Mathematical Biosciences and Engineering

http://www.aimspress.com/journal/MBE

MBE, 18(6): 8392–8414.
DOI: 0.3934/mbe.2021416
Received: 25 July 2021
Accepted: 21 September 2021
Published: 27 September 2021

Research article

Long-time behaviors of two stochastic mussel-algae models

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Abstract: In this paper, we develop two stochastic mussel-algae models: one is autonomous and the other is periodic. For the autonomous model, we provide sufficient conditions for the extinction, nonpersistent in the mean and weak persistence, and demonstrate that the model possesses a unique ergodic stationary distribution by constructing some suitable Lyapunov functions. For the periodic model, we testify that it has a periodic solution. The theoretical findings are also applied to practice to dissect the effects of environmental perturbations on the growth of mussel.

Keywords: mussel-algae models; random perturbations; stationary distribution; periodic solution

1. Introduction

A nail-cap-sized zebra mussel was first discovered in the US waters in 1988 and has a powerful reproductive capacity [1]. The invasion of zebra mussel has caused great inconvenience to people, such as blocking pipes, polluting water sources, crowding out local species and causing serious economic losses. According to an estimation from the Center for Invasive Species Research at UC Riverside [1], the US spends as much as 500 million dollars every year to manage mussel in the Great Lakes. As a result, many biologists, ecologists, and mathematicians have studied the invasion of mussel from different perspectives.

The growth and survival of mussel depend heavily on the availability of food sources for algae. A lot of literatures have revealed that the food supply of algae can limit mussel intake [2–4]. In order to uncover the relationships between mussel and algae, Koppel et al. [5] proposed a diffusive mussel-algae model, considering the corresponding nondiffusive form:

$$\begin{cases} \dfrac{\mathrm{d}M(t)}{\mathrm{d}t} = \beta c A(t)M(t) - \dfrac{\mu k}{k+M(t)}M(t), \\ \dfrac{\mathrm{d}A(t)}{\mathrm{d}t} = (A_{up} - A(t))f - \dfrac{c}{h}A(t)M(t), \end{cases} \tag{1.1}$$

where $M(t)$ and $A(t)$ respectively denote the size of mussel and algae, β represents the conversion rate of ingested algae to mussel production, c is the consumption constant, μ is the maximal per capita mussel death rate, k stands for the value of $M(t)$ at which mortality is half maximal, A_{up} denotes the concentration of algae in the upper water layer, f describes the rate of exchange between the lower and upper water layers, h is the height of the lower water layer. All the parameters are positive.

The research on model (1.1) has attracted much attention. For example, based on model (1.1), Koppel et al. [5] analyzed the scale-dependent feedback and regular spatial patterns of young mussel beds, and uncovered that the self-organization patterns would affect the emergent properties of ecosystems in large-scale space. Cangelosi et al. [6] established a mussel-algae model with Turing patterns and carried out a series of stability analyses. Song et al. [7] dissected Turing-Hopf bifurcation of model (1.1) with reaction-diffusion. Similar diffusive models to study the spatial dynamics of mussel-algae can be found in [8–11]. In addition, quite a few researchers pay attention on the control of mussel-algae. A model describing mussel bed appearance was proposed in [12] to explore the habitat suitability analysis for littoral mussel beds in the Dutch Wadden Sea. The effects of 19 macroalgal species on the settlement and metamorphosis of the mussel were investigated [13].

Considering that the growth of mussel is affected by intraspecific competition, we transform model (1.1) into the following model:

$$\begin{cases} \dfrac{\mathrm{d}M(t)}{\mathrm{d}t} = \beta c A(t)M(t) - aM^2(t) - \dfrac{\mu k}{k+M(t)}M(t), \\ \dfrac{\mathrm{d}A(t)}{\mathrm{d}t} = (A_{up} - A(t))f - \dfrac{c}{h}A(t)M(t), \end{cases} \tag{1.2}$$

where a is the intraspecific competition strength of mussel and positive. Other parameters are defined in the same as in model (1.1).

Note that the above studies are all deterministic models. However, environmental uncertainties are ubiquitous in aquatic ecosystems, the populations are always inevitably influenced by environmental noises, which is a momentous element in ecosystems [14]. Environmental stochasticity may involve water temperature, noise, salinity, depth and predators, which might affect the growth and evolution of the populations. Accordingly, stochastic models are usually more realistic, and it is essential to bring environmental stochasticity into model (1.2). Quite a few existing literatures focus on this and obtain excellent results, e.g., survival analysis [15], asymptotic stability [16], stationary distribution [17], optimal harvesting [18, 19] and so on. However, as we know, a very little bit of work has been done with stochastic mussel-algae models, especially the corresponding stochastic version of model (1.2).

For $M(t)$ and $A(t)$ in model (1.2), given $\Delta t > 0$ is a fixed step size. Define $\Gamma^{\Delta t}(p\Delta t) = (M^{\Delta t}(p\Delta t), A^{\Delta t}(p\Delta t))$, $p = 0, 1, 2, \ldots$. Let a normal distribution random variable sequence $\{\Theta_i^{\Delta t}(p)\}_{p=0}^{\infty}$ satisfy $\mathbb{E}[\Theta_i^{\Delta t}(p)] = 0, \mathbb{E}[\Theta_i^{\Delta t}(p)]^2 = \sigma_i^2 \Delta t, i = 1, 2$, where the constants σ_1^2 and σ_2^2 reflect the size of the random perturbations. In each time period $[p\Delta t, (p+1)\Delta t]$, we hypothesize that $\Gamma^{\Delta t}$ grows in the light of the discrete modification of model (1.2) as well as a stochastic amount $(M^{\Delta t}(p\Delta t)\Theta_1^{\Delta t}(p), A^{\Delta t}(p\Delta t)\Theta_2^{\Delta t}(p))$, then we get

$$\begin{cases} M^{\Delta t}((p+1)\Delta t) = M^{\Delta t}(p\Delta t) + \Big[\beta c A^{\Delta t}(p\Delta t)M^{\Delta t}(p\Delta t) - a(M^{\Delta t}(p\Delta t))^2 \\ \qquad\qquad - \dfrac{\mu k}{k + M^{\Delta t}(p\Delta t)} M^{\Delta t}(p\Delta t)\Big]\Delta t + M^{\Delta t}(p\Delta t)\Theta_1^{\Delta t}(p), \\ A^{\Delta t}((p+1)\Delta t) = A^{\Delta t}(p\Delta t) + \Big[(A_{up} - A^{\Delta t}(p\Delta t))f - \dfrac{c}{h}A^{\Delta t}(p\Delta t)M^{\Delta t}(p\Delta t)\Big]\Delta t + A^{\Delta t}(p\Delta t)\Theta_2^{\Delta t}(p). \end{cases}$$

On the basis of [20] (Theorem 7.1 and Lemma 8.2), as $\Delta t \to 0$, $\Gamma^{\Delta t}$ converges weakly to the solution of the following stochastic differential equation:

$$\begin{cases} dM(t) = \left[\beta c A(t)M(t) - aM^2(t) - \dfrac{\mu k}{k + M(t)}M(t)\right]dt + \sigma_1 M(t)dB_1(t), \\ dA(t) = \left[(A_{up} - A(t))f - \dfrac{c}{h}A(t)M(t)\right]dt + \sigma_2 A(t)dB_2(t), \end{cases} \tag{1.3}$$

where $B_1(t)$ and $B_2(t)$ are independent standard Brownian motions defined on a complete probability space $(\Omega, F, \{F\}_{t\geq 0}, P)$.

The effects of a periodically varying environment are important as populations evolve influenced by external effects, for example, seasonal changes, food supply, living habits and other factors, which changes significantly through the whole life of populations. This idea has found much attention and is incorporated into dynamical models [21–25]. Till date, to investigate whether these models will exist period solutions or not is still worth noting. Keeping given this fact, model (1.3) may need to be extended into the following periodic version:

$$\begin{cases} dM(t) = \left[\beta c(t)A(t)M(t) - a(t)M^2(t) - \dfrac{\mu(t)k(t)}{k(t) + M(t)}M(t)\right]dt + \sigma_1(t)M(t)dB_1(t), \\ dA(t) = \left[(A_{up}(t) - A(t))f(t) - \dfrac{c(t)}{h}A(t)M(t)\right]dt + \sigma_2(t)A(t)dB_2(t), \end{cases} \tag{1.4}$$

where the coefficients $c(t)$, $a(t)$, $\mu(t)$, $k(t)$, $A_{up}(t)$, $f(t)$ are positive continuous T-periodic functions.

It is well known that stability is one of the key topics in mathematical biology. For autonomous stochastic population models, scholars are concerned with the stable "stochastic positive equilibrium"—stationary distribution. For periodic stochastic population models, positive periodic solution is an attractive concept. To the best of our knowledge, however, both the stationary distribution of model (1.3) and the existence of periodic solution of model (1.4) have not been considered. The objectives of this paper are to test these two issues. The rest arrange of this paper is as follows. In the next section, the existence and uniqueness of the global positive solution are testified. In Section 3, the extinction, nonpersistent in the mean and weak persistence of model (1.3) are probed. Section 4 provides the conditions under which model (1.3) possesses a unique ergodic stationary distribution. In Section 5, we explore the existence of T-periodic solution of model (1.4). To illustrate the theoretical findings, some numerical simulations are given in Section 6. A few biological meanings of conditions and results are discussed to end Section 7.

2. Existence and uniqueness of the global positive solution

Theorem 2.1. *For arbitrary initial data* $(M(0), A(0)) \in \mathbb{R}^2_+$, *model* (1.3) *has a unique global positive solution with probability one.*

Proof. Recalling model (1.3), assign $M(t) = e^{\widetilde{M}(t)}$, we obtain

$$\begin{cases} d\widetilde{M}(t) = \left[\beta c A(t) - \dfrac{\sigma_1^2}{2} - ae^{\widetilde{M}(t)} - \dfrac{\mu k}{k + e^{\widetilde{M}(t)}}\right]dt + \sigma_1 dB_1(t), \\ dA(t) = \left[(A_{up} - A(t))f - \dfrac{c}{h}A(t)e^{\widetilde{M}(t)}\right]dt + \sigma_2 A(t)dB_2(t) \end{cases} \tag{2.1}$$

with $(\widetilde{M}(0), A(0)) = (\ln M(0), A(0))$. One can see that the coefficients of model (2.1) obey the locally Lipschitz continuous conditions, as a result, it possesses a unique solution $(\widetilde{M}(t), A(t))$ on $[0, \tau_e)$, where $\tau_e \leq +\infty$. Accordingly, model (1.3) possesses a unique positive solution $(M(t), A(t)) = (e^{\widetilde{M}(t)}, A(t))$ on $[0, \tau_e)$. To finish the proof, we only need to testify that $\tau_e = +\infty$ a.s. Choose an integer $n_0 > 0$ which obeys that $1/n_0 \leq M(0), A(0) \leq n_0$. For every $n \geq n_0$, define

$$\tau_n = \inf\{t \in [0, \tau_e] : \min\{M(t), A(t)\} \leq 1/n \text{ or} \max\{M(t), A(t)\} \geq n\}.$$

Set $\tau_\infty = \lim_{n\to+\infty} \tau_n$. As a result, $\tau_\infty \leq \tau_e$. Now we only need to testify that $\tau_\infty = +\infty$. If it is not true, then one can find two constants $T > 0$ and $\epsilon \in (0, 1)$ such that $P\{\tau_\infty \leq T\} > \epsilon$. As a result, one can set an integer $n_1 \geq n_0$ which satisfies

$$P\{\tau_n \leq T\} \geq \epsilon. \tag{2.2}$$

Define

$$V(M, A) = M + h\beta A.$$

Taking advantage of Itô's formula, one has

$$\mathrm{d}V(M, A) = LV(M, A)\mathrm{d}t + M\sigma_1 \mathrm{d}B_1(t) + h\beta A\sigma_2 \mathrm{d}B_2(t), \tag{2.3}$$

where

$$\begin{aligned} LV(M, A) &= \Big(\beta cAM - aM^2 - \frac{\mu k}{k + M} M\Big) + h\beta\Big[(A_{up} - A)f - \frac{c}{h} AM\Big] \\ &= \beta cAM - aM^2 - \frac{\mu k}{k + M} M + h\beta A_{up} f - h\beta f A - \beta cAM \\ &\leq h\beta A_{up} f = G. \end{aligned}$$

Integrating both sides of Eq (2.3) from 0 to $\tau_n \wedge T$ yields

$$\int_0^{\tau_n\wedge T} \mathrm{d}V(M, A) \leq \int_0^{\tau_n\wedge T} G\mathrm{d}t + \int_0^{\tau_n\wedge T} M\sigma_1 \mathrm{d}B_1(t) + \int_0^{\tau_n\wedge T} h\beta A\sigma_2 \mathrm{d}B_2(t).$$

Taking expectation on both sides results in

$$\begin{aligned} \mathbb{E}V(M(\tau_n \wedge T), A(\tau_n \wedge T)) &\leq V(M(0), A(0)) + G\mathbb{E}(\tau_n \wedge T) \\ &\leq V(M(0), A(0)) + GT. \end{aligned} \tag{2.4}$$

Set $\Omega_n = \{\tau_n \leq T\}$ for $n \geq n_1$. According to Eq (2.2), $P(\Omega_n) \geq \epsilon$. For any $\theta \in \Omega_n$, at least one of $M(\tau_n, \theta), A(\tau_n, \theta)$ equals to n or $1/n$. Thus, we derive

$$V(M(\tau_n, \theta), A(\tau_n, \theta)) \geq (n - 1 - \ln n) \wedge \Big(\frac{1}{n} - 1 - \ln \frac{1}{n}\Big).$$

Therefore, Eq (2.4) implies that

$$\begin{aligned} V(M(0), A(0)) + GT &\geq \mathbb{E}[1_{\Omega_n}(\theta)V(M(\tau_n, \theta), A(\tau_n, \theta))] \\ &\geq \varepsilon\Big[(n - 1 - \ln n) \wedge \Big(\frac{1}{n} - 1 - \ln \frac{1}{n}\Big)\Big], \end{aligned}$$

where 1_{Ω_n} denotes the indicator function of Ω_n. Letting $n \to +\infty$ causes the contradiction:

$$V(M(0), A(0)) + GT > +\infty.$$

This finishes the proof. □

Remark 2.1. *Similar to the proof of Theorem 2.1, one can testify that model* (1.4) *has a unique global positive solution with probability one, and the details are left out.*

3. Extinction and persistence of model (1.3)

Lemma 3.1. *Given* $(M(0), A(0)) \in \mathbb{R}_+^2$, *model* (1.3) *admits* $\limsup_{t\to+\infty}[M(t) + \beta hA(t)] < +\infty$ *and*

$$\lim_{t\to+\infty}\frac{1}{t}\int_0^t \sigma_1 M(s)\mathrm{d}B_1(s) = 0, \quad \lim_{t\to+\infty}\frac{1}{t}\int_0^t \sigma_2 A(s)\mathrm{d}B_2(s) = 0 \ a.s.$$

Proof. Denote $Z(t) = M(t) + \beta hA(t)$. From model (1.3), we have

$$\begin{aligned}
\mathrm{d}Z = &\left(\beta cAM - aM^2 - \frac{\mu k}{k+M}M + \beta h\left[(A_{up} - A)f - \frac{c}{h}AM\right]\right)\mathrm{d}t \\
&+ \sigma_1 M(t)\mathrm{d}B_1(t) + \beta h\sigma_2 A(t)\mathrm{d}B_2(t) \\
\le &\ [a(-2M+1) + \beta hA_{up}f - \beta hfA]\mathrm{d}t + \sigma_1 M(t)\mathrm{d}B_1(t) + \beta h\sigma_2 A(t)\mathrm{d}B_2(t) \\
\le &\ (a + \beta hA_{up}f - \delta Z)\mathrm{d}t + \sigma_1 M(t)\mathrm{d}B_1(t) + \beta h\sigma_2 A(t)\mathrm{d}B_2(t),
\end{aligned}$$

where $\delta = \min\{2a, f\} > 0$. Consider

$$\begin{cases} \mathrm{d}Y = (a + \beta hA_{up}f - \delta Y)\mathrm{d}t + \sigma_1 M(t)\mathrm{d}B_1(t) + \beta h\sigma_2 A(t)\mathrm{d}B_2(t), \\ \mathrm{d}Y(0) = (M(0), A(0)). \end{cases} \tag{3.1}$$

The solution of model (3.1) is

$$Y(t) = \frac{a + \beta hA_{up}f}{\delta} + \left[Y(0) - \frac{a + \beta hA_{up}f}{\delta}\right]e^{-\delta t} + N(t),$$

where

$$N(t) = \sigma_1\int_0^t e^{-\delta(t-s)}M(s)\mathrm{d}B_1(s) + \beta h\sigma_2\int_0^t e^{-\delta(t-s)}A(s)\mathrm{d}B_2(s)$$

is a local martingale satisfying $N(0) = 0$ a.s. Thus

$$Y(t) = Y(0) + Q(t) - P(t) + N(t),$$

where

$$Q(t) = \frac{a + \beta hA_{up}f}{\delta}(1 - e^{-\delta t}), \ P(t) = Y(0)(1 - e^{-\delta t})$$

with $Q(0) = P(0) = 0$. Clearly, $Q(t)$ and $P(t)$ are continuous increasing functions. By [26], we have $\lim_{t\to+\infty} Y(t) < +\infty$ a.s., then by stochastic comparison theorem, one has $\limsup_{t\to+\infty} Z(t) < +\infty$ a.s.

Let $N_1 = \int_0^t \sigma_1 M(s)\mathrm{d}B_1(s)$ and $N_2 = \int_0^t \sigma_2 A(s)\mathrm{d}B_2(s)$. Through calculation, we obtain

$$\langle N_1, N_1\rangle(t) = \sigma_1^2\int_0^t M^2(s)\mathrm{d}s,$$

then

$$\lim_{t\to+\infty}\int_0^t \frac{\sigma_1^2 M^2(s)\mathrm{d}s}{(1+s)^2} \le \sigma_1^2\sup_{t\ge0}\{M^2(t)\} < +\infty.$$

In light of [27], $\lim_{t\to+\infty} t^{-1}N_1(t) = 0$ a.s. Similarly, we have $\lim_{t\to+\infty} t^{-1}N_2(t) = 0$. □

Theorem 3.1. *If* $\lambda_0 = \beta cA_{up} - \mu - \sigma_1^2/2 < 0$ *and* $a > \mu/k$, *then* $M(t)$ *is extinct a.s.*

Proof. We deduce from model (1.3) that

$$\begin{aligned}
\mathrm{d}\Big(\frac{1}{h}M + \beta A\Big) &= \frac{1}{h}\Big(\beta c AM - aM^2 - \frac{\mu k}{k+M}M\Big)\mathrm{d}t + \beta\Big[(A_{up} - A)f - \frac{c}{h}AM\Big]\mathrm{d}t \\
&\quad + \frac{\sigma_1}{h}M\mathrm{d}B_1(t) + \beta\sigma_2 A\mathrm{d}B_2(t) \\
&= \Big[\frac{\beta c}{h}AM - \frac{a}{h}M^2 - \frac{\mu k}{h(k+M)}M + \beta(A_{up} - A)f - \frac{\beta c}{h}AM\Big]\mathrm{d}t \\
&\quad + \frac{\sigma_1}{h}M\mathrm{d}B_1(t) + \beta\sigma_2 A\mathrm{d}B_2(t) \\
&= \Big(\beta A_{up}f - \beta f A - \frac{a}{h}M^2 - \frac{\mu k}{h(k+M)}M\Big)\mathrm{d}t + \frac{\sigma_1}{h}M\mathrm{d}B_1(t) + \beta\sigma_2 A\mathrm{d}B_2(t),
\end{aligned}$$

which implies that

$$\beta A_{up}f - \frac{\beta f}{t}\int_0^t A(s)\mathrm{d}s - \frac{a}{ht}\int_0^t M^2(s)\mathrm{d}s - \frac{\mu}{ht}\int_0^t \frac{kM(s)}{k+M(s)}\mathrm{d}s = \frac{\varphi_1(t)}{t}, \tag{3.2}$$

where

$$\varphi_1(t) = \frac{1}{h}M(t) - \frac{1}{h}M(0) + \beta A(t) - \beta A(0) - \frac{1}{h}\int_0^t \sigma_1 M(s)\mathrm{d}B_1(s) - \beta\int_0^t \sigma_2 A(s)\mathrm{d}B_2(s)$$

satisfying $\lim_{t\to+\infty}\varphi_1(t)/t = 0$. In light of Eq (3.2), we have

$$\frac{1}{t}\int_0^t A(s)\mathrm{d}s = A_{up} - \frac{a}{\beta f h t}\int_0^t M^2(s)\mathrm{d}s - \frac{\mu}{\beta f h t}\int_0^t \frac{kM(s)}{k+M(s)}\mathrm{d}s - \frac{\varphi_1(t)}{\beta f t}. \tag{3.3}$$

By the first equation of model (1.3) and using Itô's formula, we obtain

$$\mathrm{d}\ln M(t) = \Big(\beta c A - aM - \frac{\mu k}{k+M} - \frac{1}{2}\sigma_1^2\Big)\mathrm{d}t + \sigma_1\mathrm{d}B_1(t),$$

then together with Eq (3.3), one has

$$\begin{aligned}
\frac{1}{t}\ln\frac{M(t)}{M(0)} &= \frac{\beta c}{t}\int_0^t A(s)\mathrm{d}s - \frac{a}{t}\int_0^t M(s)\mathrm{d}s - \frac{1}{2}\sigma_1^2 - \frac{\mu}{t}\int_0^t \frac{k}{k+M(s)}\mathrm{d}s + \frac{1}{t}\int_0^t \sigma_1\mathrm{d}B_1(s) \\
&= \beta c\Big(A_{up} - \frac{a}{\beta f h t}\int_0^t M^2(s)\mathrm{d}s - \frac{\mu}{\beta f h t}\int_0^t \frac{kM(s)}{k+M(s)}\mathrm{d}s - \frac{\varphi_1(t)}{\beta f t}\Big) \\
&\quad - \frac{a}{t}\int_0^t M(s)\mathrm{d}s - \frac{1}{2}\sigma_1^2 - \mu + \frac{1}{t}\int_0^t \frac{\mu M(s)}{k+M(s)}\mathrm{d}s + \frac{1}{t}\int_0^t \sigma_1\mathrm{d}B_1(s) \\
&= \beta c A_{up} - \mu - \frac{1}{2}\sigma_1^2 - \frac{ac}{fht}\int_0^t M^2(s)\mathrm{d}s - \frac{\mu c}{fht}\int_0^t \frac{kM(s)}{k+M(s)}\mathrm{d}s - \frac{c\varphi_1(t)}{ft} \\
&\quad - \frac{1}{t}\int_0^t \Big(a - \frac{\mu}{k+M(s)}\Big)M(s)\mathrm{d}s + \frac{1}{t}\int_0^t \sigma_1\mathrm{d}B_1(s).
\end{aligned} \tag{3.4}$$

Since the strong law of numbers implies that

$$\lim_{t\to+\infty}\frac{1}{t}\int_0^t \sigma_1\mathrm{d}B_1(s) = 0. \tag{3.5}$$

Thus, it follows from Eqs (3.4) and (3.5) that

$$\lim_{t\to+\infty} t^{-1}\ln M(t) \le \beta c A_{up} - \mu - \frac{1}{2}\sigma_1^2 < 0,$$

which implies the required assertion. □

Theorem 3.2. *If* $\lambda_0 = 0$ *and* $a > \mu/k$, *then* $M(t)$ *is nonpersistent in the mean a.s., namely,* $\lim_{t\to+\infty} t^{-1}\int_0^t M(s)\mathrm{d}s = 0$ *a.s.*

Proof. Let $\rho > 0$ be a constant which satisfies $\rho < a - \mu/k$. From Eq (3.4), we have

$$\begin{aligned}\ln M(t) - \ln M(0) &= \lambda_0 t - \frac{ac}{fh}\int_0^t M^2(s)\mathrm{d}s - \frac{\mu c}{fh}\int_0^t \frac{kM(s)}{k+M(s)}\mathrm{d}s - \frac{c\varphi_1(t)}{f} \\ &\quad - \int_0^t \Big(a - \frac{\mu}{k+M(s)}\Big)M(s)\mathrm{d}s + \int_0^t \sigma_1 \mathrm{d}B_1(s) \\ &\le - \int_0^t \rho M(s)\mathrm{d}s + \int_0^t \sigma_1 \mathrm{d}B_1(s).\end{aligned} \tag{3.6}$$

Note that for any $\varepsilon > 0$, there is $T > 0$ such that for $t \ge T$,

$$t^{-1}\ln M(0) \le \varepsilon/2,\ t^{-1}\int_0^t \sigma_1 \mathrm{d}B_1(s) \le \varepsilon/2. \tag{3.7}$$

Substituting Eq (3.7) into Eq (3.6), we have

$$\ln M(t) \le \varepsilon t - \rho\int_0^t M(s)\mathrm{d}s,\ t \ge T.$$

Set $\varrho(t) = \int_0^t M(s)\mathrm{d}s$, then we get

$$\ln(\mathrm{d}\varrho(t)/\mathrm{d}t) \le \varepsilon t - \rho\varrho(t).$$

Hence for $t > T$, we have

$$e^{\rho\varrho(t)}(\mathrm{d}\varrho(t)/\mathrm{d}t) \le e^{\varepsilon t}.$$

Integrating this inequality from T to t, one can derive that

$$\rho^{-1}(e^{\rho\varrho(t)} - e^{\rho\varrho(T)}) \le \varepsilon^{-1}(e^{\varepsilon t} - e^{\varepsilon T}).$$

That is,

$$e^{\rho\varrho(t)} \le e^{\rho\varrho(T)} + \rho\varepsilon^{-1}e^{\varepsilon t} - \rho\varepsilon^{-1}e^{\varepsilon T}. \tag{3.8}$$

Taking the logarithm of both sides of Eq (3.8) results in

$$\varrho(t) \le \rho^{-1}\ln(e^{\rho\varrho(T)} + \rho\varepsilon^{-1}e^{\varepsilon t} - \rho\varepsilon^{-1}e^{\varepsilon T}).$$

Note that $\varrho(t) = \int_0^t M(s)\mathrm{d}s$, then one can obtain that

$$\limsup_{t\to+\infty} t^{-1}\int_0^t M(s)\mathrm{d}s \le \rho^{-1}\limsup_{t\to+\infty} t^{-1}\ln\Big\{e^{\rho\varrho(T)} + \rho\varepsilon^{-1}e^{\varepsilon t} - \rho\varepsilon^{-1}e^{\varepsilon T}\Big\}.$$

Applying L'Hospital's rule leads to

$$\limsup_{t\to+\infty} t^{-1}\int_0^t M(s)\mathrm{d}s \le \varepsilon/\rho.$$

It then follows from the arbitrariness of ε that $\limsup_{t\to+\infty} t^{-1}\int_0^t M(s)\mathrm{d}s \le 0$. This proof is complete. □

Theorem 3.3. *If $\lambda_0 > 0$, then $M(t)$ is weakly persistent a.s., namely,* $\limsup_{t\to+\infty} M(t) > 0$ *a.s.*

Proof. We first testify that

$$\limsup_{t\to+\infty} t^{-1} \ln M(t) \leq 0 \; a.s. \tag{3.9}$$

By Itô's formula,

$$\begin{aligned} \mathrm{d}(e^t \ln M) &= e^t \ln M \mathrm{d}t + e^t \mathrm{d} \ln M \\ &= e^t \Big\{ \Big[\ln M + \beta c A - aM - \frac{\mu k}{k+M} - \frac{1}{2}\sigma_1^2 \Big] \mathrm{d}t + \sigma_1 \mathrm{d}B_1(t) \Big\}. \end{aligned}$$

Integrating the both sides from 0 to t, we have

$$e^t \ln M(t) - \ln M(0) = \int_0^t e^s \Big[\ln M(s) + \beta c A(s) - aM(s) - \frac{\mu k}{k+M(s)} - \frac{1}{2}\sigma_1^2 \Big] \mathrm{d}s + W(t), \tag{3.10}$$

where $W(t) = \int_0^t e^s \sigma_1 \mathrm{d}B_1(s)$ is a local martingale with the quadratic form

$$\langle W(t), W(t) \rangle = \sigma_1^2 \int_0^t e^{2s} \mathrm{d}s.$$

By the exponential martingale inequality (see [26] on page 44), for arbitrary positive constants T_0, ι and ν , one has

$$P\Big\{ \sup_{0\leq t\leq T_0} \Big[W(t) - \frac{\iota}{2} \langle W(t), W(t) \rangle \Big] > \nu \Big\} \leq e^{-\iota\nu}.$$

Choose $T_0 = \vartheta r$, $\iota = e^{-\vartheta r}$ and $\nu = \varpi e^{\vartheta r} \ln r$, then we obtain

$$P\Big\{ \sup_{0\leq t\leq \vartheta r} \Big[W(t) - 0.5 e^{-\vartheta r} \langle W(t), W(t) \rangle \Big] > \varpi e^{\vartheta r} \ln r \Big\} \leq r^{-\varpi},$$

where $\varpi > 1$, $\vartheta > 0$. By the Borel-Cantalli lemma (see [26] on page 7), for almost all $\zeta \in \Omega$, there exists a $r_0(\zeta)$ such that for $r \geq r_0(\zeta)$,

$$W(t) \leq 0.5 e^{-\vartheta r} \langle W(t), W(t) \rangle + \varpi e^{\vartheta r} \ln r, \; 0 \leq t \leq \vartheta r. \tag{3.11}$$

Combining Eq (3.10) with Eq (3.11), we obtain

$$\begin{aligned} e^t \ln M(t) - \ln M(0) &\leq \int_0^t e^s \Big[\ln M(s) + \beta c A(s) - aM(s) - \frac{1}{2}\sigma_1^2 \Big] \mathrm{d}s \\ &\quad + 0.5 e^{-\vartheta r} \sigma_1^2 \int_0^t e^{2s} \mathrm{d}s + \varpi e^{\vartheta r} \ln r \\ &= \int_0^t e^s \Big[\ln M(s) + \beta c A(s) - aM(s) - \frac{1}{2}\sigma_1^2 + 0.5 e^{s-\vartheta r} \sigma_1^2 \Big] \mathrm{d}s \\ &\quad + \varpi e^{\vartheta r} \ln r. \end{aligned} \tag{3.12}$$

Since $\ln M(t) + \beta c A(t) - aM(t) - \frac{1}{2}\sigma_1^2 + 0.5 e^{t-\vartheta r} \sigma_1^2$ is bounded, for any $0 \leq s \leq \vartheta r$, there is a constant C independent of r such that

$$\ln M(t) + \beta c A(t) - aM(t) - \frac{1}{2}\sigma_1^2 + 0.5 e^{t-\vartheta r} \sigma_1^2 \leq C. \tag{3.13}$$

Substituting Eq (3.13) into Eq (3.12), we obtain

$$e^t \ln M(t) - \ln M(0) \le C[e^t - 1] + \varpi e^{\vartheta r} \ln r. \tag{3.14}$$

Dividing the both sides of Eq (3.14) by e^t leads to

$$\ln M(t) \le e^{-t} \ln M(0) + C[1 - e^{-t}] + \varpi e^{-t} e^{\vartheta r} \ln r.$$

Consequently, if $\vartheta(r-1) \le t \le \vartheta r$ and $r \ge r_0(\zeta)$, then one can observe that

$$t^{-1} \ln M(t) \le e^{-t} t^{-1} \ln M(0) + C t^{-1}[1 - e^{-t}] + \varpi e^{-\vartheta(r-1)} e^{\vartheta r} t^{-1} \ln r,$$

which is the needed assertion Eq (3.9) by letting $r \to +\infty$.

Now let us testify $\limsup_{t\to+\infty} M(t) > 0$ a.s. If not, then we denote $S = \{\limsup_{t\to+\infty} M(t) = 0\}$, $P(S) > 0$. In light of Eq (3.4), one has

$$\begin{aligned} \frac{1}{t} \ln \frac{M(t)}{M(0)} = \lambda_0 - \frac{ac}{fht} \int_0^t M^2(s) \mathrm{d}s - \frac{\mu c}{fht} \int_0^t \frac{kM(s)}{k + M(s)} \mathrm{d}s - \frac{c\varphi_1(t)}{ft} \\ - \frac{1}{t} \int_0^t \Big(a - \frac{\mu}{k + M(s)}\Big) M(s) \mathrm{d}s + \frac{1}{t} \int_0^t \sigma_1 \mathrm{d}B_1(s). \end{aligned}$$

For all $\zeta \in S$, we have $\lim_{t\to+\infty} M(t,\zeta) = 0$, and the law of large numbers for local martingales indicates that $\lim_{t\to+\infty} \frac{1}{t} \int_0^t \sigma_1 \mathrm{d}B_1(s) = 0$. Thus we have $\limsup_{t\to+\infty} t^{-1} \ln M(t,\zeta) = \lambda_0 > 0$. By Eq (3.9), a contradiction arises. □

4. Ergodic stationary distribution (ESD) of model (1.3)

Now we dissect the stationary distribution for model (1.3) by taking advantage of Has'minskii's results [28]. Denote by $X(t)$ a time-homogeneous Markov process in $\mathbb{R}^n$ which obeys

$$\mathrm{d}X(t) = b(X)\mathrm{d}t + \sum_{r=1}^{m} \sigma_r(X) \mathrm{d}B_r(t).$$

Let $I(x) = (a_{ij}(x))$ be the diffusion matrix of $X(t)$, where

$$a_{ij}(x) = \sum_{r=1}^{m} \sigma_r^i(x) \sigma_r^j(x).$$

For any C^2- function $V_1(x)$, define

$$LV_1 = \sum_{i=1}^{l} b_i(x) \frac{\partial V_1(x)}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^{l} a_{ij}(x) \frac{\partial^2 V_1(x)}{\partial x_i \partial x_j}.$$

Lemma 4.1. *If there is a bounded domain $U \subset \mathbb{R}^d$ with regular boundary such that ([28])*

- *there is a positive number Λ which obeys*

$$\sum_{i,j=1}^{2} a_{ij}(x) \xi_i \xi_j \ge \Lambda |\xi|^2, \ x \in U, \ \xi = (\xi_1, \xi_2) \in \mathbb{R}^d,$$

- *there is a nonnegative C^2- function V_2 such that $LV_2(x) < -1$ for any $x \in \mathbb{R}^d \setminus U$,*

then $X(t)$ admits a unique ESD.

Define

$$R_0 = \frac{A_{up} f \beta c}{(\mu + \frac{1}{2}\sigma_1^2)(f + \frac{1}{2}\sigma_2^2)}.$$

Theorem 4.1. *If $\lambda_0 > 0$ and σ_2^2 is sufficiently small such that*

$$\sigma_2^2 < \min\Big\{\frac{2\lambda_0 f}{\mu + \frac{1}{2}\sigma_1^2},\ h\beta f\Big\},$$

then model (1.3) *admits a unique ESD.*

Proof. Considering the function $V_3(M, A) = -m_1 \ln M - m_2 \ln A$, and m_1, m_2 are positive constants to be chosen later, we obtain

$$\begin{aligned}
LV_3(M, A) &= -\frac{m_1}{M}\Big(\beta c A M - aM^2 - \frac{\mu k}{k + M} M\Big) - \frac{m_2}{A}\Big[(A_{up} - A) f - \frac{c}{h} A M\Big] + \frac{m_1}{2}\sigma_1^2 + \frac{m_2}{2}\sigma_2^2 \\
&= -m_1 \beta c A + a m_1 M + \frac{\mu k m_1}{k + M} - \frac{A_{up}}{A} m_2 f + m_2 f + \frac{c}{h} m_2 M + \frac{m_1}{2}\sigma_1^2 + \frac{m_2}{2}\sigma_2^2 \\
&\leq -m_1 \beta c A - \frac{A_{up}}{A} m_2 f + a m_1 M + \mu m_1 + m_2 f + \frac{c}{h} m_2 M + \frac{m_1}{2}\sigma_1^2 + \frac{m_2}{2}\sigma_2^2 \\
&\leq -2\sqrt{m_1 m_2 \beta c A_{up} f} + (\mu + \sigma_1^2/2) m_1 + (f + \sigma_2^2/2) m_2 + \Big(a m_1 + \frac{c}{h} m_2\Big) M \\
&= -2\sqrt{\frac{\beta c A_{up} f}{(\mu + \sigma_1^2/2)(f + \sigma_2^2/2)}} + 2 + \Big(a m_1 + \frac{c}{h} m_2\Big) M \\
&= -2\Big(\sqrt{\frac{\beta c A_{up} f}{(\mu + \sigma_1^2/2)(f + \sigma_2^2/2)}} - 1\Big) + \Big(a m_1 + \frac{c}{h} m_2\Big) M \\
&= -2\Big(\sqrt{R_0} - 1\Big) + \Big(a m_1 + \frac{c}{h} m_2\Big) M \\
&= -D_1 + \Big(a m_1 + \frac{c}{h} m_2\Big) M,
\end{aligned}$$

where

$$m_1 = \frac{1}{\mu + \sigma_1^2/2},\ m_2 = \frac{1}{f + \sigma_2^2/2},\ D_1 = 2\Big(\sqrt{R_0} - 1\Big) > 0.$$

Define

$$V_4(M, A) = \frac{1}{2}\Big(M + h\beta A\Big)^2 - \ln A,$$

we have

$$
\begin{aligned}
L\Big(\frac{1}{2}(M+h\beta A)^2\Big) &= (M+h\beta A)\Big(\beta cAM - aM^2 - \frac{\mu k}{k+M}M + h\beta\Big[(A_{up}-A)f - \frac{c}{h}AM\Big]\Big) \\
&\quad + \frac{\sigma_1^2}{2}M^2 + \frac{h\beta}{2}\sigma_2^2A^2 \\
&\le (M+h\beta A)(-aM^2 + h\beta A_{up}f - h\beta fA) + \frac{\sigma_1^2}{2}M^2 + \frac{h\beta}{2}\sigma_2^2A^2 \\
&= -aM^3 + h\beta A_{up}fM - h\beta fAM - ah\beta AM^2 + h^2\beta^2A_{up}fA - h^2\beta^2fA^2 \\
&\quad + \frac{\sigma_1^2}{2}M^2 + \frac{h\beta}{2}\sigma_2^2A^2 \\
&\le -aM^3 + h\beta A_{up}fM + h^2\beta^2A_{up}fA - h^2\beta^2fA^2 + \frac{\sigma_1^2}{2}M^2 + \frac{h\beta}{2}\sigma_2^2A^2 \\
&\le -\frac{a}{2}M^3 - \frac{h^2\beta^2f}{2}A^2 + D_2.
\end{aligned}
$$

Notice that $h^2\beta^2f - h\beta\sigma_2^2 > 0$, hence

$$
D_2 = \sup_{(M,A)\in\mathbb{R}_+^2}\Big\{-\frac{a}{2}M^3 + \frac{\sigma_1^2}{2}M^2 + h\beta A_{up}fM - \frac{h^2\beta^2f}{2}A^2 + \frac{h\beta}{2}\sigma_2^2A^2 + h^2\beta^2A_{up}fA\Big\} < +\infty.
$$

In addition,

$$
\begin{aligned}
L(-\ln A) &= -\frac{1}{A}\Big[(A_{up}-A)f - \frac{c}{h}AM\Big] + \frac{\sigma_2^2}{2} \\
&= -\frac{A_{up}f}{A} + f + \frac{c}{h}M + \frac{\sigma_2^2}{2} \\
&= -\frac{A_{up}f}{A} + \frac{c}{h}M + D_3,
\end{aligned}
$$

where $D_3 = f + \sigma_2^2/2$.

Therefore,

$$
\begin{aligned}
LV_4(M,A) &= L\Big(\frac{1}{2}(M+h\beta A)^2\Big) + L(-\ln A) \\
&\le -\frac{a}{2}M^3 - \frac{h^2\beta^2f}{2}A^2 - \frac{A_{up}f}{A} + \frac{c}{h}M + D_2 + D_3.
\end{aligned}
$$

Now define $V_5(M,A) = \lambda V_3(M,A) + V_4(M,A)$, where $\lambda > 0$ is sufficiently large. Hence,

$$
\liminf_{q_1\to+\infty,(M,A)\in\mathbb{R}_+^2\setminus U_{q_1}} V_5(M,A) = +\infty,
$$

where $U_{q_1} = (\frac{1}{q_1}, q_1)\times(\frac{1}{q_1}, q_1)$, q_1 is a sufficiently large number. Notice that $V_5(M,A)$ is continuous. Thus $V_5(M,A)$ has a minimum point (M_0,A_0) in $\mathbb{R}_+^2$. Define

$$
V_6(M,A) = V_5(M,A) - V_5(M_0,A_0).
$$

Thus, we can get

$$
\begin{aligned}
LV_6(M,A) &\le \lambda\Big(-D_1 + \Big(am_1 + \frac{c}{h}m_2\Big)M\Big) + \Big(-\frac{a}{2}M^3 - \frac{h^2\beta^2f}{2}A^2 - \frac{A_{up}f}{A} + \frac{c}{h}M + D_2 + D_3\Big) \\
&\le -\lambda D_1 - \frac{a}{2}M^3 - \frac{h^2\beta^2f}{2}A^2 - \frac{A_{up}f}{A} + \Big[\lambda\Big(am_1 + \frac{c}{h}m_2\Big) + \frac{c}{h}\Big]M + D_2 + D_3.
\end{aligned}
$$

Define a bounded close set:

$$U = \{\varepsilon \leq M \leq 1/\varepsilon, \varepsilon \leq A \leq 1/\varepsilon\},$$

where $0 < \varepsilon < 1$ is sufficient small. We can split $\mathbb{R}_+^2 \setminus U$ into the following four ranges,

$$U_1 = \{M < \varepsilon\},\ U_2 = \{A < \varepsilon\},\ U_3 = \{M > 1/\varepsilon\},\ U_4 = \{A > 1/\varepsilon\}.$$

Case 1. If $(M, A) \in U_1$, then we have

$$\begin{aligned} LV_6(M,A) &\leq -\lambda D_1 - \frac{h^2\beta^2 f}{2}A^2 - \frac{a}{2}M^3 - \frac{A_{up}f}{A} + \left[\lambda\left(am_1 + \frac{c}{h}m_2\right) + \frac{c}{h}\right]\varepsilon + D_2 + D_3 \\ &\leq -\lambda D_1 + \left[\lambda\left(am_1 + \frac{c}{h}m_2\right) + \frac{c}{h}\right]\varepsilon + D_2 + D_3. \end{aligned} \tag{4.1}$$

Case 2. If $(M, A) \in U_2$, then one can see that

$$\begin{aligned} LV_6(M,A) &\leq -\frac{A_{up}f}{\varepsilon} - \frac{a}{2}M^3 + \left[\lambda\left(am_1 + \frac{c}{h}m_2\right) + \frac{c}{h}\right]M + D_2 + D_3 \\ &\leq -\frac{A_{up}f}{\varepsilon} + F_1 + D_2 + D_3, \end{aligned} \tag{4.2}$$

where

$$F_1 = \sup_{M\in\mathbb{R}_+} \left\{ -\frac{a}{2}M^3 + \left[\lambda\left(am_1 + \frac{c}{h}m_2\right) + \frac{c}{h}\right]M \right\}.$$

Case 3. If $(M, A) \in U_3$, then one has

$$\begin{aligned} LV_6(M,A) &\leq -\frac{a}{4}M^3 - \frac{a}{4}M^3 + \left[\lambda\left(am_1 + \frac{c}{h}m_2\right) + \frac{c}{h}\right]M + D_2 + D_3 \\ &\leq -\frac{a}{4\varepsilon^3} + F_2 + D_2 + D_3, \end{aligned} \tag{4.3}$$

where

$$F_2 = \sup_{M\in\mathbb{R}_+} \left\{ -\frac{a}{4}M^3 + \left[\lambda\left(am_1 + \frac{c}{h}m_2\right) + \frac{c}{h}\right]M \right\}.$$

Case 4. If $(M, A) \in U_4$, then we obtain

$$\begin{aligned} LV_6(M,A) &\leq -\frac{h^2\beta^2 f}{2\varepsilon^2} - \frac{a}{2}M^3 + \left[\lambda\left(am_1 + \frac{c}{h}m_2\right) + \frac{c}{h}\right]M + D_2 + D_3 \\ &\leq -\frac{h^2\beta^2 f}{2\varepsilon^2} + F_1 + D_2 + D_3. \end{aligned} \tag{4.4}$$

In $\mathbb{R}_+^2 \setminus U$, let ε be sufficiently small which satisfies

$$\begin{aligned} &-\lambda D_1 + \left[\lambda\left(am_1 + \frac{c}{h}m_2\right) + \frac{c}{h}\right]\varepsilon + D_2 + D_3 < -1, \\ &-\frac{A_{up}f}{\varepsilon} + F_1 + D_2 + D_3 < -1, \\ &-\frac{a}{4\varepsilon^3} + F_2 + D_2 + D_3 < -1, \\ &-\frac{h^2\beta^2 f}{2\varepsilon^2} + F_1 + D_2 + D_3 < -1. \end{aligned} \tag{4.5}$$

It follows from Eqs (4.1)–(4.5) that

$$\sup_{(M,A)\in\mathbb{R}_+^2\setminus U} LV_6(M,A) < -1. \tag{4.6}$$

The diffusion matrix of model (1.3) has the form

$$I(M,A) = \begin{pmatrix} \sigma_1^2 M^2 & 0 \\ 0 & \sigma_2^2 A^2 \end{pmatrix}.$$

Choosing $\Lambda = \min_{(M,A)\in U_q}\{\sigma_1^2 M^2, \sigma_2^2 A^2\} > 0$, we have

$$\sum_{i,j=1}^{2} a_{ij}(M,A)\xi_i\xi_j = \sigma_1^2 M^2 \xi_1^2 + \sigma_2^2 A^2 \xi_2^2 \geq \Lambda|\xi|^2,\ \xi = (\xi_1, \xi_2) \in \mathbb{R}_+^2. \tag{4.7}$$

According to Eqs (4.6), (4.7) and Lemma 4.1 that we complete the proof. □

5. Existence of T-periodic solution of model (1.4)

Consider the stochastic periodic equation

$$\mathrm{d}x(t) = v(t, x(t))\mathrm{d}t + g(t, x(t))\mathrm{d}B(t), \tag{5.1}$$

where $v(t)$ and $g(t)$ are T-periodic functions in t.

Lemma 5.1. *If there exists a function $V_7(t,x) \in C^2$ which is* T*-periodic and satisfies the conditions ([28])*

- $\inf_{|x|>\Theta} V_7(t,x) \to \infty$ *as* $\Theta \to \infty$,
- $LV_7(t,x) \leq -1$ *on the outside of some compact set,*

then there exists a periodic solution to Eq (5.1)*.*

Define

$$R_1 = \frac{\langle (A_{up} f\beta c)^{\frac{1}{2}}\rangle_{\mathrm{T}}}{(\langle \mu + \sigma_1^2/2\rangle_{\mathrm{T}} \langle f + \sigma_2^2/2\rangle_{\mathrm{T}})^{\frac{1}{2}}}.$$

Define $\langle g\rangle_{\mathrm{T}} = \dfrac{1}{\mathrm{T}}\displaystyle\int_0^{\mathrm{T}} g(s)\mathrm{d}s$, where $g(t) \in [0,\infty)$ is an integrable function.

Define $g^u = \max_{t\in[0,+\infty)} g(t)$, $g^l = \min_{t\in[0,+\infty)} g(t)$, where $g(t) \in [0,+\infty)$ is a bounded function.

Theorem 5.1. *If $R_1 > 1$ and $(\sigma_2^2)^u < h\beta f^l$, then model* (1.4) *admits a positive* T*-periodic solution.*

Proof. Define

$$V_8(t, M, A) = -b_1 \ln M - b_2 \ln A,$$

and b_1, b_2 are positive constants to be chosen later. By Itô's formula, we have

$$
\begin{aligned}
LV_8(t,M,A) &= -\frac{b_1}{M}\Big(\beta c(t)AM - a(t)M^2 - \frac{\mu(t)k(t)}{k(t)+M}M\Big) - \frac{b_2}{A}\Big[(A_{up}(t)-A)f(t) - \frac{c(t)}{h}AM\Big] \\
&\quad + \frac{\sigma_1^2(t)}{2}b_1 + \frac{\sigma_2^2(t)}{2}b_2 \\
&= -b_1\beta c(t)A + a(t)b_1M + \frac{\mu(t)k(t)b_1}{k(t)+M} - \frac{A_{up}(t)}{A}b_2f(t) + b_2f(t) + \frac{c(t)}{h}b_2M \\
&\quad + \frac{\sigma_1^2(t)}{2}b_1 + \frac{\sigma_2^2(t)}{2}b_2 \\
&\le -b_1\beta c(t)A - \frac{A_{up}(t)}{A}b_2f(t) + \Big(\mu(t) + \frac{\sigma_1^2(t)}{2}\Big)b_1 + \Big(f(t) + \frac{\sigma_2^2(t)}{2}\Big)b_2 \\
&\quad + \Big(a(t)b_1 + \frac{c(t)}{h}b_2\Big)M \\
&\le -2\sqrt{b_1b_2\beta c(t)A_{up}(t)f(t)} + \Big(\mu(t) + \frac{\sigma_1^2(t)}{2}\Big)b_1 + \Big(f(t) + \frac{\sigma_2^2(t)}{2}\Big)b_2 \\
&\quad + \Big(a^ub_1 + \frac{c^u}{h}b_2\Big)M \\
&= R(t) + \Big(a^ub_1 + \frac{c^u}{h^l}b_2\Big)M,
\end{aligned}
\tag{5.2}
$$

where

$$
\begin{aligned}
R(t) &= -2\sqrt{b_1b_2\beta c(t)A_{up}(t)f(t)} + \Big(\mu(t) + \frac{\sigma_1^2(t)}{2}\Big)b_1 + \Big(f(t) + \frac{\sigma_2^2(t)}{2}\Big)b_2, \\
b_1 &= \frac{1}{\langle \mu + \sigma_1^2/2\rangle_{\mathrm{T}}}, \quad b_2 = \frac{1}{\langle f + \sigma_2^2/2\rangle_{\mathrm{T}}}.
\end{aligned}
$$

Let $\bar{\omega}(t)$ be the solution of the following equation

$$
\bar{\omega}'(t) = \langle R(t)\rangle_{\mathrm{T}} - R(t). \tag{5.3}
$$

Then $\bar{\omega}(t)$ is a T-periodic function. On the basis of Eqs (5.2) and (5.3), we can obtain

$$
\begin{aligned}
L(V_8 + \bar{\omega}(t)) &\le \langle R(t)\rangle_{\mathrm{T}} + \Big(a^ub_1 + \frac{c^u}{h}b_2\Big)M \\
&= -2\frac{\langle (A_{up}f\beta c)^{\frac{1}{2}}\rangle_{\mathrm{T}}}{(\langle \mu + \sigma_1^2/2\rangle_{\mathrm{T}}\langle f + \sigma_2^2/2\rangle_{\mathrm{T}})^{\frac{1}{2}}} + 2 + \Big(a^ub_1 + \frac{c^u}{h}b_2\Big)M \\
&= -2(R_1 - 1) + \Big(a^ub_1 + \frac{c^u}{h}b_2\Big)M \\
&= -\alpha_1 + \Big(a^ub_1 + \frac{c^u}{h}b_2\Big)M,
\end{aligned}
\tag{5.4}
$$

where $\alpha_1 = 2(R_1 - 1) > 0$.

Define

$$
V_9(t,M,A) = \frac{1}{2}(M + h\beta A)^2 - \ln A.
$$

Applying Itô's formula, one has

$$
\begin{aligned}
L\Big(\frac{1}{2}(M+h\beta A)^2\Big) &= (M+h\beta A)\Big(\beta c(t)AM - a(t)M^2 - \frac{\mu(t)k(t)}{k(t)+M}M \\
&\quad + h\beta\Big[(A_{up}(t)-A)f(t) - \frac{c(t)}{h}AM\Big]\Big) + \frac{\sigma_1^2(t)}{2}M^2 + \frac{h\beta}{2}\sigma_2^2(t)A^2 \\
&\leq (M+h\beta A)\Big(-a(t)M^2 + h\beta A_{up}(t)f(t) - h\beta f(t)A\Big) \\
&\quad + \frac{\sigma_1^2(t)}{2}M^2 + \frac{h\beta}{2}\sigma_2^2(t)A^2 \\
&= -a(t)M^3 + h\beta A_{up}(t)f(t)M - h\beta f(t)AM - a(t)h\beta AM^2 \\
&\quad + h^2\beta^2 A_{up}(t)f(t)A - h^2\beta^2 f(t)A^2 + \frac{\sigma_1^2(t)}{2}M^2 + \frac{h\beta}{2}\sigma_2^2(t)A^2 \\
&\leq -a(t)M^3 + h\beta A_{up}(t)f(t)M + h^2\beta^2 A_{up}(t)f(t)A \\
&\quad - h^2\beta^2 f(t)A^2 + \frac{\sigma_1^2(t)}{2}M^2 + \frac{h\beta}{2}\sigma_2^2(t)A^2 \\
&\leq -\frac{a^l}{2}M^3 - \frac{h^2\beta^2 f^l}{2}A^2 + \alpha_2,
\end{aligned}
\tag{5.5}
$$

where

$$
\begin{aligned}
\alpha_2 = \sup_{(M,A)\in\mathbb{R}_+^2} \Big\{ &-\frac{a^l}{2}M^3 + \frac{(\sigma_1^2)^u}{2}M^2 + h\beta A_{up}^u f^u M - \frac{h^2\beta^2 f^l}{2}A^2 \\
&+ \frac{h\beta}{2}(\sigma_2^2)^u A^2 + h^2\beta^2 A_{up}^u f^u A\Big\} < +\infty.
\end{aligned}
$$

Similarly, one deduces

$$
\begin{aligned}
L(-\ln A) &= -\frac{1}{A}\Big[(A_{up}(t)-A)f(t) - \frac{c(t)}{h}AM\Big] + \frac{1}{2}\sigma_2^2(t) \\
&= -\frac{A_{up}(t)}{A}f(t) + f(t) + \frac{c(t)}{h}M + \frac{1}{2}\sigma_2^2(t) \\
&\leq -\frac{A_{up}^l f^l}{A} + \frac{c^u}{h}M + f^u + \frac{1}{2}(\sigma_2^2)^u.
\end{aligned}
\tag{5.6}
$$

According to Eqs (5.5) and (5.6) one can get

$$
LV_9(t,M,A) \leq -\frac{h^2\beta^2 f^l}{2}A^2 - \frac{a^l}{2}M^3 - \frac{A_{up}^l f^l}{A} + \frac{c^u}{h}M + \alpha_2 + f^u + \frac{(\sigma_2^2)^u}{2}. \tag{5.7}
$$

Define

$$
V_{10}(t,M,A) = H(V_8 + \bar{\omega}) + V_9,
$$

where H is positive constant. Clearly,

$$
\liminf_{q_2\to+\infty,\ (M,A)\in\mathbb{R}_+^2\setminus U_{q_2}} V_{10}(t,M,A) \to +\infty,
$$

where $U_{q_2} = (\frac{1}{q_2}, q_2) \times (\frac{1}{q_2}, q_2)$, q_2 is a sufficiently large number. Combining with Eqs (5.4) and (5.7), we have

$$\begin{aligned} LV_{10} &\leq -H\alpha_1 + \Big(a^u b_1 + \frac{c^u}{h} b_2\Big) HM - \frac{h^2\beta^2 f^l}{2} A^2 - \frac{a^l}{2} M^3 - \frac{A_{up}^l f^l}{A} + \frac{c^u}{h} M \\ &\quad + \alpha_2 + f^u + \frac{(\sigma_2^2)^u}{2} \\ &= -H\alpha_1 - \frac{h^2\beta^2 f^l}{2} A^2 - \frac{a^l}{2} M^3 - \frac{A_{up}^l f^l}{A} + \Big[H\Big(a^u b_1 + \frac{c^u}{h} b_2\Big) + \frac{c^u}{h}\Big] M \\ &\quad + \alpha_2 + f^u + \frac{(\sigma_2^2)^u}{2}. \end{aligned}$$

Define a bounded close set:

$$\mathcal{K} = \{(M, A) \in \mathbb{R}_+^2 : \varepsilon \leq M \leq 1/\varepsilon, \varepsilon \leq A \leq 1/\varepsilon\},$$

where $0 < \varepsilon < 1$ is a sufficient small number. We divide $\mathbb{R}_+^2 \setminus \mathcal{K}$ into the following four ranges

$$\mathcal{K}_1 = \{M < \varepsilon\},\ \mathcal{K}_2 = \{A < \varepsilon\},\ \mathcal{K}_3 = \{M > 1/\varepsilon\},\ \mathcal{K}_4 = \{A > 1/\varepsilon\}.$$

Case 1'. If $(M, A) \in \mathcal{K}_1$, then we get

$$\begin{aligned} LV_{10}(t, M, A) &\leq -H\alpha_1 - \frac{h^2\beta^2 f^l}{2} A^2 - \frac{a^l}{2} M^3 - \frac{A_{up}^l f^l}{A} + \Big[H\Big(a^u b_1 + \frac{c^u}{h} b_2\Big) + \frac{c^u}{h}\Big]\varepsilon \\ &\quad + \alpha_2 + f^u + (\sigma_2^2)^u/2 \\ &\leq -H\alpha_1 + \Big[H\Big(a^u b_1 + \frac{c^u}{h} b_2\Big) + \frac{c^u}{h}\Big]\varepsilon + \alpha_2 + f^u + (\sigma_2^2)^u/2. \end{aligned} \tag{5.8}$$

Case 2'. If $(M, A) \in \mathcal{K}_2$, then we have

$$\begin{aligned} LV_{10}(t, M, A) &\leq -\frac{A_{up}^l f^l}{\varepsilon} - \frac{a^l}{2} M^3 + \Big[H\Big(a^u b_1 + \frac{c^u}{h} b_2\Big) + \frac{c^u}{h}\Big] M + \alpha_2 + f^u + (\sigma_2^2)^u/2 \\ &\leq -\frac{A_{up}^l f^l}{\varepsilon} + J_1 + \alpha_2 + f^u + (\sigma_2^2)^u/2, \end{aligned} \tag{5.9}$$

where

$$J_1 = \sup_{M \in \mathbb{R}_+} \Big\{ -\frac{a^l}{2} M^3 + \Big[H\Big(a^u b_1 + \frac{c^u}{h} b_2\Big) + \frac{c^u}{h}\Big] M \Big\}.$$

Case 3'. If $(M, A) \in \mathcal{K}_3$, then we derive

$$\begin{aligned} LV_{10}(t, M, A) &\leq -\frac{a^l}{4} M^3 - \frac{a^l}{4} M^3 + \Big[H\Big(a^u b_1 + \frac{c^u}{h} b_2\Big) + \frac{c^u}{h}\Big] M + \alpha_2 + f^u + (\sigma_2^2)^u/2 \\ &\leq -\frac{a^l}{4\varepsilon^3} + J_2 + \alpha_2 + f^u + (\sigma_2^2)^u/2, \end{aligned} \tag{5.10}$$

where

$$J_2 = \sup_{M \in \mathbb{R}_+} \Big\{ -\frac{a^l}{4} M^3 + \Big[H\Big(a^u b_1 + \frac{c^u}{h} b_2\Big) + \frac{c^u}{h}\Big] M \Big\}.$$

Case 4'. If $(M, A) \in \mathcal{K}_4$, then one has

$$\begin{aligned} LV_{10}(t, M, A) &\leq -\frac{h^2\beta^2 f^l}{2\varepsilon^2} - \frac{a^l}{2} M^3 + \Big[H\Big(a^u b_1 + \frac{c^u}{h} b_2\Big) + \frac{c^u}{h}\Big] M + \alpha_2 + f^u + (\sigma_2^2)^u/2 \\ &\leq -\frac{h^2\beta^2 f^l}{2\varepsilon^2} + J_1 + \alpha_2 + f^u + (\sigma_2^2)^u/2. \end{aligned} \tag{5.11}$$

In the set $\mathbb{R}_+^2 \setminus \mathcal{K}$, we choose ε sufficiently small such that

$$-H\alpha_1 + \left[H\left(a^u b_1 + \frac{c^u}{h} b_2\right) + \frac{c^u}{h}\right]\varepsilon + \alpha_2 + f^u + (\sigma_2^2)^u/2 < -1, \tag{5.12}$$

$$-\frac{A_{up}^l f^l}{\varepsilon} + J_1 + \alpha_2 + f^u + (\sigma_2^2)^u/2 < -1, \tag{5.13}$$

$$-\frac{a^l}{4\varepsilon^3} + J_2 + \alpha_2 + f^u + (\sigma_2^2)^u/2 < -1, \tag{5.14}$$

$$-\frac{h^2\beta^2 f^l}{2\varepsilon^2} + J_1 + \alpha_2 + f^u + (\sigma_2^2)^u/2 < -1. \tag{5.15}$$

It then follows from Eqs (5.8)–(5.15) that $LV_{10}(t, M, A) < -1$ for all $(M, A) \in \mathbb{R}_+^2 \setminus \mathcal{K}$. □

6. Discussion and simulations

In this section, we take advantage of some real data (see Table 1) and the Euler-Maruyama method [34] to illustrate the above results. For model (1.3), we pay attention to the discretization equation:

$$\begin{cases} M_{n+1} = M_n + \left[\beta c A_n M_n - a M_n^2 - \dfrac{\mu k}{k + M_n} M_n\right]\Delta t + \sigma_1 M_n \zeta_{1n}\sqrt{\Delta t} \\ \qquad + \dfrac{1}{2}\sigma_1^2 M_n^2 (\zeta_{1n}^2 - 1)\Delta t, \\ A_{n+1} = A_n + \left[(A_{up} - A_n) f - \frac{c}{h} A_n M_n\right]\Delta t + \sigma_2 A_n \zeta_{2n}\sqrt{\Delta t} \\ \qquad + \dfrac{1}{2}\sigma_2^2 A_n^2 (\zeta_{2n}^2 - 1)\Delta t, \end{cases}$$

where ζ_{1n}, ζ_{2n} mean independent Gaussian random variable.

Table 1. Parameter values used in the simulation.

Symbol	Value	Unit	Source
a	0.01	$g/g/h$	Estimated
f	0.4	$m^3/m^3/h$	Estimated
μ	0.015	$g/g/h$	Estimated
A_{up}	1	g/m^3	[29]
h	0.1	m	[2,30]
c	0.1	$m^3/g/h$	[31,32]
β	0.2	g/g	[33]
k	150	g/m^2	[5]

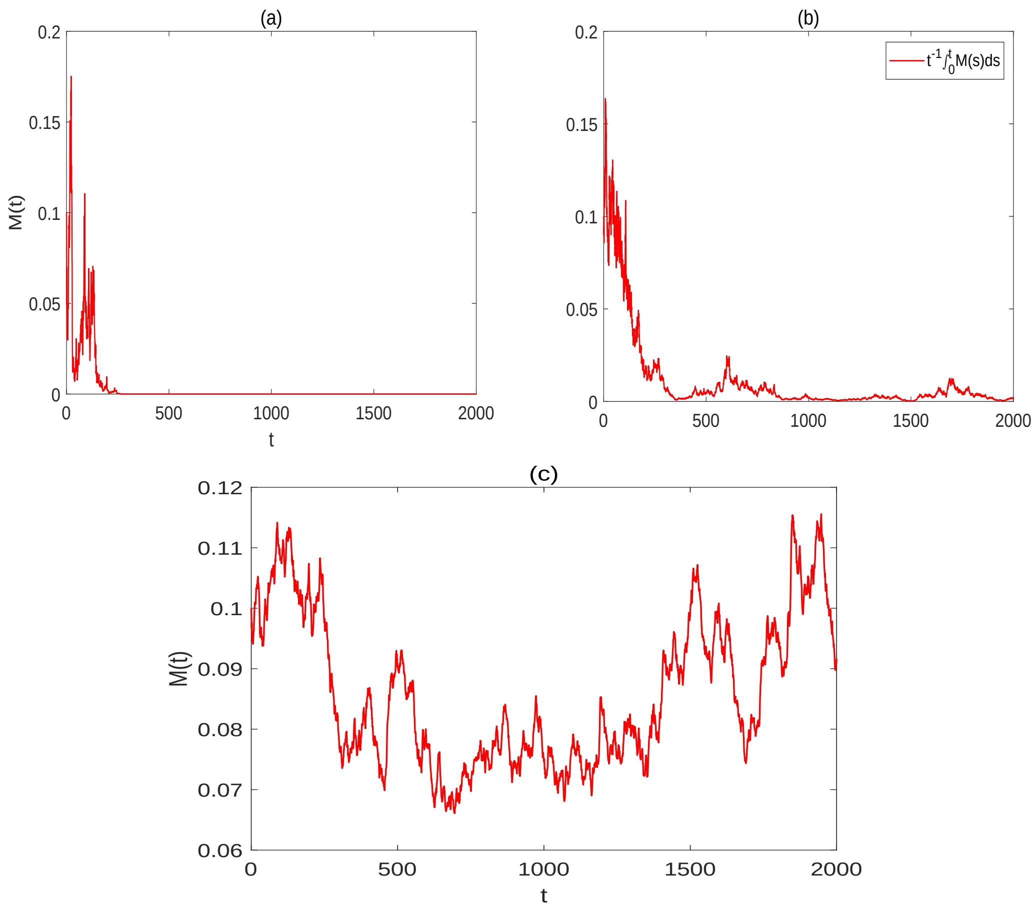

Figure 1. (a) The mussel species of model (1.3) dies out; (b) the mussel species of model (1.3) is nonpersistent in the mean; (c) the mussel species of model (1.3) is weakly persistent.

From Theorems 3.1–3.3, one can find out that under the assumption $a > \mu/k$, λ_0 is sufficient conditions determining the persistence and extinction of the mussel species. More precisely, under the assumption $a > \mu/k$, if $\lambda_0 < 0$, then the mussel species dies out; if $\lambda_0 = 0$, then the mussel species is nonpersistent in the mean. If $\lambda_0 > 0$, then the mussel species is weakly persistent. Note that $\lambda_0 = \beta c A_{up} - \mu - \frac{1}{2}\sigma_1^2$, which suggests that white noise can greatly influence the survival of mussel: when the intensity of the noise is large enough, it could make mussel become extinct.

Theorem 4.1 suggests that if $\lambda_0 > 0$ and σ_2^2 is sufficiently small such that

$$\sigma_2^2 < \min\left\{\frac{2\lambda_0 f}{\mu + \frac{1}{2}\sigma_1^2},\ h\beta f\right\},$$

then model (1.3) possesses a unique ESD on $\mathbb{R}_+^2$. This ESD can be used to estimate the outbreak possibility of mussel.

Figure 1(a)–(c) characterize the persistence and extinction of the mussel species in model (1.3) with different σ_1. We choose $\sigma_2 = 0.3$, initial value $M(0) = 0.1$ and $A(0) = 0.12$. Figure 1(a) is with $\sigma_1 = 0.3$, which reflects that the mussel species dies out with probability one; Figure 1(b) is with $\sigma_1 = 0.1$, which suggests that the mussel species is nonpersistent in the mean; Figure 1(c) is with

$\sigma_1 = 0.01$, which shows that the mussel species is weakly persistent. Comparing Figure 1(c) with Figure 1(a), one can observe that with the increasing value of σ_1, the mussel species tends to go to extinction. In other words, large noise could lead to the extinction of mussel.

Figure 2 plots the probability density function (PDF) of the stationary distribution of model (1.3) with $\sigma_1 = 0.005$ and $\sigma_2 = 0.0048$.

For model (1.4), we focus on the following discretization equation:

$$\begin{cases} M_{n+1} = M_n + \left[\beta c(n\,\Delta t)A_n M_n - a(n\,\Delta t)M_n^2 - \dfrac{\mu(n\,\Delta t)k(n\,\Delta t)}{k(n\,\Delta t) + M_n}M_n\right]\Delta t \\ \qquad + \sigma_1(n\,\Delta t)M_n\zeta_{1n}\sqrt{\Delta t} + \dfrac{1}{2}\sigma_1^2(n\,\Delta t)M_n(\zeta_{1n}^2 - 1)\Delta t, \\ A_{n+1} = A_n + \left[(A_u p(n\,\Delta t) - A_n)f(n\,\Delta t) - \dfrac{c(n\,\Delta t)}{h}A_n M_n\right]\Delta t \\ \qquad + \sigma_2(n\,\Delta t)A_n\zeta_{2n}\sqrt{\Delta t} + \dfrac{1}{2}\sigma_2^2(n\,\Delta t)A_n(\zeta_{2n}^2 - 1)\Delta t, \end{cases}$$

where ζ_{1n}, ζ_{2n} mean independent Gaussian random variable.

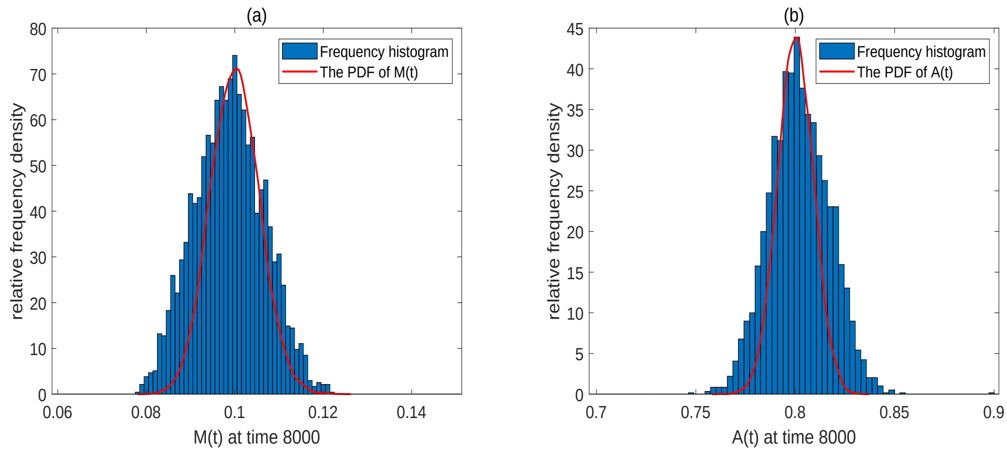

Figure 2. Probability density function of the stationary distribution of model (1.3).

Theorem 5.1 provides sufficient conditions (i.e., $R_1 > 1$ and $(\sigma_2^2)^u < h\beta f^l$) under which model (1.4) admits a positive periodic solution. This offers some insightful understanding on how environmental fluctuations affect the survival of mussel and algae.

The initial values $M(0) = 0.1$ and $A(0) = 0.12$ are kept the same as in Fig.1, and we choose $\beta = 0.2$, $h = 1$, $A_{up}(t) = 10 + 0.1\sin(\pi t)$, $a(t) = 0.2 + 0.1\sin(\pi t)$, $f(t) = 1 + 0.1\sin(\pi t)$, $c(t) = 0.5 + 0.1\sin(\pi t)$, $\mu(t) = 0.1 + 0.1\sin(\pi t)$, $k(t) = 10 + 0.1\sin(\pi t)$, $\sigma_1(t) = 0.03 + 0.01\sin(\pi t)$, $\sigma_2(t) = 0.04 + 0.01\sin(\pi t)$. It follows from Theorem 5.1 that model (1.4) has a T-periodic solution, see Figsure 3(a),(c). Moreover, Figsure 3(a),(b) show that $M(t)$ and $A(t)$ fluctuate periodically, that is, mussel and algae will not die out. In particular, the effect of environmental noises can be easily found by comparing Figsure 3(a),(c) with Figsure 3(b),(d).

7. Conclusions

Understanding the effect of random perturbations on the evolution of the mussel is useful for managing this species. This paper proposed two stochastic mussel-algae models (one is autonomous, and the other is non-autonomous) to test the effect of environmental fluctuations on the evolution of mussel. For the autonomous model, the critical value between extinction and weak persistence was obtained. In addition, sufficient conditions for the existence of an ESD were established. For the non-autonomous model, the existence of a positive periodic solution was examined. Some vital impacts of environmental fluctuations on the evolution of mussel were uncovered.

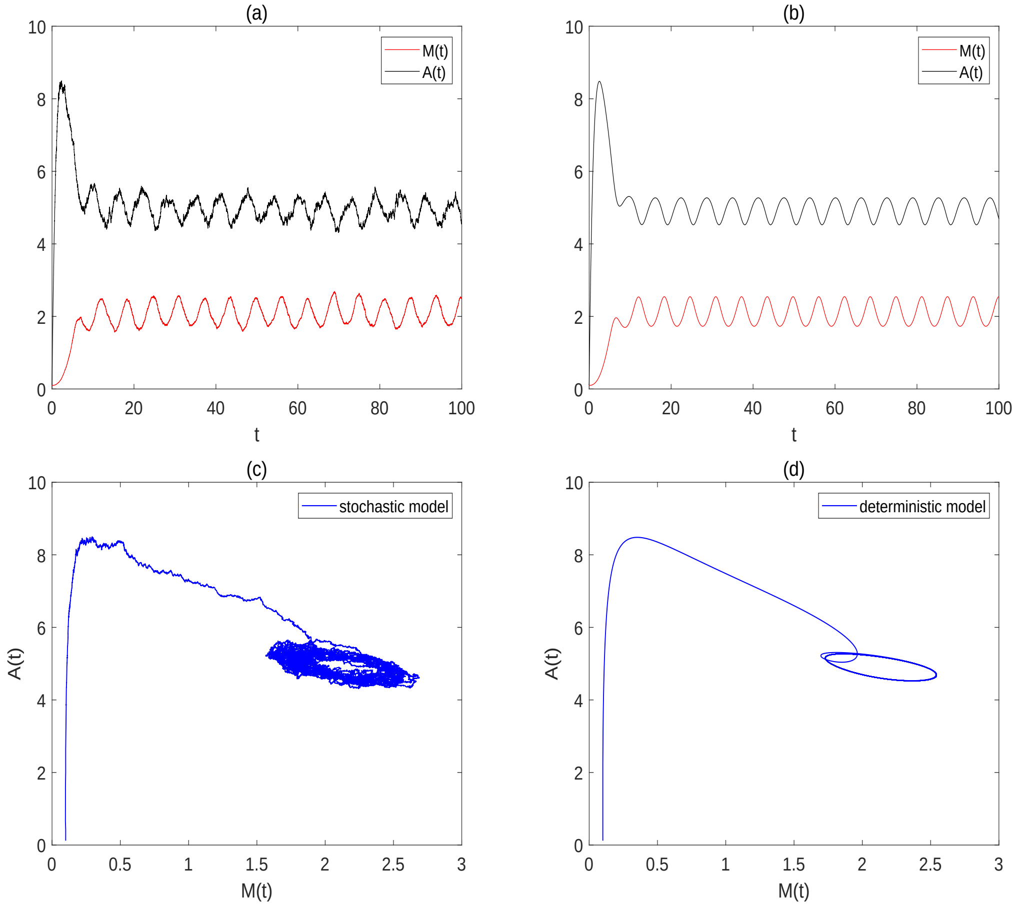

Figure 3. (a) Periodic solution of model (1.4); (b) periodic solution of model (1.4) with $\sigma_1(t) = \sigma_2(t) \equiv 0$; (c) the phase portrait of (a); (d) the phase portrait of (b).

In comparison with the existing papers, this research has the following contributions:

- Our models consider the environmental fluctuations which are more reasonable. Actually, to the best of our knowledge, this research is the first attempt to dissect the stochastic mussel-algae models.
- We obtain the critical value between extinction and weak persistence for the mussel, and uncover that environmental fluctuations can significantly affect the extinction/persistence of the mussel.
- We give some conditions under which model (1.3) has an ESD. This ESD is useful to estimate the outbreak probability of the mussel.
- We provide sufficient conditions for existence of a positive periodic solution of model (1.4). This positive periodic solution is helpful for the understanding how environmental fluctuations affects the survival of mussel and algae.

Some studies on mussel-algal models are worth further investigations. Actually, Theorem 3.1 and Theorem 3.2 have an assumption $a > \mu/k$, what happens when $a < \mu/k$ is still unclear. In addition, Theorem 4.1 testifies that if $\lambda_0 > 0$ and

$$\sigma_2^2 < \min\left\{\frac{2\lambda_0 f}{\mu + \frac{1}{2}\sigma_1^2}, h\beta f\right\},$$

then model (1.3) possesses a unique ESD on $\mathbb{R}_+^2$. It is interesting to relax the restriction on σ_2^2. Finally, one may put forward some more realistic and meaningful models, such as considering the effects of Lévy jump [35, 36], impulsive perturbations [37, 38], time delay [39, 40] or fractional order [41, 42]. We will leave these for future works.

Acknowledgments

The authors thank the editor and referees for their careful reading and valuable comments. This work is supported by the National Natural Science Foundations of China (Nos.11771174, 11871201).

Conflict of interest

The authors declare there is no conflict of interest.

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