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Research article

Global attractors, extremal stability and periodicity for a delayed population model with survival rate on time scales

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Abstract: In this paper, we investigate the existence of global attractors, extreme stability, periodicity and asymptotically periodicity of solutions of the delayed population model with survival rate on isolated time scales given by

$$x^{\Delta}(t) = \gamma(t)x(t) + \frac{x(d(t))}{\mu(t)} e^{r(t)\mu(t)\left(1-\frac{x(d(t))}{\mu(t)}\right)}, \quad t \in \mathbb{T}.$$

We present many examples to illustrate our results, considering different time scales.

Keywords: delayed population model; time scales; global attractors; stability; periodicity

1. Introduction

In this paper, we are interested to investigate the delayed population model with survival rate on isolated time scales given by

$$x^{\Delta}(t) = \gamma(t)x(t) + \frac{x(d(t))}{\mu(t)} e^{r(t)\mu(t)\left(1-\frac{x(d(t))}{\mu(t)}\right)}, \quad t \in \mathbb{T} \tag{1.1}$$

where $\gamma : \mathbb{T} \to (-\infty, 0)$, $r, k : \mathbb{T} \to (0, \infty)$ describe, respectively, the intrinsic growth rate and the carrying capacity of the habitat, and $d : \mathbb{T} \to \mathbb{T}$ is the delay function such that $\rho^{\alpha}(t) \leqslant d(t) \leqslant t$ for some $\alpha \in \mathbb{N}$. This model is equivalent to

$$x(\sigma(t)) = \widetilde{\gamma}(t)x(t) + x(d(t))e^{r(t)\mu(t)\left(1-\frac{x(d(t))}{\mu(t)}\right)}, \quad t \in \mathbb{T}$$

where the function $\widetilde{\gamma}(t) = 1 + \mu(t)\gamma(t)$ belongs to $(0, 1)$. This is a generalization of the model considered in [1] for any isolated time scales. Clearly, in the particular case $\mathbb{T} = \mathbb{Z}$, our model reaches the one found in [1].

A quick look at the formulation of the model described by equation (1.1) may seem different, since in its formulation appears the graininess function in the denominator of the second term on the right–hand side of the equation. However, in [2], the authors show that this formulation is necessary when we are dealing with quantum calculus (which is also encompassed here), since depending on the formulation of the model and the assumptions, one cannot even ensure the existence of ω–periodic solutions without considering this term for the quantum case (see [2] for details). But it is important to mention that our model reaches the model for the case $\mathbb{T} = \mathbb{Z}$ considered in the literature, showing that this formulation is appropriate and unifies all the cases.

We point out that our model is valid for all isolated time scales, which includes many important examples such as $\mathbb{T} = \mathbb{Z}$, $\mathbb{T} = \mathbb{N}_0^2 = \{n^2 : n \in \mathbb{N}_0\}$, $\mathbb{T} = q^{\mathbb{N}_0} = \{q^n : n \in \mathbb{N}_0\}$, $q > 1$. This last one is known as quantum scale and it has been investigated by many authors [3], mainly concerning the ω–periodicity (see [4] and [5]). This quantum scale has several applications in many fields of physics such as cosmic strings and black holes [6], conformal quantum mechanics, nuclear and high energy physics, fractional quantum Hall effect, and high-T_c superconductors [7]. Thermostatistics of q-bosons and q-fermions can be established using basic numbers and employing the quantum calculus [8]. On the other hand, it worths mentioning the importance of time scales to describe population models, since it allows to consider a variety of scenery and many possibilities in the behavior of different populations (see, for instance, [9]). Also, the population models for quantum calculus play important role, bringing relevant applications (see [10] and [11]).

The formulation of this model for its analogue for $\mathbb{T} = \mathbb{Z}$ without delays was investigated by many authors. See the references [12], [13] and [14] for instance. In particular, in [14], the authors investigated the extreme stability of the following discrete logistic equation

$$x(t+1) = x(t)e^{r(t)\left(1-\frac{x(t)}{k(t)}\right)}, \quad t \in \mathbb{Z}_+. \tag{1.2}$$

In [1], the author considered a version of the model with delays

$$x(t+1) = \gamma(t)x(t) + x(\tau(t))e^{r(t)\left(1-\frac{x(\tau(t))}{k(t)}\right)}, \quad t \in \mathbb{Z}_+. \tag{1.3}$$

The formulation considered here in this present paper generalizes (1.2) and (1.3). We are interested to investigate the asymptotic behavior of the solutions of (1.1) on isolated time scales, including global attractor, extremely stability, asymptotic periodicity and periodicity.

This paper is divided as follows. In the second section, we present some preliminary results on theory of time scale and explain the delayed model that will be investigated. In the third section, we investigate the stability of equation (1.1). The fourth section is devoted to study the extremal stability of (1.1) and to present some examples to illustrate our main results. Finally, the goal of last section is to investigate the periodicity and asymptotically periodicity of solutions of (1.1), and to present examples.

2. Preliminaries

In this section, our goal is to recall some basic definitions and results from time scale theory. For more details, we refer [15] and [16].

A time scale $\mathbb{T}$ is any closed and nonempty subset of $\mathbb{R}$ endowed with the topology inherited from $\mathbb{R}$.

Definition 2.1. The *forward jump operator* $\sigma : \mathbb{T} \to \mathbb{T}$ is defined by $\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$ and the *backward jump operator* $\rho : \mathbb{T} \to \mathbb{T}$ by $\rho(t) = \sup\{s \in \mathbb{T} : s > t\}$, provided $\inf \emptyset = \sup \mathbb{T}$ and $\sup \emptyset = \inf \mathbb{T}$.

If $\sigma(t) > t$, then t is called *right–scattered*. Otherwise, t is called *right–dense*. Similarly, if $\rho < t$, then t is said to be *left–scattered*, while if $\rho(t) = t$, then t is called *left–dense*.

From now on, we only consider isolated time scales, i.e., all points are right–scattered and all points are left–scattered.

Moreover we denote the composition $\underbrace{\sigma \circ \ldots \circ \sigma}_{n \text{ times}}$ by σ^n. The same notation we use for the composition of operator ρ.

Definition 2.2. The *graininess function* $\mu : \mathbb{T} \to [0, \infty)$ is defined by $\mu(t) = \sigma(t) - t$.

The delta (or Hilger) derivative of $f : \mathbb{T} \to \mathbb{R}$ at a point $t \in \mathbb{T}^\kappa$, where

$$\mathbb{T}^\kappa = \begin{cases} \mathbb{T} \backslash (\rho(\sup \mathbb{T}), \sup \mathbb{T}], & \text{if } \sup \mathbb{T} < \infty \\ \mathbb{T}, & \text{if } \sup \mathbb{T} = \infty \end{cases}$$

is defined in the following way:

Definition 2.3 ([15])**.** The *delta derivative of function f at a point t*, denoted by $f^\Delta(t)$, is the number (provided it exists) with the property that given any $\varepsilon > 0$, there is a neighborhood U of t (i.e., $U = (t - \delta, t + \delta) \cap \mathbb{T}$ for some $\delta > 0$) such that

$$\left| (f(\sigma(t)) - f(s)) - f^\Delta(t)(\sigma(t) - s) \right| \leqslant \varepsilon |\sigma(t) - s| \quad \text{for all } s \in U.$$

We say that a function f is *delta (or Hilger) differentiable* on $\mathbb{T}^\kappa$ provided $f^\Delta(t)$ exists for all $t \in \mathbb{T}^\kappa$. The function $f^\Delta : \mathbb{T}^\kappa \to \mathbb{R}$ is then called *the (delta) derivative of f* on $\mathbb{T}^\kappa$.

Throughout this paper, we assume that $\mathbb{T}$ is an isolated time scale such that

$$\sup \mathbb{T} = \infty, \ \ \inf \mathbb{T} = t_0 \ \text{ and } \ \inf_{t \in \mathbb{T}} \mu(t) > 0. \tag{2.1}$$

By [15, Theorem 1.16], for any function $f : \mathbb{T} \to \mathbb{R}$, its derivative is given by

$$f^\Delta(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)} \quad \text{for all } t \in \mathbb{T}^\kappa.$$

We consider the delayed population model of the form

$$\begin{cases} x^\Delta(t) = \gamma(t)x(t) + \frac{x(d(t))}{\mu(t)} e^{r(t)\mu(t)\left(1 - \frac{x(d(t))}{k(t)}\right)}, \ t \geqslant t_0 \\ x(t_0) = x_0, \end{cases} \tag{2.2}$$

with $\gamma : \mathbb{T} \to (-\infty, 0)$, $r, k : \mathbb{T} \to (0, \infty)$ and $d : \mathbb{T} \to \mathbb{T}$ such that

$$\rho^\alpha(t) \leqslant d(t) \leqslant t \ \text{ for some } \alpha \in \mathbb{N}. \tag{2.3}$$

The functions r and k describe, respectively, the intrinsic growth rate and the carrying capacity of the habitat. The delay is introduced to this model by the function d. From (2.3), it is clear that $\lim_{t\to\infty} d(t) = \infty$.

By solution of equation (2.2) with initial value x_0, we mean function $x : \mathbb{T} \to \mathbb{R}$ which satisfies (2.2) for $t > t_0$ and $x(t_0) = x_0$.

Our aim is to study the stability, existence of a global attractor and the extreme stability, as well as periodicity and asymptotically periodicity of (2.2), according to the notion of periodicity for isolated time scales given in [17] by Bohner et al.

Remark 2.4. Let us emphasize that solution of equation (2.2) depends on only one initial value $x(t_0)$, since the delay function d can be expressed in terms of iterations of the backward jump operator ρ.

Example 2.5. Suppose that the time scale $\mathbb{T} = \{t_0, t_1, t_2, \ldots\}$ satisfies condition (2.1). Consider a delay function of the form

$$d(t_i) = \begin{cases} \rho^2(t_i) & \text{if } i \text{ is even} \\ \rho^3(t_i) & \text{if } i \text{ is odd.} \end{cases}$$

By equation (2.2), we obtain

$$\begin{aligned} x(t_1) &= (1 + \mu(t_0)\gamma(t_0))x(t_0) + x(\rho^2(t_0))e^{r(t_0)\mu(t_0)\left(1-\frac{x(\rho^2(t_0))}{k(t_0)}\right)} \\ &= (1 + \mu(t_0)\gamma(t_0))x(t_0) + x(t_0)e^{r(t_0)\mu(t_0)\left(1-\frac{x(t_0)}{k(t_0)}\right)} \\ x(t_2) &= (1 + \mu(t_1)\gamma(t_1))x(t_1) + x(\rho^3(t_1))e^{r(t_1)\mu(t_1)\left(1-\frac{x(\rho^3(t_1))}{k(t_1)}\right)} \\ &= (1 + \mu(t_1)\gamma(t_1))x(t_1) + x(t_0)e^{r(t_1)\mu(t_1)\left(1-\frac{x(t_0)}{k(t_1)}\right)} \end{aligned}$$

and so on.

Remark 2.6. We can also consider the model of population of the form

$$\begin{gathered} x^{\Delta}(t) = \gamma(t)x(t) + \frac{x(d(t))}{\mu(t)}e^{r(t)\mu(t)\left(1-\frac{x(d(t))}{k(t)}\right)},\ t \geqslant t_\beta \\ x(t_0) = x_0,\ x(t_1) = x_1,\ \ldots,\ x(t_{\beta-1}) = x_{\beta-1} \end{gathered} \tag{2.4}$$

i.e., with β initial values, where β depends on the delay function d.

Throughout this paper, we consider the following general assumptions on equation (2.2):

(A1) There exist γ_0 and γ_1 in $(0, 1)$ such that

$$\inf_{t\in\mathbb{T}} \Big(1 + \mu(t)\gamma(t)\Big) = \gamma_0 \ \text{ and } \ \sup_{t\in\mathbb{T}} \Big(1 + \mu(t)\gamma(t)\Big) = \gamma_1.$$

(A2) There exist constants r_i, k_i in $(0, \infty)$ for $i = 0, 1$, such that

$$\inf_{t\in\mathbb{T}} r(t)\mu(t) = r_0,\ \ \sup_{t\in\mathbb{T}} r(t)\mu(t) = r_1,\ \ \inf_{t\in\mathbb{T}} k(t) = k_0 \ \text{ and } \ \sup_{t\in\mathbb{T}} k(t) = k_1.$$

In sequel, we introduce the following notation which will be important to our purposes:

(A3) Let the functions L and U be defined as follows

$$L(u) = ue^{r_0 - \frac{r_1 u}{k_0}} \text{ and } U(u) = ue^{r_1 - \frac{r_0 u}{k_1}} \text{ for } u \geqslant 0$$

and the constants M and m be given by

$$M = U\Big(\frac{k_1}{r_0}\Big) = \frac{k_1}{r_0}e^{r_1 - 1} \text{ and } m = L\Big(\frac{M}{1-\gamma_1}\Big).$$

(A4) For $\delta \geqslant 1$, we set a constant B as follows

$$B = \min\Big(L\Big(\frac{\delta M}{1-\gamma_1}\Big), \frac{r_0 k_0}{r_1}\Big).$$

(A5) Let constant $\widetilde{m}$ be given by

$$\widetilde{m} = \min\left\{m, \frac{r_0 k_0}{r_1}\right\}.$$

It is not difficult to check that M is the maximum value of the function U and $\frac{r_0 k_0}{r_1}$ is the fixed point of the function L.

3. Result

3.1. Stability

In this section, our goal is to investigate the stability of (2.2). We start by recalling some important definitions.

Definition 3.1. A set $S \subset \mathbb{R}$ is said to be *invariant relative* to (2.2) if for any positive value $x(t_0)$ such that $x(t_0) \in S$, the solution x of (2.2) satisfies $x(t) \in S$ for all $t \geqslant t_0$.

Definition 3.2. A set $S \subset \mathbb{R}$ is said to be a *global attractor* of (2.2) if for any $\varepsilon > 0$ and positive value of $x(t_0)$, there exists an element $T(\varepsilon, x(t_0)) \in \mathbb{T}$ such that the solution x of (2.2) satisfies

$$\min_{s\in S} |x(t) - s| < \varepsilon \text{ for all } t \geqslant T(\varepsilon, x(t_0)).$$

Definition 3.3. Equation (2.2) is said to be *extremely stable* if for any two positive solutions x and y of (2.2), we have

$$\lim_{t\to\infty} |x(t) - y(t)| = 0.$$

Remark 3.4. If (2.1) is fulfilled, any function x can be represented as a sequence $\{x(\sigma^n(t_0))\}_{n\in\mathbb{N}}$, so we can reformulate the above definitions as follows.

A set $S \subset \mathbb{R}$ is said to be *invariant relative to* (2.2) if for any positive value $x(t_0)$ belonging to S, the solution $\{x(\sigma^n(t_0))\}_{n\in\mathbb{N}}$ of (2.2) satisfies

$$x(\sigma^n(t_0)) \in S \text{ for all } n \in \mathbb{N}.$$

A set $S \subset \mathbb{R}$ is said to be a *global attractor* of (2.2) if for any $\varepsilon > 0$ and positive value of $x(t_0)$, there exists a natural number $N(\varepsilon, x(t_0))$ such that the solution $\{x(\sigma^n(t_0))\}_{n\in\mathbb{N}}$ of (2.2) satisfies

$$\min_{s\in S} |x(\sigma^n(t_0)) - s| < \varepsilon \text{ for all } n \geqslant N(\varepsilon, x(t_0)).$$

Equation (2.2) is said to be *extremely stable* if for any two positive solutions $\{x(\sigma^n(t_0))\}_{n\in\mathbb{N}}$ and $\{y(\sigma^n(t_0))\}_{n\in\mathbb{N}}$ of (2.2) we have

$$\lim_{n\to\infty} |x(\sigma^n(t_0)) - y(\sigma^n(t_0))| = 0.$$

Lemma 3.5. *Any solution x of* (2.2) *satisfies*

$$\begin{aligned} x(\sigma^n(t)) &= \prod_{k=0}^{n-1}\Big(1+\mu(\sigma^k(t))\gamma(\sigma^k(t))\Big)x(t) \\ &+ \sum_{k=0}^{n-1}\Big[\prod_{j=k+1}^{n-1}\Big(1+\mu(\sigma^j(t))\gamma(\sigma^j(t))\Big)\Big]g(\sigma^k(t), x(d(\sigma^k(t)))), \end{aligned} \tag{3.1}$$

where

$$g(v,u) = ue^{r(v)\mu(v)\left(1-\frac{u}{k(v)}\right)}. \tag{3.2}$$

Proof. Let us prove by induction. Since x is a solution of (2.2), we have

$$\frac{x(\sigma(t)) - x(t)}{\mu(t)} = \gamma(t)x(t) + \frac{x(d(t))}{\mu(t)}e^{r(t)\mu(t)\left(1-\frac{x(d(t))}{k(t)}\right)}.$$

It implies immediately that (3.1) holds for $n = 1$.

Suppose now that (3.1) holds for n. Let us show that it also happens for $n + 1$. Hence,

$$\begin{aligned} x(\sigma^{n+1}(t)) &= x(\sigma^n(\sigma(t))) \\ &= \prod_{k=0}^{n-1}\Big(1+\mu(\sigma^{k+1}(t))\gamma(\sigma^{k+1}(t))\Big)x(\sigma(t)) \\ &+ \sum_{k=0}^{n-1}\Big[\prod_{j=k+1}^{n-1}\Big(1+\mu(\sigma^{j+1}(t))\gamma(\sigma^{j+1}(t))\Big)\Big]g(\sigma^{k+1}(t), x(d(\sigma^{k+1}(t)))) \\ &= \prod_{k=1}^{n}\Big(1+\mu(\sigma^k(t))\gamma(\sigma^k(t))\Big)x(\sigma(t)) \\ &+ \sum_{k=1}^{n}\Big[\prod_{j=k+2}^{n}\Big(1+\mu(\sigma^j(t))\gamma(\sigma^j(t))\Big)\Big]g(\sigma^k(t), x(d(\sigma^k(t)))). \end{aligned}$$

Using the definition of $x(\sigma(t))$ given by the case $n = 1$ and replacing in the above equation, we have

$$\begin{aligned} x(\sigma^{n+1}(t)) &= \prod_{k=1}^{n}\Big(1+\mu(\sigma^k(t))\gamma(\sigma^k(t))\Big)\Big[\Big(1+\mu(t)\gamma(t)\Big)x(t) + g(t, x(d(t)))\Big] \\ &+ \sum_{k=1}^{n}\Big[\prod_{j=k+2}^{n}\Big(1+\mu(\sigma^j(t))\gamma(\sigma^j(t))\Big)\Big]g(\sigma^k(t), x(d(\sigma^k(t)))) \\ &= \prod_{k=0}^{n}\Big(1+\mu(\sigma^k(t))\gamma(\sigma^k(t))\Big)x(t) \\ &+ \sum_{k=0}^{n}\Big[\prod_{j=k+1}^{n}\Big(1+\mu(\sigma^j(t))\gamma(\sigma^j(t))\Big)\Big]g(\sigma^k(t), x(d(\sigma^k(t)))), \end{aligned}$$

getting the desired result. □

From assumption (A1) and (3.1), it follows that positive value of $x(t_0)$ implies solution x of (2.2) takes positive values only.

By assumption (A2), we get

$$L(u) \leqslant g(v,u) \leqslant U(u) \quad \text{for} \quad v \in [t_0,\infty)_{\mathbb{T}} \quad \text{and} \quad u \geqslant 0. \tag{3.3}$$

In consequence,

$$g(v,u) \leqslant M \quad \text{for} \quad v \in [t_0,\infty)_{\mathbb{T}} \quad \text{and} \quad u \geqslant 0. \tag{3.4}$$

From (3.1) and (3.4), due to assumption (A1), we have

$$\begin{aligned} x(\sigma^n(t_0)) &\leqslant \gamma_1^n x(t_0) + M \sum_{k=0}^{n-1} \gamma_1^{n-k-1} \\ &= \gamma_1^n x(t_0) + M \frac{1-\gamma_1^n}{1-\gamma_1} \end{aligned} \tag{3.5}$$

for any $n \in \mathbb{N}$. Note that for $x(t_0) \in \left(0, \frac{\delta M}{1-\gamma_1}\right]$ with $\delta \geqslant 1$, we get from (3.5)

$$\begin{aligned} x(\sigma^n(t_0)) &\leqslant \frac{\gamma_1^n \delta M}{1-\gamma_1} + \frac{M - M\gamma_1^n}{1-\gamma_1} \\ &= \frac{\gamma_1^n(\delta M - M) + M}{1-\gamma_1} \\ &\leqslant \frac{\delta M - M + M}{1-\gamma_1} \\ &= \frac{\delta M}{1-\gamma_1}. \end{aligned} \tag{3.6}$$

Using the notations from (A3)–(A4) and by properties of function L, we obtain

$$L(u) \geqslant B \quad \text{for} \quad u \in \Big[\frac{B}{1-\gamma_0}, \frac{\delta M}{1-\gamma_1}\Big],$$

where B is defined in (A4). By inequality (3.3), we get

$$g(v,u) \geqslant B \quad \text{for} \quad v \in [t_0,\infty)_{\mathbb{T}} \quad \text{and} \quad u \in \Big[\frac{B}{1-\gamma_0}, \frac{\delta M}{1-\gamma_1}\Big]. \tag{3.7}$$

Assuming $x(t_0) \geqslant \frac{B}{1-\gamma_0}$, it follows by Lemma 3.5, inequality (3.7) and assumption (A1), the following inequality

$$\begin{aligned} x(\sigma^n(t_0)) &\geqslant \left(\prod_{k=0}^{n-1} \gamma_0\right) x(t_0) + \left(\sum_{k=0}^{n-1} \left(\prod_{j=k+1}^{n-1} \gamma_0\right) g(\sigma^k(t_0), x(d(\sigma^k(t_0))))\right) \\ &\geqslant \prod_{k=0}^{n-1} \gamma_0 \frac{B}{1-\gamma_0} + \sum_{k=0}^{n-1} \gamma_0^{n-1-k} B \\ &= \gamma_0^n \frac{B}{1-\gamma_0} + B\left(\frac{1-\gamma_0^n}{1-\gamma_0}\right) \end{aligned}$$

$$= \frac{B}{1-\gamma_0} \tag{3.8}$$

for $n \in \mathbb{N}$. Hence, inequalities (3.6) and (3.8) allow us to formulate the following theorem.

Theorem 3.6. *If conditions* (A1)–(A4) *hold, then the set* $\left[\frac{B}{1-\gamma_0}, \frac{\delta M}{1-\gamma_1}\right]$ *is invariant relative to* (2.2) *for positive values of* $x(t_0)$*, where* $\delta \geqslant 1$ *and* B *is given in* (A4).

The next theorem brings the statement on a global attractor.

Theorem 3.7. *Under assumptions* (A1)–(A3) *and* (A5)*, the set* $\left[\frac{\widetilde{m}}{1-\gamma_0}, \frac{M}{1-\gamma_1}\right]$ *is a global attractor of* (2.2) *for positive values of* $x(t_0)$.

Proof. Since γ_0 and γ_1 belong to $(0,1)$, for any $\varepsilon > 0$ and positive value $x(t_0)$, there exists an integer $N(\varepsilon, x(t_0)) > 0$ such that

$$\gamma_1^n \left| x(t_0) - \frac{M}{1-\gamma_1} \right| < \varepsilon \text{ and } \gamma_0^n \left| x(t_0) - \frac{\widetilde{m}}{1-\gamma_0} \right| < \varepsilon \text{ for any } n \geqslant N(\varepsilon, x(t_0)).$$

It implies

$$\gamma_1^n |x(t_0)| < \varepsilon + \left| \frac{\gamma_1^n M}{1-\gamma_1} \right| = \varepsilon + \frac{M\gamma_1^n}{1-\gamma_1}.$$

Applying the above to (3.5), we obtain

$$\begin{aligned} x(\sigma^n(t_0)) &< \varepsilon + \frac{M\gamma_1^n}{1-\gamma_1} + \frac{M}{1-\gamma_1} - \frac{M\gamma_1^n}{1-\gamma_1} \\ &= \varepsilon + \frac{M}{1-\gamma_1} \text{ for any } n \geqslant N(\varepsilon, x(t_0)). \end{aligned}$$

In analogous way, we get

$$x(\sigma^n(t_0)) > \frac{\widetilde{m}}{1-\gamma_0} - \varepsilon \text{ for } n \geqslant N(\varepsilon, x(t_0)),$$

which concludes the proof. □

3.2. Extreme stability

In this section, we are interested to investigate the extreme stability of equation (2.2).

Lemma 3.8. *Let the assumptions* (A1)–(A3) *and* (A5) *hold. If* x *is a solution of* (2.2) *such that* $x(t_0)$ *is a positive value and*

$$\frac{\widetilde{m}}{k_1(1-\gamma_0)} > 1, \tag{3.9}$$

then

$$\limsup_{t\to\infty} \left| 1 - \frac{r(t)\mu(t)x(d(t))}{k(t)} \right| e^{r(t)\mu(t)(1-\frac{x(d(t))}{k(t)})} \leqslant \max\left\{ \left| 1 - \frac{r_0\widetilde{m}}{k_1(1-\gamma_0)} \right|, \left| 1 - \frac{r_1 M}{k_0(1-\gamma_1)} \right| \right\}.$$

Proof. By Theorem 3.7, for any given solution x of (2.2) and for every $\varepsilon > 0$, there exists $T = T(\varepsilon, x(t_0)) \in \mathbb{T}$ such that

$$\frac{\widetilde{m}}{1-\gamma_0} - \varepsilon < x(d(t)) < \frac{M}{1-\gamma_1} + \varepsilon \ \text{ for } \ t \geqslant T. \tag{3.10}$$

It implies the following estimates

$$1 - \frac{r(t)\mu(t)x(d(t))}{k(t)} < 1 - \frac{r_0}{k_1}\Big(\frac{\widetilde{m}}{1-\gamma_0} - \varepsilon\Big) \tag{3.11}$$

and

$$1 - \frac{r(t)\mu(t)x(d(t))}{k(t)} > 1 - \frac{r_1}{k_0}\Big(\frac{M}{1-\gamma_1} + \varepsilon\Big) \tag{3.12}$$

for $t \geqslant T$ such that $t \in \mathbb{T}$. Since ε is arbitrary, we can write

$$\limsup_{t\to\infty}\Big|1 - \frac{r(t)\mu(t)x(d(t))}{k(t)}\Big| \leqslant \max\Big\{\Big|1 - \frac{r_0\widetilde{m}}{k_1(1-\gamma_0)}\Big|, \Big|1 - \frac{r_1 M}{k_0(1-\gamma_1)}\Big|\Big\}.$$

Now, it remains to show that

$$\limsup_{t\to\infty} e^{r(t)\mu(t)(1-\frac{x(d(t))}{k(t)})} \leqslant 1 \tag{3.13}$$

to conclude the proof. Observe that the left–hand side of inequality in formula (3.10) combined with (3.9) implies that

$$\frac{x(d(t))}{k(t)} > \frac{\widetilde{m}}{k_1(1-\gamma_0)} - \frac{\varepsilon}{k_0} > 1 - \frac{\varepsilon}{k_0} \ \text{ for any } t \geqslant T.$$

In consequence,

$$\liminf_{t\to\infty} \frac{x(d(t))}{k(t)} \geqslant 1.$$

This ends the proof. □

Theorem 3.9. *Let assumptions* (A1)–(A3) *and* (A5) *hold. If condition* (3.9) *is satisfied and*

$$\max\Big\{\Big|1 - \frac{r_0\widetilde{m}}{k_1(1-\gamma_0)}\Big|, \Big|1 - \frac{r_1 M}{k_0(1-\gamma_1)}\Big|\Big\} < 1 - \gamma_1, \tag{3.14}$$

then equation (2.2) *is extremely stable.*

Proof. Let x and y be arbitrary positive solutions of (2.2). Since x and y satisfy (2.2) for all $t \in \mathbb{T}$, we have by Lemma 3.5

$$\begin{aligned} x(\sigma^n(t)) \ &= \ \prod_{k=0}^{n-1}\Big(1 + \mu(\sigma^k(t))\gamma(\sigma^k(t))\Big)x(t) \\ &+ \ \sum_{k=0}^{n-1}\Big[\prod_{j=k+1}^{n-1}\Big(1 + \mu(\sigma^j(t))\gamma(\sigma^j(t))\Big)\Big]g(\sigma^k(t), x(d(\sigma^k(t)))) \end{aligned}$$

and

$$\begin{aligned} y(\sigma^n(t)) &= \prod_{k=0}^{n-1}\Big(1+\mu(\sigma^k(t))\gamma(\sigma^k(t))\Big)y(t) \\ &+ \sum_{k=0}^{n-1}\Big[\prod_{j=k+1}^{n-1}\Big(1+\mu(\sigma^j(t))\gamma(\sigma^j(t))\Big)\Big]g(\sigma^k(t),y(d(\sigma^k(t)))) \end{aligned}$$

for any $t \in \mathbb{T}$ and $n \in \mathbb{N}$. In consequence, we have

$$\begin{aligned} x(\sigma^n(t)) - y(\sigma^n(t)) &= \prod_{k=0}^{n-1}\Big(1+\mu(\sigma^k(t))\gamma(\sigma^k(t))\Big)(x(t)-y(t)) \\ &+ \sum_{i=0}^{n-1}\Big[\prod_{j=i+1}^{n-1}\Big(1+\mu(\sigma^j(t))\gamma(\sigma^j(t))\Big)\Big]\Big\{x(d(\sigma^i(t)))e^{\widetilde{r}(\sigma^i(t))\left(1-\frac{x(d(\sigma^i(t)))}{k(\sigma^i(t))}\right)} \\ &\quad - y(d(\sigma^i(t)))e^{\widetilde{r}(\sigma^i(t))\left(1-\frac{y(d(\sigma^i(t)))}{k(\sigma^i(t))}\right)}\Big\} \text{ for all } n \in \mathbb{N}, \end{aligned}$$

where $\widetilde{r} = r\mu$. Applying assumption (A1) and Mean Value Theorem to the above, we get the following estimate

$$\begin{aligned} &|x(\sigma^n(t)) - y(\sigma^n(t))| \leqslant \gamma_1^n|x(t)-y(t)| + \\ &+ \sum_{i=0}^{n-1}\gamma_1^{n-i-1}\left|1-\frac{\widetilde{r}(\sigma^i(t))\eta(d(\sigma^i(t)))}{k(\sigma^i(t))}\right|e^{\widetilde{r}(\sigma^i(t))\left(1-\frac{\eta(d(\sigma^i(t)))}{k(\sigma^i(t))}\right)}|x(d(\sigma^i(t))) - y(d(\sigma^i(t)))| \end{aligned} \tag{3.15}$$

for all $n \in \mathbb{N}$, where $\eta(d(\sigma^i(t)))$ is between $x(d(\sigma^i(t)))$ and $y(d(\sigma^i(t)))$ for $i = 0, 1, \ldots, n-1$. By condition (3.14), there exists real number M_1 such that

$$\max\left\{\left|1-\frac{r_0\widetilde{m}}{k_1(1-\gamma_0)}\right|, \left|1-\frac{r_1 M}{k_0(1-\gamma_1)}\right|\right\} < M_1 < 1-\gamma_1. \tag{3.16}$$

Hence, by Lemma 3.8 and by the definition of η, there exists $t_2 \in \mathbb{T}$ such that for $t \geqslant t_2$, we have

$$\left|1-\frac{\widetilde{r}(\sigma^i(t))\eta(d(\sigma^i(t)))}{k(\sigma^i(t))}\right|e^{\widetilde{r}(\sigma^i(t))\left(1-\frac{\eta(d(\sigma^i(t)))}{k(\sigma^i(t))}\right)} \leqslant M_1 \quad \text{for all } i \in \mathbb{N}. \tag{3.17}$$

On the other hand, for any $t \in \mathbb{T}$, due to Theorems 3.6 and 3.7, the sequence

$$\{x(\sigma^n(t)) - y(\sigma^n(t))\}_{n\in\mathbb{N}}$$

is bounded. Hence, there exists $a \geqslant 0$ such that

$$\limsup_{n\to\infty}|x(\sigma^n(t)) - y(\sigma^n(t))| = a. \tag{3.18}$$

In conclusion, for every $\varepsilon > 0$, there exists $t_3 \in \mathbb{T}$ such that

$$|x(\sigma^n(t)) - y(\sigma^n(t))| < a + \varepsilon \text{ for all } n \in \mathbb{N} \text{ and } t_3 \leqslant t \in \mathbb{T}. \tag{3.19}$$

It is convenient to choose t_3 such that $t_3 \geqslant t_2$, since it implies condition (3.17) also holds. Combining inequalities (3.15), (3.17) and (3.19), we obtain for $t \geqslant t_3$ and for all $n \in \mathbb{N}$

$$|x(\sigma^n(t)) - y(\sigma^n(t))| \leqslant \gamma_1^n |x(t) - y(t)| + \frac{1-\gamma_1^n}{1-\gamma_1} M_1(a+\varepsilon). \tag{3.20}$$

Taking lim sup when $n \to +\infty$ on both sides of the above inequality, we get

$$a \leqslant \frac{1}{1-\gamma_1} M_1(a+\varepsilon).$$

Since inequality (3.16) is satisfied, we have

$$a \leqslant \frac{M_1\varepsilon}{1-\gamma_1-M_1}.$$

By the arbitrariness of ε, we obtain that $a = 0$, obtaining the desired result. □

Remark 3.10. Notice that since $\mathbb{T}$ is an isolated time scale such that $\sup\mathbb{T} = +\infty$, it is clear that $\lim_{n\to\infty} \sigma^n(t) = +\infty$. From this, we can infer by the properties of lim sup that (3.18) also holds for t sufficiently large, obtaining (3.19).

In sequel, we present some examples to illustrate the above results.

Example 3.11. Let

$$\mathbb{T} = \left\{3n + k : n \in \mathbb{N}_0, k \in \left\{0, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, 1\right\}\right\},$$

where $\mathbb{N}_0$ is the set of nonnegative integers. Then $t_0 = 0$ and $\min_{t\in\mathbb{T}} \mu(t) = \frac{1}{5}$. Consider equation (2.2) with $d = \rho^2$,

$$\gamma(t) = \begin{cases} -0.4 & \text{if } t \in \left\{3n+1 : n \in \mathbb{N}_0\right\} \\ -3.4 & \text{if } t \in \left\{3n+k : n \in \mathbb{N}_0, k \in \left\{0, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}\right\}\right\}, \end{cases}$$

$$r(t) = \begin{cases} 0.425 & \text{if } t \in \left\{3n+1 : n \in \mathbb{N}_0\right\} \\ 0.405 & \text{if } t \in \left\{3n+k : n \in \mathbb{N}_0, k \in \left\{0, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}\right\}\right\}, \end{cases}$$

and

$$k(t) = \begin{cases} 10 & \text{if } t \in \left\{3n+1 : n \in \mathbb{N}_0\right\} \\ 9 + t - [t] & \text{if } t \in \left\{3n+k : n \in \mathbb{N}_0, k \in \left\{0, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}\right\}\right\}. \end{cases}$$

Hence

$$\gamma_0 = 0.2,\ \gamma_1 = 0.32,\ r_0 = 0.81,\ r_1 = 0.85,\ k_0 = 9,\ k_1 = 10.$$

Calculating

$$\begin{cases} M = U\left(\frac{k_1}{r_0}\right) = \frac{k_1}{r_0} e^{r_1 - 1} \approx 10.62602; \\ m = L\left(\frac{M}{1-\gamma_1}\right) \approx 8.02958; \\ \frac{r_0 k_0}{r_1} \approx 8.57647; \end{cases}$$

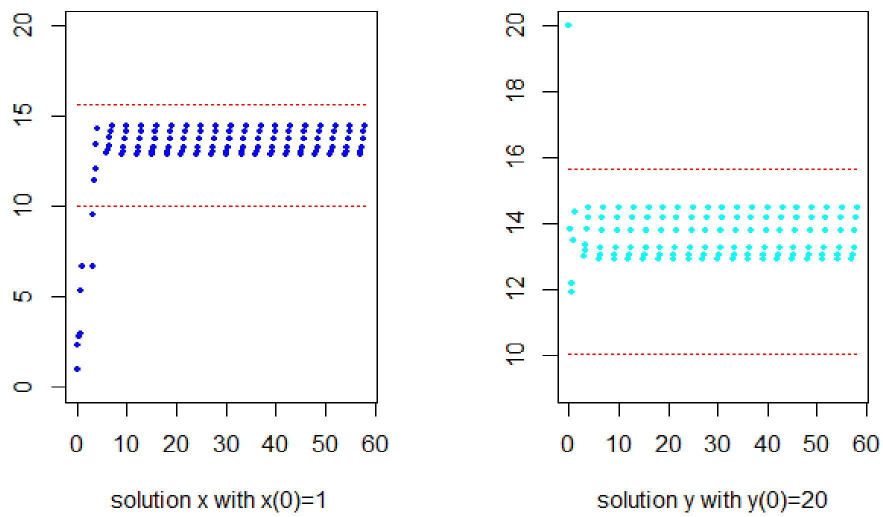

Figure 1. Example 3.11 - the part of plot of two chosen solutions of (2.2) for $n = 120$ points.

we get $\widetilde{m} = \min\left\{m, \frac{r_0 k_0}{r_1}\right\} = m$. By Theorem 3.7, interval $\left[\frac{\widetilde{m}}{1-\gamma_0}, \frac{M}{1-\gamma_1}\right] \approx \left[10.03697, 15.62651\right]$ is a global attractor of (2.2). Figure 1 shows behavior of two solutions x and y with positive initial values $x(t_0) = 1$ and $y(t_0) = 20$, respectively. The range of the global attractor is illustrated by red dotted lines. Hence,

$$\frac{\widetilde{m}}{k_1(1-\gamma_0)} \approx 1.00370$$

and

$$\max\left\{\left|1 - \frac{r_0 \widetilde{m}}{k_1(1-\gamma_0)}\right|, \left|1 - \frac{r_1 M}{k_0(1-\gamma_1)}\right|\right\} \approx 0.47584 < 0.68 = 1 - \gamma_1.$$

Therefore, conditions (3.9) and (3.14) are satisfied. Theorem 3.9 implies that (2.2) is extremely stable. In Figure 2, difference of two solutions x and y with initial conditions $x(t_0) = 1$ and $y(t_0) = 20$ is shown, confirming that (2.2) is extremly stable.

Example 3.12. Let $\mathbb{T} = q^{\mathbb{N}}$, where $q = 1.1$, and consider equation (2.2) with

$$d(t) = \begin{cases} \rho^2(t) & \text{if } t \in \left\{q^{2n} : n \in \mathbb{N}\right\} \\ \rho(t) & \text{if } t \in \left\{q^{2n-1} : n \in \mathbb{N}\right\}, \end{cases}$$

$$\gamma(t) = \begin{cases} \frac{-0.75}{t(q-1)} & \text{if } t \in \left\{q^{2n-1} : n \in \mathbb{N}\right\} \\ \frac{-0.65}{t(q-1)} & \text{if } t \in \left\{q^{2n} : n \in \mathbb{N}\right\}, \end{cases}$$

$$r(t) = \begin{cases} \frac{0.35}{t(q-1)} & \text{if } t \in \left\{q^{2n-1} : n \in \mathbb{N}\right\} \\ \frac{0.45}{t(q-1)} & \text{if } t \in \left\{q^{2n} : n \in \mathbb{N}\right\}, \end{cases}$$

and

$$k(t) = 14 + \sin(t\pi).$$

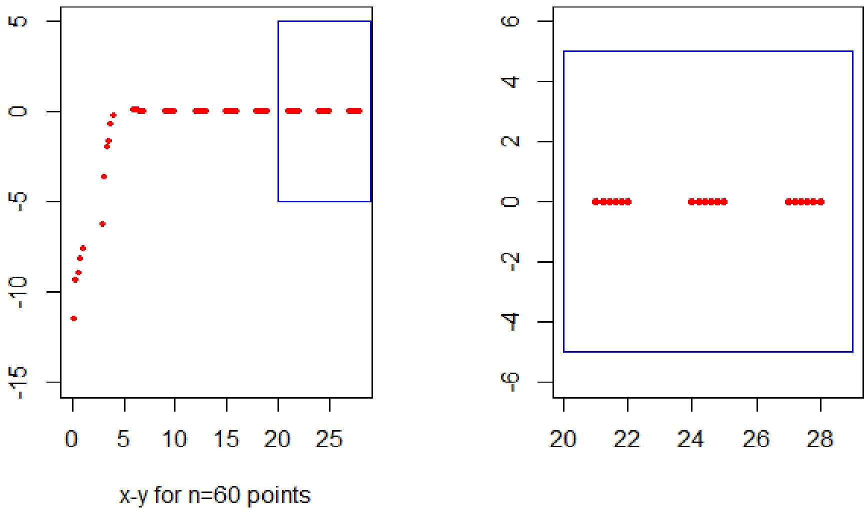

Figure 2. Example 3.11 - difference of solutions x and y.

Then constants introduced by assumption (A1)–(A2) are following

$$\gamma_0 = 0.25, \ \gamma_1 = 0.35, \ r_0 = 0.35, \ r_1 = 0.45, \ k_0 = 13, \ k_1 = 15.$$

Hence

$$M \approx 20.21769, \quad m \approx 14.74683 \quad \text{and} \quad \widetilde{m} = 11.7.$$

By Theorem 3.7, interval $\left[\frac{\widetilde{m}}{1-\gamma_0}, \frac{M}{1-\gamma_1}\right] \approx \left[15.6, 31.10414\right]$ is a global attractor of (2.2). We check that

$$\frac{\widetilde{m}}{k_1(1-\gamma_0)} = 1.04 > 1,$$

and

$$\max\left\{\left|1 - \frac{r_0\widetilde{m}}{k_1(1-\gamma_0)}\right|, \left|1 - \frac{r_1 M}{k_0(1-\gamma_1)}\right|\right\} = 0.532 < 0.65 = 1 - \gamma_1.$$

Thus assumptions of Theorem 3.9 are satisfied. Hence equation (2.2) is extremely stable. Figure 3 shows behavior of the solutions x and y with initial values $x(t_0) = 1$ and $y(t_0) = 37$ (for 120 points from the time scale). Difference of those solutions is illustrated in Figure 4.

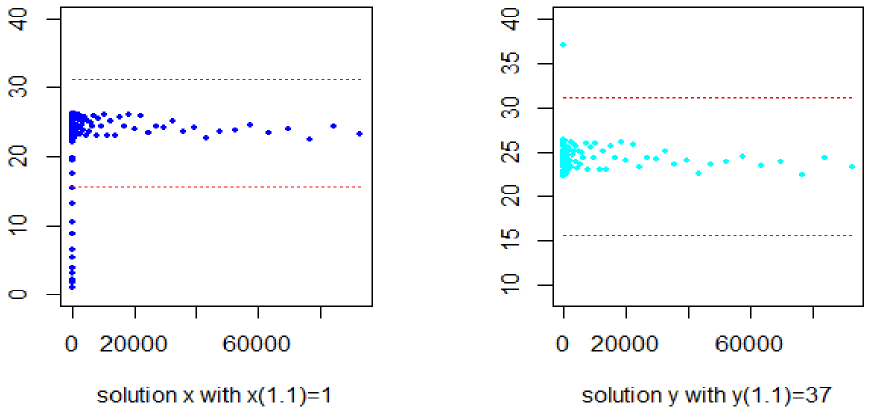

Figure 3. Example 3.12 - the part of plot of two chosen solutions of (2.2) for $n = 120$ points.

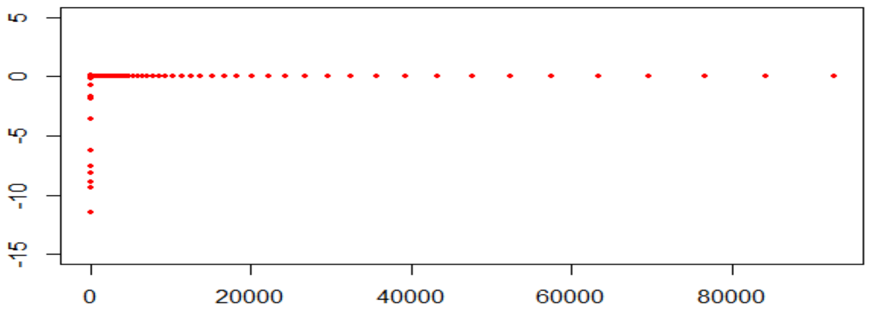

Figure 4. Example 3.12 - the plot of $x - y$ for $n = 120$ points.

3.3. Periodicity

In this section, our goal is to investigate the existence of ω–periodic solutions and asymptotically ω–periodic solutions of (3.1), using the new concept of periodicity on isolated time scales introduced in [17].

Let us start by recalling the idea of periodicity on isolated time scales introduced in [17].

Definition 3.13. A function $f : \mathbb{T} \to \mathbb{R}$ is called ω*–periodic* if

$$\nu^{\Delta} f^{\nu} = f,$$

where $\nu = \sigma^{\omega}$.

Since condition (2.1) is satisfied, $\mathbb{T}$ contains only isolated points and

$$\nu^{\Delta}(t) = \frac{\nu(\sigma(t)) - \nu(t)}{\mu(t)} = \frac{\sigma(\nu(t)) - \nu(t)}{\mu(t)} = \frac{\mu(\nu(t))}{\mu(t)}$$

(see [17]). Therefore, we can formulate the following equivalent condition of $\omega-$periodicity which can be found in [17].

Lemma 3.14. *A function $f:\mathbb{T}\to\mathbb{R}$ is $\omega-$periodic if and only if $(\mu f)^{\nu}=\mu f$.*

Remark 3.15. Observe that for $\mathbb{T}=\mathbb{Z}$ we have $\mu(t)=1$, $\nu(t)=t+\omega$ and in this case, $\omega-$periodicity condition given in Lemma 3.14 takes the known form

$$f(t+\omega)=f(t)\ \text{ for all }\ t\in\mathbb{Z}.$$

When $\mathbb{T}=2^{\mathbb{N}_0}$, then one can check that function f is $\omega-$periodic if

$$2^{\omega}f(2^{\omega}t)=f(t)\ \text{ for all }\ t\in 2^{\mathbb{N}_0},$$

reaching the ω–periodicity for the quantum case. See [4, 5, 10, 11].

Definition 3.16. A function $f:\mathbb{T}\to\mathbb{R}$ is said to be *asymptotically $\omega-$periodic* (or *asymptotically $\omega-$periodic* for $t\geqslant t_1$) if there exist two functions $p,q:\mathbb{T}\to\mathbb{R}$ such that

$$f(t)=p(t)+q(t),$$

where $p(t)$ is $\omega-$periodic (or $\omega-$periodic for $t\geqslant t_1$) and $q(t)\to 0$ as $t\to\infty$.

As in the previous section, assume $\widetilde{m}=\min\{m,\frac{r_0k_0}{r_1}\}$. The next result follows the same way as the proof of Lemma 3.8. Therefore, we omit its proof here.

Lemma 3.17. *Assume* (A1)–(A3) *are satisfied. If $x:\mathbb{T}\to\mathbb{R}$ is such that*

$$\sup_{t\in\mathbb{T}}|x(t)|\in\Big[\frac{\widetilde{m}}{1-\gamma_0},\frac{M}{1-\gamma_1}\Big]$$

and

$$\frac{\widetilde{m}}{k_1(1-\gamma_0)}\geqslant 1, \tag{3.21}$$

then for all $t\in\mathbb{T}$, the inequality

$$\Big|1-\frac{r(t)\mu(t)x(d(t))}{k(t)}\Big|e^{r(t)\mu(t)\left(1-\frac{x(d(t))}{k(t)}\right)}\leqslant\max\Big\{\Big|1-\frac{r_0\widetilde{m}}{k_1(1-\gamma_0)}\Big|,\Big|1-\frac{r_1M}{k_0(1-\gamma_1)}\Big|\Big\}$$

holds.

Lemma 3.18. *Suppose conditions* (A1)–(A3), (A5), (3.14) *and* (3.21) *hold. If $x:\mathbb{T}\to\mathbb{R}$ is an asymptotically $\omega-$periodic function, $r,k:\mathbb{T}\to\mathbb{R}$ are $\omega-$periodic functions and there exists $t_1\in\mathbb{T}$ such that for any $\omega-$periodic function $p:\mathbb{T}\to\mathbb{R}$, $p\circ d$ is also $\omega-$periodic for all $t\geqslant t_1$, then $g(t,x(d(t)))$ defined by* (3.2) *is an asymptotically $\omega-$periodic function for $t\geqslant t_1$.*

Proof. Since x is asymptotically $\omega-$periodic, it can be decomposed by

$$x(t)=p(t)+q(t),$$

where p is $\omega-$periodic and $\lim_{t\to\infty} q(t) = 0$. Applying Mean Value Theorem, we obtain the following inequality

$$\begin{aligned} |g(t, x(d(t))) - g(t, p(d(t)))| &= |g(t, x(d(t))) - p(d(t))e^{r(t)\mu(t)\left(1-\frac{p(d(t))}{k(t)}\right)}| \\ &\leqslant \left|1 - \frac{r(t)\mu(t)\xi}{k(t)}\right| e^{r(t)\mu(t)\left(1-\frac{\xi}{k(t)}\right)} |x(d(t)) - p(d(t))|, \end{aligned} \tag{3.22}$$

where ξ is between $x(d(t))$ and $p(d(t))$. By Lemma 3.17, inequality (3.14) and asymptotic $\omega-$periodicity of x, the right hand side of (3.22) tends to 0 if $t \to \infty$. On the other hand, notice that

$$\begin{aligned} &\mu(\nu(t))p(d(\nu(t)))e^{\mu(\nu(t))r(\nu(t))\left(1-\frac{p(d(\nu(t)))}{k(\nu(t))}\right)} \\ &= \mu(\nu(t))p(d(\nu(t)))e^{\mu(\nu(t))r(\nu(t))\left(1-\frac{p(d(\nu(t)))\mu(\nu(t))}{k(\nu(t))\mu(\nu(t))}\right)} \\ &= \mu(t)p(d(t))e^{\mu(t)r(t)\left(1-\frac{p(d(t))\mu(t)}{k(t)\mu(t)}\right)} \\ &= \mu(t)p(d(t))e^{\mu(t)r(t)\left(1-\frac{p(d(t))}{k(t)}\right)} \end{aligned}$$

for $t \geqslant t_1$ since by assumption there exists $t_1 \in \mathbb{T}$ such that $p \circ d$ is $\omega-$periodic for $t \geqslant t_1$. Thus, by Lemma 3.14, the function $p(d(t))e^{r(t)\mu(t)\left(1-\frac{p(d(t))}{k(t)}\right)}$ is $\omega-$periodic for $t \geqslant t_1$, proving the lemma. □

The proof of the next result follows the same way as the proof of the previous result. Thus, we omit it here.

Corollary 3.19. *Suppose $r, k : \mathbb{T} \to \mathbb{R}$ are ω–periodic functions and for any ω–periodic function $p : \mathbb{T} \to \mathbb{R}$, $p \circ d$ is also an ω–periodic function. If $x : \mathbb{T} \to \mathbb{R}$ is an ω–periodic function, then $g(t, x(d(t)))$ is also an ω–periodic function.*

Lemma 3.20. *Assume* (A1) *holds. If $x : \mathbb{T} \to \mathbb{R}$ is an asymptotically $\omega-$periodic function and $\gamma : \mathbb{T} \to \mathbb{R}$ is an $\omega-$periodic function, then function $\mu\gamma x$ is an asymptotically $\omega-$periodic function.*

Proof. Since x is an asymptotically ω–periodic function, it can be decomposed by

$$x(t) = p(t) + q(t),$$

where p is $\omega-$periodic and $\lim\limits_{t\to\infty} q(t) = 0$, which implies

$$\mu\gamma x = \mu\gamma p + \mu\gamma q.$$

By (A1), $\mu\gamma$ is a bounded function, which implies that there exists $\lim\limits_{t\to\infty} \mu(t)\gamma(t)q(t) = 0$. Thus, it remains to show that function $\mu\gamma p$ is $\omega-$periodic. By Lemma 3.14, we obtain that $(\mu\gamma)^{\nu} = \mu\gamma$ and $(\mu p)^{\nu} = \mu p$. It implies the following equality

$$(\mu\mu\gamma p)^{\nu} = (\mu\gamma)^{\nu}(\mu p)^{\nu} = \mu\mu\gamma p.$$

Applying Lemma 3.14 again, we get the desired result. □

In the same manner, we can prove the following result.

Corollary 3.21. *Suppose* (A1) *holds. If $x : \mathbb{T} \to \mathbb{R}$ is an asymptotically $\omega-$periodic for $t \geqslant t_1$ and $\gamma : \mathbb{T} \to \mathbb{R}$ is an $\omega-$periodic function, then $\mu\gamma x$ is an asymptotically $\omega-$periodic function for $t \geqslant t_1$.*

To guarantee the existence of an asymptotically $\omega-$periodic solution of (2.2), we apply the Krasnoselskii Fixed Point Theorem.

Theorem 3.22 ([18])**.** *Let $\mathcal{B}$ be a Banach space, let Ω be a bounded, convex and closed subset of $\mathcal{B}$ and let F, G be maps of Ω into $\mathcal{B}$ such that*

(i) $Fx + Gy \in \Omega$ for any $x, y \in \Omega$,
(ii) F is a contraction,
(iii) G is completely continuous.

Then operator $F + G$ has a fixed point in Ω.

Theorem 3.23. *Let conditions* (A1)–(A3)*,* (A5)*,* (3.14) *and* (3.21) *hold. If γ, r and k are $\omega-$periodic functions and there exists $t_1 \in \mathbb{T}$ such that for any $\omega-$periodic function $p : \mathbb{T} \to \mathbb{R}$, $p \circ d$ is also $\omega-$periodic for $t \geqslant t_1$ then there exists $t^* \in \mathbb{T}$ such that equation* (2.2) *has a unique $\omega-$periodic (for $t \geqslant t_1$) solution x and all other solutions are asymptotically $\omega-$periodic.*

Proof. Let $\mathcal{B}(\mathbb{T})$ denote a Banach space of the form

$$\mathcal{B}(\mathbb{T}) := \{x = \{x(t)\}_{t \geqslant t_0} : \sup_{t \in \mathbb{T}} |x(t)| < \infty\}$$

equipped with the norm defined by $\|x\| = \sup_{t \in \mathbb{T}} |x(t)|$. It is not difficult to show that the set

$$\mathcal{B}(\mathbb{T})_{ap} := \{x \in \mathcal{B}(\mathbb{T}) : x \text{ is asymptotically } \omega\text{–periodic for } t \geqslant t_1\}$$

with the supremum norm defined above is also a Banach space. Let us introduce the following subset of $\mathcal{B}(\mathbb{T})_{ap}$

$$\Omega_{ap} := \left\{ x \in \mathcal{B}(\mathbb{T})_{ap} : \frac{\widetilde{m}}{1 - \gamma_0} \leqslant \|x\| \leqslant \frac{M}{1 - \gamma_1} \right\}.$$

Observe that Ω_{ap} is a bounded, convex and closed subset in $\mathcal{B}(\mathbb{T})_{ap}$. Let us define two operators $F, G : \Omega_{ap} \to \mathcal{B}(\mathbb{T})_{ap}$ in the following way

$$(Fx)(t) = \begin{cases} 0, & \text{if } t = t_0 \\ (1 + \mu(\rho(t))\gamma(\rho(t)))x(\rho(t)) + g(\rho(t), x(d(\rho(t)))), & \text{if } t > t_0, \end{cases}$$

where function g is given by (3.2), and

$$(Gx)(t) = \begin{cases} x(t) & \text{if } t = t_0 \\ 0 & \text{if } t > t_0. \end{cases}$$

By (A1), for any $x, y \in \Omega_{ap}$ and for $t > t_0$, we get

$$\begin{aligned} (Fx)(t) + (Gy)(t) &= (1 + \mu(\rho(t))\gamma(\rho(t)))x(\rho(t)) + g(\rho(t), x(d(\rho(t)))) \\ &\leqslant \gamma_1 \frac{M}{1 - \gamma_1} + M \end{aligned}$$

$$= \frac{M}{1-\gamma_1} \tag{3.23}$$

and

$$(Fx)(t) + (Gy)(t) \geqslant \gamma_0 \frac{\widetilde{m}}{1-\gamma_0} + \widetilde{m} = \frac{\widetilde{m}}{1-\gamma_0}. \tag{3.24}$$

Clearly, (3.23) and (3.24) also remain valid for $t = t_0$. In consequence,

$$\frac{\widetilde{m}}{1-\gamma_0} \leqslant \|Fx + Gy\| \leqslant \frac{M}{1-\gamma_1}.$$

By Lemma 3.18 and Corollary 3.21, we obtain that function $Fx + Gy$ is asymptotically $\omega-$periodic for $t \geqslant t_1$, hence $Fx + Gy \in \Omega_{ap}$.

The next step is to show that F is a contraction. Taking any $x, y \in \Omega_{ap}$, we have

$$\begin{aligned} |(Fx)(t) - (Fy)(t)| \;&\leqslant\; |1 + \mu(\rho(t))\gamma(\rho(t))||x(\rho(t)) - y(\rho(t))| \\ &\quad + |g(\rho(t), x(d(\rho(t)))) - g(\rho(t), y(d(\rho(t))))|. \end{aligned}$$

By condition (A1) and Mean Value Theorem, for any $t \in \mathbb{T}$, we get

$$|(Fx)(t) - (Fy)(t)| \leqslant \gamma_1 \|x - y\| + \left|1 - \frac{r(\rho(t))\mu(\rho(t))\xi(d(\rho(t)))}{k(\rho(t))}\right| e^{r(\rho(t))\mu(\rho(t))\left(1 - \frac{\xi(d(\rho(t)))}{k(\rho(t))}\right)} \|x - y\|,$$

where $\xi(d(\rho(t)))$ is between $x(d(\rho(t)))$ and $y(d(\rho(t)))$. By condition (3.14), we can choose an $\varepsilon_0 > 0$ such that

$$\max\left\{\left|1 - \frac{r_0\widetilde{m}}{k_1(1-\gamma_0)}\right|, \left|1 - \frac{r_1 M}{k_0(1-\gamma_1)}\right|\right\} \leqslant 1 - \gamma_1 - \varepsilon_0.$$

Finally, by the arbitrariness of ε_0, condition (3.14) and Lemma 3.17 lead to estimate

$$\|Fx - Fy\| \leqslant (1-\gamma_1)\|x - y\| \;\text{ for all } t \in \mathbb{T},$$

which means that F is a contraction.

To be able to use the Krasnoselskii Fixed Point Theorem, it remains to verify that G is completely continuous. It is evident that $G\Omega_{ap}$ is a bounded subset in $\mathbb{R}$ and this implies that it is relatively compact. Thus, G is completely continuous.

Theorem 3.22 implies the existence of $\widetilde{x} \in \Omega_{ap}$ such that

$$\widetilde{x}(t) = (F\widetilde{x})(t) + (G\widetilde{x})(t) \text{ for all } t \in \mathbb{T}.$$

It can be equivalently rewritten as

$$\widetilde{x}(\sigma(t)) = (1 + \mu(t)\gamma(t))\widetilde{x}(t) + g(t, \widetilde{x}(d(t))) \;\text{ for } t > t_0. \tag{3.25}$$

This means that $\widetilde{x}$ is an asymptotically $\omega-$periodic (for $t \geqslant t_1$) solution of (2.2). Thus, $\widetilde{x}$ has the following decomposition

$$\widetilde{x}(t) = \widetilde{p}(t) + \widetilde{q}(t), \tag{3.26}$$

where $\widetilde{p}(t)$ is $\omega-$periodic for $t \geqslant t_1$ and $\lim_{t\to\infty} \widetilde{q}(t) = 0$. Combining (3.25) and (3.26), we obtain

$$(1 + \mu(t)\gamma(t))\widetilde{x}(t) + g(t, \widetilde{x}(d(t))) = \widetilde{p}(\sigma(t)) + \widetilde{q}(\sigma(t))$$

which implies

$$(1 + \mu(t)\gamma(t))(\widetilde{p}(t) + \widetilde{q}(t)) + g(t, \widetilde{x}(d(t))) - g(t, \widetilde{p}(d(t))) + g(t, \widetilde{p}(d(t))) = \widetilde{p}(\sigma(t)) + \widetilde{q}(\sigma(t)).$$

We claim that

$$\widetilde{p}(\sigma(t)) = (1 + \mu(t)\gamma(t))\widetilde{p}(t) + g(t, \widetilde{p}(d(t))) \tag{3.27}$$

and

$$\widetilde{q}(\sigma(t)) = (1 + \mu(t)\gamma(t))\widetilde{q}(t) + g(t, \widetilde{x}(d(t))) - g(t, \widetilde{p}(d(t))).$$

Firstly, notice that $(1 + \mu(t)\gamma(t))\widetilde{p}(t) + g(t, \widetilde{p}(d(t)))$ is ω–periodic. Indeed, by Lemma 3.14, we get for $t \geqslant t_1$

$$\begin{aligned}
&(\mu(t)[(1 + \mu(t)\gamma(t))\widetilde{p}(t) + g(t, \widetilde{p}(d(t)))])^{\nu} \\
&= (\mu(t)\widetilde{p}(t))^{\nu} + (\mu(t)\gamma(t))^{\nu}(\mu(t)\widetilde{p}(t))^{\nu} + (\mu(t)g(t, \widetilde{p}(d(t))))^{\nu} \\
&= \mu(t)\widetilde{p}(t) + \mu(t)\gamma(t)\mu(t)\widetilde{p}(t) + \mu(t)g(t, \widetilde{p}(d(t))) \\
&= \mu(t)(1 + \gamma(t)\mu(t))\widetilde{p}(t) + \mu(t)g(t, \widetilde{p}(d(t))),
\end{aligned}$$

since $g(t, \widetilde{p}(d(t)))$ is ω–periodic, by Corollary 3.19. On the other hand, proceeding the same way as in (3.22) we obtain by applying Mean Value Theorem

$$\begin{aligned}
|g(t, \widetilde{x}(d(t))) - g(t, \widetilde{p}(d(t)))| &= |g(t, \widetilde{x}(d(t))) - \widetilde{p}(d(t))e^{r(t)\mu(t)\left(1-\frac{\widetilde{p}(d(t))}{k(t)}\right)}| \\
&\leqslant \left|1 - \frac{r(t)\mu(t)\xi}{k(t)}\right| e^{r(t)\mu(t)\left(1-\frac{\xi}{k(t)}\right)}|\widetilde{x}(d(t)) - \widetilde{p}(d(t))|,
\end{aligned}$$

where ξ is between $x(d(t))$ an $p(d(t))$. Hence, we get

$$\lim_{t\to\infty} |g(t, \widetilde{x}(d(t))) - g(t, \widetilde{p}(d(t)))| = 0$$

and also, it implies that

$$\lim_{t\to\infty} (1 + \mu(t)\gamma(t))\widetilde{q}(t) + g(t, \widetilde{x}(d(t))) - g(t, \widetilde{p}(d(t))) = 0,$$

since $\widetilde{q}(t) \to 0$ as $t \to \infty$ and $1 + \mu\gamma$ is bounded. Therefore, by the uniqueness of decomposition, the claim follows. By the equality (3.27), we obtain $\widetilde{p}$ is an $\omega-$periodic (for $t \geqslant t_1$) solution of (2.2).

To prove the uniqueness, assume $\widetilde{y}$ is another $\omega-$periodic (for $t \geqslant t_1$) solution of (2.2), then by Theorem 3.9, we have

$$\lim_{t\to\infty} |\widetilde{p}(t) - \widetilde{y}(t)| = 0.$$

This clearly forces $\widetilde{p}(t) = \widetilde{y}(t)$ for $t \geqslant t_1$.

Finally, let x be an arbitrary solution of (2.2), then applying again Theorem 3.9, we have

$$\lim_{t\to\infty} |x(t) - \widetilde{x}(t)| = 0.$$

It implies that

$$x(t) = \widetilde{p}(t) + q(t),$$

with $\lim_{t\to\infty} q(t) = 0$, hence x is an asymptotically $\omega-$periodic solution of (2.2), proving the result. □

Let us illustrate the above result returning to the equation considered in Example 3.11.

Example 3.24. Let

$$\mathbb{T} = \left\{3n + k : n \in \mathbb{N}_0, k \in \left\{0, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, 1\right\}\right\},$$

where $\mathbb{N}_0$ is the set of nonnegative integers. Consider equation (2.2) with $d = \rho^5$ and functions γ, r, k defined as in Example 3.11.

It is easy to check that functions γ, r and k are 6−periodic, since for all $t \in \mathbb{T}$ we have $\mu(t) = \mu(\sigma^6(t))$. This property of graininess function implies that for any 6−periodic function p holds $p = p(\sigma^6)$. In consequence, the composition of any 6−periodic function p with the delay function d, i.e., $p \circ d$ is 6−periodic for $t \geqslant \sigma^5(t_0)$. Since all assumptions of Theorem 3.23 are satisfied equation (2.2) admits unique ω−periodic (for $t \geqslant \sigma^5(t_0)$) solution and all other solutions are asymptotically ω−periodic (see Figure 5).

In general, assuming $d = \rho^j$, we have that if p is 6−periodic function, then $p \circ d$ is 6−periodic for $t \geqslant \sigma^j(t_0)$.

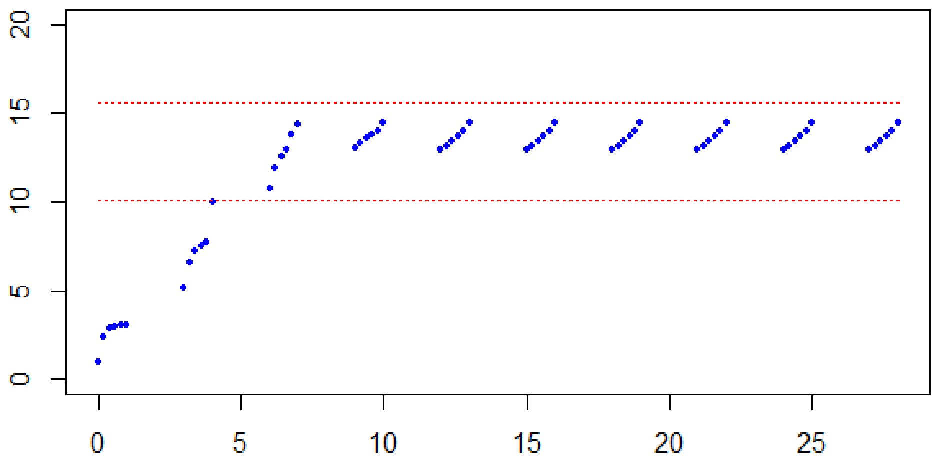

Figure 5. Example 3.24 - the part of plot of solution (with initial value $x(0) = 1$) of (2.2) for $n = 60$ points.

4. Conclusion

In this paper we conducted an analysis of stability, extreme stability and periodicity on all solutions of (2.2). Under certain assumptions, the global attractor of (2.2) for positive initial value is determined. Sufficient conditions for extreme stability of considered equations are given. We also presented conditions under which equation (2.2) has a unique ω-periodic solution and all other solutions are asymptotically ω-periodic.

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Conflict of interest

The authors declare that there are no conflict of interest associated with this publication.

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