(1+f)A_w(t-\tau(t))A_w^{-1}(t))Z^2(x,t-\tau(t))dtdx \\
& + \int_\Gamma (1 + A_w(t)\ln(1+I_{kw}) + \alpha(1+f)A_w(t-\tau(t))A_w^{-1}(t))Z^2(x,t-\tau(t) + \rho_3^2\sigma_3^2)Z^2(x,t)dtdx \\
& + \int_\Gamma\int_{\mathbb{Y}} \rho_1 y(\rho_1 y + 2\mathbf{m}(\Gamma) - 2)\lambda(dy)X(x,t)^2dx + \int_\Gamma\int_{\mathbb{Y}} \rho_2 y(\rho_2 y + 2\mathbf{m}(\Gamma) - 2)\lambda(dy)Y(x,t)^2dx \\
& + \int_\Gamma\int_{\mathbb{Y}} \rho_3 y(\rho_3 y + 2\mathbf{m}(\Gamma) - 2)\lambda(dy)Z(x,t)^2dx \\
\leq & K_0(1 + \int_\Gamma X^2(x,t)dx + \int_\Gamma Y^2(x,t)dx + \int_\Gamma Z^2(x,t)dx + \int_\Gamma Z^2(x,t-\tau(t))dx)
\end{aligned}$$

where

$$
\begin{aligned}
K_0 = \max\Big\{ &\sup_{t\in[0,\tau_k\wedge T)} (1 + A_u(t)\ln(1 + I_{ku}) + \rho_1^2\sigma_1^2 + \gamma A_v(t)^{-1}A_u(t) + L_1),\\
&\sup_{t\in[0,\tau_k\wedge T)} (A_v(t)^{-1}(1 + f)A_w(t - \tau(t)) + \alpha(1 + f) + \rho_2^2\sigma_2^2 + \gamma A_v(t)^{-1}A_u(t) + L_2),\\
&\sup_{t\in[0,\tau_k\wedge T)} (1 + A_w(t)\ln(1+I_{kw}) + \alpha(1 + f)A_w(t - \tau(t))A_w^{-1}(t)) + \rho_3^2\sigma_3^2 + L_3),\\
&\sup_{t\in[0,\tau_k\wedge T)} (\alpha A_v(t)^{-1}(1 + f)A_w(t - \tau(t)) + \alpha(1 + f)A_w(t - \tau(t))A_w^{-1}(t)),\\
&\sup_{t\in[0,\tau_k\wedge T)} (RA_w(t)^{-1}\mathbf{m}(\Gamma)).\Big\}.
\end{aligned}
$$

Therefore, we can know that

$$
\begin{aligned}
dV(t) = \mathcal{L}V(t)dt - 2\int_\Gamma &\rho_1\sigma_1 X(x,t)^2dB_1(t) + \rho_2\sigma_2 Y(x,t)^2dB_2(t) + \rho_3\sigma_3 Z(x,t)^2dB_3(t)dx\\
&+ \int_{\mathbb{Y}} \rho_1 y(\rho_1 y - 2)\tilde{N}(dt,dy)X^2(x,t) + \int_{\mathbb{Y}} \rho_2 y(\rho_2 y - 2)\tilde{N}(dt,dy)Y^2(x,t)\\
&+ \int_{\mathbb{Y}} \rho_3 y(\rho_3 y - 2)\tilde{N}(dt,dy)Z^2(x,t)
\end{aligned}
\tag{6.5}
$$

Integrating bosh sides of (6.5) from 0 to $t_1 \wedge \tau_k$ and taking expectations gives that

$$
\begin{aligned}
EV(t_1 \wedge \tau_K) &= V(0) + E\int_0^{t_1\wedge\tau_k} (K_0(1 + V(s) + \int_\Gamma Z^2(x, s - \tau(s))dx))ds\\
&\le V(0)+E\int_0^{t_1\wedge\tau_k} K_0 ds+E\int_0^{t_1\wedge\tau_k} \frac{K_0}{1-\eta}\int_\Gamma Z^2(x, s - \tau(s))dxd(s - \tau(s))+E\int_0^{t_1\wedge\tau_k} K_0V(s)ds\\
&\le V(0) + E\int_{-\bar{\tau}}^0 \frac{K_0}{1-\eta}(\int_\Gamma Z^2(x,s)dx)ds + E\int_0^{t_1\wedge\tau_k} \frac{K_0}{1-\eta}(\int_\Gamma Z^2(x,s)dx)ds\\
&\quad + K_0T + E\int_0^{t_1\wedge\tau_k} K_0V(s)ds\\
&\le C_1 + E\int_0^{t_1\wedge\tau_k}\int_\Gamma K_0X^2(x,s)dx + \int_\Gamma K_0Y^2(x,s)dx + \int_\Gamma K_0(1 + \frac{1}{1-\eta})Z^2(x,s)dxds\\
&\le C_1 + K_1\int_0^{t_1\wedge\tau_k} V(s)ds
\end{aligned}
$$

where $C_1 = V(0) + E\int_{-\bar{\tau}}^0 \frac{K}{1-\eta}(\int_\Gamma Z^2(x,s)dx)ds + KT < \infty$, $K_1 = \max\{K, K + \frac{K}{1-\eta}\}$. Further, we can drive that

$$
EV(t_1 \wedge \tau_k) \le C_1 + K_1E\int_0^{t_1\wedge\tau_k} V(t)dt \le C_1 + K_1E\int_0^{t_1} V(t\wedge\tau_k)dt \le C_1 + K_1\int_0^{t_1} EV(t\wedge\tau_k)dt.
\tag{6.6}
$$

For $\forall\, t_1 \in [0,T]$, (6.6) holds, then, it follows from Gronwall inequalities [43] that

$$
EV(t_1 \wedge \tau_k) \le C_1e^{K_1T}, \quad 0 \le t_1 \le T,
\tag{6.7}
$$

for any $k \ge k_0$. Particularly,

$$
EV(T \wedge \tau_k) \le C_1e^{K_1T}, \quad \forall\, k \ge k_0.
\tag{6.8}
$$

Define

$$\beta(k) = \inf_{\min\{u(x,t),v(x,t),v(x,t)\}\geq k,\ 0\leq t\leq\infty} V(t), \quad \forall\, k \geq k_0.$$

Thus, (6.8) implies that

$$\beta(k)\mathcal{P}(\tau_k \leq T) \leq E(V(\tau_k)I_{\tau_k\leq T}) \leq EV(\tau_k \wedge T) \leq C_1 e^{K_1 T}. \tag{6.9}$$

However, we can easily see that

$$\lim_{k\to\infty} \beta(k) = \infty.$$

Letting $k \to \infty$ in (6.9), one can deduce that $\mathcal{P}(\tau_\infty \leq T) = 0$, that is

$$\mathcal{P}(\tau_\infty \geq T) = 1. \tag{6.10}$$

For the arbitrariness of T, we must have $\tau_\infty = \infty$. Then, the system (6.1) has a unique global positive solution. Therefore, we complete the proof.

Algorithm

Step 1: **for** $i = 1 : N_x$
  for $j = -N_{tau} : 0$
    $u_{i,j} = u_0;\ v_{i,j} = v_0;\ w_{i,j} = w_0;$
  end
  for $j = N_t + 1 : N_t + N_{tau}$
    $p^1_{i,j} = 0;\ p^2_{i,j} = 0;\ p^3_{i,j} = 0;$
  end
end
$o = [o_1, o_2, o_3, \cdots]$ $\qquad \tau(j) = tau$
Step 2: **for** $i = 1 : N_x - 1$
  for $j = 0 : N_t - 1$
    $u_{i,j+1} = u_{i,j} + State_1;\ v_{i,j+1} = v_{i,j} + State_2;\ w_{i,j+1} = w_{i,j} + State_3;$
  for $k = N_t - j + 1$
    $p^1_{i,k-1} = p^1_{i,k} - Adjoint_1;\ p^2_{i,k-1} = p^2_{i,k+1} - Adjoint_2;\ p^3_{i,k-1} = p^3_{i,k} - Adjoint_3;$
    for $m = 1 : length(o)$
      if $j + 1 = o(m)$
        $u_{i,j+1} = (1 + I_{ku}\pi^4_{i,j})u_{i,j+1};\ v_{i,j+1} = (1 + I_{kv}\pi^5_{i,j})v_{i,j+1};\ w_{i,j+1} = (1 + I_{kw}\pi^6_{i,j})w_{i,j+1};$
      else
        $u_{i,j+1} = u_{i,j+1};\ v_{i,j+1} = v_{i,j+1};\ w_{i,j+1} = w_{i,j+1};$
      end
  end
      if $k - 1 = o(m)$
        $p^1_{i,k-1} = (1 - I_{ku}\pi^4_{i,j})p^1_{i,k-1} + \bar{P}_1;\ p^2_{i,k-1} = (1 - I_{kv}\pi^5_{i,j})p^2_{i,k-1} + \bar{P}_2;$
        $p^3_{i,k-1} = (1 - I_{kw}\pi^6_{i,j})p^3_{i,k-1} + \bar{P}_3;$
      else
        $p^1_{i,k-1} = p^1_{i,k-1};\ p^1_{i,k-1} = p^1_{i,k-1};\ p^1_{i,k-1} = p^1_{i,k-1};$
      end
    end
  $\pi^1_{i,j} = \frac{-p^1_{i,k}u_{i,j}}{Q_1};\ \pi^2_{i,j} = \frac{-p^2_{i,k}v_{i,j}}{Q_2};\ \pi^3_{i,j} = \frac{-p^3_{i,k}w_{i,j}}{Q_3};$
  $\pi^4_{i,j} = \frac{-p^4_{i,k}I_{ku}u_{i,j}}{Q_4};\ \pi^5_{i,j} = \frac{-p^5_{i,k}I_{kv}v_{i,j}}{Q_5};\ \pi^6_{i,j} = \frac{-p^6_{i,k}I_{kw}w_{i,j}}{Q_6};$
    end
  $u_{1,j} = u_{2,j};\ v_{1,j} = v_{2,j};\ w_{1,j} = w_{2,j}; u_{N_x,j} = u_{N_x-1,j};\ v_{N_x,j} = v_{N_x-1,j};\ w_{N_x,j} = w_{N_x-1,j};$
  $p^1_{1,j} = p^1_{2,j};\ p^2_{1,j} = p^2_{2,j};\ p^3_{1,j} = p^3_{2,j}; p^1_{N_x,j} = p^1_{N_x-1,j};\ p^2_{N_x,j} = p^2_{N_x-1,j};\ p^3_{N_x,j} = p^3_{N_x-1,j};$
  end
  end

where

$$State_1 = [d_u \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{\Delta_x^2} + \pi_1 u_{i,j} + \frac{v_{i,j}u_{i,j}}{v_{i,j} + 1} - l_1 u_{i,j}]\Delta_t$$

$$
\begin{aligned}
&-\rho_1\sigma_1 u_{i,j} rand\sqrt{\Delta_t} - \frac{\rho_1^2}{2}\sigma_1 u_{i,j}^2(\mathrm{rand}^2 - 1)\Delta_t - Z(n)\\
State_2 &= [d_v \frac{v_{i+1,j} - 2v_{i,j} + v_{i-1,j}}{\Delta_x^2} + \alpha\frac{u_{i,j} + f}{u_{i,j} + 1} w_{i,j-\tau(j)} - \gamma\frac{v_{i,j}}{v_{i,j} + 1} u_{i,j} - (l_2 - \pi_2) v_{i,j}]\Delta_t\\
&-\rho_2\sigma_2 v_{i,j} \mathrm{rand}\sqrt{\Delta_t} - \frac{\rho_2^2}{2}\sigma_2 v_{i,j}^2(\mathrm{rand}^2 - 1)\Delta_t - Z(n)\\
State_3 &= [d_w \frac{w_{i+1,j} - 2w_{i,j} + w_{i-1,j}}{\Delta_x^2} + R - \alpha\frac{u_{i,j} + f}{u_{i,j} + 1} w_{i,j-\tau(j)} - (l_3 - \pi_3) w_{i,j}]\Delta_t\\
&-\rho_3\sigma_3 w_{i,j} rand\sqrt{\Delta_t} - \frac{\rho_3^2}{2}\sigma_3 w_{i,j}^2(rand^2 - 1)\Delta_t - Z(n)\\
Adjoint_1 &= [d_u \frac{p_{i+1,k}^1 - 2p_{i,k}^1 + p_{i-1,k}^1}{\Delta_x^2} + (\pi_1 - l_1) p_{i,k}^1 + \frac{v_{i,j}}{v_{i,j} + 1}(p^1 i, k - \gamma p_{i,k}^2)\\
&+ \alpha\frac{1 - f}{(u_{i,j} + 1)^2} w_{i,j-\tau(j)}(p_{i,k}^2 - p_{i,k}^3) - \rho_1\sigma_1 q_1 - \rho_1 r_1 - P_1]\Delta_t + q_1 \mathrm{rand}\sqrt{\Delta_t} - Z(n)\\
Adjoint_2 &= [d_v \frac{p_{i+1,k}^2 - 2p_{i,k}^2 + p_{i-1,k}^2}{\Delta_x^2} + (l_2 - \pi_2) p_{i,k}^2 + \frac{u_{i,j}}{(v_{i,j} + 1)^2}(p^1 i, k - \gamma p_{i,k}^2) - \rho_2\sigma_2 q_2 - \rho_2 r_2 - P_2]\Delta_t\\
&+ q_2 \mathrm{rand}\sqrt{\Delta_t} - Z(n)\\
Adjoint_3 &= [d_w \frac{p_{i+1,k}^3 - 2p_{i,k}^3 + p_{i-1,k}^3}{\Delta_x^2} + \chi_{[0,N_t-\tau(N_t)]}(p_{i,k+\tau}^2 - p_{i,k+\tau}^3)\alpha\frac{u_{i,j} + f}{u_{i,j} + 1} + (\pi_{i,j}^3 - l_3) p_{i,k}^3 - \rho_3\sigma_3 q_3\\
&- \rho_3 r_3 - P_3]\Delta_t + q_3 \mathrm{rand}\sqrt{\Delta_t} - Z(n)
\end{aligned}
$$

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