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Research article

Modeling the trajectory of motion of a linear dynamic system with multi-point conditions

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Abstract: The motion of the linear dynamic system with given properties is modeled; conditions for system state at various arbitrarily points in time are given. Simulated movement carried out due to the calculated input vector function. The method of undefined coefficients is used to construct the input vector function and the corresponding trajectory. The proposed method consists in the formation of the state vector function, the trajectory of motion and the input vector function in exponential-polynomial form, that is, in the form of linear combinations of the powers of the time parameter with vector coefficients. This linear combination is complemented by a scalar exponential function with an additional parameter in the exponent to change the type of trajectory. To find the introduced coefficients, formulas and a linear algebraic system are formed. To find the introduced coefficients, the formed linear combinations are substituted directly into the equations describing the dynamic system and into the given multipoint conditions for finding the entered coefficients. All this leads to obtaining algebraic formulas and linear algebraic systems. Only the matrices included in the system that describe the dynamics of the model (and similar matrices with higher exponents) are the coefficients for the unknown parameters of the resulting algebraic system. It is proved that the fulfillment of the condition Kalman is sufficient for the solvability of the resulting system. To substantiate the solvability of the system, the properties of finite-dimensional mappings are used: decomposition of spaces into subspaces, projectors on subspaces, semi-inverse operators. But for the practical use of the proposed method, it is sufficient to solve the obtained linear algebraic system and use the obtained linear formulas. The correctness of the obtained model is investigated. Due to the non-uniqueness of the solution to the problem posed, the trajectory of motion can be unstable. It is revealed which components of the desired coefficients are arbitrary. It is showed which ones to choose, to make the movement steady, that is, so that small changes in the given multi-point values, as well as a small change parameters of the dynamic system corresponded to a small change in the trajectory of motion. An example is given of constructing trajectories of a material point in a vertical plane under the action of a reactive force in order to hit a given point with a given speed.

Keywords: dynamic system; the model of a multi-point motion; undetermined coefficients method;

the implementation of the process

1. Introduction

The system

$$\dot{x}(t) = Ax(t) + Bv(t) \tag{1.1}$$

is considered, where $x(t) \in \mathbb{R}^n$, $v(t) \in \mathbb{R}^m$, A and B − matrices of appropriate sizes, $t \in [t_0, t_k]$, $\nexists B^{-1}$. The vector function $x(t)$ is called the state, the trajectory of the system, $v(t)$ − input vector function, input.

Such systems simulate dynamic processes in biology. (change in the population of animals, etc. [1]), in medicine (distribution infectious diseases [2], [3], etc.), in the economy (dynamic model of input-output balance [1], [4] etc.), in electrical engineering [3], etc.); the mechanical motion is described by the basic laws of dynamics [3], [5] and many others processes are also modeled by such systems.

It is known [6] that the complete condition for the possibility of transferring the state of the system (1.1) from an arbitrary initial state

$$x(t_0) = x_0 \tag{1.2}$$

to an arbitrary final state

$$x(t_k) = x_k \tag{1.3}$$

for an arbitrary time interval $t \in [t_0, t_k]$, that is, the existence of an input vector-function $v(t)$ such that the state $x(t)$ of the system with this realized vector-function and with the realized initial state Eq (1.2) takes at the moment t_k exactly the value x_k, is the condition

$$\operatorname{rank}(B\,AB\,...\,A^{n-1}B) = n. \tag{1.4}$$

This condition was used by Acad. Krylov N.N. [7] when calculating shipbuilding systems, acad. Pontryagin L.D. [8] when constructing optimal control processes. Later, after R. Kalman's report at the IFAC congress in 1959, this condition was called the Kalman condition [9].

Since that time (since 1960), a huge number of scientific research works and practical solutions of the problem Eqs (1.1)–(1.3) of finding the input $v(t)$ for the existence or to find the corresponding state $x(t)$.

As a rule, the practical solution of the problem is carried out only by approximate methods.

We know of only two methods for constructing $v(t)$ $x(t)$ for the problem Eqs (1.1)–(1.3) in analytical form:

1) construction [5] by the formulas

$$W = \int_0^{t_1} e^{-sA}BB^* e^{sA^*} ds, \quad z = W^{-1}\left(e^{-t_1 A}x_1 - x_0\right),$$

$$v(t) = B^* e^{tA^*} z, \quad x(t) = e^{tA}x_0 + e^{tA}\int_0^t e^{-sA}Bv(s)ds;$$

2) the method of cascade decomposition [10–13].

The disadvantages of the first method are the presence in the formulas of matrix exponentials, that is, matrix series; narrowness of the class of solutions (these are the series by powers of t); the method is applicable only for solving a two-point problem (the initial position of the system and the final one are given). The cascade decomposition method is applicable for solving multi-point problems; $v(t)$ and $x(t)$ are constructed as linear combinations of linearly independent functions of various types. For the practical application of the method, a fairly simple solution algorithm was developed in [13]; however, a lot of work is required with the given conditions at each stage of the system decomposition. Construction of the corresponding $v(t)$ and $x(t)$ in an analytical form remains an urgent task. In this paper, the following motion model is considered: the trajectory of the system (1.1) leaving the given point Eq (1.2) must pass at arbitrarily specified times t_i, $i = 0, 1, , \ldots, k, ;$ $t_0 < t_1 < \ldots < t_k$, through the points (t_i, x_i) with any given values of $x_i \in \mathbb{R}^n$. That is, the conditions

$$x(t_i) = x_i, \quad i = 0, 1, \ldots, k \tag{1.5}$$

on $x(t)$ are imposed.

For example, linearized models of the movement of aircraft: airplanes, helicopters, satellites, including objects moving under the action of reactive forces [3], [5], have the form Eq (1.1). For such systems, it is important when moving to be at the right time t_i in the right place (t_i, x_i).

In [10–13], it was proved that the condition Eq (1.4) is a complete condition for the existence of the input $v(t)$, under the realization of which the trajectory of the system (1.1) has the properties Eq (1.5). It is established that under the condition Eq (1.4) there exist $x(t)$ and $v(t)$ in the form of linear combinations of some linearly independent scalar functions with vector coefficients. The results of these works make it possible to simulate multi-point motion for stationary systems by a simpler method of indefinite coefficients, which is proposed below.

In this paper, we propose to calculate $x(t)$ and $v(t)$ in the form

$$x(t) = \sum_{j=0}^{r-1} \alpha_j \varphi_j(t), \tag{1.6}$$

$$v(t) = \sum_{j=0}^{r-1} \beta_j \varphi_j(t), \tag{1.7}$$

$\varphi_j(t) = e^{-at} \cdot \dfrac{t^j}{j!}$, $a > 0$, with undefined vector coefficients $\alpha_j \in \mathbb{R}^n$, $\beta_j \in \mathbb{R}^m$. The number r depends on the number of control points k and on the number p such, that

$$\operatorname{rank}(B\,AB \ldots A^p B) = n, \quad p \leq n - 1. \tag{1.8}$$

It follows from the results of [10–13] that

$$r = (p + 1)(k + 1). \tag{1.9}$$

$t_0 = 0$ is assumed.

The idea of searching for $v(t)$ and $x(t)$ in the form of linear combinations of linearly independent functions is not new.

In the monograph [14] (with reference to other primary sources for particular dynamical systems) are constructed $v(t)$ in the form of feedback, if $x(t)$ − is a linear combination of functions $\varphi_i(t)$, where $\varphi_i(t) = t^i$ or $\varphi_i(t) = e^{\lambda_i t}$. In monograph [5], when solving a boundary value problem for a nonstationary system (1) with one-dimensional $v(t)$ in the case of known linearly independent impulse transition functions of the object $h^i(t, \tau)$, $i = \overline{0, n}$, the function $v(t)$ is constructed as a linear combination of these functions.

In [15], the existence of $v(t)$ in polynomial form in the problem of constructing for system (1.1) the trajectory $x(t)$ with properties Eq (1.5).

In some cases, decreasing functions of t are more preferable than polynomial functions. If the system fails (the switch did not work at the time t_k, the technological modes were not switched, ...) the system control and, accordingly, the state of the system tend to zero in this case at $t \longrightarrow \infty$, in contrast from polynomial vector functions tending at $t \longrightarrow \infty$ to ∞. Therefore, in this work, we use exponential-polynomial functions with a negative exponent at the exponent.

The proposed method consists in the substitution of Eqs (1.6) and (1.7) directly into the Eq (1.1) and the given conditions Eq (1.5), and the determination of the coefficients α_j and β_j from the obtained relations. The system obtained as a result of these substitutions can be underdetermined, the ”extra” components can be set equal to zero or used to obtain $v(t)$ and $x(t)$ with additional properties. For this purpose, the number of terms in Eqs (1.6) and (1.7) can be increased.

Wherein, to calculate vector coefficients, it is important to obtain linear algebraic systems in which the coefficients are only matrices A^iB, $i = 0, 1, ...,$,in contrast to works by other authors, for example [15], where other matrix coefficients are used to prove the existence of the corresponding $v(t)$.

The advantage of the proposed method over the cascade method developed in [10–13] is that it does not require the derivation of multipoint conditions for the vector functions of the last step decomposition, and also in the fact that in the resulting formulas and systems for finding the coefficients there are no derivatives of functions, which leads to a minimum calculation inaccuracies.

2. Exponential-polynomial model of multi-point motion

For a dynamical system Eq (1.1) with conditions Eq (1.5) the problem of constructing the trajectory of motion in the exponential-polynomial form

$$x(t) = e^{-at} \sum_{j=0}^{r-1} \alpha_j \frac{t^j}{j!}, \ a > 0 \tag{2.1}$$

is set, for which the input vector function is sought in the form

$$v(t) = e^{-at} \sum_{j=0}^{r-1} \beta_j \frac{t^j}{j!}. \tag{2.2}$$

To find the vector coefficients $\alpha_j \in \mathbb{R}^n, \beta_j \in \mathbb{R}^m$, Eqs (2.1) and (2.2) are substituted directly into the Eq (1.1). Reducing the received expression on e^{-at} and comparing the coefficients at the same degrees of

t, we obtain the system

$$\begin{cases} \alpha_1 = (A + aI)\alpha_0 + B\beta_0, \\ \alpha_2 = (A + aI)\alpha_1 + B\beta_1, \\ \alpha_3 = (A + aI)\alpha_2 + B\beta_2, \\ \dots\dots\dots\dots\dots\dots \\ \alpha_{r-1} = (A + aI)\alpha_{r-2} + B\beta_{r-2}, \\ 0 = (A + aI)\alpha_{r-1} + B\beta_{r-1}. \end{cases} \tag{2.3}$$

Here $\alpha_0 = x_0$ is determined from the condition Eq (1.5) $x(t_0) = x(0) = x_0$ and this makes it possible to express the coefficients α_j through x_0 and β_j using the relations Eq (2.3).

From the first relation in system (2.3) we have:

$$\alpha_1 = B\beta_0 + (A + aI)x_0.$$

From the second relation of the system (2.3), taking into account the first relation, we obtain:

$$\alpha_2 = B\beta_1 + (A + aI)B\beta_0 + (A + aI)^2 x_0.$$

The third equality in system (2.3) leads to the expression

$$\alpha_3 = B\beta_2 + (A + aI)B\beta_1 + (A + aI)^2 B\beta_0 + (A + aI)^3 x_0$$

etc ... .

$$\alpha_j = \sum_{s=0}^{j-1} (A + aI)^s B\beta_{j-s-1} + (A + aI)^j x_0. \tag{2.4}$$

Substituting the last expression into the last equality of the system (2.3), we obtain one of the equations for finding the coefficients β_j:

$$B\beta_{r-1} + (A + aI)B\beta_{r-2} + \dots, (A + aI)^{r-1} B\beta_0 = -(A + aI)^r x_0. \tag{2.5}$$

Relations Eqs (2.1) and (2.4) are substituted into conditions Eq (1.5):

$$\sum_{s=0}^{r-2} (A + aI)^s B \sum_{j=0}^{r-s-2} \frac{t_i^{s+j+1}}{(s + j + 1)!} \beta_j = e^{at_i} x_i - \sum_{j=0}^{r-1} \frac{t_i^j}{j!} (A + aI)^j x_0, \tag{2.6}$$

$i = 1, 2, \dots, k$.

So, to construct the multipoint motion of the dynamic system (1.1) , it is proposed to search for the input vector function $v(t)$ in the form Eq (2.2) with coefficients β_j, for finding which a system (2.5) and (2.6) of linear algebraic equations with coefficients $(A + aI)^s B$, $s = 0, 1, \dots, r-1$ has been compiled. In this case, the trajectory $x(t)$ of motion of the system (1.1) has the form Eq (2.1) with coefficients α_j, determined by the Eq (2.4).

3. On the solvability of the systems (2.5) and (2.6)

3.1. About the systems (2.5) and (2.6)

The systems (2.5) and (2.6) consists of $(k + 1)n$ scalar equations for the rm sought components β_j^s of vectors $\beta_j = (\beta_j^1, \beta_j^2, \dots, \beta_j^m) \in \mathbb{R}^m$, $s = 1, 2, \dots, m$.

Lemma 1. *There are no more scalar equations in the systems (2.5) and (2.6) than the number of scalar unknowns:* $(k+1)n \leq r \cdot m$.

Since $r = (k+1)(p+1)$, it should be proved that $(k+1)n \leq (k+1)(p+1)m$, or $n \leq (p+1)m$.

Matrix $W = (BAB \ldots A^pB)$ of the condition (1.8) consists of n rows and $(p+1)m$ columns , and since rank W=n, then $n \leq (p+1)m$.

Lemma 1 is proved.

To prove the solvability of systems (2.5) and (2.6) we use the mapping property $B : \mathbb{R}^m \to \mathbb{R}^n$ [16] (the mappings and the corresponding matrices are denoted in the same way):

$$\mathbb{R}^m = \mathrm{Coim}B \dot{+} \mathrm{Ker}B, \quad \mathbb{R}^n = \mathrm{Im}B \dot{+} \mathrm{Coker}B, \tag{3.1}$$

where Ker B − is the kernel B, Im B − is the image of B; Coker B − defect subspace, Coim B − direct complement to subspace Ker B in $\mathbb{R}^m$. The restriction of $\widetilde{B}$ onto Coim B has the inverse mapping $\widetilde{B}^{-1}$. By P we denote the projector onto Ker B, by Q − the projector onto Coker B, corresponding to the decompositions (3.1). By I we denote the unit operator at any space. Through B^- − we denote a semi-inverse operator to B, that is $B^- = \widetilde{B}^-(I - Q)$. The following lemma is known [11].

Lemma 2. *Equality*

$$Bz = h, \quad z \in \mathbb{R}^m, \ h \in \mathbb{R}^n$$

equivalent to the system

$$\begin{cases} Qh = 0, \\ z = B^- h + \varphi, \ \forall \varphi \in KerB \ (\varphi = Pz). \end{cases}$$

Systems (2.5) and (2.6) is considered under condition Eq (1.8), which means that equation

$$Bu_1 + ABu_2 + \ \ldots \ + A^pBu_{p+1} = w$$

has a solution $u_1, \ u_2, \ \ldots, u_{p+1} \in \mathbb{R}^m$ for all $w \in \mathbb{R}^n$. The equation is considered for a simple case $p = 1$:

$$Bu_1 + ABu_2 = w. \tag{3.2}$$

It's solution $u_1, \ u_2$ will be given as it will be needed later.

3.2. The solution of Eq (3.2)

By Lemma 2, the Eq (3.2) is equivalent to the system

$$\begin{cases} QABu_2 = Qw, \\ u_1 = -B^-ABu_2 + B^-w + \varphi_1, \quad \forall \varphi_1 \in \mathrm{Ker}B \ (\varphi_1 = Pu_1). \end{cases} \tag{3.3}$$

The first equality in Eq (3.3) − is $QA((I - Q) + Q)Bu_2 = Qw$, but $QB = 0$, and $QA(I - Q)$, denoted by B_1, (here B_1 :Im $B \to$Coker B), splits Im B and Coker B into direct sums:

$$\mathrm{Im}B = \mathrm{Coim}B_1 \dot{+} \mathrm{Ker}B_1, \quad \mathrm{Coker}B = \mathrm{Im}B_1 \dot{+} \mathrm{Coker}B_1,$$

with the corresponding projectors Q_1 and P_1, and semi-inverse operator B_1^- : Im $B_1 \to$Coim B_1.

Now the first equality in Eq (3.3): $B_1Bu_2 = Qw$, due to Lemma 2, it is equivalent to the system

$$\begin{cases} Q_1Qw = 0, \\ Bu_2 = B_1^{-}Qw + \varphi_2, \quad \forall \varphi_2 \in \mathrm{Ker}B_1. \end{cases} \tag{3.4}$$

If $Q_1 \neq (0)$, then the system (3.1) is solvable not $\forall w \in \mathbb{R}^n$, rank $(B\ AB) < n$.

If $Q_1 = (0)$, then the second equation in Eq (3.4) has decision

$$u_2 = B^{-}B_1^{-}Qw + B^{-}\varphi_2 + \varphi_3, \quad \forall \varphi_3 \in \mathrm{Ker}B,$$

due to the fact that $B_1Qw + \varphi_2 \in \mathrm{Im}\, B$.

Thus, rank $(B\ AB) = n$ if and only if $Q_1 = (0)$, i.e., $B_1 -$ is surjection. In this case the Eq (3.2) has a solution

$$u_2 = B^{-}B_1^{-}Qw + B^{-}\varphi_2 + \varphi_3, \quad \forall \varphi_2 \in \mathrm{Ker}B_1, \quad \forall \varphi_3 \in \mathrm{Ker}B;$$

$$u_1 = -B^{-}ABu_2 + B^{-}w + \varphi_1, \quad \forall \varphi_1 \in \mathrm{Ker}B. \tag{3.5}$$

Thus, the following result is obtained.

Lemma 3. *If rank$(B\ AB) = n$ then equation*

$$Bu_1 + ABu_2 = w$$

has solution u_1, u_2 of the form Eq (3.5) for all $w \in \mathbb{R}^n$.

Remark 1. *System*

$$Bu_1 + (A + aI)Bu_2 = w, \quad u_1, u_2 \in \mathbb{R}^m, \ w \in \mathbb{R}^n, \tag{3.6}$$

or, equivalently,

$$B(u_1 + au_2) + ABu_2 = w,$$

has solution

$$u_1 = -(aI + B^{-}AB)u_2 + B^{-}w + \varphi_1, \quad \forall \varphi_1 \in KerB,$$

$$u_2 = B^{-}B_1^{-}Qw + B^{-}\varphi_2 + \varphi_3, \quad \forall \varphi_2 \in KerB_1, \quad \forall \varphi_3 \in KerB.$$

3.3. The solvability of the systems (2.5) and (2.6)

The following theorem holds.

Theorem 1. *The systems (2.5) and (2.6) is solvable with respect to the vectors β_j, $j = 0, 1, \ldots, r-1$ for any x_i, $i = 0, 1, \ldots, k$.*

In view of the cumbersomeness of the system under consideration, we present a detailed proof in two special cases. In the next case (and in the general case) the proof is schematic, since the basic principles of the proof are given in the previous cases.

3.3.1. Case $k = 1$, $p = 1$

Such a motion is considered further in example 1 - this is the motion of a material point moving in vertical plane under the action of a reactive force from the initial position to the final position.

For $k = 1$, $p = 1$, systems (2.5) and (2.6) is this:

$$B\beta_3 + (A + aI)B\beta_2 + (A + aI)^2 B\beta_1 + (A + aI)^3 B\beta_0 = -(A + aI)^4 x_0, \tag{3.7}$$

$$B(t_1\beta_0 + \frac{t_1^2}{2!}\beta_1 + \frac{t_1^3}{3!}\beta_2) + (A + aI)B(\frac{t_1^2}{2!}\beta_0 + \frac{t_1^3}{3!}\beta_1) + (A + aI)^2 B(\frac{t_1^3}{3!}\beta_0) = f_1, \tag{3.8}$$

where

$$f_1 = e^{at_1}x_1 - \sum_{j=0}^{3} \frac{t_1^j}{j!}(A + aI)^j x_0.$$

The Eq (3.7) should be converted to the form Eq (3.2), for which from the Eq (3.7), multiplied by $\frac{t_1^3}{3!}$, subtract Eq (3.8), left multiplied by $(A + aI)$:

$$B(\frac{t_1^3}{3!}\beta_3) + (A + aI)B(-t_1\beta_0 - \frac{t_1^2}{2!}\beta_1) + (A + aI)^2 B(-\frac{t_1^2}{2!}\beta_0) = f_2, \tag{3.9}$$

(hereinafter, for brevity, the right-hand sides of the obtained relations denote by f_i). Further to the Eq (3.8) we add Eq (3.9) multiplied by $\frac{t_1}{3}$:

$$B(\frac{t_1^4}{3 \cdot 3!}\beta_3 + \frac{t_1^3}{3!}\beta_2 + \frac{t_1^2}{2!}\beta_1 + t_1\beta_0) + (A + aI)B(\frac{t_1^3}{3!}\beta_0) = f_4. \tag{3.10}$$

Hence, by Remark 1:

$$\frac{t_1^3}{3!}\beta_0 = B^- B_1^- Q f_4 + B^-\psi_2 + \psi_3, \quad \forall\psi_2 \in \mathrm{Ker}B_1, \tag{3.11}$$

$\forall\psi_3 \in \mathrm{Ker}B$,

$$\frac{t_1^4}{3 \cdot 3!}\beta_3 + \frac{t_1^3}{3!}\beta_2 + \frac{t_1^2}{2!}\beta_1 + t_1\beta_0 = -(a \cdot I + B^- AB)(\frac{t_1^3}{3!}\beta_0) + B^- w + \psi_1, \quad \forall\psi_1 \in \mathrm{Ker}B. \tag{3.12}$$

Substituting the value β_0 found in Eq (3.11) into Eq (3.8), we get:

$$B(\frac{t_1^2}{2!}\beta_1 + \frac{t_1^3}{3!}\beta_2) + (A + aI)B(\frac{t_1^3}{3!}\beta_1) = f_6, \tag{3.13}$$

whence, by Remark 1, the values of β_1 and β_2 are found. Then β_3 is defined from Eq (3.12). For the case $p = 1$ and $k = 1$ Theorem 1 is proved.

3.3.2. The case of arbitrary k and $p = 1$

In the systems (2.5) and (2.6) now there are $k + 1$ equations and $2(k + 1)$ unknown vectors β_j:

$$\sum_{j=0}^{2k+1} (A + aI)^j B\beta_{2k-j+1} = -(A + aI)^{2(k+1)} x_0, \tag{3.14}$$

$$\sum_{s=0}^{2k}(A+aI)^s B\sum_{j=0}^{2k-s}\frac{t_i^{s+j+1}}{(s+j+1)!}\beta_j = e^{at_i}x_i - \sum_{j=0}^{2(k+1)}\frac{t_i^j}{j!}(A+aI)^j x_0, \tag{3.15}$$

$i = 1, 2, \dots, k$.

Each equation contains blocks $Bu_1 + (A + aI)Bu_2$ with unknowns u_1, u_2, and from each Eq (3.6) two required u_1, u_2, are defined, therefore the systems (3.14) and (3.15) is solvable with respect to β_j, $j = 0, 1, \dots, 2k + 1$.

3.3.3. Case $k = 1$, $p = 2$

System (2.5) and (2.6) is a system

$$\begin{gathered} B\beta_5 + (A+aI)B\beta_4 + (A+aI)^2B\beta_3 + (A+aI)^3B\beta_2 + (A+aI)^4B\beta_1 + \\ +(A+aI)^5B\beta_0 = -(A+aI)^6x_0, \end{gathered} \tag{3.16}$$

$$\begin{gathered} B(t_1\beta_0 + \frac{t_1^2}{2!}\beta_1 + \dots + \frac{t_1^5}{5!}\beta_4) + (A+aI)B(\frac{t_1^2}{2!}\beta_0 + \frac{t_1^3}{3!}\beta_1 + \frac{t_1^4}{4!}\beta_2 + \frac{t_1^5}{5!}\beta_3) + \\ +(A+aI)^2B(\frac{t_1^3}{3!}\beta_0 + \frac{t_1^4}{4!}\beta_1 + \frac{t_1^5}{5!}\beta_2) + (A+aI)^3B(\frac{t_1^4}{4!}\beta_0 + \frac{t_1^5}{5!}\beta_1) + \\ +(A+aI)^4B(\frac{t_1^5}{5!}\beta_0) = f_1, \end{gathered} \tag{3.17}$$

where $f_1 = e^{at_1}x_1 - \sum_{j=0}^{5}\frac{t_1^j}{j!}(A+aI)^j x_0$.

The system contains terms

$Bu_1 + (A + aI)Bu_2 + (A + aI)^2Bu_3$, and if $p = 2$, then the equation

$$Bu_1 + ABu_2 + A^2Bu_3 = w, \quad u_1, u_2, u_3 \in \mathbb{R}^m, \ w \in \mathbb{R}^n$$

is also solvable with respect to u_1, u_2, u_3, since $\operatorname{rank}(B\ AB\ A^2B) = n$. Then the equation

$$Bu + (A+aI)Bu_2 + (A+aI)^2Bu_3 = w$$

is also solvable with respect to u_1, u_2, u_3, hence the systems (3.16) and (3.17) is solvable with respect to β_j, $j = 0, 1, \dots 5$.

3.3.4. The case of arbitrary k and arbitrary $p < n$

In this case, the systems (2.5) and (2.6) consists of $k + 1$ equation and contains $r = (p + 1)(k + 1)$ unknown vectors b_j.

The equations contain terms of the form $Bu_1 + (A + aI)Bu_2 + \dots + (A + aI)^pBu_{p+1}$. The system is solvable due to the fact that $\operatorname{rank}(B\ AB \dots A^pB) = n$.

4. About the correctness of the model

It is required to prove that the presented model of motion of a dynamic system is correct, that is, with small changes in the parameters of the system, the trajectory $x(t)$, calculated by Eqs (2.1) and (2.4), also changes little; where β_j, $j = 0, 1, \dots, r - 1$ is determined from system (2.5) and (2.6).

The parameters of the model are the coefficients A, B of system (1.1) and the specified values of x_i in condition Eq (1.5).

4.1. On the continuous dependence of the trajectory on multipoint values

The continuous dependence of the trajectory of the simulated motion on the specified values of x_i, $i = 0, 1, ..., k$, is obvious, since x_i are contained in the right-hand sides of linear algebraic Eqs (2.5) and (2.6).

Small changes of x_i will entail small changes of the coefficients β_j. From Eq (2.4) it follows that α_j, $j = 0,\ 1,\ ...,\ r-1$ changes little in this case; Eq (2.1) shows that $x(t)$ changes little.

4.2. On a small change in the input vector function under small perturbations A and B

The coefficients $\beta_j(t)$ with some j in the Eq (2.2) for $v(t)$ are defined as solutions of equations of the form

$$Bu_1 + ABu_2 + ... + A^pBu_{p+1} = w_1,$$

therefore, we study the change in the coefficients $\beta_j(t)$ (due to cumbersomeness) for the particular case $p = 1,\ \ k = 1$.

Let's show that the solutions u_1 and u_2, obtained in the system (3.2), differ little from the solution $\widetilde{u}_1$ and $\widetilde{u}_2$ of the perturbed system

$$(B + G_2)\widetilde{u}_1 + (A + G_1)(B + G_2)\widetilde{u}_2 = w + g, \tag{4.1}$$

$G_1 : \mathbb{R}^n \to \mathbb{R}^n$, $G_2 : \mathbb{R}^m \to \mathbb{R}^n$, $g \in \mathbb{R}^n$, if G_1, G_2 and g are small in norms.

Equation (4.1)

$$(B + G_2)\widetilde{u}_1 + AB\widetilde{u}_2 + (AG_2 + G_1(B + G_2))\widetilde{u}_2 = w + g,$$

by Lemma 2, it is equivalent to the system

$$QG_2\widetilde{u}_1 + B_1B\widetilde{u}_2 + Q(AG_2 + G_1(B + G_2))\widetilde{u}_2 = Q(w + g), \tag{4.2}$$

$$(I + B^-G_2)\widetilde{u}_1 + B^-AB\widetilde{u}_2 + B^-(AG_2 + G_1(B + G_2))\widetilde{u}_2 = B^-(w + g) \tag{4.3}$$

(see notation in Section 3.2).

From Eq (4.3) we find $\widetilde{u}_1$:

$$\widetilde{u}_1 = (I + B^-G_2)^{-1}(-B^-AB - B^-G_3)\widetilde{u}_2 + B^-(w + g) + \psi_1,\ \forall\psi_1 \in KerB,$$

where $G_3 = B^-AG_2 + G_1(B + G_2) -$ is small due to the smallness of G_1 G_2. And since $(I + B^-G_2)^{-1} = I - B^-G_2(I + B^-G_2)^{-1}$,then

$$\widetilde{u}_1 = -B^-AB\widetilde{u}_2 + B^-(w + g) + \psi_1 + G_4\widetilde{u}_2 \tag{4.4}$$

with small G_4.

From the relation Eq (4.2) we find

$$\widetilde{u}_2 = B^-B_1^-Q(w + g) - B^-B_1^-QG_2\widetilde{u}_1 - B^-B_1^-QG_3\widetilde{u}_2 + B^-\psi_2 + \psi_3,$$

$\forall\psi_2 \in KerB_1,\ \ \forall\psi_1 \in KerB$, i.e.,

$$(I + B^-B_1^-QG_3)\widetilde{u}_2 = B^-B_1^-Q(w + g) - B^-B_1^-QG_2\widetilde{u}_1 + B^-\psi_2 + \psi_3,$$

from where

$$\widetilde{u}_2 = (I + B^- B_1^- QG_3)^{-1}(B^- B_1^- Q(w + g) - B^- B_1^- QG_2\widetilde{u}_1 + B^-\psi_2 + \psi_3), \tag{4.5}$$

if $\|B^- B_1^- QG_3\| < 1$ due to the smallness of G_3.

Replacing $(I + B^- B_1^- QG_3)^{-1}$ with $I - B^- B_1^- QG_3(I + B^- B_1^- QG_3)^{-1}$, or with $I - G_4$, where G_4 is small due to the smallness of G_3, we get:

$$\widetilde{u}_2 = B^- B_1^- Q(w + g) + B^-\psi_2 + \psi_3 + G_5\widetilde{u}_1 - G_4(B^-\psi_2 + \psi_3) + G_6 Qw, \tag{4.6}$$

with small G_6.

Substituting Eq (4.6) in Eq (4.4), inverting the matrix $(I + B^- ABG_5)$ and replacing $(I + B^- ABG_5)^{-1}$ with $I - B^- ABG_5(I + B^- ABG_5)^{-1}$, that is, with $I - G_7$, where G_7 is small, we get:

$$\widetilde{u}_1 = -B^- AB(B^- B_1^- Q(w + g) + B^-\psi_2 + \psi_3) + B^-(w + g) + \psi_1 + G_8\psi_2 + G_9\psi_3,$$

where G_8 and G_9 are small.

The vector functions $\varphi_i(t)$ and $\psi_i(t)$, $i = 1, 2, 3,$ – are arbitrary from the kernels of the corresponding mappings, and if we choose $\varphi_i(t) = \psi_i(t)$, then $\|u_1(t) - \widetilde{u}_1(t)\|$ and $\|u_2(t) - \widetilde{u}_2(t)\|$ are small.

Thus, the solutions β_j of the system (2.5) and (2.6) react little to small perturbations of the parameters A and B, therefore, the input vector function changes little for small change in the coefficients A and B of the system (1.1).

4.3. *On the continuous dependence of the trajectory motion from the perturbation of the parameters A and B*

Small change of the parameters A and B of the system (1.1) there corresponds a small change of the trajectory of movement. That is, the trajectory $\widetilde{x}(t)$ of the system

$$\dot{\widetilde{x}}(t) = (A + G_1)\widetilde{x}(t) + (B + G_2)\widetilde{v}(t), \tag{4.7}$$

$$\dot{\widetilde{x}}(t_i) = x_i,$$

where $\|G_1\|$, $\|G_2\|$ – are small and $\widetilde{v}(t)$ differs little from the input vector function $v(t)$ of the unperturbed system (1.1), differs little from the trajectory $x(t)$.

Indeed, subtracting the relation Eq (1.1) from Eq (4.7), we obtain

$$\frac{d}{dt}(\widetilde{x}(t) - x(t)) = (A + G_1)(\widetilde{x}(t) - x(t)) + B(\widetilde{v}(t) - v(t)) - G_1 x(t) + G_2\widetilde{v}(t),$$

whence, due to the uniqueness of the solution to the initial problem and the notation $G_2\widetilde{v}(t) - G_1 x(t) = h(t)$ follows:

$$\widetilde{x}(t) - x(t) = \int_0^t e^{(t - s)(A + G_1)} (B(\widetilde{v}(s) - v(s)) + h(s))\, ds.$$

i.e.,

$$\|\widetilde{x}(t) - x(t)\| \leq \int_0^t e^{(t - s)\|A + G_1\|} \cdot (\|B\|\|\widetilde{v}(s) - v(s)\| + \|h(s)\|)\, ds$$

for $t \in [0, t_k]$, $t_k < \infty$. Due to the smallness of G_1, G_2, $h(t)$ and the difference $\widetilde{v}(t) - v(t)$, the trajectory of the perturbed system differs little from the trajectory of the unperturbed dynamical system.

Thus, the simulated movement is correct.

5. Example

It is required to carry out the movement of the rocket in the vertical plane from point A_0 to point A_1; with given speeds at points A_0 and A_1.

We neglect the size of the projectile. The movement of a material point with mass m under the action of a reactive force (arising as a result of the separation of particles with elementary mass dm_1 from the material point) is considered. The monograph [5] contains a system describing such a motion. If we neglect the force of gravity of the Earth, then the system has the form

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = v_1, \\ \dot{x}_3 = x_4, \\ \dot{x}_4 = v_2, \end{cases} \tag{5.1}$$

where $x_1(t)$ is the horizontal component of the matherial point position (we will use the notation MP for a material point), $x_3(t)$ – vertical component of MT, $x_2(t)$ $x_4(t)$ are components of the MP velocity; $v_1 = \sigma \cdot \cos\alpha \cdot \dfrac{\dot{m}}{m}$, $v_2 = \sigma \cdot \cos\beta \cdot \dfrac{\dot{m}}{m}$, $\sigma = |a|$, $a(t) = s(t) - v(t)$ – the vector of the relative velocity of the separating particle; $s(t)$ – the velocity vector of the particle dm_1 at the moment $t + dt$ after its separation; v – vector of absolute speed of MT; $m = m(t) = m_0 + m_1(t)$, $m_0 = const$ – constant part of the MT mass; $m_1(t)$ – the reactive mass of the MT; α and β – are the angles of the vector a with the axes OX and OY.

The starting position and starting speed of MP are given: $x_i(0) = x_{0i}$.

It is required to get to the point $(x_1(t_1),\ x_3(t_1))$ in time t_1 with speed $(x_2(t_1),\ x_4(t_1))$.

The system (5.1) consists of two identical subsystems, so it is enough to consider the solution of the problem for one subsystem

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= v, \end{aligned} \tag{5.2}$$

with the conditions

$$\begin{aligned} x_1(0) = x_{01},\ \ x_1(t_1) = x_{11}, \\ x_2(0) = x_{02},\ \ x_2(t_1) = x_{12}. \end{aligned} \tag{5.3}$$

The system (5.2) is the system (1.1) with $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $n = 2$, $m = 1$; rank $(B\ AB)$ = rank $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = 2$, therefore, $p = 1$ and $r = 4$. The trajectory of movement is constructed in the form Eq (2.1):

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = e^{-at} \sum_{j=0}^{3} \begin{pmatrix} \alpha_{j1} \\ \alpha_{j2} \end{pmatrix} \cdot \frac{t^j}{j!}, \tag{5.4}$$

the input function is constructed in the form

$$v(t) = e^{-at} \sum_{j=0}^{3} \beta_j \cdot \frac{t^j}{j!},\ \ a > 0,\ \ \beta_j \in \mathbb{R}^1. \tag{5.5}$$

The coefficients β_j are found from the system (2.5) and (2.6), where $(A+aI)B = \begin{pmatrix} 1 \\ a \end{pmatrix}$, $(A+aI)^2B = \begin{pmatrix} 2a \\ a^2 \end{pmatrix}$, $(A+aI)^3B = \begin{pmatrix} 3a^2 \\ a^3 \end{pmatrix}$.

The system (2.5) and (2.6) is as follows:

$$\begin{pmatrix} 1 & a & a^2 & a^3 \\ 0 & 1 & 2a & 3a^2 \\ 0 & \frac{t_1^3}{3!} & (\frac{t_1^2}{2!} + a\frac{t_1^3}{3!}) & (t_1 + a\frac{t_1^2}{2!} + a^2\frac{t_1^3}{3!}) \\ 0 & 0 & \frac{t_1^3}{3!} & (\frac{t_1^2}{2!} + 2a\frac{t_1^3}{3!}) \end{pmatrix} \begin{pmatrix} \beta_3 \\ \beta_2 \\ \beta_1 \\ \beta_0 \end{pmatrix} = \begin{pmatrix} g_1 \\ g_2 \\ g_3 \\ g_4 \end{pmatrix}, \tag{5.6}$$

where $g_1 = -a^4x_{02},\ g_2 = -a^4x_{01} - 4a^3x_{02},\ g_3 = e^{at_1}x_{12} - x_{02}(1 + at_1 + \frac{a^2t_1^2}{2!} + \frac{a^3t_1^3}{3!}),$

$$g_4 = e^{at_1}x_{11} - x_{01}(1 + at_1 + \frac{a^2t_1^2}{2!} + \frac{a^3t_1^3}{3!}) - x_{02}(t_1 + 2a\frac{t_1^2}{2!} + 3a^2\frac{t_1^3}{3!}).$$

The main determinant of the system (5.6) after some transformations takes the form

$$\triangle = \begin{vmatrix} 1 & a & a^2 & a^3 \\ 0 & 1 & 2a & 3a^2 \\ 0 & 0 & \frac{t_1^2}{2!} & (t_1 + at_1^2) \\ 0 & 0 & 0 & \frac{t_1^2}{2!} \end{vmatrix}.$$

And since $\triangle = \frac{t_1^4}{4} \neq 0$, the system (5.6) has a unique solution $\beta_j,\ j = 3,2,1,0$. The uniqueness of the solution is due to the fact that $KerD = 0$, $KerD_1 = 0$, since $Q = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $I - Q = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, $D_1 = QB(I-Q) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, D_1 acts one-to-one from Im $D = \left\{\begin{pmatrix} 0 \\ c \end{pmatrix}\right\}$ to Coker $D = \left\{\begin{pmatrix} d \\ 0 \end{pmatrix}\right\}$, $c, d \in \mathbb{R}^1$.

Next, substitution of β_j values into Eq (2.4), then into Eq (2.1) produces the result:

$$x_1(t) = e^{-at}(x_{01} + \frac{t}{1!}(ax_{01} + x_{02}) + \frac{t^2}{2!}(\beta_0 + a^2x_{01} + 2ax_{02}) + \frac{t^3}{3!}(\beta_1 + 2a\beta_0 + a^3x_{01} + 3a^2x_{02})),$$

$$x_3(t) = e^{-at}(x_{03} + \frac{t}{1!}(ax_{03} + x_{04}) + \frac{t^2}{2!}(\gamma_0 + a^2x_{03} + 2ax_{04}) + \frac{t^3}{3!}(\gamma_1 + 2a\gamma_0 + a^3x_{03} + 3a^2x_{04})),$$

where γ_0, γ_1 are found from the system (5.6) with x_{01} replaced by x_{03}; x_{02} replaced by x_{04}; x_{11} replaced by x_{13}; x_{12} replaced by x_{14}; with β_j replaced by γ_j; with g_j replaced by h_j; where $h_1 = g_1,\ h_2 = g_2,\ h_3 = g_3 - e^{at_1} \times x_{14},\ h_4 = g_4 + e^{at_1} \times x_{11}$.

The collection of points $\{(x_1(t),\ x_3(t))\}$ the required trajectory of the system (5.1) with the conditions $x_i(0) = x_{0i},\ x_i(t_1) = x_{1i},\ i = 1-4$.

The speed of movement of MP $(x_2(t),\ x_4(t))$ is calculated by the formulas

$$x_2(t) = e^{-at}(x_{02} + t(\beta_0 + ax_{02}) + \frac{t^2}{2!}(\beta_1 + a\beta_0 + a^2 x_{02}) + \frac{t^3}{3!}(\beta_2 + a\beta_1 + a^2\beta_0 + a^3 x_{02})),$$

$$x_4(t) = e^{-at}(x_{04} + t(\gamma_0 + ax_{04}) + \frac{t^2}{2!}(\gamma_1 + a\gamma_0 + a^2 x_{04}) + \frac{t^3}{3!}(\gamma_2 + a\gamma_1 + a^2\gamma_0 + a^3 x_{04})).$$

The movement of of MP is carried out due to action of $(v_1,\ v_2)$, where

$$v_1 = e^{-at}\sum_{j=0}^{3}\beta_j \cdot \frac{t^j}{j!}, \quad v_2 = e^{-at}\sum_{j=0}^{3}\gamma_j \cdot \frac{t^j}{j!}.$$

In particular, for values

$$\begin{aligned} &x_{01} = 0\,m, \;\; x_{11} = 3000\,m, \\ &x_{02} = 500\,m/sec, \;\; x_{12} = 0\,m/sec, \\ &x_{03} = 0\,m, \;\; x_{13} = 0\,m, \\ &x_{04} = 500\,m/sec, \;\; x_{14} = 500\,m/sec, \\ &t = 3\,sec; \end{aligned} \tag{5.7}$$

and for $a = -0,9$; $a = -1$; $a = -1,1$; $a = -1,2$, the calculations made in the program of the MathCad show the following.

For $a = -0,9$ MT reaches the ground approximately at the point $(2,955;\ 0)$; therefore the value $a = -0,9$ is excluded.

For $a = -1$ an exact hit on the target is achieved; the values x_{1i}, $i = 1,\ 2,\ 3,\ 4$ given in Eq (5.7) are realized without error.

For $a = -1,1$ MT will hit the point $(3000 + 0,2 \times 10^{-13};\ 0,3 \times\ e^{-14})$ with the speed $(-0,5 \times e^{-14};\ 500 + 0,1 \times 10^{-13})$ at the moment $t = 3\,sec$.

With $a = -1.2$ at the moment $t = 3\,sec$, MT will hit the point $(3000 + 0,5 \times 10^{-13};\ 0,4\times\ e^{-14})$ with a speed $(2,4 \times 10^{-11};\ 500 + 0,4 \times 10^{-14})$.

In this case, the trajectories of the MP are different (of different types):

for $a = -1$ maximum height of the trajectory $\approx 873,7\,m$ is reached at $t \approx 1,7\,sec$ and $x_1 \approx 1508,7\,m$;

for $a = -1,1$ maximum height of the trajectory $\approx 985,6\,m$ is reached at $t \approx 1,7\,sec$ and $x_1 \approx 1795,6\,m$;

for $a = -1,2$ maximum height of the trajectory $\approx 1197,5\,m$ m is reached at $t \approx 1,8\,sec$ and $x_1 \approx 2077\,m$.

Thus, the task is completely fulfilled for $a = -1$; and it is fulfilled with a subtle error for $a = -1,1$ and for $a = -1,2$.

6. Conclusions

In this paper, for the first time, the method of indefinite coefficients is applied to simulate the multipoint motion of a linear dynamic system. This method is as follows.

1). The type of the modeled trajectory (state) of the system is selected; the type of the input vector-function is also selected, under the influence of which the movement is carried out. In this paper, the exponential-polynomial form is chosen from the following considerations. Polynomial vector functions are easy to calculate, but they have large amplitudes and grow rapidly over time. To neutralize these shortcomings, an exponential function with an arbitrary negative exponent was introduced into these vector functions.

This indicator can be selected based on the characteristics of a specific practical task. To determine the number of terms in the formulated expressions, the information obtained earlier in [10–12] is used.

2). The generated state and input vector functions are substituted directly into the equations of the dynamic system and the specified multipoint conditions. As a result, the following are displayed:

– system of linear algebraic equations for finding vector coefficients, with the help of which the input vector function is then constructed;

– formulas for determining the vector coefficients involved in the construction of the trajectory of the system.

Both the algebraic system and the named formulas are simple, the coefficients in them are only the products of the matrix coefficients of the dynamical system under consideration. But the proof of the existence of a solution to the obtained algebraic system has certain difficulties. For the proof, we use the properties of finite-dimensional mappings: split spaces into subspaces, projection onto subspaces, semi-inversion of mappings, ...

In view of the cumbersomeness of the proof, here we give a complete proof in a particular case and outline ideas for evidence in other cases.

The correctness of the obtained motion model is proved when some parameters of the coefficients are fixed. The need to fix the parameters is due to the nonuniqueness of the trajectory of motion of the dynamic system under given multipoint conditions.

In the case of non-uniqueness of the solution of the obtained algebraic system, the ”extra” undefined coefficients can be used solving additional problems, for example, for constructing a suboptimal input vector function or for performing some other properties of the system state.

An example of obtaining formulas for constructing a trajectory in the vertical plane of a material point under the action of a reactive force is given.

Calculations for a specific task performed in the MathCad system show the high efficiency of the proposed method for modeling the trajectory of a linear dynamic system.

Conflict of interest

All authors declare no conflicts of interest in this article.

References

1. V. I. Gurmn, Degenerate optimal control problems, *IFAC Proc. Vol.*, **11** (1978), 1101–1106..
2. P. U. Nelson, A. S. Perelson, Mathematical analysis of models differential equation of HIV-1 infection delay, *Math. Biosci.*, **179** (2002), 73–94.
3. R. Dorf, R. Bishop, *Modern Control Systems*, Pearson Prentice Hall, 2008.
4. E. F. Baranov, S. S. Shatalin, Problems of developing a dynamic model of of interbranch balance, *Econ. Math. Methods*, **1** (1968).

5. N. N. Krasovsky, *Theory of Motion Control*, Moscow, 1968.
6. R. E. Kalman, P. Falb, M. Arbib, Essays on the mathematical theory of systems, Editorial, Moscow, 2004.
7. A. N. Krylov, *Selected Works*, Publishing house of Academy of Sciences of the USSR, 1958.
8. L. S. Pontryagin, *Optimal processes of regulation*, UMN, **14** (1959), 3–20.
9. R. E. Kalman, On the general theory of control systems, in *Proceedings First International Conference on Automatic Control*, Moscow, (1960), 481–492.
10. S. P. Zubova, Solution of inverse problems for linear dynamical systems by the cascade method, in *Doklady Mathematics*, Pleiades Publishing, **86** (2012), 846–849.
11. S. P. Zubova, On full controllability criteria of a descriptor system. The polynomial solution of a control problem with checkpoints, *Autom. Remote Control*, **72** (2011), 23–37.
12. S. P. Zubova, E. V. Raetskaya, L. H. Trung, On polynomial solutions of the linear stationary control system, *Autom. Remote Control*, **69** (2008), 1852–1858.
13. S. P. Zubova, E. V. Raetskaya, Algorithm to solve linear multipoint problems of control by the method of cascade decomposition, *Autom. Remote Control*, **78** (2017), 1189–1202.
14. P. D. Krutko, *Inverse Problems of the Dynamics of Controlled Systems. Linear models*, Moscow, 1987.
15. A. Ailon, G. Langholz, More on the controllability of linear time-invariant systems, *Int. J. Control*, **44** (1986), 1161–1176.
16. F. R. Gantmakher, *Matrix Theory*, Moscow, 1988.